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REGULAR ARTICLE

Classical and Bayesian estimation of $P(X < Y)$ using upper record values from Kumaraswamy's distribution

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Abstract In this paper, maximum likelihood and Bayesian approaches have been used to obtain the estimation of $P(X < Y)$ based on a set of upper record values from Kumaraswamy distribution. The existence and uniqueness of the maximum likelihood estimates of the Kumaraswamy distribution parameters are obtained. Confidence intervals, exact and approximate, as well as Bayesian credible intervals are constructed. Bayes estimators have been developed under symmetric (squared error) and asymmetric (LINEX) loss functions using the conjugate and non informative prior distributions. The approximation forms of Lindley (Trabajos de Estadistica 3:281–288, 1980) and Tierney and Kadane (J Am Stat Assoc 81:82–86, 1986) are used for the Bayesian cases. Monte Carlo simulations are performed to compare the different proposed methods.

Keywords Kumaraswamy distribution · Stres-strength model · Record values · Bayes estimation · Symmetric and asymmetric loss functions

Mathematics Subject Classification (2000) 62F10 · 62F15 · 62N05

1 Introduction

Let X and Y be independent random variables, the quantity of $R = P(X < Y)$ is commonly referred as stress-strength parameter or reliability. In the simplest terms this can be described as an assessment of reliability of a component in terms of random

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variables X representing stress experienced by the component and Y representing the strength of the component available to overcome the stress. If the stress exceeds the strength, i.e. $X > Y$, then the component will fail. The main idea was introduced by Birnbaum (1956) and developed by Birnbaum and McCarty (1958). The problem of estimating of R on random samples has been extensively studied under various distributional assumptions on X and Y. A comprehensive account of this topic is presented by Kotz et al. (2003). It is provided an excellent review of the development of the stress-strength under classical and Bayesian point of views up to the year 2003. For most recent results on the topic see Kundu and Gupta (2005), Mokhlis (2005), Baklizi (2008), Rezaei et al. (2010), Nadar et al. (2012) and the references therein.

Record values arise naturally many real life applications involving data relating to meteorology, hydrology, sports and life-tests. In industry and reliability studies, many products may fail under stress. For example, a wooden beam breaks when sufficient perpendicular force is applied to it, an electronic component ceases to function in an environment of too high temperature, and a battery dies under the stress of time. But the precise breaking stress or failure point varies even among identical items. Hence, in such experiments, measurements may be made sequentially and only values smaller (or larger) than all previous ones are recorded. Data of this type are called “Record Data” or “Records”. Thus, the number of measurements made is considerably smaller than the complete sample size. This “measurement saving” can be important when the measurements of these experiments are costly if the entire sample was destroyed. For more examples, see Gulati and Padgett (1994).

Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (iid) random variables with common cumulative distribution function (cdf) $F(x;\theta)$ and probability density function (pdf) $f(x;\theta)$, where $\theta \in \Theta$ could be a vector parameter and Θ is the parameter space. An observation X_j is called an upper record value if it exceeds that of than all previous observations. Thus, X_j is an upper record value if $X_j > X_i$ for all $i < j$. The record time sequence $\{T_n, n \geq 1\}$, at which the records appear, is defined as: $T_n = \min\left\{j : j > T_{n-1}, X_j > X_{T_{n-1}}\right\}, n > 1$, and $T_1 = 1$ with probability 1. By definition X_1 is an upper, as well as a lower, record value. Then the sequence $R_n = X_{T_n}, n \geq 1$ defines a sequence of upper record values. We can give an analogous definition for the lower record values. For more details and references, see Ahsanullah (1995), Arnold et al. (1998) and Nevzorov (2001).

The theory of record values have been extensively studied in the literature. It was first introduced by Chandler (1952). Feller (1966) gave some examples of record values with respect to gambling problems. In recent years there has been a growing interest in the study of inference problems associated with record data. When the underlying distribution is generalized exponential distribution, Bayes and empirical Bayes estimators of the parameter were derived by Jaheen (2004) based on record values. Ahmadi et al. (2006) considered Bayesian estimation for the two parameters of some life distributions, including exponential, Weibull, Pareto and Burr Type XII, based on upper record values. Statistical inference based on record values from the two parameter Pareto distribution was studied by Raqab et al. (2007). Baklizi (2008) studied likelihood and Bayesian estimation of the stress- strength reliability based on lower record values from generalized exponential distribution. Statistical analysis of record values from the Kumaraswamy distribution was done by Nadar et al. (2013).

Ahmadi and Arghami (2001) compared the Fisher information contained in a set of n upper (lower) record values with the Fisher information contained a random sample which consists of n iid observations from the original distribution. They showed that the information contained in the first n record values is greater than that of n iid observations for some families of distributions. Moreover, the comparison of the Shannon information was considered by Madadi and Tata (2011) based on records and random samples.

A random variable X said to have a Kumaraswamy distribution, denoted by $X \sim Kum(a, b)$, if its cdf is

$$F(x; a, b) = 1 - (1 - x^a)^b, \quad 0 < x < 1, \tag{1}$$

and hence the pdf is given by

$$f(x; a, b) = abx^{a-1}(1 - x^a)^{b-1}, \quad 0 < x < 1, \tag{2}$$

where $a > 0$ and $b > 0$ are the shape parameters. It is known that X is Kumaraswamy then $-\ln X$ is the two parameter generalized exponential distribution. Kumaraswamy (1980) developed a more general pdf for double bounded random process with hydrological applications, which is known as Kumaraswamy distribution. Nadarajah (2008) has pointed out that many papers in the hydrological literature have used Kumaraswamy's distribution because it is deemed as a "better alternative" to the beta distribution, see Koutsoyiannis and Xanthopoulos (1989). Jones (2009) explored the background and genesis of the Kumaraswamy distribution, and more importantly, made clear some similarities and differences between the beta and Kumaraswamy distributions. Kumaraswamy distribution has some advantages over the beta distribution in terms of tractability. For example, its cdf has a closed form, the quantile functions are easily obtainable and one can easily generate random variables from Kumaraswamy distribution. This distribution has been studied many authors in hydrology and related areas, see Sundar and Subbiah (1989), Fletcher and Ponnamblam (1996), Seifi et al. (2000), Ponnambalam et al. (2001), and Ganji et al. (2006).

For most statisticians, interested mainly in controlling the amount of variability, it has become standard practice to consider a squared error (SE) loss function, which is symmetrical and gives equal weight to overestimation as well as underestimation. It is well known that the use of symmetric loss functions may be inappropriate in many circumstances, particularly when positive and negative errors have different consequences. The use of asymmetrical loss function, which associates greater importance to overestimation or underestimation, can be considered for the estimation of the parameter. A number of asymmetric loss functions are proposed for use, among these, one of the most popular asymmetric loss functions is linear-exponential loss function (LINEX), was introduced by Varian (1975). The LINEX loss function rises approximately exponentially on one side of zero and approximately linearly on the other side. The LINEX loss function can be expressed as

$$L(\theta, \delta) = e^{v(\delta-\theta)} - v(\delta - \theta) - 1, \quad v \neq 0 \tag{3}$$

where δ is an estimator of θ. The sign and magnitude of v represents the direction and degree of asymmetry, respectively. If $v > 0$, the overestimation is more serious than underestimation, and vice versa. For v close to zero, the LINEX loss is approximately the SE loss and therefore almost symmetric. It is easily seen that the value of $\delta(X)$ that minimizes $E_{\theta|X}[L(\theta, \delta(X))]$ in Eq. 3 is

$$\widehat{\delta}_{BL} = -\frac{1}{v} \log\left(E_{\theta|X}(e^{-v\theta})\right), \tag{4}$$

provided $E_{\theta|X}(e^{-v\theta})$ exists and is finite. Here $E_{\theta|X}(.)$ denotes the expected value with respect to the posterior density function of θ given X.

Our aim in this paper is to improve inference procedures for the stress-strength model when the measurements follow the Kumaraswamy distribution with the first shape parameters are same based on upper record values. Different estimators of R are obtained, namely, maximum likelihood estimator (MLE), uniformly minimum variance unbiased estimator (UMVUE), and Bayesian and empirical Bayesian estimators with SE and LINEX loss functions corresponding to conjugate and non informative priors. Moreover, exact, asymptotic and Bayesian credible intervals of R are also obtained.

The rest of the paper is organized as follows. In Sect. 2, we derive the ML and Bayesian estimation of R with common first shape parameters. The existence and uniqueness of the MLEs of the parameters are proved. The asymptotic confidence interval is obtained. The Tierney and Kadane (1986) approximation is used for the Bayes estimation of R. It is obtained under the SE and LINEX loss functions for the conjugate prior case. In Sect. 3, estimation of R is discussed when the first shape parameter is known. In this section the MLE and UMVUE of R are derived. The Bayes estimators of R are obtained by using series expansion and Lindley's approximation under the SE and LINEX loss functions for the conjugate and non informative prior cases. The empirical Bayes estimators of R are also derived by using two different ways. Moreover, approximate, exact and Bayesian credible intervals of R are constructed. In Sect. 4, the different proposed methods have been compared using Monte Carlo simulations and their results have been reported. Finally, we conclude the paper in Sect. 5.

2 Estimation of R with common first shape parameter

In this section, we investigate the properties of $R = P(X < Y)$, when the first shape parameter a is same for both distributions. MLEs and its existence and uniqueness, asymptotic distributions and confidence intervals for R are obtained.

2.1 Maximum likelihood estimator of R

Let $X \sim Kum(a, b_1)$ and $Y \sim Kum(a, b_2)$, where they are independent. Therefore,

$$
\begin{aligned}
R = P(X < Y) &= \int_0^1 f_Y(y) P(X < Y \mid Y = y) dy \\
&= \int_0^1 a b_2 y^{a-1} (1 - y^a)^{b_2 - 1} \left(1 - (1 - y^a)^{b_1}\right) dy \\
&= \frac{b_1}{b_1 + b_2}. \qquad (5)
\end{aligned}
$$

Our interest is in estimating R based on upper record data on both variables. Let $R_1, \ldots, R_n$ be the first n upper record values observed from $Kum(a, b_1)$ and $S_1, \ldots, S_m$ be an m upper record values observed from $Kum(a, b_2)$ independently from the first sample. The likelihood functions are given by, see Arnold et al. (1998),

$$
l_1(b_1, a \mid \underline{r}) = f(r_n; a, b_1) \prod_{i=1}^{n-1} \frac{f(r_i; a, b_1)}{1 - F(r_i; a, b_1)}, \quad -\infty < r_1 < \cdots < r_n < \infty,
$$

and

$$
l_2(b_2, a \mid \underline{s}) = g(s_m; a, b_2) \prod_{j=1}^{m-1} \frac{g(s_j; a, b_2)}{1 - G(s_j; a, b_2)}, \quad -\infty < s_1 < \cdots < s_m < \infty,
$$

where $\underline{r} = (r_1, \ldots, r_n)$, $\underline{s} = (s_1, \ldots, s_m)$, f and F are the pdf and cdf of X follows $Kum(a, b_1)$, respectively and g and G are the pdf and cdf of Y follows $Kum(a, b_2)$, respectively. Substituting f, F, g and G in the likelihood functions, we obtain the likelihood functions

$$
l_1(b_1, a \mid \underline{r}) = a^n b_1^n h_1(\underline{r}; a) e^{-b_1 T_1(r_n; a)},
$$

and

$$
l_2(b_2, a \mid \underline{s}) = a^m b_2^m h_2(\underline{s}; a) e^{-b_2 T_2(s_m; a)},
$$

where

$$
h_1(\underline{r}; a) = \prod_{i=1}^{n} \frac{r_i^{a-1}}{1 - r_i^a}, \; h_2(\underline{s}; a) = \prod_{j=1}^{m} \frac{s_j^{a-1}}{1 - s_j^a},
$$
$$
T_1(r_n; a) = -\ln\left(1 - r_n^a\right) \text{ and } T_2\left(s_m; a\right) = -\ln(1 - s_m^a\right).
$$

The joint likelihood and the joint log-likelihood functions are

$$
l(b_1, b_2, a \mid \underline{r}, \underline{s}) = l_1(b_1, a \mid \underline{r}) l_2(b_2, a \mid \underline{s}),
$$

and

$$L(b_1, b_2, a \mid \underline{r}, \underline{s}) = (n+m)\ln a + n\ln b_1 + m\ln b_2 + \ln h_1(\underline{r}; a) + \ln h_2(\underline{s}; a) - b_1 T_1(r_n; a) - b_2 T_2(s_m; a), \tag{6}$$

respectively. The ML estimators of b_1, b_2 and a, say $\widehat{b}_1$, $\widehat{b}_2$ and $\widehat{a}$ respectively, can be obtained as a solution of

$$\frac{\partial L}{\partial b_1} = \frac{n}{b_1} - T_1(r_n; a) = 0,$$
$$\frac{\partial L}{\partial b_2} = \frac{m}{b_2} - T_2(s_m; a) = 0,$$
$$\frac{\partial L}{\partial a} = \left[\begin{array}{l} \frac{n+m}{a} + \sum_{i=1}^{n} \frac{\ln r_i}{1-r_i^a} + \sum_{j=1}^{m} \frac{\ln s_j}{1-s_j^a} + \left(\frac{n}{\ln(1-r_n^a)}\right) \frac{r_n^a \ln r_n}{1-r_n^a} \\ + \left(\frac{m}{\ln(1-s_m^a)}\right) \frac{s_m^a \ln s_m}{1-s_m^a} \end{array}\right] = 0.$$

