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Fourier Type Super Convergence Study on DDGIC and Symmetric DDG Methods

Mengping Zhang[1] · **Jue Yan**[2]

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Abstract In this paper, using Fourier analysis technique, we study the super convergence property of the DDGIC (Liu and Yan in Commun Comput Phys 8(3):541–564, 2010) and the symmetric DDG (Vidden and Yan in J Comput Math 31(6):638–662, 2013) methods for diffusion equation. With kth degree piecewise polynomials applied, the convergence to the solution's spatial derivative is kth order measured under regular norms. On the other hand when measuring the error in the weak sense or in its moment format, the error is super convergent with $(k + 2)$th and $(k + 3)$th orders for its first two moments with even order degree polynomial approximations. We carry out Fourier type (Von Neumann) error analysis and obtain the desired super convergent orders for the case of P^2 quadratic polynomial approximations. The theoretical predicted errors agree well with the numerical results.

Keywords Discontinuous Galerkin method · Diffusion equation · Stability · Consistency · Convergence · Super convergence

1 Introduction

In this article we apply Fourier analysis technique to investigate solution gradient's super convergence property for diffusion equations. We focus on the simple Heat equation under one-dimensional setting,

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✉ Jue Yan
jyan@iastate.edu

Mengping Zhang
mpzhang@ustc.edu.cn

1 Department of Mathematics, University of Science and Technology of China, Hefei 230026, Anhui, People's Republic of China

2 Department of Mathematics, Iowa State University, Ames, IA 50011, USA

$$u_t - u_{xx} = 0, \quad \text{for } x \in [0, 2\pi], \tag{1.1}$$

with zero Neumann boundary condition and initial condition $u(x, 0) = \cos(x)$. Understanding the analysis for the Heat equation is important for other applications such as the Keller–Segal equations of chemotaxis.

Recently in [11] we carry out the study on chemotaxis Keller–Segel equations with DDGIC [14] and symmetric DDG [20] methods and obtain some interesting results. Let's write out the one-dimensional Keller–Segel equations,

$$\begin{cases} \rho_t + (\chi c_x \rho)_x = \rho_{xx} \\ c_t = c_{xx} - c + \rho \end{cases}, \quad x \in (a, b) \text{ and } t > 0, \tag{1.2}$$

to illustrate the major difficulties and issues solving (1.2) which in return motivate the super convergence study in this article. We have two solution variables with $\rho(x, t)$ denoting the cell density and $c(x, t)$ denoting the chemoattractant concentration. And $\chi = 1$ is the chemotactic sensitivity constant. We have zero Neumann boundary conditions associated with problem (1.2). We see the equation for $\rho(x, t)$ involves the spatial derivative of the concentration variable $c_x(x, t)$, thus direct spatial discretization of (1.2) leads to one order loss for the approximation of the cell density $\rho(\cdot, t)$. It is natural to introduce one extra variable to approximate $c_x(\cdot, t)$ separately and plug it into the equation for $\rho(x, t)$ to obtain uniform optimal convergence of (1.2), see [7,8,12].

In [11] we apply the DDGIC method [14] and the symmetric DDG method [20] to directly discretize the system (1.2) with no extra variable introduced and numerically we obtain optimal convergence for both $\rho(\cdot, t)$ and $c(\cdot, t)$. Notice that the classical SIPG method [1] directly applied to (1.2) leads to the expected one order loss, see [7]. It turns out that our diffusion solver the DDGIC method or the symmetric DDG method have the hidden super convergence property on it's approximation to the solution's spatial derivative $c_x(\cdot, t)$. Even the super convergence is in the weak sense or in its moment format, it is sufficient to guarantee the optimal convergence of $\rho(x, t)$ in (1.2).

In this paper we consider to perform Fourier type error analysis to study solution gradient's super convergence property of the DDGIC and symmetric DDG methods. Fourier analysis is a technique to study stability and error estimates for discontinuous Galerkin method and other related schemes, especially in some cases where standard finite element technique can not be applied. The Fourier analysis does have several advantages over the standard finite element techniques. It can be used to analyze some of the bad schemes [22]; it can be used for stability analysis for some of the non-standard methods such as the special volume (SV) method [23], which belongs to the class of Petrov–Galerkin methods and cannot be easily amended to the standard finite element analysis framework; it can provide quantitative error comparisons among different schemes [18,19]; and it can be used to prove superconvergence and time evolution of errors for the DG method [5,6,9,25].

In [24] we apply Fourier analysis technique to study the original DDG method [13] and prove that optimal order convergence can be obtained with the proper choice of (β_0, β_1) coefficients chosen in the numerical flux. In [24] the Fourier type analysis also shows that non-symmetric DDG method [21] converges with optimal order, which improves the order loss of Baummann–Oden method [2] and NIPG method [15].

