A PROPERTY OF ALGEBRAIC UNIVOQUE NUMBERS

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Abstract. Consider the set $\mathcal{U}$ of real numbers $q \geq 1$ for which only one sequence (c_i) of integers $0 \leq c_i \leq q$ satisfies the equality $\sum_{i=1}^{\infty} c_i q^{-i} = 1$. We show that the set of algebraic numbers in $\mathcal{U}$ is dense in the closure $\overline{\mathcal{U}}$ of $\mathcal{U}$.

1. Introduction

Given a real number $q \geq 1$, a q -*expansion* (or simply *expansion*) is a sequence $(c_i) = c_1 c_2 \dots$ of integers satisfying $0 \leq c_i \leq q$ for all $i \geq 1$ such that

$$\frac{c_1}{q} + \frac{c_2}{q^2} + \frac{c_3}{q^3} + \dots = 1.$$

The *greedy expansion*, denoted by $(\gamma_i(q)) = (\gamma_i)$, is obtained by performing the *greedy algorithm* of Rényi [11]: if γ_i is already defined for $i < n$, then γ_n is the largest integer satisfying

$$\sum_{i=1}^n \frac{\gamma_i}{q^i} \leq 1.$$

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Equivalently, (γ_i) is the largest expansion in lexicographical order.

If $q > 1$, then *quasi-greedy expansion*, denoted by $(\alpha_i(q)) = (\alpha_i)$, is obtained by performing the *quasi-greedy algorithm*: if α_i is already defined for $i < n$, then α_n is the largest integer satisfying

$$\sum_{i=1}^n \frac{\alpha_i}{q^i} < 1.$$

An expansion is called *infinite* if it contains infinitely many nonzero terms; otherwise it is called *finite*. Observe that there are no infinite expansions if $q = 1$: the only 1-expansions are given by $1(0)^\infty, 01(0)^\infty, 001(0)^\infty, \dots$. On the other hand, if $q > 1$, then (α_i) is the largest infinite expansion in lexicographical order.

For any given $q > 1$, the following relations between the quasi-greedy expansion and the greedy expansion are straightforward. The greedy expansion is finite if and only if (α_i) is periodic. If (γ_i) is finite and γ_m is its last nonzero term, then m is the smallest period of (α_i) , and

$$\alpha_i = \gamma_i \quad \text{for } i = 1, \dots, m-1, \quad \text{and} \quad \alpha_m = \gamma_m - 1.$$

Erdős, Horváth and Joó [4] discovered that for some real numbers $q > 1$ there exists only one q -expansion. Subsequently, the set $\mathcal{U}$ of such *univoque numbers* was characterized in [5], [6], [9] (see Theorem 2.1). Using this characterization, Komornik and Loreti showed in [7] that $\mathcal{U}$ has a smallest element $q' \approx 1.787$ and the corresponding expansion (τ_i) is given by the truncated Thue–Morse sequence, defined by setting $\tau_{2^N} = 1$ for $N = 0, 1, \dots$ and

$$\tau_{2^N+i} = 1 - \tau_i \quad \text{for } 1 \leq i < 2^N, \quad N = 1, 2, \dots$$

Allouche and Cosnard [1] proved that the number q' is transcendental. This raised the question whether there exists a smallest algebraic univoque number. Komornik, Loreti and Pethő [8] answered this question in the negative by constructing a decreasing sequence (q_n) of algebraic univoque numbers converging to q' .

It is the aim of this note to show that for each $q \in \mathcal{U}$ there exists a sequence of algebraic univoque numbers converging to q :

THEOREM 1.1. *The set $\mathcal{A}$ consisting of all algebraic univoque numbers is dense in $\overline{\mathcal{U}}$.*

Our proof of Theorem 1.1 relies on a characterization of the closure $\overline{\mathcal{U}}$ of $\mathcal{U}$, recently obtained by Komornik and Loreti in [9] (see Theorem 2.2).

2. Proof of Theorem 1.1

In the sequel, a sequence always means a sequence of nonnegative integers. We use systematically the lexicographical order between sequences; we write $(a_i) < (b_i)$ if there exists an index $n \geq 1$ such that $a_i = b_i$ for $i < n$ and $a_n < b_n$. This definition extends in the obvious way to sequences of finite length.