Then, we obtain

$$\widehat{b}_1 = -\frac{n}{\ln\left(1 - r_n^{\widehat{a}}\right)}, \tag{7}$$

$$\widehat{b}_2 = -\frac{m}{\ln\left(1 - s_m^{\widehat{a}}\right)}, \tag{8}$$

and $\widehat{a}$ can be obtained as a solution of the non-linear equation

$$\left[\begin{array}{l} \frac{n+m}{a} + \sum_{i=1}^{n} \frac{\ln r_i}{1-r_i^a} + \sum_{j=1}^{m} \frac{\ln s_j}{1-s_j^a} + \left(\frac{n}{\ln(1-r_n^a)}\right) \frac{r_n^a \ln r_n}{1-r_n^a} \\ + \left(\frac{m}{\ln(1-s_m^a)}\right) \frac{s_m^a \ln s_m}{1-s_m^a} \end{array}\right] = 0.$$

Therefore, $\widehat{a}$ can be obtained as a solution of the non-linear equation of the form $h(a) = a$ where

$$h(a) = -(n+m)\left[\begin{array}{l} \sum_{i=1}^{n} \frac{\ln r_i}{1-r_i^a} + \sum_{j=1}^{m} \frac{\ln s_j}{1-s_j^a} + \left(\frac{n}{\ln(1-r_n^a)}\right) \frac{r_n^a \ln r_n}{1-r_n^a} \\ + \left(\frac{m}{\ln(1-s_m^a)}\right) \frac{s_m^a \ln s_m}{1-s_m^a} \end{array}\right]^{-1}. \tag{9}$$

Since, $\widehat{a}$ is a fixed point solution of the non-linear Eq. 9, its value can be obtained using an iterative scheme as: $h(a_{(j)}) = a_{(j+1)}$, where $a_{(j)}$ is the jth iterate of $\widehat{a}$. The iteration procedure should be stopped when $\left|a_{(j+1)} - a_{(j)}\right|$ is sufficiently small. After $\widehat{a}$ is obtained, $\widehat{b}_1$ and $\widehat{b}_2$ can be obtained from Eqs. 7 and 8, respectively. Therefore, the MLE of R is given as

$$\widehat{R}_{MLE} = \frac{\widehat{b}_1}{\widehat{b}_1 + \widehat{b}_2}. \tag{10}$$

2.2 Existence and uniqueness of the MLEs

In the following theorem we establish the existence and uniqueness of the MLEs.

Theorem 1 *The MLEs of the parameters b_1, b_2 and a are unique and are given by*

$$\widehat{b}_1 = -\frac{n}{\ln(1-r_n^{\widehat{a}})},\ \widehat{b}_2 = -\frac{m}{\ln(1-s_m^{\widehat{a}})},$$

where $\widehat{a}$ is the solution of the non-linear equation:

$$G(a) = \left[\begin{array}{l} \frac{n+m}{a} + \sum_{i=1}^{n} \frac{\ln r_i}{1-r_i^a} + \sum_{j=1}^{m} \frac{\ln s_j}{1-s_j^a} + \left(\frac{n}{\ln(1-r_n^a)}\right) \frac{r_n^a \ln r_n}{1-r_n^a} \\ \quad + \left(\frac{m}{\ln(1-s_m^a)}\right) \frac{s_m^a \ln s_m}{1-s_m^a} \end{array}\right] = 0.$$

Proof $G(a)$ can be rewritten as

$$G(a) = \frac{n}{a}\left[1 + G_1(a) + \frac{G_2(a)}{G_3(a)}\right] + \frac{m}{a}\left[1 + H_1(a) + \frac{H_2(a)}{H_3(a)}\right],$$

where

$$G_1(a) = \frac{1}{n}\sum_{i=1}^{n} \frac{\ln v_i}{1-v_i}, \quad G_2(a) = \frac{1}{n}\frac{v_n \ln v_n}{1-v_n}, \quad G_3(a) = \frac{1}{n}\ln(1-v_n),$$

$$H_1(a) = \frac{1}{m}\sum_{j=1}^{m} \frac{\ln w_j}{1-w_j}, \quad H_2(a) = \frac{1}{m}\frac{w_m \ln w_m}{1-w_m}, \quad H_3(a) = \frac{1}{m}\ln(1-w_m),$$

$v_i = r_i^a$, $i = 1, \ldots, n$ and $w_j = s_j^a$, $j = 1, \ldots, m$. We investigate the limit of $G(a)$ as $a \to 0^+$ and $a \to \infty$. We obtain that $\lim_{a\to 0^+} G(a) = \infty$ and $\lim_{a\to\infty} G(a) < 0$. By the intermediate value theorem $G(a)$ has at least one root in $(0, \infty)$. If it can be shown that $G'(a) < 0$ then the proof will be completed. Since $r_i < r_n$, $\frac{1}{1-r_n^a} > \frac{1}{1-r_i^a}$, $i = 1, \ldots, n-1$ and $s_j < s_m$, $\frac{1}{1-s_m^a} > \frac{1}{1-s_j^a}$, $j = 1, \ldots, m-1$ for $a > 0$,

$$G'(a) < \frac{-(n+m)}{a^2} + \frac{n r_n^a}{a^2}\left(\frac{\ln r_n^a}{1-r_n^a}\right)^2 \left[1 + \frac{r_n^a + \ln(1-r_n^a)}{\left(\ln(1-r_n^a)\right)^2}\right]$$
$$+ \frac{m s_m^a}{a^2}\left(\frac{\ln s_m^a}{1-s_m^a}\right)^2 \left[1 + \frac{s_m^a + \ln(1-s_m^a)}{\left(\ln(1-s_m^a)\right)^2}\right]$$

$$
\begin{aligned}
&= \frac{-(n+m)}{a^2} + \frac{nv_n}{a^2}\left(\frac{\ln v_n}{1-v_n}\right)^2\left[1+\frac{v_n+\ln(1-v_n)}{(\ln(1-v_n))^2}\right] \\
&\quad + \frac{mw_m}{a^2}\left(\frac{\ln w_m}{1-w_m}\right)^2\left[1+\frac{w_m+\ln(1-w_m)}{(\ln(1-w_m))^2}\right] \\
&= \frac{n}{a^2}h(v_n) + \frac{m}{a^2}h(w_m),
\end{aligned}
$$

where

$$
h(x) = -1 + x\left(\frac{\ln x}{1-x}\right)^2\left(1+\frac{x+\ln(1-x)}{(\ln(1-x))^2}\right), \quad 0 < x < 1.
$$

It can be easily shown that $h(x)$ is a monotone increasing function and $h(x) < 0$ for all $0 < x < 1$. Hence, $G^{'}(a) < 0$ is obtained.

Finally, we will show that the MLEs of (b_1, b_2, a) maximizes the log-likelihood function $L(b_1, b_2, a \mid \underline{r}, \underline{s})$. Let $H(b_1, b_2, a)$ be the Hessian matrix of $L(b_1, b_2, a \mid \underline{r}, \underline{s})$ at (b_1, b_2, a). We know that if $\det(H) \neq 0$ for the critical point (b_1, b_2, a) and $\det(H_1) < 0$, $\det(H_2) > 0$, $\det(H_3) < 0$ at (b_1, b_2, a) then it is a local maximum of $L(b_1, b_2, a \mid \underline{r}, \underline{s})$, where $H_1 = \frac{\partial^2 L}{\partial b_1^2}$, $H_2 = \begin{pmatrix} \frac{\partial^2 L}{\partial b_1^2} & \frac{\partial^2 L}{\partial b_1 \partial b_2} \\ \frac{\partial^2 L}{\partial b_2 \partial b_1} & \frac{\partial^2 L}{\partial b_2^2} \end{pmatrix}$, $H_3 = H$. It can be easily seen that

$$
\det(H_1(\widehat{b}_1, \widehat{b}_2, \widehat{a})) = \frac{-\left(\ln(1-r_n^{\widehat{a}})\right)^2}{n} < 0,
$$

$$
\det(H_2(\widehat{b}_1, \widehat{b}_2, \widehat{a})) = \frac{\left(\ln(1-r_n^{\widehat{a}})\right)^2}{n}\frac{\left(\ln(1-s_m^{\widehat{a}})\right)^2}{m} > 0,
$$

and

$$
\det(H(\widehat{b}_1, \widehat{b}_2, \widehat{a})) = G^{'}(\widehat{a})\frac{\left(\ln(1-r_n^{\widehat{a}})\right)^2}{n}\frac{\left(\ln(1-s_m^{\widehat{a}})\right)^2}{m} < 0.
$$

Since there is no singular point of $L(b_1, b_2, a \mid \underline{r}, \underline{s})$ and it has a single critical point then it is enough to show that the absolute maximum of the function is indeed the local maximum. Assume that there exist an $\widehat{a}_0$ in the domain in which $L^*(\widehat{a}_0) > L^*(\widehat{a})$, where $L^*(\widehat{a}) = L(\widehat{b}_1, \widehat{b}_2, \widehat{a} \mid \underline{r}, \underline{s})$. Since $\widehat{a}$ is the local maximum there should be some point b in the neighborhood of $\widehat{a}$ such that $L^*(b) < L^*(\widehat{a})$. Let $K(a) = L^*(a) - L^*(\widehat{a})$ then $K(\widehat{a}_0) > 0$, $K(b) < 0$ and $K(\widehat{a}) = 0$. This implies that b is a local minimum of the $L^*(a)$, but $\widehat{a}$ is the only critical point so it is a contradiction. Therefore, $(\widehat{b}_1, \widehat{b}_2, \widehat{a})$ is the absolute maximum of $L(b_1, b_2, a \mid \underline{r}, \underline{s})$. □

2.3 Asymptotic distribution and confidence intervals for R

We denote the Fisher information matrix of $\theta = (b_1, b_2, a)$ as $I(\theta) = \left(I_{i,j}(\theta)\right)$, $i, j = 1, 2, 3$ and

$$I(\theta) = -\begin{pmatrix} E\left(\frac{\partial^2 L}{\partial b_1^2}\right) & E\left(\frac{\partial^2 L}{\partial b_1 \partial b_2}\right) & E\left(\frac{\partial^2 L}{\partial b_1 \partial a}\right) \\ E\left(\frac{\partial^2 L}{\partial b_2 \partial b_1}\right) & E\left(\frac{\partial^2 L}{\partial b_2^2}\right) & E\left(\frac{\partial^2 L}{\partial b_2 \partial a}\right) \\ E\left(\frac{\partial^2 L}{\partial a \partial b_1}\right) & E\left(\frac{\partial^2 L}{\partial a \partial b_2}\right) & E\left(\frac{\partial^2 L}{\partial a^2}\right) \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}.$$

The elements of the matrix are obtained as,

$$I_{11} = \frac{n}{b_1^2}, \quad I_{22} = \frac{m}{b_2^2}, \quad I_{12} = I_{21} = 0,$$
$$I_{13} = I_{31} = \int_0^1 \frac{r_n^a \ln r_n}{1 - r_n^a} f_{R_n}(r_n) dr_n, \quad I_{23} = I_{32} = \int_0^1 \frac{s_m^a \ln s_m}{1 - s_m^a} g_{S_m}(s_m) ds_m,$$

where $f_{R_n}(r_n)$ is a pdf of nth upper record value from $Kum(a, b_1)$ and $g_{S_m}(s_m)$ is a pdf of mth upper record value from $Kum(a, b_2)$,

$$I_{33} = \frac{n+m}{a^2} - \sum_{i=1}^{n} \int_0^1 r_i^a \left(\frac{\ln r_i}{1 - r_i^a}\right)^2 f_{R_i}(r_i) dr_i - \sum_{j=1}^{m} \int_0^1 s_j^a \left(\frac{\ln s_j}{1 - s_j^a}\right)^2 g_{S_j}(s_j) ds_j$$
$$+ b_1 \int_0^1 r_n^a \left(\frac{\ln r_n}{1 - r_n^a}\right)^2 f_{R_n}(r_n) dr_n + b_2 \int_0^1 s_m^a \left(\frac{\ln s_m}{1 - s_m^a}\right)^2 g_{S_m}(s_m) ds_m,$$

where $f_{R_i}(r_i)$ is a pdf of ith upper record value from $Kum(a, b_1)$ and $g_{S_j}(s_j)$ is a pdf of jth upper record value from $Kum(a, b_2)$. After making suitable transformations we obtain

$$I_{13} = \frac{b_1^n}{a} \sum_{i=1}^{\infty} \frac{1}{i} \left[\frac{1}{(b_1 + i)^n} - \frac{1}{(b_1 + i - 1)^n}\right],$$
$$I_{32} = \frac{b_2^m}{a} \sum_{j=1}^{\infty} \frac{1}{j} \left[\frac{1}{(b_2 + j)^m} - \frac{1}{(b_2 + j - 1)^m}\right],$$

and

$$I_{33} = \frac{n+m}{a^2} - \frac{2}{a^2} \left[\sum_{i=1}^{n} b_1^i A_i(b_1) + \sum_{j=1}^{m} b_2^j B_j(b_2) - b_1^{n+1} A_n(b_1) - b_2^{m+1} B_m(b_2)\right]$$

where

$$A_i(b_1) = \left[\sum_{k=1}^{\infty} \frac{1}{k+1} \left(\frac{1}{(b_1+k-1)^i} - \frac{1}{(b_1+k)^i} \right) \left(\sum_{q=1}^{k} \frac{1}{q} \right) \right],$$

and

$$B_j(b_2) = \left[\sum_{k=1}^{\infty} \frac{1}{k+1} \left(\frac{1}{(b_2+k-1)^j} - \frac{1}{(b_2+k)^j} \right) \left(\sum_{q=1}^{k} \frac{1}{q} \right) \right],$$

see Gradshteyn and Ryzhik (1994) (formula 1.516(1), 4.272(6)).