In this article we carry out Fourier type error estimate on approximating the solution's spatial derivative, namely $u_x(\cdot, t)$ of (1.1) with the SIPG method [1], the DDGIC method [14] and the symmetric DDG method [20]. Due to limited symbolic computing power, we focus on the case of P^2 quadratic polynomial approximations which is good enough to illustrate the super convergence properties of the DDG methods. Here we use $u_h(\cdot, t)$ to denote the

DG polynomial numerical solution. We carry out Fourier type error analysis on each method and analytically calculate the following moment error:

$$ME_m\left[u_x-(u_h)_x\right]=\max_{1\le j\le N}\left|\frac{\int_{I_j}(u_x-(u_h)_x)\,v(x)\,dx}{\|v\|_{L_1}}\right|, \tag{1.3}$$

between $u_x(\cdot,t)$ and $(u_h)_x(\cdot,t)$. Here I_j is the jth indexed computational cell and $v(x)$ is a m degree polynomial on cell I_j. With P^2 approximations, we expect the errors of (1.3) are on the level of 2nd (kth) order. For $m=0$ of the moment error (1.3), corresponding to the average error, we obtain the expected 2nd order convergence with the SIPG method [1]. On the other hand, we obtain $(k+2)=4$th order super convergence with the DDGIC [14] and the symmetric DDG method [20] methods. For $m=1$ case of (1.3), we obtain $(k+3)=5$th order super convergence with the DDGIC and the symmetric DDG methods. All super convergence errors and orders are theoretically verified through Fourier analysis technique and the predicted errors match well with the numerical results. We should highlight that our diffusion solver the DDGIC method [14] is closely related to the SIPG method [1]. The DDGIC method is different to the SIPG method only when high order polynomials (P^k with $k\ge 2$) are considered.

Notice that the errors of $\|u_x-(u_h)_x\|$ measured under strong sense, for example under the standard $\|\cdot\|_{L^2}$, $\|\cdot\|_{L^1}$ and $\|\cdot\|_{L^\infty}$ norms, are all of 2nd order with P^2 polynomial approximations. The super convergence to the solution's spatial derivative, even in the weak sense as listed in (1.3), is sufficient to guarantee the optimal convergence of the cell density variable $\rho(\cdot,t)$ in the Keller–Segel system (1.2). For the average error corresponding to $m=0$ in (1.3), numerically we obtain $(k+2)th$ order super convergence with all $k=\textit{even}$ degree polynomial approximations. The super convergence result is observed with $\beta_1=1/2k(k+1)$ chosen in the numerical flux. With $k=\textit{odd}$ degree polynomials approximations, we obtain $(k+1)th$ order super convergence for the average error and the super convergence is not sensitive to the choice of β_1 coefficient. In a word, any admissible coefficients in the numerical flux lead to the super convergence phenomena.

This rest of the paper is organized as follows. In Sect. 2 we describe the three DG methods for the model Heat equation (1.1). In Sect. 3 we write out the Fourier analysis technique and lay out the details of the ODE system for each method. Then we symbolically calculate the moment errors and show the analytical predictions match well with the numerical simulations. Finally some concluding remarks are given in Sect. 4.

2 Three Discontinuous Galerkin Methods

Let's first introduce the notations. We have $I_j=[x_{j-\frac12},x_{j+\frac12}]$, $j=1,\dots,N$ denoting a partition of the domain $[0,2\pi]$, with $x_{\frac12}=0$ and $x_{N+\frac12}=2\pi$. We denote the center of each cell by $x_j=\frac12\left(x_{j-\frac12}+x_{j+\frac12}\right)$ and the size of each cell by $h_j=x_{j+\frac12}-x_{j-\frac12}$. The computational cells are not supposed to be uniform for the numerical methods. For simplicity of analysis we only consider uniform meshes in this paper and we use h to denote the uniform mesh size. We have $P^k(I_j)$ representing the polynomial space on cell I_j with degree at most k. The DG solution space is defined as:

$$\mathbb{V}_h^k:=\left\{v\in L^2(0,2\pi)\colon v|_{I_j}\in P^k(I_j),\ \ j=1,\dots,N\right\}.$$

For $v \in \mathbb{V}_h^k$, we adopt following notations to denote the jump and average of v across the cell interface $x_{j+1/2}$:

$$v^{\pm} = v(x \pm 0, t), \quad [\![v]\!] = v^+ - v^-, \quad \{\{v\}\} = \frac{v^+ + v^-}{2}.$$

Now we are ready to formulate the three DG schemes, namely the SIPG method [1], the DDGIC method [14] and the symmetric DDG method [20] to solve (1.1). In [13] we introduce a numerical flux concept $\widehat{(u_h)_x}$ to approximate $u_x(\cdot, t)$ at the cell interface $x_{j+1/2}$ and obtain a new diffusion solver called direct discontinuous Galerkin (DDG) method. It is hard to identify suitable coefficients in the numerical flux formula with higher order approximations ($k \geq 4$), thus we modify the DDG method with extra interface terms included and obtain the DDGIC method in [14]. To obtain $L^2(L^2)$ error estimate, we further introduce same format numerical flux for the test function $\widetilde{v_x}$ and obtain the symmetric DDG method in [20]. The symmetric DDG method is the one that have symmetric structure for the bilinear form.

