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Integrated Production Inventory Policies for Multistage Multiproduct Batch Production Systems

M. G. KORGAONKER
Indian Institute of Management, Ahmedabad

An integrated production inventory model is considered in this paper, for a flow shop type multiproduct batch production system, with a multifacility structure. Instantaneous production is allowed in each facility. The model aims to determine simultaneously the optimal manufacturing cycle for the multiple products and the corresponding optimal procurement policies for the raw material. The cycle concept of multiproduct batch processing is extended to multifacility system and is integrated with the concept of production-inventory system for a single product, single facility system.

INTRODUCTION

Goyal[1] derived an integrated production inventory policy for the elementary case of a single product processed on a single facility. The model simultaneously determines the economic batch size for the product and economic order quantities for raw materials, that will minimise the total variable cost of the production inventory system. Such a policy takes into account the fact that raw materials are consumed only during the production time of the batch, and not uniformly as commonly assumed. This policy is more important in a situation where multiple products are processed on multifacility production systems on a batch basis. Although models have been proposed separately for production of multiple products in a cycle,[2–7] and for optimal lot sizes on multiple processors,[8–11] these do not take into account the particular consumption pattern of raw materials in batch production.

In this paper, an integrated production inventory model is developed for multiproduct multifacility production. In this way, optimal production cycle for products and corresponding optimal procurement policies for raw materials are simultaneously determined. The objective is to minimise the total variable cost of production inventory system.

THE MATHEMATICAL MODEL

The mathematical model presented is based on the following assumptions:

(1) A flow shop type production system is assumed, where all products are processed on the same facilities, in the same sequence, on a batch basis. The facility set up costs are not sequence dependent.
(2) The facility production rates are assumed very high, so that instantaneous production results. This assumption is, however, not restrictive and could be dropped, for instance by the alternative consideration of goods transfer during processing.
(3) Products are demanded at the final processing facility and all demands are known and uniform.
(4) Raw materials are fed to the production system as required at any facility.
(5) All the other cost factors pertaining to set ups, order processing and inventory carrying are assumed known with certainty.

The following notation is used:

n number of products.
λ_i demand/year of product i.
T Manufacturing cycle time of products, on the last processing facility m.
m number of facilities. These are serially numbered from 1 to m.
A_{ij} Set up cost for product i on facility j.

$h_{ij} =$ $V_{ij}I =$ Value added inventory cost per unit per year, for product i at facility j.
$V_{ij} =$ Value added per unit for product i at facility j.
$I =$ Annual inventory carrying charges factor.

k_{ij} A positive integer such that product i is processed on facility j once every $\prod_{q=j}^{m} k_{iq}$ cycles.

Q_{ij} Optimal lot size of product i on facility j

Raw material l *of product* i, *facility* j

m_{ij} Number of raw materials.
A_{ijl} Ordering cost per order.
h_{ijl} Inventory carrying cost per unit per year.
x_{ijl} Demand per year.
k_{ijl} A positive integer such that the raw material is ordered once every $k_{ijl}\left(\prod_{q=j}^{m} k_{iq}\right)$ cycles.
Q_{ijl} Optimal order quantity.

Costs

C Total variable cost per year of the multistage multiproduct production-inventory system.
C_i Total variable cost per year of the production inventory system of product i.
C_{ij} Total variable cost per year of the production inventory system of product i, facility j.
C_{ijl} Total variable cost per year of the inventory system of raw material l, of product i, facility j.

Following Crowston *et al.*[11] we shall use the concept of echelon stock. Echelon stock is the number of units in the system which have passed through the facility i but have not as yet been sold. This concept allows consideration of value added inventory of a production facility and permits the inventory holding cost to be considered as a function of the facility concerned only. Also, our assumption of instantaneous production guarantees schedule feasibility and our cycles will be repeatable in the long run sense. Similar analysis would hold for the case of finite production rates, with the feasibility of goods transfer between stages, during batch processing.

Crowston *et al.*[11] have proved that for the single stage production system, the economic batch size of a given facility is an integral multiple of economic batch size of its successor facility. This fact permits consideration of only batch sizes that are integer multiples. With schedule feasibility guaranteed, the result clearly holds for each individual member of the multiproduct family.