Theorem 2 *As $n \to \infty$ and $m \to \infty$ and $\frac{n}{m} \to p$ then*

$$\left[\sqrt{n}(\widehat{b}_1 - b_1), \sqrt{m}(\widehat{b}_2 - b_2), \sqrt{n}(\widehat{a} - a) \right] \to N_3(0, U^{-1}(b_1, b_2, a)),$$

where

$$U(b_1, b_2, a) = \begin{pmatrix} u_{11} & 0 & u_{13} \\ 0 & u_{22} & u_{23} \\ u_{31} & u_{32} & u_{33} \end{pmatrix},$$

and

$$u_{11} = \lim_{n,m\to\infty} \frac{1}{n} I_{11}, u_{13} = u_{31} = \lim_{n,m\to\infty} \frac{1}{n} I_{13}, u_{22} = \lim_{n,m\to\infty} \frac{1}{m} I_{22},$$
$$u_{23} = u_{32} = \lim_{n,m\to\infty} \frac{\sqrt{p}}{n} I_{23}, u_{33} = \lim_{n,m\to\infty} \frac{1}{n} I_{33}.$$

Proof The proof follows from the asymptotic normality of MLE. □

Theorem 3 *As $n \to \infty$ and $m \to \infty$ and $\frac{n}{m} \to p$ then*

$$\sqrt{n}(\widehat{R}_{MLE} - R) \to N(0, \sigma^2) \tag{11}$$

where

$$\sigma^2 = \frac{1}{k(b_1+b_2)^4} \left[b_1^2 p(u_{11}u_{33} - u_{13}^2) - 2b_1 b_2 \sqrt{p} u_{13} u_{23} + b_2^2 (u_{22}u_{33} - u_{23}^2) \right],$$

and

$$k = u_{11}u_{22}u_{33} - u_{11}u_{23}u_{32} - u_{13}u_{22}u_{31}.$$

Proof $\sqrt{n}\widehat{R}_{MLE}$ is asymptotically normal with mean $\sqrt{n}R$ and variance

$$\sigma^2 = \lim_{n,m\to\infty} n \sum_{j=1}^{3}\sum_{i=1}^{3} \frac{\partial R}{\partial b_i}\frac{\partial R}{\partial b_j} I_{ij}^{-1}$$

where I_{ij}^{-1} is the (i, j) th element of the inverse of the $I(\theta)$, see Rao (1965). Since $\frac{\partial R}{\partial b_3} = \frac{\partial R}{\partial a} = 0$,

$$\begin{aligned}\sigma^2 &= \lim_{n,m\to\infty} n\left[\frac{\partial R}{\partial b_1}\frac{\partial R}{\partial b_1}I_{11}^{-1} + \frac{\partial R}{\partial b_2}\frac{\partial R}{\partial b_1}I_{21}^{-1} + \frac{\partial R}{\partial b_1}\frac{\partial R}{\partial b_2}I_{12}^{-1} + \frac{\partial R}{\partial b_2}\frac{\partial R}{\partial b_2}I_{22}^{-1}\right]\\ &= \lim_{n,m\to\infty} n\left[\frac{b_1^2(I_{11}I_{33} - I_{13}^2) - 2b_1b_2I_{13}I_{23} + b_2^2(I_{22}I_{33} - I_{23}^2)}{(b_1+b_2)^4(I_{11}I_{22}I_{33} - I_{11}I_{23}^2 - I_{22}I_{13}^2)}\right]\end{aligned}$$

When this expression is multiplied by $\dfrac{1}{n^2m}n^2m$ a suitable form is obtained, considering $\frac{n}{m} \to p$ as $n \to \infty$ and $m \to \infty$, then the desired result is obtained. □

Remark 1 Theorem 3 can be used to construct the asymptotic confidence interval of R. The variance σ^2 needs to be estimated to compute the confidenceinterval of R. The empirical Fisher information matrix and the MLEs of b_1, b_2 and a are used to estimate σ^2 as follows

$$\widehat{u}_{11} = \frac{1}{\widehat{b}_1^2}, \widehat{u}_{22} = \frac{1}{\widehat{b}_2^2},$$

$$\widehat{u}_{13} = \frac{\widehat{b}_1^n}{n\widehat{a}}\sum_{i=1}^{\infty}\frac{1}{i}\left[\frac{1}{(\widehat{b}_1 + i)^n} - \frac{1}{(\widehat{b}_1 + i - 1)^n}\right],$$

$$\widehat{u}_{23} = \frac{\sqrt{p}}{n}\frac{\widehat{b}_2^m}{\widehat{a}}\sum_{j=1}^{\infty}\frac{1}{j}\left[\frac{1}{(\widehat{b}_2 + j)^m} - \frac{1}{(\widehat{b}_2 + j - 1)^m}\right],$$

$$\widehat{u}_{33} = \frac{n+m}{n\widehat{a}^2} - \frac{2}{n\widehat{a}^2}\left[\sum_{i=1}^{n}\widehat{b}_1^i A_i(\widehat{b}_1) + \sum_{j=1}^{m}\widehat{b}_2^j B_j(\widehat{b}_2) - \widehat{b}_1^{n+1}A_n(\widehat{b}_1) - \widehat{b}_2^{m+1}B_m(\widehat{b}_2)\right].$$

2.4 Bayes estimation of R

In this subsection, we investigate the Bayes estimation of R when b_1, b_2 and a have independent priors with $b_1 \sim Gamma(\alpha_1, \beta_1)$, $b_2 \sim Gamma(\alpha_2, \beta_2)$ and $a \sim Gamma(\alpha_3, \beta_3)$. A gamma random variable X with the shape and scale parameters $\alpha > 0$ and $\beta > 0$, respectively, has a density function

$$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-x\beta}.$$

The joint prior density function for $\theta = (b_1, b_2, a)$ is $g(b_1, b_2, a) = \pi(b_1)\pi(b_2)\pi(a)$, and the joint posterior density function of $\theta = (b_1, b_2, a)$ given $(\underline{r}, \underline{s})$ is given by

$$\begin{aligned}\pi\left(b_1, b_2, a \mid \underline{r}, \underline{s}\right) &= \frac{l(b_1, b_2, a \mid \underline{r}, \underline{s})g(b_1, b_2, a)}{\int_0^\infty \int_0^\infty \int_0^\infty l(b_1, b_2, a \mid \underline{r}, \underline{s})g(b_1, b_2, a)db_1 db_2 da} \\ &= \frac{h_1(\underline{r}; a)h_2(\underline{s}; a)b_1^{n+\alpha_1-1} b_2^{m+\alpha_2-1} a^{n+m+\alpha_3-1}}{\Gamma(n+\alpha_1)\Gamma(m+\alpha_2)I_0(\underline{r}, \underline{s})} \\ &\quad \times e^{-b_1(\beta_1+T_1(r_n;a))} e^{-b_2(\beta_2+T_2(s_m;a))} e^{-a\beta_3},\end{aligned} \tag{12}$$

where

$$I_0(\underline{r}, \underline{s}) = \int_0^\infty \frac{a^{n+m+\alpha_3-1} h_1(\underline{r}; a)h_2(\underline{s}; a)e^{-a\beta_3}}{(\beta_1 + T_1(r_n; a))^{n+\alpha_1}(\beta_2 + T_2(s_m; a))^{m+\alpha_2}} da. \tag{13}$$

It is well known that, under SE loss function, the Bayes estimator of any arbitrary function, say $u(b_1, b_2, a)$, is the posterior mean of the function and is given by a ratio of two integrals which may be written as

$$\begin{aligned}E\left[u(b_1, b_2, a) \mid \underline{r}, \underline{s}\right] &= \int_0^\infty \int_0^\infty \int_0^\infty u(b_1, b_2, a)\pi(b_1, b_2, a \mid \underline{r}, \underline{s})db_1 db_2 da \\ &= \frac{\int_0^\infty \int_0^\infty \int_0^\infty u(b_1,b_2,a)l(b_1,b_2,a|\underline{r},\underline{s})g(b_1,b_2,a)db_1 db_2 da}{\int_0^\infty \int_0^\infty \int_0^\infty l(b_1,b_2,a|\underline{r},\underline{s})g(b_1,b_2,a)db_1 db_2 da}.\end{aligned} \tag{14}$$

The ratio of two integrals Eq. 14 cannot be solved analytically. We may use a numerical integration method to calculate the integrals or use approximate methods such as the approximate form due to Lindley (1980) or that of Tierney and Kadane (1986). Lindley (1980) has proposed approximations for moments that capture the first-order error terms of the normal approximation. This is generally accurate enough, but, as Lindley points out, the required evaluation of third derivatives of the posterior can be rather tedious, especially, in problems with several parameters. Moreover, the error of Tierney and Kadane's approximate is of the order $O(n^{-2})$ while the error in using Lindley's approximate form is of the order $O(n^{-1})$. Therefore, we prefer the Tierney and Kadane (1986) approximation for our case. The regularity condition required for using Tierney–Kadane's form is that the posterior density function should be unimodal.

To show that the posterior density function is unimodal, it suffices to show that the function $Q(b_1, b_2, a) \equiv \ln \pi\left(b_1, b_2, a \mid \underline{r}, \underline{s}\right)$ has the unique mode. The extremum points of $Q(b_1, b_2, a)$ are given by

$$\widetilde{b}_1 = \frac{n+\alpha_1-1}{\beta_1 + T_1(r_n; \widetilde{a})}, \quad \widetilde{b}_2 = \frac{m+\alpha_2-1}{\beta_2 + T_2(s_m; \widetilde{a})},$$

and $\widetilde{a}$ is the solution of the non-linear equation:

$$P(a) = \left[\begin{array}{l} \frac{n+m+\alpha_3-1}{a} - \frac{n+\alpha_1-1}{\beta_1+T_1(r_n;a)} \frac{r_n^a \ln r_n}{1-r_n^a} - \frac{m+\alpha_2-1}{\beta_2+T_2(s_m;a)} \frac{s_m^a \ln s_m}{1-s_m^a} \\ -\beta_3 \end{array} \right] = 0.$$

$P(a)$ can be rewritten as

$$P(a) = \frac{1}{a} \left[\begin{array}{l} (n+m+\alpha_3-1) - \frac{n+\alpha_1-1}{\beta_1-\ln(1-v_n)} \frac{v_n \ln v_n}{1-v_n} - \frac{m+\alpha_2-1}{\beta_2-\ln(1-w_m)} \\ \times \frac{w_m \ln w_m}{1-w_m} \end{array} \right] - \beta_3,$$

where $v_n = r_n^a$ and $w_m = s_m^a$. It is easily seen that $\lim_{a\to 0^+} P(a) = \infty$ and $\lim_{a\to\infty} P(a) < 0$. If it can be shown that $P(a)$ is monotone decreasing for all a then the equation $P(a) = 0$ has a unique solution in $(0, \infty)$.

$$\begin{aligned} P^{'}(a) &= -\frac{1}{a^2} \left[\begin{array}{l} (n+m+\alpha_3-1) \\ +(n+\alpha_1-1)v_n \left(\frac{\ln v_n}{1-v_n}\right)^2 \left\{\frac{1}{\beta_1-\ln(1-v_n)} - \frac{v_n}{(\beta_1-\ln(1-v_n))^2}\right\} \\ +(m+\alpha_2-1)w_m \left(\frac{\ln w_m}{1-w_m}\right)^2 \left\{\frac{1}{\beta_2-\ln(1-w_m)} - \frac{w_m}{(\beta_2-\ln(1-w_m))^2}\right\} \end{array} \right] \\ &= -\frac{1}{a^2}\left[(n+m+\alpha_3-1) + (n+\alpha_1-1)h_1(v_n) + (m+\alpha_2-1)h_1(w_m)\right], \end{aligned}$$

where

$$h_1(x) = x \left(\frac{\ln x}{1-x}\right)^2 \left\{\frac{1}{\beta_1 - \ln(1-x)} - \frac{x}{(\beta_1 - \ln(1-x))^2}\right\}, \quad 0 < x < 1.$$

Let $f_1(x) = \beta_1 - \ln(1-x) - x$, then $f_1(0) > 0$ and $f_1(x)$ is a monotone increasing function for all $0 < x < 1$. It can be easily shown that $h_1(x) > 0$ for all $0 < x < 1$, by noticing $h_1(x) = x \left(\frac{\ln x}{1-x}\right)^2 \left(\frac{f_1(x)}{(\beta_1-\ln(1-x))^2}\right)$. Hence, $P^{'}(a) < 0$ is obtained. Now, we want to show that the function $Q(b_1, b_2, a)$ is the maximum at the point $(\widetilde{b}_1, \widetilde{b}_2, \widetilde{a})$. Let $H^*(b_1, b_2, a)$ be the Hessian matrix of $Q(b_1, b_2, a)$. We obtain that

$$\det(H_1^*(\widetilde{b}_1, \widetilde{b}_2, \widetilde{a})) = -\frac{(\beta_1 - \ln(1 - r_n^{\widetilde{a}}))^2}{n + \alpha_1 - 1} < 0,$$

$$\det(H_2^*(\widetilde{b}_1, \widetilde{b}_2, \widetilde{a})) = \frac{(\beta_1 - \ln(1 - r_n^{\widetilde{a}}))^2}{n + \alpha_1 - 1} \frac{(\beta_2 - \ln(1 - s_m^{\widetilde{a}}))^2}{m + \alpha_2 - 1} > 0,$$

and

$$\det(H^*(\widetilde{b}_1, \widetilde{b}_2, \widetilde{a})) = P^{'}(\widetilde{a}) \frac{(\beta_1 - \ln(1 - r_n^{\widetilde{a}}))^2}{n + \alpha_1 - 1} \frac{(\beta_2 - \ln(1 - s_m^{\widetilde{a}}))^2}{m + \alpha_2 - 1} < 0.$$

Therefore $Q(b_1, b_2, a)$ has unique mode and so the posterior density function is unimodal. Consequently, Tierney and Kadane's approximation can be applied to our case.