The following algebraic characterization of the set $\mathcal{U}$ can be found in [5], [6], [9]:

THEOREM 2.1. *The map $q \mapsto (\gamma_i(q))$ is a strictly increasing bijection between the set $\mathcal{U}$ and the set of all sequences (γ_i) satisfying*

$$(2.1) \quad \gamma_{j+1}\gamma_{j+2}\dots < \gamma_1\gamma_2\dots \quad \text{for all } j \geq 1$$

and

$$(2.2) \quad \overline{\gamma_{j+1}\gamma_{j+2}\dots} < \gamma_1\gamma_2\dots \quad \text{for all } j \geq 1$$

where we use the notation $\overline{\gamma_n} := \gamma_1 - \gamma_n$.

REMARK. It was essentially shown by Parry [10] that a sequence (γ_i) is the greedy q -expansion for some $q \geq 1$ if and only if (γ_i) satisfies the condition (2.1).

Using the above result, Komornik and Loreti [9] investigated the topological structure of the set $\mathcal{U}$. In particular they showed that $\overline{\mathcal{U}} \setminus \mathcal{U}$ is dense in $\overline{\mathcal{U}}$. Hence the set $\overline{\mathcal{U}}$ is a perfect set. Moreover, they established an analogous characterization of the closure $\overline{\mathcal{U}}$ of $\mathcal{U}$:

THEOREM 2.2. *The map $q \mapsto (\alpha_i(q))$ is a strictly increasing bijection between the set $\overline{\mathcal{U}}$ and the set of all sequences (α_i) satisfying*

$$(2.3) \quad \alpha_{j+1}\alpha_{j+2}\dots \leq \alpha_1\alpha_2\dots \quad \text{for all } j \geq 1$$

and

$$(2.4) \quad \overline{\alpha_{j+1}\alpha_{j+2}\dots} < \alpha_1\alpha_2\dots \quad \text{for all } j \geq 1$$

where we use the notation $\overline{\alpha_n} := \alpha_1 - \alpha_n$.

REMARKS. 1. It was shown in [3] that a sequence (α_i) is the quasi-greedy q -expansion for some $q > 1$ if and only if (α_i) is infinite and satisfies (2.3). Note also that a sequence satisfying (2.3) and (2.4) is automatically infinite.

2. If $q \in \overline{\mathcal{U}} \setminus \mathcal{U}$, then we must have equality in (2.3) for some $j \geq 1$, i.e., the greedy q -expansion is finite for each $q \in \overline{\mathcal{U}} \setminus \mathcal{U}$. On the other hand, it follows from Theorems 2.1 and 2.2 that a sequence of the form $(1^n 0)^\infty$ ($n \geq 2$) is the quasi-greedy q -expansion for some $q \in \overline{\mathcal{U}} \setminus \mathcal{U}$. Hence the set $\overline{\mathcal{U}} \setminus \mathcal{U}$ is countably infinite.

The following technical lemma is a direct consequence of Theorem 2.2 and Lemmas 3.4 and 4.1 in [9]:

LEMMA 2.3. *Let (α_i) be a sequence satisfying (2.3) and (2.4). Then*

(i) *there exist arbitrary large integers m such that*

$$(2.5) \quad \overline{\alpha_{j+1} \dots \alpha_m} < \alpha_1 \dots \alpha_{m-j} \quad \text{for all } 0 \leq j < m;$$

(ii) *for all positive integers $m \geq 1$,*

$$(2.6) \quad \overline{\alpha_1 \dots \alpha_m} < \alpha_{m+1} \dots \alpha_{2m}.$$

PROOF OF THEOREM 1.1. Since the set $\overline{\mathcal{U}} \setminus \mathcal{U}$ is dense in $\overline{\mathcal{U}}$, it is sufficient to show that $\overline{\mathcal{A}} \supset \overline{\mathcal{U}} \setminus \mathcal{U}$. In order to do so, fix $q \in \overline{\mathcal{U}} \setminus \mathcal{U}$. Then, according to Theorem 2.2, the quasi-greedy q -expansion (α_i) satisfies (2.3) and (2.4). Let k be a positive integer for which equality holds in (2.3), i.e., $(\alpha_i) = (\alpha_1 \dots \alpha_k)^\infty$. According to Lemma 2.3 there exists an integer $m \geq k$ such that (2.5) is satisfied. Let N be a positive integer such that $kN \geq m$ and consider the sequence

$$(\gamma_i) = (\gamma_i^N) = (\alpha_1 \dots \alpha_k)^N (\alpha_1 \dots \alpha_m \overline{\alpha_1 \dots \alpha_m})^\infty.$$

For ease of exposition we suppress the dependence of (γ_i) on N . Note that $\gamma_i = \alpha_i$ for $1 \leq i \leq m + kN$. In particular, we have

$$(2.7) \quad \gamma_i = \alpha_i \quad \text{for } 1 \leq i \leq 2m.$$

Since (γ_i) has a periodic tail, the number q_N determined by

$$1 = \sum_{i=1}^{\infty} \frac{\gamma_i}{q_N^i}$$

is an algebraic number and $q_N \rightarrow q$ as $N \rightarrow \infty$.