The batch size Q_{im} of the product i on the last facility is given by the formula

$$Q_{im} = k_{im}\lambda_i T.$$

We can now write

$$Q_{ij} = k_{ij}k_{ij+1},\ldots,k_{im}\lambda_i T$$

$$= \left(\prod_{q=j}^{m} k_{iq}\right)\lambda_i T,$$

$$Q_{ijl} = k_{ijl}Q_{ij}.$$

In these formulae, all k_{ij} and k_{ijl} are positive integers. Our problem is reduced to one of determining T^*_{opt}, $k^*_{ij\text{opt}}$ $(i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m)$ and $k^*_{ijl\text{opt}}$ $(i = 1, 2, \ldots, n;$ $j = 1, 2, \ldots, m; l = 1, 2, \ldots, m_{ij})$ such that all $k^*_{ij\text{opt}}$ and $k^*_{ijl\text{opt}}$ are positive integers.

ANALYSIS OF VARIABLE COST

The total variable cost per year of procuring raw material l of product i, facility j, is

$$C_{ijl} = \frac{A_{ijl}}{\left(\prod_{q=j}^{m} k_{iq}\right) k_{ijl} T} + \frac{k_{ijl}\left(\prod_{q=j}^{m} k_{iq}\right) x_{ijl} h_{ijl} T}{2}.$$

Next, the total variable cost per year of the production inventory system of product i, facility j, is

C_{ij} = Variable production system cost + variable procurement system cost.

$$C_{ij} = \frac{A_{ij}}{\left(\prod_{q=j}^{m} k_{iq}\right) T} + \frac{\left(\prod_{q=j}^{m} k_{iq}\right) \lambda_i T h_{ij}}{2} + \sum_{l=1}^{m_{ij}} C_{ijl}.$$

We now write the total variable cost per year of the production inventory system of product i, as follows:

$$C_i = \sum_j C_{ij}$$

$$= \sum_j \left(\frac{A_{ij}}{\left(\sum_{q=j}^{m} k_{iq}\right)} + \frac{\left(\prod_{q=j}^{m} k_{iq}\right) \lambda_i T h_{ij}}{2} \right) + \sum_j \sum_l C_{ijl}.$$

Hence the total variable cost of the entire production-inventory system is

$$C = \sum_i C_i$$

$$= \sum_i \sum_j \left(\frac{A_{ij}}{\left(\prod_{q=j}^{m} k_{iq}\right) T} + \frac{\left(\prod_{q=j}^{m} k_{iq}\right) \lambda_i T h_{ij}}{2} \right)$$

$$+ \sum_i \sum_j \sum_l \left(\frac{A_{ijl}}{\left(\prod_{q=j}^{m} k_{iq}\right) k_{ijl} T} + \frac{k_{ijl}\left(\prod_{q=j}^{m} k_{iq}\right) x_{ijl} h_{ijl} T}{2} \right).$$

The optimal production inventory policies will be obtained by first setting to zero partial derivatives of C with reference to T, k_{iq} and k_{ijl} respectively.

$$\text{i.e.} \quad \frac{\partial C}{\partial T} = 0 \quad \frac{\partial C}{\partial k_{iq}} = 0 \quad \frac{\partial C}{\partial k_{ijl}} = 0$$

yielding

$$T^* = \left\{ \frac{2\left[\sum_i \sum_j \frac{A_{ij}}{\left(\prod_{q=j}^{m} k_{iq}\right)} + \sum_i \sum_j \sum_l \frac{A_{ijl}}{\left(\prod_{q=j}^{m} k_{iq}\right) k_{ijl}} \right]}{\sum_i \sum_j \left(\prod_{q=j}^{m} k_{iq}\right) \lambda_i h_{ij} + \sum_i \sum_j \sum_l \left(\prod_{q=j}^{m} k_{iq}\right) x_{ijl} h_{ijl} k_{ijl}} \right\}^{1/2}$$

$$k_{ij}^* = \frac{1}{T}\left\{\frac{2\left[\sum_{s=1}^{j}\frac{A_{is}}{\left(\prod_{q=s}^{j=1}k_{iq}\right)\left(\prod_{q=j+1}^{m}k_{iq}\right)} + \sum_{s=1}^{j}\sum_{l=1}^{m_{ij}}\frac{A_{isl}}{\left(\prod_{q=s}^{j-1}k_{iq}\right)\left(\prod_{q=j+1}^{n}k_{iq}\right)k_{isl}}\right]}{\sum_{s=1}^{j}\left(\prod_{q=s}^{j-1}k_{iq}\right)\left(\prod_{q=j+1}^{m}k_{iq}\right)\lambda_i h_{is} + \sum_{s=1}^{j}\sum_{l=1}^{m_{ij}}k_{isl}\left(\prod_{q=s}^{j-1}k_{iq}\right)\left(\prod_{q=j+1}^{m}k_{iq}\right)x_{isl}h_{isl}}\right\}^{1/2}$$

$$k_{ijl}^* = \frac{1}{T\left(\prod_{q=j}^{m}k_{iq}\right)}\left(\frac{2A_{ijl}}{x_{ijl}h_{ijl}}\right)^{1/2}$$

INTEGRALITY OF k_{ij}^*, k_{ijl}^*

The values of k_{ij}^* and k_{ijl}^* obtained above may not be integers because of the underlying continuity assumption. We examine below the question of integrality of k_{ij}^* and k_{ijl}^*.