The posterior mean of the $u(b_1, b_2, a)$, Eq. 14, can be rewritten as

$$E\left[u(b_1, b_2, a) \mid \underline{r}, \underline{s}\right] = \frac{\int_0^\infty \int_0^\infty \int_0^\infty e^{n\Lambda^*(b_1,b_2,a)} db_1 db_2 da}{\int_0^\infty \int_0^\infty \int_0^\infty e^{n\Lambda(b_1,b_2,a)} db_1 db_2 da}, \tag{15}$$

where

$$\Lambda(b_1, b_2, a) = \frac{\left[\ln(l(b_1, b_2, a \mid \underline{r}, \underline{s})) + \ln(g(b_1, b_2, a))\right]}{n}, \tag{16}$$

and

$$\Lambda^*(b_1, b_2, a) = \Lambda(b_1, b_2, a) + \frac{1}{n}\ln(u(b_1, b_2, a)). \tag{17}$$

Following the Tierney and Kadane (1986), Eq. 15 can be approximated in the form

$$\widehat{u}_{BT}(b_1, b_2, a) = \left[\frac{\det \Sigma^*}{\det \Sigma}\right]^{1/2} \exp\left(n\left[\Lambda^*(\widetilde{b}_1^*, \widetilde{b}_2^*, \widetilde{a}^*) - \Lambda(\widetilde{b}_1, \widetilde{b}_2, \widetilde{a})\right]\right), \tag{18}$$

where $(\widetilde{b}_1^*, \widetilde{b}_2^*, \widetilde{a}^*)$ and $(\widetilde{b}_1, \widetilde{b}_2, \widetilde{a})$ maximize $\Lambda^*(b_1, b_2, a)$ and $\Lambda(b_1, b_2, a)$, respectively, and Σ^* and Σ are the negatives of the inverse Hessians of $\Lambda^*(b_1, b_2, a)$ and $\Lambda(b_1, b_2, a)$ at $(\widetilde{b}_1^*, \widetilde{b}_2^*, \widetilde{a}^*)$ and $(\widetilde{b}_1, \widetilde{b}_2, \widetilde{a})$, respectively.

In our case, we have

$$\Lambda(b_1, b_2, a) = \frac{1}{n}\left[\begin{array}{l} L(b_1, b_2, a \mid \underline{r}, \underline{s}) + \ln C + (\alpha_1 - 1)\ln b_1 + (\alpha_2 - 1)\ln b_2 \\ +(\alpha_3 - 1)\ln a - b_1\beta_1 - b_2\beta_2 - a\beta_3 \end{array}\right],$$

where $C = \frac{\beta_1^{\alpha_1}\beta_2^{\alpha_2}\beta_3^{\alpha_3}}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}$. $(\widetilde{b}_1, \widetilde{b}_2, \widetilde{a})$ can be obtained by solving the following equations

$$\Lambda_1 = \frac{\partial\Lambda(b_1, b_2, a)}{\partial b_1} = 0, \quad \Lambda_2 = \frac{\partial\Lambda(b_1, b_2, a)}{\partial b_2} = 0, \quad \Lambda_3 = \frac{\partial\Lambda(b_1, b_2, a)}{\partial a} = 0,$$

and are given by

$$\widetilde{b}_1 = \frac{n + \alpha_1 - 1}{\beta_1 + T_1(r_n; \widetilde{a})},$$
$$\widetilde{b}_2 = \frac{m + \alpha_2 - 1}{\beta_2 + T_2(s_m; \widetilde{a})},$$

and $\widetilde{a}$ is the solution of the non-linear equation

$$\frac{n+m+\alpha_3-1}{a}+\sum_{i=1}^{n}\frac{\ln r_i}{1-r_i^a}+\sum_{j=1}^{m}\frac{\ln s_j}{1-s_j^a}-\left(\frac{n+\alpha_1-1}{\beta_1+T_1(r_n;a)}\right)\frac{r_n^a\ln r_n}{1-r_n^a}$$
$$-\left(\frac{m+\alpha_2-1}{\beta_2+T_2(s_m;a)}\right)\frac{s_m^a\ln s_m}{1-s_m^a}-\beta_3=0.$$

The fixed point method is applied as in the ML estimation of a. The units of the Hessian matrix of $\Lambda(b_1, b_2, a)$ are obtained as

$$\Lambda_{11}=\frac{\partial^2\Lambda(b_1,b_2,a)}{\partial b_1^2}=\frac{1}{n}\left(-\frac{n+\alpha_1-1}{b_1^2}\right),\quad \Lambda_{12}=\Lambda_{21}=\frac{\partial^2\Lambda(b_1,b_2,a)}{\partial b_1\partial b_2}=0,$$
$$\Lambda_{13}=\Lambda_{31}=\frac{\partial^2\Lambda(b_1,b_2,a)}{\partial b_1\partial a}=\frac{1}{n}\left(-\frac{r_n^a\ln r_n}{1-r_n^a}\right),$$
$$\Lambda_{22}=\frac{\partial^2\Lambda(b_1,b_2,a)}{\partial b_2^2}=\frac{1}{n}\left(-\frac{m+\alpha_2-1}{b_2^2}\right),$$
$$\Lambda_{23}=\Lambda_{32}=\frac{\partial^2\Lambda(b_1,b_2,a)}{\partial b_2\partial a}=\frac{1}{n}\left(-\frac{s_m^a\ln s_m}{1-s_m^a}\right),$$
$$\Lambda_{33}=\frac{\partial^2\Lambda(b_1,b_2,a)}{\partial a^2}$$
$$=\frac{1}{n}\left[\begin{array}{l}-\frac{n+m+\alpha_3-1}{a^2}+\sum_{i=1}^{n}r_i^a\left(\frac{\ln r_i}{1-r_i^a}\right)^2+\sum_{j=1}^{m}s_j^a\left(\frac{\ln s_j}{1-s_j^a}\right)^2-b_1r_n^a\left(\frac{\ln r_n}{1-r_n^a}\right)^2\\ -b_2s_m^a\left(\frac{\ln s_m}{1-s_m^a}\right)^2\end{array}\right].$$

Hence,

$$\sum=-\begin{pmatrix}\Lambda_{11} & 0 & \Lambda_{13}\\ 0 & \Lambda_{22} & \Lambda_{23}\\ \Lambda_{13} & \Lambda_{23} & \Lambda_{33}\end{pmatrix}^{-1}$$

and the determinant of Σ is evaluated at $(\widetilde{b}_1, \widetilde{b}_2, \widetilde{a})$.

We get the Bayes estimator of R under the SE loss function when $u(b_1, b_2, a) = R$. Equation 17 takes the form

$${}_{BS}\Lambda^*(b_1,b_2,a)=\Lambda(b_1,b_2,a)+\frac{1}{n}\ln R.$$

The maximum value of the function ${}_{BS}\Lambda^*(b_1, b_2, a)$, say at $({}_{BS}\widetilde{b}_1^*, {}_{BS}\widetilde{b}_2^*, {}_{BS}\widetilde{a}^*)$, is a solution of the non-linear equation system

$$\frac{n+\alpha_1-1}{b_1} - \beta_1 - T_1(r_n; a) + \frac{b_2}{b_1(b_1+b_2)} = 0,$$
$$\frac{m+\alpha_2-1}{b_2} - \beta_2 - T_2(s_m; a) + \frac{1}{b_1+b_2} = 0,$$

and

$$\frac{n+m+\alpha_3-1}{a} + \sum_{i=1}^{n} \frac{\ln r_i}{1-r_i^a} + \sum_{j=1}^{m} \frac{\ln s_j}{1-s_j^a} - b_1 r_n^a \frac{\ln r_n}{1-r_n^a} - b_2 s_m^a \frac{\ln s_m}{1-s_m^a} - \beta_3 = 0.$$

The solution of the system can be obtained by using the fixed point method. We can compute the Hessian matrix of ${}_{BS}\Lambda^*(b_1, b_2, a)$ following the same arguments as in the first case. Therefore, the value of $\det({}_{BS}\Sigma^*)$ at $({}_{BS}\widetilde{b}_1^*, {}_{BS}\widetilde{b}_2^*, {}_{BS}\widetilde{a}^*)$ is obtained. The Bayes estimator of R under the SE loss function is obtained by using Eq. 18 and is given by

$$\widehat{R}_{BS} = \left[\frac{\det_{BS}\Sigma^*}{\det \Sigma}\right]^{1/2} \exp\left(n\left[{}_{BS}\Lambda^*({}_{BS}\widetilde{b}_1^*, {}_{BS}\widetilde{b}_2^*, {}_{BS}\widetilde{a}^*) - \Lambda(\widetilde{b}_1, \widetilde{b}_2, \widetilde{a})\right]\right). \quad (19)$$

If we choose $u(b_1, b_2, a) = e^{-\upsilon R}$, we obtain the Bayes estimator of R under LINEX loss function. Similar to the SE loss function case, we get

$${}_{BL}\Lambda^*(b_1, b_2, a) = \Lambda(b_1, b_2, a) - \frac{\upsilon R}{n},$$

from Eq. 17.

The maximum value of the function ${}_{BL}\Lambda^*(b_1, b_2, a)$, say at $({}_{BL}\widetilde{b}_1^*, {}_{BL}\widetilde{b}_2^*, {}_{BL}\widetilde{a}^*)$, is a solution of the non-linear equation system

$$\frac{n+\alpha_1-1}{b_1} - \beta_1 - T_1(r_n; a) - \frac{\upsilon b_2}{(b_1+b_2)^2} = 0,$$
$$\frac{m+\alpha_2-1}{b_2} - \beta_2 - T_2(s_m; a) + \frac{\upsilon b_1}{(b_1+b_2)^2} = 0,$$

and

$$\frac{n+m+\alpha_3-1}{a} + \sum_{i=1}^{n} \frac{\ln r_i}{1-r_i^a} + \sum_{j=1}^{m} \frac{\ln s_j}{1-s_j^a} - b_1 r_n^a \frac{\ln r_n}{1-r_n^a} - b_2 s_m^a \frac{\ln s_m}{1-s_m^a} - \beta_3 = 0.$$

The Bayes estimator of R under the LINEX loss function is obtained by using Eq. 18 and is given by

$$\widehat{R}_{BL} = \left[\frac{\det_{BL} \Sigma^*}{\det \Sigma}\right]^{1/2} \exp\left(n\left[{}_{BL}\Lambda^*({}_{BL}\widetilde{b}_1^*, {}_{BL}\widetilde{b}_2^*, {}_{BL}\widetilde{a}^*) - \Lambda(\widetilde{b}_1, \widetilde{b}_2, \widetilde{a})\right]\right). \quad (20)$$

3 Estimation of *R* when a is known

In this section, we consider the estimation of R when a is known. Without loss of generality, we assume that $a = 1$. Therefore, $R_1, \ldots, R_n$ be a set of upper records from $Kum(1, b_1)$ and $S_1, \ldots, S_m$ be a independent set of upper records from $Kum(1, b_2)$.

3.1 MLE estimation and confidence intervals of R

Based on the above samples, the MLE of R, say $\widehat{R}_{MLE}$ will be

$$\widehat{R}_{MLE} = \frac{\widehat{b}_1}{\widehat{b}_1 + \widehat{b}_2} = \frac{n\ln(1 - s_m)}{n\ln(1 - s_m) + m\ln(1 - r_n)}. \quad (21)$$

In this case the Fisher information matrix of $\theta = (b_1, b_2)$ is given by

$$I(\theta) = -\begin{pmatrix} E\left(\frac{\partial^2 L}{\partial b_1^2}\right) & E\left(\frac{\partial^2 L}{\partial b_1 \partial b_2}\right) \\ E\left(\frac{\partial^2 L}{\partial b_2 \partial b_1}\right) & E\left(\frac{\partial^2 L}{\partial b_2^2}\right) \end{pmatrix} = \begin{pmatrix} n/b_1^2 & 0 \\ 0 & m/b_2^2 \end{pmatrix}.$$

The MLE estimate of R, $\widehat{R}_{MLE}$ is approximately distributed as normal with mean R and variance

$$\sigma^2 = \sum_{j=1}^{2}\sum_{i=1}^{2} \frac{\partial R}{\partial b_i}\frac{\partial R}{\partial b_j} I_{ij}^{-1}$$

where I_{ij}^{-1} is the (i, j) th element of the inverse of the $I(\theta)$, see Rao (1965). Hence an approximate $100(1-\alpha)\%$ confidence interval for R can be obtained as

$$\left[\widehat{R}_{MLE} - cz_{\alpha/2}\widehat{R}_{MLE}(1 - \widehat{R}_{MLE}), \widehat{R}_{MLE} + cz_{\alpha/2}\widehat{R}_{MLE}(1 - \widehat{R}_{MLE})\right], \quad (22)$$

where $z_{\alpha/2}$ is the upper $\frac{\alpha}{2}$th percentile points of a standard normal distribution and $c = \sqrt{\frac{1}{n} + \frac{1}{m}}$.