According to Theorem 2.1 it remains to verify the inequalities (2.1) and (2.2). First we verify (2.1) and (2.2) for $j \geq kN$. For those values of j the inequality (2.1) for $j + m$ is equivalent to (2.2) for j and (2.2) for $j + m$ is equivalent to (2.1) for j . Therefore it suffices to verify the inequalities (2.1) and (2.2) for $kN \leq j < kN + m$. Fix $kN \leq j < kN + m$. From (2.3), (2.6) and (2.7) we have

$$\begin{aligned} \gamma_{j+1} \dots \gamma_{kN+2m} &= \alpha_{j-kN+1} \dots \alpha_m \overline{\alpha_1 \dots \alpha_m} \\ &< \alpha_{j-kN+1} \dots \alpha_m \alpha_{m+1} \dots \alpha_{2m} \leq \alpha_1 \dots \alpha_{kN+2m-j} = \gamma_1 \dots \gamma_{kN+2m-j} \end{aligned}$$

and from inequality (2.5) we have

$$\overline{\gamma_{j+1} \cdots \gamma_{kN+m}} = \overline{\alpha_{j-kN+1} \cdots \alpha_m} < \alpha_1 \cdots \alpha_{kN+m-j} = \gamma_1 \cdots \gamma_{kN+m-j}.$$

Now we verify (2.1) for $j < kN$. If $m \leq j < kN$, then by (2.3) and (2.6),

$$\gamma_{j+1} \cdots \gamma_{kN+2m} < \alpha_{j+1} \cdots \alpha_{kN+2m} \leq \alpha_1 \cdots \alpha_{kN+2m-j} = \gamma_1 \cdots \gamma_{kN+2m-j}.$$

If $1 \leq j < m$, then by (2.3) and (2.5),

$$\begin{aligned} \gamma_{j+1} \cdots \gamma_{kN+m+j} &= \alpha_{j+1} \cdots \alpha_{kN+m} \overline{\alpha_1 \cdots \alpha_j} \\ &\leq \alpha_1 \cdots \alpha_{kN+m-j} \overline{\alpha_1 \cdots \alpha_j} < \alpha_1 \cdots \alpha_{kN+m-j} \alpha_{m-j+1} \cdots \alpha_m = \gamma_1 \cdots \gamma_{kN+m}. \end{aligned}$$

Finally, we verify (2.2) for $j < kN$. Write $j = k\ell + i$, $0 \leq \ell < N$ and $0 \leq i < k$. If $i = 0$, then (2.2) follows from the relation

$$\overline{\gamma_{j+1}} = \overline{\alpha_1} = 0 < \alpha_1 = \gamma_1.$$

If $1 \leq i < k$, then applying Lemma 2.3(ii) we get $\overline{\alpha_{i+1} \cdots \alpha_{2i}} < \alpha_1 \cdots \alpha_i$. Hence

$$\overline{\gamma_{j+1} \cdots \gamma_{j+k}} = \overline{\alpha_{j+1} \cdots \alpha_{j+k}} = \overline{\alpha_{i+1} \cdots \alpha_{i+k}} < \alpha_1 \cdots \alpha_k = \gamma_1 \cdots \gamma_k.$$

(In order for the first equality to hold in case $\ell = N - 1$, we need the condition $m \geq k$.) $\square$

REMARKS. 1. Since the set $\overline{\mathcal{U}}$ is a perfect set and $\overline{\mathcal{U}} \setminus \mathcal{U}$ is countable, each neighborhood of $q \in \mathcal{U}$ contains uncountably many elements of $\mathcal{U}$. Hence the set of transcendental univoque numbers is dense in $\overline{\mathcal{U}}$ as well.

2. Recently, Allouche, Frougny and Hare [2] proved that there also exist univoque Pisot numbers. In particular they determined the smallest three univoque Pisot numbers.

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