The total variable cost per year of the production-inventory system which depends on k_{ij} is

$$C'_{ij} = \sum_{s=1}^{j}\left(\frac{A_{is}}{\left(\prod_{q=s}^{m}k_{iq}\right)T} + \frac{\left(\prod_{q=s}^{m}k_{iq}\right)\lambda_i T h_{is}}{2}\right) + \sum_{s=1}^{j}\sum_{l=1}^{m_{ij}}\left(\frac{A_{isl}}{\left(\prod_{q=s}^{m}k_{iq}\right)k_{isl}T} + \frac{k_{isl}x_{isl}Th_{isl}\left(\prod_{q=s}^{m}k_{iq}\right)}{2}\right).$$

Let $C'_{ij}(k_{ij})$, $C'_{ij}(k_{ij}^*)$ and $C'_{ij}(k_{ij}+1)$ be the values of C'_{ij} at k_{ij}, k_{ij}^* and $k_{ij}+1$, respectively.

Let k_{ij} be the largest integer less than or equal to k_{ij}^*, so that we can write $k_{ij}^* = k_{ij} + \delta_{ij}$.

We wish to derive conditions for rounding off k_{ij}^* to either k_{ij} or $k_{ij}+1$. k_{ij}^* is rounded off to k_{ij}, if

$$C'_{ij}(k_{ij}) < C'_{ij}(k_{ij}+1).$$

Similarly, k_{ij}^* is rounded off to $k_{ij}+1$, if

$$C'_{ij}(k_{ij}) > C'_{ij}(k_{ij}+1).$$

Finally, k_{ij}^* is rounded off to either k_{ij} or $k_{ij}+1$ if

$$C'_{ij}(k_{ij}) = C'_{ij}(k_{ij}+1).$$

In the appendix, the conditions are derived in detail.

The integrality of k_{ijl}^* can be treated in an identical manner. We first let $C_{ijl}(k_{ijl})$, $C_{ijl}(k_{ijl}^*)$ and $C_{ijl}(k_{ijl}+1)$ be the values of C_{ijl} at k_{ijl}, k_{ijl}^* and $(k_{ijl}+1)$, respectively. Next, we allow $k_{ijl}^* = k_{ijl} + \delta_{ijl}$, so that k_{ijl} is the largest integer less than or equal to k_{ijl}^*.

We round off k_{ijl}^* to k_{ijl}, if

$$C_{ijl}(k_{ijl}) < C_{ijl}(k_{ijl}+1).$$

Round off k_{ijl}^* to $k_{ijl}+1$, if

$$C_{ijl}(k_{ijl}+1) > C_{ijl}(k_{ijl}).$$

Round off k_{ijl}^* to either k_{ijl} or $k_{ijl}+1$, if

$$C_{ijl}(k_{ijl}) = C_{ijl}(k_{ijl}+1).$$

If $\delta_{ij} < \delta^*_{ij}$, round off k^*_{ij} to k_{ij}

$> \delta^*_{ij}$, round off k^*_{ij} to $k_{ij} + 1$

$= \delta^*_{ij}$, round off k^*_{ij} to either k_{ij} or $k_{ij} + 1$,

and if $\delta_{ijl} < \delta^*_{ijl}$, round off k^*_{ijl} to k_{ijl}

$> \delta^*_{ijl}$, round off k^*_{ijl} to $k_{ijl} + 1$

$= \delta^*_{ijl}$, round off to either k_{ijl} or $k_{ijl} + 1$.

SUGGESTED COMPUTATIONAL SCHEME

The steps of the computational scheme, for determining the optimal production-inventory policy are stated below.