It is easy to see that $-2b_1\ln(1 - r_n) \sim \chi^2(2n)$ and $-2b_2\ln(1 - s_m) \sim \chi^2(2m)$. Therefore,

$$F^* = \left(\frac{R}{1 - R}\right)\left(\frac{1 - \widehat{R}_{MLE}}{\widehat{R}_{MLE}}\right)$$

is an F distributed random variable with $(2n, 2m)$ degrees of freedom. The pdf of $\widehat{R}_{MLE}$ is as follows;

$$f_{\widehat{R}_{MLE}}(r) = \frac{1}{r^2 B(m,n)} \left(\frac{nb_1}{mb_2}\right)^n \frac{(\frac{1-r}{r})^{n-1}}{\left(1+\frac{nb_1(1-r)}{mb_2 r}\right)^{n+m}},$$

where $0 < r < 1$. The $100(1-\alpha)\%$ confidence interval for R can be obtained as

$$\left[\frac{1}{1+F_{2m,2n;\frac{\alpha}{2}}\left(\frac{1-\widehat{R}_{MLE}}{\widehat{R}_{MLE}}\right)}, \frac{1}{1+F_{2m,2n;1-\frac{\alpha}{2}}\left(\frac{1-\widehat{R}_{MLE}}{\widehat{R}_{MLE}}\right)}\right], \tag{23}$$

where $F_{2m,2n;\frac{\alpha}{2}}$ and $F_{2m,2n;1-\frac{\alpha}{2}}$ are the lower and upper $\frac{\alpha}{2}$th percentile points of a F distribution with $(2m, 2n)$ degrees of freedom.

3.2 UMVUE of R

The joint pdf of records is

$$f(b_1, b_2 \mid \underline{r}, \underline{s}) = h_1(\underline{r}) h_2(\underline{s}) b_1^n b_2^m e^{-b_1 T_1(r_n)} e^{-b_2 T_2(s_m)}, \tag{24}$$

where $h_1(\underline{r}) = \prod_{i=1}^{n} \frac{1}{1-r_i}$, $h_2(\underline{s}) = \prod_{j=1}^{m} \frac{1}{1-s_j}$, $T_1(r_n) = -\ln(1-r_n)$ and $T_2(s_m) = -\ln(1-s_m)$. It is clear that $(T_1(r_n), T_2(s_m))$ is a sufficient statistic for (b_1, b_2). It can be shown that it is also a complete sufficient statistic by using Theorem 10-9 in Arnold (1990). Let us define

$$\phi(R_1, S_1) = \begin{cases} 1 & \text{if } R_1 < S_1 \\ 0 & \text{if } R_1 \geq S_1 \end{cases}.$$

We have $E(\phi(R_1, S_1)) = R$ so it is an unbiased estimator of R. Let $P_1 = -\ln(1-R_1)$ and $P_2 = -\ln(1-S_1)$. Using Rao-Blackwell and Lehmann-Scheffe's Theorems, see Arnold (1990) the UMVUE of R, say $\widehat{R}_U$, can be obtained as

$$\begin{aligned} \widehat{R}_U &= E(\phi(P_1, P_2) \mid (T_1, T_2)) \\ &= \int\limits_{P_2} \int\limits_{P_1} \phi(P_1, P_2)\, f(p_1, p_2 \mid T_1, T_2) dp_1 dp_2 \\ &= \int\limits_{P_2} \int\limits_{P_1} \phi(P_1, P_2)\, f_{P_1|T_1}(p_1 \mid T_1) f_{P_2|T_2}(p_2 \mid T_2) dp_1 dp_2, \end{aligned}$$

where $(T_1, T_2) = (T_1(r_n), T_2(s_m))$, $f(p_1, p_2 \mid T_1, T_2)$ is the conditional pdf of (P_1, P_2) given (T_1, T_2). Using the joint pdf of (R_1, R_n) and (S_1, S_m) and after making a simple transformation, we obtain the $f_{P_1|T_1}(p_1 \mid T_1)$ and $f_{P_2|T_2}(p_2 \mid T_2)$, and are given by

$$f_{P_1|T_1}(p_1 \mid T_1) = (n-1)\frac{(t_1 - p_1)^{n-2}}{t_1^{n-1}}, \quad 0 < p_1 < t_1,$$
$$f_{P_2|T_2}(p_2 \mid T_2) = (m-1)\frac{(t_2 - p_2)^{m-2}}{t_2^{m-1}}, \quad 0 < p_2 < t_2.$$

Therefore,

$$\begin{aligned}
\widehat{R}_U &= \iint\limits_{P_1<P_2} f_{P_1|T_1}(p_1 \mid T_1) f_{P_2|T_2}(p_2 \mid T_2) dp_1 dp_2 \\
&= \begin{cases} \int_0^{t_1}\int_{p_1}^{t_2}(n-1)(m-1)\frac{(t_1-p_1)^{n-2}}{t_1^{n-1}}\frac{(t_2-p_2)^{m-2}}{t_2^{m-1}}dp_2dp_1 & \text{if } t_2 \geq t_1 \\ \int_0^{t_2}\int_0^{p_2}(n-1)(m-1)\frac{(t_1-p_1)^{n-2}}{t_1^{n-1}}\frac{(t_2-p_2)^{m-2}}{t_2^{m-1}}dp_2dp_1 & \text{if } t_2 < t_1 \end{cases} \\
&= \begin{cases} {}_2F_1(1, 1-m; n; t_1/t_2) & \text{if } t_2 \geq t_1 \\ 1 -{}_2F_1(1, 1-n; m; t_2/t_1) & \text{if } t_2 < t_1 \end{cases},
\end{aligned} \tag{25}$$

where ${}_2F_1(.,.;.;.)$ is Gauss hypergeometric function, see Gradshteyn and Ryzhik (1994)) (formula 3.196(1)).

3.3 Bayesian estimation of R

The Bayes estimators of R with respect to the SE and LINEX loss functions are obtained for the conjugate and non informative prior distributions.

3.3.1 Conjugate prior distributions

We assume that b_1 and b_2 have independent gamma priors with the parameters $b_1 \sim Gamma(\alpha_1, \beta_1)$, $b_2 \sim Gamma(\alpha_2, \beta_2)$. The joint prior density function is obtained multiplying $\pi(b_1)$ by $\pi(b_2)$, and the joint posterior density function of b_1 and b_2 given $(\underline{r}, \underline{s})$ is given by

$$\begin{aligned}
\pi\left(b_1, b_2 \mid \underline{r}, \underline{s}\right) &= \frac{l(b_1, b_2 \mid \underline{r}, \underline{s})\pi(b_1)\pi(b_2)}{\int_0^\infty\int_0^\infty l(b_1, b_2 \mid \underline{r}, \underline{s})\pi(b_1)\pi(b_2)db_1db_2} \\
&= \frac{\lambda_1^{\delta_1}\lambda_2^{\delta_2}}{\Gamma(\delta_1)\Gamma(\delta_2)} b_1^{\delta_1-1} b_2^{\delta_2-1} e^{-b_1\lambda_1} e^{-b_2\lambda_2},
\end{aligned}$$

where $\lambda_1 = \beta_1 + T_1(r_n)$, $\lambda_2 = \beta_2 + T_2(s_m)$, $\delta_1 = n + \alpha_1$, $\delta_2 = m + \alpha_2$. We can obtain the posterior pdf of R using the joint posterior density function and is given by

$$f_R(r) = \frac{\lambda_1^{\delta_1}\lambda_2^{\delta_2}}{B(\delta_1, \delta_2)} \frac{r^{\delta_1-1}(1-r)^{\delta_2-1}}{(r\lambda_1 + (1-r)\lambda_2)^{\delta_1+\delta_2}}, \qquad 0 < r < 1. \tag{26}$$

The Bayes estimator of R, say $\widehat{R}_{BS}$, under the SE loss function is given by

$$\widehat{R}_{BS} = \int_0^1 r f_R(r) dr.$$

After making a suitable transformations and simplifications by using formula 3.197(3) of Gradshteyn and Ryzhik (1994), we get

$$\widehat{R}_{BS} = \begin{cases} c_1 (\frac{\lambda_1}{\lambda_2})^{\delta_1} \ {}_2F_1(\delta_1 + \delta_2, \delta_1 + 1; \delta_1 + \delta_2 + 1; 1 - \frac{\lambda_1}{\lambda_2}) & \text{if } \lambda_1 < \lambda_2 \\ c_1 (\frac{\lambda_2}{\lambda_1})^{\delta_2} \ {}_2F_1(\delta_1 + \delta_2, \delta_2; \delta_1 + \delta_2 + 1; 1 - \frac{\lambda_2}{\lambda_1}) & \text{if } \lambda_2 \le \lambda_1 \end{cases}. \tag{27}$$

where $c_1 = \frac{\delta_1}{\delta_1 + \delta_2}$. The Bayes estimator of R, say $\widehat{R}_{BL}$, under the LINEX loss function is given by

$$\widehat{R}_{BL} = -\frac{1}{v} \ln E_R(e^{-vR}),$$

where $E_R(.)$ denotes posterior expectation with respect to the posterior density of R. It can be easily obtained that

$$\begin{aligned} E(e^{-vR}) &= \int_0^1 e^{-vr} f_R(r) dr \\ &= \begin{cases} (\frac{\lambda_1}{\lambda_2})^{\delta_1} \Phi_1(\delta_1, \delta_1 + \delta_2, \delta_1 + \delta_2, 1 - \frac{\lambda_1}{\lambda_2}, -v) & \text{if } \lambda_1 < \lambda_2 \\ (\frac{\lambda_2}{\lambda_1})^{\delta_2} e^{-v} \Phi_1(\delta_2, \delta_1 + \delta_2, \delta_1 + \delta_2, 1 - \frac{\lambda_2}{\lambda_1}, v) & \text{if } \lambda_2 \le \lambda_1 \end{cases}, \end{aligned}$$

where $\Phi_1(., ., ., ., .)$ is confluent hypergeometric series of two variables, see Gradshteyn and Ryzhik (1994) (formula 3.385 and 9.261(1)). Therefore,

$$\widehat{R}_{BL} = \begin{cases} -\frac{1}{v} \left(c_2 + \ln \left[\Phi_1(\delta_1, \delta_1 + \delta_2, \delta_1 + \delta_2, 1 - \frac{\lambda_1}{\lambda_2}, -v) \right] \right) & \text{if } \lambda_1 < \lambda_2 \\ -\frac{1}{v} \left(c_3 + \ln \left[\Phi_1(\delta_2, \delta_1 + \delta_2, \delta_1 + \delta_2, 1 - \frac{\lambda_2}{\lambda_1}, v) \right] \right) & \text{if } \lambda_2 \le \lambda_1 \end{cases}, \tag{28}$$

where $c_2 = \delta_1 \ln(\frac{\lambda_1}{\lambda_2})$ and $c_3 = \delta_2 \ln(\frac{\lambda_2}{\lambda_1}) - v$.

Alternatively, we consider using the approximation of Lindley (1980) and following the approach of Jaheen (2005), it can be easily seen that the approximate Bayes estimate of R under the SE and LINEX loss functions, say $\widehat{R}^*_{BS}$ and $\widehat{R}^*_{BL}$ respectively, are

$$\widehat{R}^*_{BS} = \widetilde{R} \left(1 + \frac{(1 - \widetilde{R})^2}{n + \alpha_1 - 1} - \frac{\widetilde{R}(1 - \widetilde{R})}{m + \alpha_2 - 1} \right), \tag{29}$$

and

$$\widehat{R}^{*}_{BL} = \widetilde{R} - \frac{1}{v}\ln\left[1 + \frac{v\widetilde{R}(1-\widetilde{R})^2(v\widetilde{R}-2)}{2(n+\alpha_1-1)} + \frac{v\widetilde{R}^2(1-\widetilde{R})(v-v\widetilde{R}+2)}{2(m+\alpha_2-1)}\right], \quad (30)$$

where $\widetilde{R} = \frac{\widetilde{b}_1}{\widetilde{b}_1+\widetilde{b}_2}$, $\widetilde{b}_1 = \frac{n+\alpha_1-1}{\beta_1+T_1(r_n)}$ and $\widetilde{b}_2 = \frac{m+\alpha_2-1}{\beta_2+T_2(s_m)}$.

3.3.2 Non informative prior distributions

We use the Jeffrey's non informative prior which is given by $\sqrt{\det I(b_1, b_2)}$. It is easily seen that the joint prior density function is

$$\pi(b_1, b_2) \propto \frac{1}{b_1 b_2}.$$

Therefore, the joint posterior density function of b_1 and b_2 given $(\underline{r}, \underline{s})$ is given by

$$\pi(b_1, b_2 \mid \underline{r}, \underline{s}) = \frac{(T_1(r_n))^n\,(T_2(s_m))^m}{\Gamma(n)\Gamma(m)} b_1^{n-1} b_2^{m-1} e^{-b_1 T_1(r_n)} e^{-b_2 T_2(s_m)},$$

and the posterior pdf of R is given by

$$f_R(r) = \frac{(T_1(r_n))^n\,(T_2(s_m))^m}{B(n,m)} \frac{r^{n-1}(1-r)^{m-1}}{(rT_1(r_n)+(1-r)T_2(s_m))^{n+m}}, \quad 0 < r < 1.$$

The Bayes estimator of R under the SE and LINEX loss function, say $\widehat{R}_{BS}$ and $\widehat{R}_{BL}$ respectively, are

$$\widehat{R}_{BS} = \begin{cases} (\frac{T_1}{T_2})^n (\frac{n}{n+m}) {}_2F_1(n+m, n+1; n+m+1; 1-\frac{T_1}{T_2}) & \text{if } T_1 < T_2 \\ (\frac{T_2}{T_1})^m (\frac{n}{n+m}) {}_2F_1(n+m, m; n+m+1; 1-\frac{T_2}{T_1}) & \text{if } T_2 \leq T_1 \end{cases}, \quad (31)$$

and

$$\widehat{R}_{BL} = \begin{cases} -\frac{1}{v}\left(c_4 + \ln\left[\Phi_1(n, n+m, n+m, 1-\frac{T_1}{T_2}, -v)\right]\right) & \text{if } T_1 < T_2 \\ -\frac{1}{v}\left(c_5 + \ln\left[\Phi_1(m, n+m, n+m, 1-\frac{T_2}{T_1}, v)\right]\right) & \text{if } T_2 \leq T_1 \end{cases}, \quad (32)$$

where $c_4 = n\ln(\frac{T_1}{T_2})$, $c_5 = m\ln(\frac{T_2}{T_1}) - v$, $T_1 = T_1(r_n)$ and $T_2 = T_2(s_m)$.

Using the non informative prior, based on Lindley's approximation, the approximate Bayes estimate of R under the SE and LINEX loss functions, say $\widehat{R}^{*}_{BS}$ and $\widehat{R}^{*}_{BL}$ respectively, are

$$\widehat{R}^{*}_{BS} = \widetilde{R}\left(1 + \frac{(1-\widetilde{R})^2}{n-1} - \frac{\widetilde{R}(1-\widetilde{R})}{m-1}\right), \quad (33)$$

and

$$\widehat{R}^{*}_{BL}=\widetilde{R}-\frac{1}{v}\ln\left[1+\frac{v\widetilde{R}(1-\widetilde{R})^2(v\widetilde{R}-2)}{2(n-1)}+\frac{v\widetilde{R}^2(1-\widetilde{R})(v-v\widetilde{R}+2)}{2(m-1)}\right], \quad (34)$$

where $\widetilde{R}=\frac{\widetilde{b}_1}{\widetilde{b}_1+\widetilde{b}_2}$, $\widetilde{b}_1=\frac{n-1}{T_1(r_n)}$ and $\widetilde{b}_2=\frac{m-1}{T_2(s_m)}$.

3.4 Empirical Bayes estimation of R

We obtained the Bayes estimator of R using two different ways described in Sect. 3.3.1. It is clear that these estimators depend on the parameters of the prior distributions of b_1 and b_2. However, the Bayes estimators can be obtained independently of the prior parameters.

Firstly, these parameters could be estimated by means of an empirical Bayes procedure, see Lindley (1969) and Awad and Gharraf (1986). Let $R_1,\ldots,R_n$ and $S_1,\ldots,S_m$ be two independent random samples from $Kum(1,b_1)$ and $Kum(1,b_2)$, respectively. For fixed $\underline{r}$, the function $l(b_1,1\mid\underline{r})$ of b_1 can be considered as a gamma density with parameters $(n+1,T_1(r_n))$. Therefore, it is proposed to estimate the prior parameters α_1 and β_1 from the samples as $n+1$ and $T_1(r_n)$, respectively. Similarly, α_2 and β_2 could be estimated from the samples as $m+1$ and $T_2(s_m)$, respectively. Hence, the empirical Bayes estimator of R with respect to SE and LINEX loss functions, say $\widehat{R}_{EBS}$ and $\widehat{R}_{EBL}$, respectively, could be given as

$$\widehat{R}_{EBS}=\begin{cases} c_6c_7\ {}_2F_1(2n+2m+2,2n+2;2n+2m+3;c_9) & \text{if } T_1<T_2\\ c_6c_8\ {}_2F_1(2n+2m+2,2m+1;2n+2m+3;c_{10}) & \text{if } T_2\le T_1\end{cases}, \quad (35)$$

and

$$\widehat{R}_{EBL}=\begin{cases} -\frac{1}{v}\left((2n+1)\ln(\frac{T_1}{T_2})+\ln c_{11}\right) & \text{if } T_1<T_2\\ -\frac{1}{v}\left((2m+1)\ln(\frac{T_2}{T_1})-v+\ln c_{12}\right) & \text{if } T_2\le T_1\end{cases}. \quad (36)$$

where $c_6=\frac{2n+1}{2n+2m+2}$, $c_7=(\frac{T_1}{T_2})^{2n+1}$, $c_8=(\frac{T_2}{T_1})^{2m+1}$, $c_9=1-\frac{T_1}{T_2}$, $c_{10}=1-\frac{T_2}{T_1}$, $c_{11}=\Phi_1(2n+1,2n+2m+2,2n+2m+2,c_9,-v)$ and $c_{12}=\Phi_1(2m+1,2n+2m+2,2n+2m+2,c_{10},v)$.

Moreover, the estimation of the parameters α_i and β_i, $i=1,2$ can be obtained by using the past estimates of the parameters b_1 and b_2. Then using these in the Bayes estimate of R gives us the empirical Bayes estimate of R, see Ahmad et al. (1997) and Jaheen (2004). When the current sample is observed, suppose that the past samples $R_{l,1},\ldots,R_{l,n}$, $l=1,\ldots,N$ are available with the past realizations $b_{1,l}$, $l=1,\ldots,N$ of the random variable b_1. Each sample is supposed to follow the $Kum(1,b_1)$ distribution. The MLE of $b_{1,l}$ for a past sample l, $l=1,\ldots,N$, is given in Eq. 7 and it can be rewritten as

$$\widehat{b}_{1,l}\equiv z_l=\frac{n}{T_{1,l}(r_n)},$$

where $T_{1,l}(r_n) = -\ln(1 - r_{l,n})$, $l = 1, \ldots, N$. For a given $b_{1,l}$, $l = 1, \ldots, N$ the conditional pdf of $T_{1,l}(r_n)$, $l = 1, \ldots, N$ is $Gamma(n, b_{1,l})$ and then z_l, $l = 1, \ldots, N$ has the inverted gamma pdf in the form

$$f(z_l \mid b_{1,l}) = \frac{(nb_{1,l})^n}{\Gamma(n)} \frac{1}{z_l^{n+1}} e^{-nb_{1,l}/z_l}, \quad z_l > 0. \tag{37}$$

Using the prior distribution of $b_{1,l}$, $l = 1, \ldots, N$ and Eq. 37, the marginal pdf of z_l, $l = 1, \ldots, N$ has the inverted beta pdf in the form

$$f(z_l) = \frac{n^n \beta_1}{B(n, \alpha_1)} \frac{(\beta_1 z_l)^{\alpha_1 - 1}}{(n + \beta_1 z_l)^{n+\alpha_1}}, \quad z_l > 0. \tag{38}$$

The moments estimates of the α_1 and β_1 are obtained by using Eq. 38, and are given by

$$\widehat{\alpha}_1 = \frac{s_1^2}{s_2 - s_1^2}, \quad \widehat{\beta}_1 = \frac{s_1}{s_2 - s_1^2}, \tag{39}$$

where

$$s_1 = \frac{(n-1)}{nN} \sum_{l=1}^{N} \widehat{b}_{1,l}, \quad s_2 = \frac{(n-1)(n-2)}{n^2 N} \sum_{l=1}^{N} \widehat{b}_{1,l}^2.$$

Similarly, the prior parameters α_2 and β_2 estimated by using the past estimates $\widehat{b}_{2,k}$, $k = 1, \ldots, M$ from the past samples $S_{k,1}, \ldots, S_{k,m}$ and are given by

$$\widehat{\alpha}_2 = \frac{s_1^{*2}}{s_2^* - s_1^{*2}}, \quad \widehat{\beta}_2 = \frac{s_1^*}{s_2^* - s_1^{*2}}, \tag{40}$$

where

$$s_1^* = \frac{(m-1)}{mM} \sum_{k=1}^{M} \widehat{b}_{2,k}, \quad s_2^* = \frac{(m-1)(m-2)}{m^2 M} \sum_{k=1}^{M} \widehat{b}_{2,k}^2.$$

Substituting $\widehat{\alpha}_1$, $\widehat{\beta}_1$, $\widehat{\alpha}_2$ and $\widehat{\beta}_2$ given in Eqs. 39 and 40 into Eqs. 29 and 30 yields the empirical Bayes estimators of R.

3.5 Bayesian credible intervals for R

The Bayesian credible intervals are obtained by using the posterior distributions of b_1 and b_2.

3.5.1 Conjugate prior distributions

Assuming that b_1 and b_2 are independent, we have obtained in Sect. 3.3.1 that the posterior distributions of b_1 and b_2 have gamma distributions with parameters $(n+\alpha_1, \beta_1+T_1(r_n))$ and $(m+\alpha_2, \beta_2+T_2(s_m))$, respectively. It can be easily seen that $2(\beta_1+T_1(r_n))b_1 \mid \underline{r} \sim \chi^2(2(n+\alpha_1))$ and $2(\beta_2+T_2(s_m))b_2 \mid \underline{s} \sim \chi^2(2(m+\alpha_2))$. Therefore,

$$W = \frac{2(\beta_2+T_2(s_m))b_2 \mid \underline{s}/2(m+\alpha_2)}{2(\beta_1+T_1(r_n))b_1 \mid \underline{r}/2(n+\alpha_1)}$$

is an F distributed random variable with $(2(m+\alpha_2), 2(n+\alpha_1))$ degrees of freedom and the $100(1-\alpha)\%$ Bayesian credible interval for R can be obtained as

$$\left[\frac{1}{1+C_1\left(F_{2(m+\alpha_2),2(n+\alpha_1);\frac{\alpha}{2}}\right)}, \frac{1}{1+C_1\left(F_{2(m+\alpha_2),2(n+\alpha_1);1-\frac{\alpha}{2}}\right)}\right] \tag{41}$$

where $C_1 = \frac{(m+\alpha_2)(\beta_1+T_1(r_n))}{(n+\alpha_1)(\beta_2+T_2(s_m))}$, $F_{2(m+\alpha_2),2(n+\alpha_1);\frac{\alpha}{2}}$ and $F_{2(m+\alpha_2),2(n+\alpha_1);1-\frac{\alpha}{2}}$ are the lower and upper $\frac{\alpha}{2}$th percentile points of a F distribution with $(2(m+\alpha_2), 2(n+\alpha_1))$ degrees of freedom. This interval depends on the prior parameters.

Moreover, this interval can be obtained independently of these parameters by using the empirical method given in Sect. 3.4. In this case the posterior distributions of b_1 and b_2 have gamma distributions with parameters $(2n+1, 2T_1(r_n))$ and $(2m+1, 2T_2(s_m))$, respectively and the $100(1-\alpha)\%$ Bayesian credible interval for R can be obtained as

$$\left[\frac{1}{1+C_2\left(F_{(4m+2),(4n+2);\frac{\alpha}{2}}\right)}, \frac{1}{1+C_2\left(F_{(4m+2),(4n+2);1-\frac{\alpha}{2}}\right)}\right] \tag{42}$$

where $C_2 = \frac{(4m+2)T_1(r_n)}{(4n+2)T_2(s_m)}$, $F_{(4m+2),(4n+2);\frac{\alpha}{2}}$ and $F_{(4m+2),(4n+2);1-\frac{\alpha}{2}}$ are the lower and upper $\frac{\alpha}{2}$th percentile points of a F distribution with $(4m+2, 4n+2)$ degrees of freedom.

3.5.2 Non informative prior distributions

Under the assumption of the independency and non informative prior distributions for b_1 and b_2 we obtain the posterior distributions of b_1 and b_2. They have gamma distributions with parameters $(n+, T_1(r_n))$ and $(m, T_2(s_m))$, respectively. It is easy to see that $2T_1(r_n)b_1 \mid \underline{r} \sim \chi^2(2n)$ and $2T_2(s_m)b_2 \mid \underline{s} \sim \chi^2(2m)$. Therefore, the $100(1-\alpha)\%$ Bayesian credible interval for R is exactly the same as in Eq. 23.

4 Simulation study

In this section, we present the results of simulation study for comparing the risk of different estimators based on Monte Carlo simulations. All computations are performed at the Gebze Institute of Technology. All the programs are written in Matlab R2007a.