1. Let all k^*_{ij} and k^*_{ijl} be 1.
2. Compute T^*
3. Compute k^*_{ij}, round off k^*_{ij} and compute k^*_{ij-1}. Round off k^*_{ij-1}. Perform this step for $j = n, n-1, \ldots, 2$.
4. Compute all k^*_{ijl}. Round off all k^*_{ijl}.
5. Use the rounded off values of all k^*_{ij} and k^*_{ijl} and go to step 2.
6. Repeat 2–5 till convergence is obtained.
7. Compute $Q_{ij} = \left(\prod_{q=j}^{m} k^*_{iq\,\text{opt}}\right) \lambda_i T, \quad i = 1, 2, \ldots, n; \; j = 1, 2, \ldots, m$

$$Q_{ijl} = k^*_{ijl\,\text{opt}} Q_{ij} \qquad i = 1, 2, \ldots, n; \quad j = 1, 2, \ldots, m; \quad l = 1, 2, \ldots, m_{ij}$$

WORKED EXAMPLE

Number of products $n = 2$.
Number of production facilities $m = 2$.
Number of raw materials of product i, at facility j, $m_{ij} = 1$, $(i = 1,2; j = 1, 2)$.

Demand/year of product:	$\lambda_1 = 40000$ units $\lambda_2 = 80000$ units.
Inventory carrying cost/ unit/year of product:	$h_{11} = h_{12} = \$2$; $h_{21} = h_{22} = \$0.2$.
Set up cost of product:	$A_{11} = A_{12} = \$100$; $A_{21} = A_{22} = \$80$.
Demand/year of raw materials:	$x_{111} = x_{121} = 40000$ units.
	$x_{211} = x_{221} = 80000$ units.
Inventory carrying cost of raw material/unit/year:	$h_{111} = \$0.3$ $h_{121} = \$0.4$
	$h_{211} = \$0.1$ $h_{221} = \$0.2$.
Ordering cost of raw materials:	$A_{111} = \$80$, $A_{121} = \$30$
	$A_{211} = \$60$, $A_{221} = \$40$.

Solution

The solution to the problem, using the computational scheme is obtained in three iterations. The values of all k^*_{ij}'s and k^*_{ijl}'s obtained at different iterations are given in table 1.

In applying the scheme, we first set all k^*_{ij} and all k^*_{ijl} equal to unity, and determine manufacturing cycle time $T^*_1 = 0.0648$ years. Using this value, all k^*_{ij}'s and k^*_{ijl}'s are recomputed. Substitution of the latter into expression for T^*, yields new value of manufacturing cycle time, namely, $T^*_2 = 0.0581$ years. All the values of k^*_{ij}'s and k^*_{ijl}'s converge at third iteration; giving the optimal value of manufacturing cycle time T equal to

TABLE 1. RESULTS OF COMPUTATIONAL SCHEME

Iteration 1	$k_{ij}^* = 1$	$i = 1,2$ $j = 1,2$	$k_{ijl}^* = 1$	$i = 1,2$ $j = 1,2$ $l = 1$	$T_1^* = 0.0648$
Iteration 2	$k_{ij}^* = 1$	$i = 1,2$ $j = 1,2$	$k_{121}^* = k_{221}^* = 1$ $k_{111}^* = k_{211}^* = 2$		$T_2^* = 0.0581$
Iteration 3 (convergence)	$k_{ij}^* = 1$	$i = 1,2$ $j = 1,2$	$k_{121}^* = k_{221}^* = 1$ $k_{111}^* = k_{211}^* = 2$		$T_3^* = 0.0581$

$Q_{11} = Q_{12} = 40000 \times 0.0581 = 2324$ units.
$Q_{21} = Q_{22} = 80000 \times 0.0581 = 4648$ units.
$Q_{111} = 2 \times 2324$ units.
$Q_{121} = 1 \times 2324 = 2324$ units.
$Q_{211} = 2 \times 4648 = 9296$ units.
$Q_{221} = 1 \times 4648 = 4648$ units.

0.0581 years. Application of step 7 of the scheme, results in optimal values of lot sizes for products and optimal order quantities for raw materials. For example, optimal lot size for product 1 at facility 1, $Q_{11} = 1 \times 40000 \times 0.0581 = 2324$ units; and optimal order quantity of raw material 1, of product 1 at facility 1, $Q_{111} = 2 \times 2324 = 4648$ units. Other results are given in table 1.

CONCLUDING REMARKS

An integrated production inventory model is developed in this paper, for a flow shop type multiproduct batch production system comprising of multiple facilities in series. The model simultaneously determines the optimal manufacturing cycle for the multiple products and the corresponding optimal procurement policies. The model assumes instantaneous production and no capacity bottleneck. Production rates and capacities are however important parameters of the production system and their implications to the production-inventory policy need further examination. Also, sensitivity analysis should help reveal the extent of gain accruing from the integrated model compared to the use of simple EOQ and EBQ models. Likewise, in situations where joint replenishment of raw materials from suppliers, is feasible, further economies could result from such a policy. Minor modification of the theoretical treatment presented in this paper should enable consideration of the possibility of joint replenishment.