We consider two cases separately to draw inference on R, namely when the common first shape parameter a is unknown and known. Without loss of generality we take $a = 1$ when a is known. In both cases we generate the record values with the sample sizes; $(n, m) = (5, 5), (8, 8), (10, 10), (12, 12)$ from Kumaraswamy distribution. All the results are based on 2,500 replications. The estimated risk (ER) of θ, when θ is estimated by $\widehat{\theta}$, is given by

$$ER(\theta) = \frac{1}{K}\sum_{i=1}^{K}\left(\widehat{\theta_i} - \theta_i\right)^2,$$

under the SE loss function. Moreover, the estimated risk of θ under the LINEX loss function is given by

$$ER(\theta) = \frac{1}{K}\sum_{i=1}^{K}\left(e^{v\left(\widehat{\theta_i}-\theta_i\right)} - v\left(\widehat{\theta_i} - \theta_i\right) - 1\right).$$

where K is the number of replication.

Case 1 a is unknown

From the sample, the estimate of a is computed by using the iterative algorithm which is given in Sect. 2.1. We have used the initial estimate of a be 1 and the iterative process stops when the difference between the two consecutive iterates are less than 10^{-6}. Once we estimate a, we estimate b_1 and b_2 using Eqs. 7 and 8, respectively. Finally, we obtain the MLE of R using Eq. 10. The Bayes estimations under the SE and LINEX loss functions are obtained by using the Tierney and Kadane (1986) approximation. The prior parameters $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3) = (8, 10, 5, 4, 5, 5)$ and $(9, 5, 7, 1, 6, 5)$ are used to tabulate the estimates in Table 1 when the true value of R are 0.501731 and 0.908896. Moreover, the average length of approximate confidence intervals and their coverage probabilities (cp) are computed based on the asymptotic distribution of $\widehat{R}_{MLE}$ and is denoted by $\overline{L}_{AMLE}$. The nominal α values is 0.05.

In Table 1, it is observed that as the sample size increases in all the cases the average ERs of the estimators decrease, as expected. It verifies the consistency properties of all the estimates. The average length of the approximate confidence intervals also decrease as the sample size increases while the coverage probability is around 0.95. The ERs of the MLE and the Bayes estimation of R under the SE and LINEX loss functions are denoted by $ER(\widehat{R}_{MLE})$, $ER(\widehat{R}_{BS})$ and $ER(\widehat{R}_{BL})$, respectively. It is observed that the ER of Bayes estimator is smaller than that of ML estimator. Heuristically, in the Bayes approach we have extra information or data based on accumulated knowledge about the parameters as opposed to the MLE approach, therefore the Bayes estimator to be better than the MLE, in the sense that it has smaller ER.

Table 1 Estimations of R when a is unknown and the priors $(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3)$ are chosen to be (8,10,5,4,5,5) and (9,5,7,1,6,5) for the true values of R = 0.501731 and 0.908896, respectively

(n, m)	R	$\widehat{R}_{MLE}$	$\widehat{R}_{BS}$	$\widehat{R}_{BL}$	CI_{AMLE}	cp	$\overline{L}_{AMLE}$
(5,5)	0.501731	0.505022	0.501740	0.608414	(0.226617,0.783427)	0.942400	0.556810
		0.013672	0.007156	0.016118			
(8,8)		0.501802	0.505712	0.605696	(0.275885,0.727719)	0.964000	0.451834
		0.009306	0.006937	0.015867			
(10,10)		0.503607	0.508404	0.603941	(0.300070,0.707145)	0.958400	0.407074
		0.007870	0.006655	0.015828			
(12,12)		0.500559	0.509592	0.603050	(0.313687,0.687430)	0.967200	0.373742
		0.006952	0.006593	0.015653			
(5,5)	0.908896	0.874885	0.892723	0.410055	(0.737286,1.012484)	0.887600	0.275197
		0.007655	0.001766	0.106927			
(8,8)		0.878471	0.888055	0.411905	(0.771516,0.985427)	0.942000	0.213910
		0.004302	0.001676	0.106247			
(10,10)		0.882239	0.886410	0.412547	(0.789373,0.975104)	0.950800	0.185731
		0.003039	0.001604	0.106019			
(12,12)		0.885163	0.885177	0.413023	(0.802291,0.968034)	0.959200	0.165743
		0.002315	0.001570	0.105845			

The first and second rows represent the average estimates and estimated risks for the estimators

Case 2 a is known

In Table 2, the MLE and UMVUE of R, denoted by $\widehat{R}_{MLE}$ and $\widehat{R}_U$, are obtained by using the Eqs. 21 and 25 . Moreover, the Bayes estimators of R , denoted by $\widehat{R}_{BS}$, $\widehat{R}_{BL}$, $\widehat{R}^*_{BS}$ and $\widehat{R}^*_{BL}$, are obtained by using Eqs. 27, 28, 29 and 30, respectively. The first two Bayes estimators are based on series expansion and the other two based on Lindley's approximation for the conjugate prior distributions. In addition, the empirical Bayes estimates denoted by $\widehat{R}_{EBS}$ and $\widehat{R}_{EBL}$ are also obtained by using Eqs. 35 and 36. The prior parameters $(\alpha_1, \alpha_2, \beta_1, \beta_2) = (6, 8, 3, 5)$ and $(10, 6, 1, 8)$ are used to tabulate the estimates in Table 2 when the true value of R are 0.548264 and 0.925025. Furthermore, we obtained the approximate and the exact confidence intervals for R by using Eqs. 22 and 23. Finally, the Bayesian credible intervals are also obtained by using Eq. 42. The average length of the interval, denoted by $\overline{L}_{Bayes}$, and average length of exact confidence interval, denoted by $\overline{L}_{MLE}$, along with their cp's are reported in Table 2.

The average ERs decrease as the sample size increases in all the cases. The Bayes estimate of R has the smallest ER. The Bayes estimates for series expansion and Lindley's methods are very close to each other. From this, we can infer that when the Bayes estimation can not be obtained in the closed form, the Lindley approximation is a good alternative. When the true value of R is 0.548264 we have $ER(\widehat{R}_{BS}) < ER(\widehat{R}_{EBS}) < ER(\widehat{R}_{MLE}) < ER(\widehat{R}_U)$. On the other hand, when the true value of R is 0.925025 we have $ER(\widehat{R}_{BS}) < ER(\widehat{R}_U) < ER(\widehat{R}_{MLE}) < ER(\widehat{R}_{EBS})$. Moreover, it is observed that the average confidence interval lengths decrease as the sample size increases. When the true value of R are 0.548264 and 0.925025, we have $\overline{L}_{MLE} < \overline{L}_{AMLE}$ and $\overline{L}_{AMLE} < \overline{L}_{MLE}$ while the cp is around 0.95. The Bayesian intervals have the smallest cp and is far from 0.95. Sometimes, the cp for the Bayesian

Table 2 Estimations of R when a is known and the priors $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ are chosen to be (6,8,3,5) and (10,6,1,8) for the true values of R = 0.548264 and 0.925025, respectively

(n, m)	R	$\widehat{R}_{MLE}$	$\widehat{R}_U$	$\widehat{R}_{BS}$	$\widehat{R}_{BL}$	$\widehat{R}^*_{BS}$	$\widehat{R}^*_{BL}$	$\widehat{R}_{EBS}$	$\widehat{R}_{EBL}$	$\overline{L}_{AMLE}/cp$	$\overline{L}_{MLE}/cp$	$\overline{L}_{Bayes}/cp$
(5,5)	0.548264	0.544393	0.548398	0.557316	0.552500	0.557118	0.551725	0.542989	0.538888	0.537462	0.518719	0.357006
		0.018763	0.022050	0.009936	0.004983	0.009945	0.004981	0.017737	0.008854	0.903600	0.963200	0.825200
(8,8)		0.543087	0.545404	0.562618	0.558775	0.562463	0.558268	0.542159	0.539442	0.434061	0.423938	0.294623
		0.013143	0.014515	0.008814	0.004427	0.008820	0.004423	0.012645	0.006323	0.916000	0.952400	0.828800
(10,10)		0.550509	0.552673	0.567317	0.563939	0.567182	0.563538	0.549606	0.547795	0.389725	0.382397	0.266601
		0.010863	0.011784	0.008384	0.004215	0.008388	0.004210	0.010509	0.005338	0.910800	0.944800	0.816000
(12,12)		0.545903	0.547493	0.567477	0.564453	0.567368	0.564132	0.545219	0.544703	0.359160	0.353201	0.246897
		0.009507	0.010134	0.007955	0.004000	0.007958	0.003996	0.009258	0.005256	0.911600	0.946400	0.802400
(5,5)	0.925025	0.911806	0.924280	0.920444	0.921894	0.920252	0.919861	0.906515	1.206026	0.187805	0.217568	0.135508
		0.003336	0.002786	0.001016	0.000508	0.001016	0.000515	0.003635	0.813203	0.919000	0.956000	0.830000
(8,8)		0.915128	0.922831	0.917583	0.917401	0.917454	0.917109	0.911635	1.152133	0.146492	0.162202	0.105474
		0.001722	0.001492	0.000965	0.000484	0.000966	0.000488	0.001862	0.554699	0.949000	0.962000	0.837000
(10,10)		0.914794	0.920968	0.915886	0.915650	0.915782	0.915462	0.911941	1.053810	0.132102	0.143450	0.094836
		0.001422	0.001237	0.000951	0.000478	0.000953	0.000481	0.001531	0.345217	0.938000	0.945000	0.804000
(12,12)		0.914524	0.919657	0.914662	0.916367	0.914577	0.914279	0.912120	0.971850	0.121081	0.129777	0.086744
		0.001264	0.001116	0.000908	0.000433	0.000909	0.000458	0.001350	0.173744	0.963000	0.938000	0.823000

The first and second rows represent the average estimates and estimated risks for the estimators. But, for the last three columns, the first row represents the average length and the second row represents the coverage probabilities.

interval based on Eq. 41 is not reasonable, because it contains prior parameters. That is why, they are not reported in the table.

In Table 3, the Bayes estimations of R are also obtained for the non informative prior case. The MLE, UMVUE, Bayes estimations and confidence intervals of R are computed for $R = 0.25, 0.33, 0.5, 0.7, 0.90, 0.92$. The Bayes estimations under SE and LINEX loss functions are obtained by using both series expansion and Lindley's methods as in Table 2. Moreover, the average length of approximate and exact confidence intervals and their cp's of R are evaluated.

The ERs decrease for all the estimates when the sample size increases, as expected. It is clear that the Bayes estimates under SE loss function for the non informative prior are similar to the corresponding MLEs. The Bayes estimates for the Jeffrey's non informative prior case are very similar to the corresponding MLEs. More specifically, the Bayes estimator given in Eq. 33 is very close to the ML estimator after some algebraic operation in which they have suitable form for comparison. For $R = 0.25, 0.33, 0.5,\ 0.7$ the UMVUE has the greatest ER and we have $ER(\widehat{R}_{BS}) < ER(\widehat{R}_{MLE}) < ER(\widehat{R}_U)$. For $R = 0.90, 0.92$, we have $ER(\widehat{R}_U) < ER(\widehat{R}_{MLE}) < ER(\widehat{R}_{BS})$. Moreover, the average lengths of the intervals also decrease as the sample size increases. When $\widehat{R}^*_{BS} < \widehat{R}_{MLE} < R$, this is the case for bigger values of R such as 0.90, 092, it can be shown that $ER(\widehat{R}_{MLE}) < ER(\widehat{R}_{BS})$ for $n = m$. When $R = 0.25, 0.90, 0.92$, we have $\overline{L}_{AMLE} < \overline{L}_{MLE}$. On the other hand, when $R = 0.33, 0.50, 0.70$, we have $\overline{L}_{AMLE} > \overline{L}_{MLE}$. The cp for exact and approximate is around 0.95.

We provide an algorithm for the empirical Bayes estimation which is considered in Table 4. The empirical Bayes estimation of R is derived by using the past estimates of b_1 and b_2 as follows:

1. $\widehat{b}_{1,l},\ l = 1, \ldots, N$ is generated from Eq. 38 for a given values of α_1 and β_1. Then $b_{1,N+1}$ is generated from the gamma prior density of b_1 and $T_{1,N+1}(r_n)$ is generated from the conditional pdf of $T_{1,l}(r_n)$ which is $Gamma(n, b_{1,l})$.
2. Similarly, for the given values of α_2 and β_2, the past estimates $\widehat{b}_{2,k}$, $k = 1, \ldots, M$ can be generated. Moreover, $b_{2,M+1}$ and $T_{2,M+1}(s_m)$ are generated from their respective densities.
3. For the current samples (the samples order $N+1$ and $M+1$), the MLEs of b_1 and b_2 are computed from Eqs. 7 and 8 with T_1 and T_2 being replaced by $T_{1,N+1}(r_n)$ and $T_{2,M+1}(s_m)$, respectively. Hence, the MLE of R is obtained. The Bayes estimates of R is evaluated from Eqs. 29 and 30 for the current samples.
4. The estimates of the prior parameters $\alpha_1, \alpha_2, \beta_1$and β_2 are computed from Eqs. 39 and 40 by using the past estimates. Substituting these estimates in Eqs. 29 and 30 yields the empirical Bayes estimate of R under SE and LINEX loss functions.

They are denoted by $\widehat{R}^*_{EBS}$ and $\widehat{R}^*_{EBL}$. The prior parameters $(\alpha_1, \alpha_2, \beta_1, \beta_2) = (4, 3, 2, 5)$ and $(9, 2, 2, 6)$ are used to tabulate the estimates in Table 4 when the true value of R are 0.753370 and 0.926196.

It is observed that the average ERs of the estimators decrease as the sample size increases in all the cases. The Bayes estimates have the smallest ERs. Moreover, when the number of repeated past sample sizes (N, M), given in the algorithm are small the ER of empirical Bayes is worst than that of MLE's. In particular, when N, $M \leq 10$, we have $ER(\widehat{R}^*_{BS}) < ER(\widehat{R}_{MLE}) < ER(\widehat{R}^*_{EBS})$. On the other hand, for

Table 3 Estimations of R for the non informative prior case when a is known

(n, m)	R	$\widehat{R}_{MLE}$	$\widehat{R}_U$	$\widehat{R}_{BS}$	$\widehat{R}_{BL}$	$\widehat{R}^*_{BS}$	$\widehat{R}^*_{BL}$	$\overline{L}_{AMLE}/cp$	$\overline{L}_{MLE}/cp$
(5,5)	0.25(0.33)	0.266423(0.341186)	0.247968(0.326509)	0.281039(0.352284)	0.274082(0.343648)	0.284963(0.355540)	0.276210(0.344467)	0.450388(0.514840)	0.453386(0.502883)
		0.014043(0.017174)	0.014965(0.019907)	0.013617(0.015422)	0.006594(0.007581)	0.013721(0.015052)	0.006438(0.007197)	0.898400(0.906000)	0.960000(0.966400)
	0.50(0.70)	0.498926(0.685468)	0.498778(0.701779)	0.499025(0.672944)	0.488998(0.664462)	0.499061(0.669370)	0.486032(0.659295)	0.566406(0.495294)	0.540191(0.488108)
		0.021534(0.016031)	0.026191(0.017984)	0.018434(0.014843)	0.009221(0.007607)	0.017520(0.014675)	0.008832(0.007770)	0.895600(0.912000)	0.958400(0.965200)
	0.90(0.92)	0.883283(0.909039)	0.899104(0.922619)	0.868514(0.895774)	0.870454(0.904289)	0.865887(0.893757)	0.863164(0.891947)	0.244504(0.197887)	0.276317(0.231064)
		0.004750(0.003040)	0.003934(0.002399)	0.005789(0.003915)	0.005073(0.008279)	0.006146(0.004172)	0.003177(0.002144)	0.909200(0.914000)	0.954800(0.968400)
(8,8)	0.25(0.33)	0.258155(0.342902)	0.246412(0.334032)	0.268267(0.350278)	0.263823(0.344579)	0.269846(0.351560)	0.264850(0.344911)	0.359416(0.418494)	0.362350(0.411523)
		0.008198(0.011890)	0.008528(0.012909)	0.008069(0.011158)	0.003957(0.005501)	0.008109(0.011070)	0.003923(0.005392)	0.929200(0.913600)	0.967200(0.953200)
	0.50(0.70)	0.500203(0.689251)	0.500203(0.699365)	0.500200(0.680734)	0.493515(0.675326)	0.500201(0.679309)	0.492308(0.673352)	0.460892(0.397948)	0.445768(0.394544)
		0.014846(0.011260)	0.016760(0.012021)	0.013380(0.010748)	0.006695(0.005441)	0.013116(0.010711)	0.006580(0.005493)	0.913600(0.908400)	0.951600(0.948000)
	0.90(0.92)	0.890575(0.913068)	0.900090(0.921253)	0.881358(0.904929)	0.882153(0.908535)	0.880492(0.904281)	0.879087(0.903326)	0.186129(0.152204)	0.203301(0.169266)
		0.002573(0.001801)	0.002258(0.001521)	0.003010(0.002187)	0.002223(0.002282)	0.003089(0.002244)	0.001575(0.001139)	0.921600(0.920000)	0.957600(0.956800)
(10,10)	0.25(0.33)	0.256511(0.339056)	0.247177(0.331922)	0.264786(0.345207)	0.261269(0.340642)	0.265795(0.346043)	0.261900(0.340865)	0.322379(0.374726)	0.324724(0.369908)
		0.006859(0.010372)	0.007054(0.011087)	0.006789(0.009830)	0.003332(0.004861)	0.006815(0.009779)	0.003321(0.004803)	0.922800(0.903600)	0.952800(0.945600)
	0.50(0.70)	0.499460(0.691922)	0.499449(0.700009)	0.499472(0.684879)	0.493939(0.680577)	0.499472(0.683957)	0.493135(0.679302)	0.419019(0.356717)	0.407135(0.354505)
		0.010976(0.009746)	0.012118(0.010276)	0.010061(0.009371)	0.005018(0.004724)	0.009928(0.009350)	0.004960(0.004745)	0.924400(0.906800)	0.953200(0.942400)
	0.90(0.92)	0.890843(0.914273)	0.898366(0.920758)	0.883502(0.907798)	0.883779(0.908572)	0.882975(0.907412)	0.881880(0.906693)	0.166387(0.135063)	0.178772(0.147460)
		0.002411(0.001396)	0.002165(0.001209)	0.002735(0.001654)	0.001758(0.001039)	0.002779(0.001682)	0.001404(0.000850)	0.903600(0.937200)	0.938800(0.955200)
(12,12)	0.25(0.33)	0.259423(0.337613)	0.251695(0.331606)	0.266398(0.342910)	0.263532(0.339048)	0.267102(0.343507)	0.263852(0.339234)	0.298084(0.344073)	0.299745(0.340346)
		0.005944(0.008643)	0.006029(0.009140)	0.005944(0.008259)	0.002914(0.004101)	0.005965(0.008230)	0.002924(0.004064)	0.930800(0.911200)	0.954800(0.942800)
	0.50(0.70)	0.499460(0.694379)	0.499417(0.701201)	0.499494(0.688311)	0.494822(0.684715)	0.499500(0.687654)	0.494266(0.683813)	0.384216(0.327169)	0.374950(0.325513)
		0.009910(0.007805)	0.010754(0.008150)	0.009209(0.007560)	0.004621(0.003826)	0.009124(0.007552)	0.004585(0.003837)	0.920000(0.912400)	0.948800(0.942400)
	0.90(0.92)	0.892941(0.913177)	0.899198(0.918626)	0.886793(0.907736)	0.886150(0.907885)	0.886442(0.907472)	0.885586(0.906874)	0.150435(0.125032)	0.160191(0.134584)
		0.001642(0.001235)	0.001488(0.001076)	0.001857(0.001446)	0.000971(0.000837)	0.001880(0.001464)	0.000951(0.000737)	0.930800(0.941600)	0.958000(0.950400)

The first and second rows represent the average estimates and estimated risks for the estimators. But, for the last two columns, the first row represents the average length and second row represents the coverage probabilities. The corresponding results are reported within bracket in each cell for R=0.33, 0.70 and 0.92

Table 4 Estimations of of R when a is known and the priors $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ are chosen to be (4,3,2,5) and (9,2,2,6) for the true values of R = 0.753370 and 0.926196, respectively

(n, N)	(m, M)	R	$\widehat{R}_{MLE}$	$\widehat{R}^*_{BS}$	$\widehat{R}^*_{BL}$	$\widehat{R}^*_{EBS}$	$\widehat{R}^*_{EBL}$
(10,5)	(10,5)	0.753370(0.926196)	0.740441(0.921217)	0.747307(0.926171)	0.744792(0.925813)	0.767509(0.934350)	0.768272(0.938533)
			0.006371(0.001336)	0.004824(0.000742)	0.002443(0.000373)	0.045465(0.011783)	0.052946(0.008616)
(15,5)	(15,5)		0.747501(0.922613)	0.751149(0.925925)	0.749371(0.925646)	0.756391(0.933780)	0.759520(0.934368)
			0.004190(0.000968)	0.003248(0.000612)	0.001633(0.000309)	0.036742(0.004355)	0.022985(0.002977)
(20,5)	(20,5)		0.751462(0.923009)	0.753842(0.925505)	0.752462(0.925276)	0.759026(0.933176)	0.758518(0.933266)
			0.003188(0.000688)	0.002777(0.000487)	0.001392(0.000246)	0.015045(0.009771)	0.009411(0.006747)
(10,10)	(10,10)	0.753370(0.926196)	0.746477(0.920828)	0.751744(0.926125)	0.749277(0.925765)	0.759985(0.929991)	0.759501(0.929797)
			0.006602(0.001498)	0.004862(0.000789)	0.002448(0.000397)	0.017838(0.002515)	0.023709(0.001211)
(15,10)	(15,10)		0.747954(0.922634)	0.751539(0.926042)	0.749759(0.925764)	0.757170(0.929356)	0.755666(0.929151)
			0.004638(0.000925)	0.003689(0.000594)	0.001860(0.000299)	0.005296(0.001032)	0.002686(0.000530)
(20,10)	(20,10)		0.748092(0.923841)	0.751063(0.926223)	0.749673(0.925998)	0.754217(0.928333)	0.752968(0.928151)
			0.003246(0.000673)	0.002688(0.000495)	0.001350(0.000249)	0.003402(0.000690)	0.001717(0.000349)
(10,15)	(10,15)	0.753370(0.926196)	0.746294(0.920424)	0.751976(0.925742)	0.749509(0.925379)	0.757484(0.930571)	0.755453(0.930314)
			0.006370(0.001453)	0.004508(0.000784)	0.002273(0.000395)	0.006151(0.001409)	0.003144(0.000727)
(15,15)	(15,15)		0.746696(0.922597)	0.750481(0.925923)	0.748698(0.925645)	0.753517(0.927839)	0.751903(0.927606)
			0.004381(0.000881)	0.003396(0.000578)	0.001709(0.000290)	0.003952(0.000715)	0.001994(0.000361)
(20,15)	(20,15)		0.750309(0.923289)	0.752787(0.925816)	0.751407(0.925589)	0.755298(0.927529)	0.754012(0.927332)
			0.003112(0.000657)	0.002657(0.000486)	0.001337(0.000245)	0.003010(0.000592)	0.001520(0.000300)
(10,20)	(10,20)	0.753370(0.926196)	0.744855(0.919652)	0.750797(0.925192)	0.748313(0.924823)	0.754378(0.928102)	0.752216(0.927813)
			0.006294(0.001446)	0.004530(0.000741)	0.002284(0.000373)	0.005422(0.000976)	0.002747(0.000495)
(15,20)	(15,20)		0.749011(0.922117)	0.752559(0.925609)	0.750788(0.925327)	0.754262(0.927155)	0.752614(0.926908)
			0.004019(0.000887)	0.003233(0.000545)	0.001626(0.000274)	0.003616(0.000633)	0.001825(0.000319)
(20,20)	(20,20)		0.750127(0.924137)	0.752826(0.926401)	0.751445(0.926177)	0.754444(0.927561)	0.753127(0.927359)
			0.003071(0.000597)	0.002594(0.000458)	0.001305(0.000230)	0.002825(0.000527)	0.001423(0.000267)

The first and second rows represent the average estimates and estimated risks. The average estimates and estimated risks for R = 0.926196 are reported within bracket in each cell the larger sample size (N, M) the ER of empirical Bayes is better than that of MLE's. In particular, when $N, M > 10$, we have $ER(\widehat{R}^*_{BS}) < ER(\widehat{R}^*_{EBS}) < ER(\widehat{R}_{MLE})$.

5 Conclusions

In this paper, we compare different methods of estimations of $P(X < Y)$ when X and Y are two independent Kumaraswamy distributions with the common first shape parameters.

When the first shape parameter is unknown, it is observed that the Bayesian estimators have a smaller ER. And this result does not change for the different values of the prior parameters. Nominal coverage probabilities are attained for the asymptotic confidence intervals.

When the first shape parameter is known, we compare the different estimators, namely MLE, UMVUE with Bayes and empirical Bayes estimators. The Bayesian estimators of R are obtained by using series expansion and Lindley's approximation method for both conjugate and non informative prior cases. Under both of these methods the ER are quite similar. Furthermore, ER of the empirical Bayes estimators for the conjugate prior case are better than that of MLE's when past sample sizes (N, M) are greater than 10. The different confidence intervals of R, namely approximate, exact and Bayesian are compared. Even though, the prior parameters are not known it is observed that the Bayesian interval discussed in Eq. 42 is quite satisfactory.

Kotz et al. (2003) show that MLE, UMVUE, Bayesian estimator as well as confidence interval for R are invariant with respect to a monotone transformation on (X, Y). If X is Kumaraswamy then $-\ln X$ is the two parameter generalized exponential distribution. Therefore, all the estimators for R, mentioned above, under the Kumaraswamy distribution is the same as the two parameter generalized exponential distribution.

The MLE, UMVUE, Bayesian estimators of R in random samples depends on all the observation, but in record case they only depend on the last record value. Moreover, we considered the non informative case (a is known) when the number of random samples and the number of record values are taken to be equal as in the work of Ahmadi and Arghami (2001). In this case, Monte Carlo simulation reveals out that the record case produces smaller ER for the Bayes estimation of R (when cp's are similar) for the large sample sizes.

On the other hand, we may use Theorem 3.1 in Ahmadi and Arghami (2001) to say that (Fisher) information in record values is no different from that of random samples case under the assumption of $X_i,\ i = 1, \ldots, n$ and $Y_j,\ j = 1, \ldots, m$ distributes as $Kum(1, b)$, and the number of record values are the same as the number of random samples. When distribution involves more than one parameters, comparing the information in records with random samples is a subject of future studies.

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