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APPENDIX

Derivation of conditions for rounding off k_{ij}^*

The part of the total cost which depends upon k_{ij} is given by

$$C'_{ij} = \sum_{s=1}^{j} \left(\frac{A_{is}}{\left(\prod_{q=s}^{j} k_{iq}\right) T} + \frac{\left(\prod_{q=s}^{m} k_{iq}\right) \lambda_i T h_{is}}{2} \right)$$

$$+ \sum_{s=1}^{j} \sum_{l=1}^{m_{ij}} \left(\frac{A_{isl}}{\left(\prod_{q=s}^{m} k_{iq}\right) k_{isl} T} + \frac{k_{isl} x_{isl} h_{isl} \left(\prod_{q=s}^{m} k_{iq}\right)}{2} \right).$$

From the above, we can write expressions for $C'_{ij}(k_{ij})$, $C'_{ij}(k_{ij}+1)$ and $C'_{ij}(k^*_{ij})$, given below:

$$C'_{ij}(k_{ij}) = \left[\sum_{s=1}^{j} (F_{s1} + F_{s3})\right]\frac{1}{k_{ij}} + \left[\sum_{s=1}^{j} (F_{s2} + F_{s4})\right]k_{ij}$$

$$C'_{ij}(k^*_{ij}) = \left[\sum_{s=1}^{j} (F_{s1} + F_{s3})\right]\frac{1}{k^*_{ij}} + \left[\sum_{s=1}^{j} (F_{s2} + F_{s4})\right]k^*_{ij}$$

$$C'_{ij}(k_{ij}+1) = \left[\sum_{s=1}^{j} (F_{s1} + F_{s3})\right]\frac{1}{k_{ij}+1} + \left[\sum_{s=1}^{j} (F_{s2} + F_{s4})\right](k_{ij}+1).$$

In the above expressions, F_{s1}, F_{s2}, F_{s3} and F_{s4} are the following expressions:

$$F_{s1} = \frac{A_{is}}{\left(\prod_{q=s}^{j-1} k_{iq}\right)\left(\prod_{q=j+1}^{m} k_{iq}\right)T},$$

$$F_{s2} = \frac{\left(\prod_{q=s}^{j-1} k_{iq}\right)\left(\prod_{q=j+1}^{m} k_{iq}\right)\lambda_i T h_{is}}{2},$$

$$F_{s3} = \sum_{l=1}^{m_{ij}} \frac{A_{isl}}{\left(\prod_{q=s}^{j-1} k_{iq}\right)\left(\prod_{q=j+1}^{m} k_{iq}\right)k_{isl}T}$$

$$F_{s4} = \sum_{l=1}^{m_{ij}} \frac{k_{isl}x_{isl}h_{isl}\left(\prod_{q=s}^{j-1} k_{iq}\right)\left(\prod_{q=j+1}^{m} k_{iq}\right)T}{2}.$$

Now k^*_{ij} *is rounded off to* k_{ij},

i.e.

$$C'_{ij}(k_{ij}) - C'_{ij}(k_{ij}+1) < 0.$$

Substituting from the expressions developed above, this condition reduces to

$$k_{ij}(k_{ij}+1) > \frac{\sum_{s=1}^{j} (F_{s1} + F_{s3})}{\sum_{s=1}^{j} (F_{s2} + F_{s4})},$$

i.e.

$$k_{ij}(k_{ij}+1) > k^{*2}_{ij};$$

i.e.

$$k_{ij}(k_{ij}+1) > (k_{ij} + \delta_{ij})^2;$$

i.e.

$$\delta^2_{ij} + 2k_{ij}\delta_{ij} - k_{ij} < 0.$$

Clearly

$$\delta_{ij} < (-k_{ij} + \sqrt{k^2_{ij} + k_{ij}}).$$

Let

$$\delta_{ij}^* = -k_{ij} + \sqrt{k_{ij}^2 + k_{ij}}.$$

Therefore,

$$C'_{ij}(k_{ij}) < C'_{ij}(k_{ij} + 1) \quad \text{implies that } \delta_{ij} < \delta_{ij}^*$$

and

$$C'_{ij}(k_{ij}) > C'_{ij}(k_{ij} + 1) \quad \text{implies that } \delta_{ij} > \delta_{ij}^*.$$

Similarly,

$$C'_{ij}(k_{ij}) = C'_{ij}(k_{ij} + 1) \quad \text{implies that } \delta_{ij} = \delta_{ij}^*.$$

The conditions for rounding off k_{ijl}^* can be derived in a similar manner.

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