It turns out that the DDGIC method is closely related to the classical SIPG method [1]. The major difference of the two is that we have the numerical flux $\widehat{(u_h)_x}$ introduced and formulated as:

$$\widehat{(u_h)_x} = \beta_0 \frac{[\![u_h]\!]}{h} + \{\{(u_h)_x\}\} + \beta_1 \Delta x [\![(u_h)_{xx}]\!]. \tag{2.1}$$

In [14] we show any admissible coefficient pair (β_0, β_1) leads to the optimal convergence of the DDGIC method. With the second derivative jump term $[\![(u_h)_{xx}]\!]$ dropped out from (2.1), the DDGIC method degenerates to the SIPG method. On the other hand, with the second derivative jump term included in the numerical flux, we do obtain quite a few advantages of DDGIC method over the SIPG method. Numerically we observe that only a small fixed penalty coefficient ($\beta_0 = 2$) is needed to stabilize the DDGIC scheme for P^k ($k \leq 9$) polynomial approximations. It is well known that the penalty coefficient [β_0 in (2.1)] of the SIPG method needs to be large enough, roughly $\beta_0 \approx k^2/2$, to stabilize the scheme. Under the topic of maximum principle, third order DDGIC numerical solution can be proved to satisfy strict maximum principle even on unstructured triangular meshes [4], while for SIPG method only second order piecewise linear polynomial solution can be proved to satisfy maximum principle.

Including the second derivative jump term $[\![(u_h)_{xx}]\!]$ in (2.1) does give DDGIC or symmetric DDG methods extra flexibility. In this paper we show that with the proper choice of β_1 coefficient in the numerical flux (2.1), extra super convergence property can be obtained with DDGIC and symmetric DDG methods. Now we adopt the notation of the numerical flux $\widehat{(u_h)_x}$ of (2.1) to uniformly write out the SIPG, DDGIC and symmetric DDG methods solving (1.1).

2.1 SIPG Method [1]

We multiply the Heat equation (1.1) with an arbitrary test function $v \in \mathbb{V}_{\Delta x}$, integrate over the cell I_j, have the integration by parts and add test function interface terms to symmetrize the diffusion term, formally we have the SIPG method written out as:

$$\begin{cases} \int_{I_j} (u_h)_t v dx - \widehat{(u_h)_x} v^- \big|_{j+\frac{1}{2}} + \widehat{(u_h)_x} v^+ \big|_{j-\frac{1}{2}} + \int_{I_j} (u_h)_x v_x dx \\ \qquad + \frac{1}{2} [\![u_h]\!] (v_x)^-_{j+\frac{1}{2}} + \frac{1}{2} [\![u_h]\!] (v_x)^+_{j-\frac{1}{2}} = 0, \\ \widehat{(u_h)_x} = \beta_0 \frac{[\![u_h]\!]}{h} + \{\{(u_h)_x\}\}, \quad \text{at } x_{j\pm 1/2}. \end{cases} \tag{2.2}$$

Summing (2.2) over all I_j, we have the primary formulation of the SIPG method as,

$$\int_0^{2\pi} (u_h)_t v dx + \mathbb{B}(u_h, v) = 0,$$

with the bilinear form laid out as:

$$\mathbb{B}(u_h, v) = \sum_{j=1}^{N} \int_{I_j} (u_h)_x v_x \, dx + \sum_{j=1}^{N} \left(\{\{(u_h)_x\}\} [\![v]\!] + \{\{v_x\}\} [\![u_h]\!] + \beta_0 \frac{[\![u_h]\!][\![v]\!]}{h} \right)_{j+1/2}.$$

Suppose periodic boundary condition is considered here. Obviously the bilinear form is symmetric with $\mathbb{B}(u_h, v) = \mathbb{B}(v, u_h)$.

2.2 DDGIC Method [14]

To solve the model Heat equation (1.1) with the DDGIC method [14], we multiply the equation with test function $v \in \mathbb{V}_{\Delta x}$ and integrate over the computational cell I_j. We perform the integration by parts, add interface terms involving test function derivative $\{\{v_x\}\}$ and we obtain the DDGIC scheme formulation as below,

$$\begin{cases} \int_{I_j} (u_h)_t v dx - \widehat{(u_h)_x} v^- \big|_{j+\frac{1}{2}} + \widehat{(u_h)_x} v^+ \big|_{j-\frac{1}{2}} + \int_{I_j} (u_h)_x v_x dx \\ \qquad + \frac{1}{2} [\![u_h]\!] (v_x)^-_{j+\frac{1}{2}} + \frac{1}{2} [\![u_h]\!] (v_x)^+_{j-\frac{1}{2}} = 0, \\ \widehat{(u_h)_x} = \beta_0 \frac{[\![u_h]\!]}{h} + \{\{(u_h)_x\}\} + \beta_1 h [\![(u_h)_{xx}]\!], \quad \text{at} \quad x_{j\pm1/2}. \end{cases} \tag{2.3}$$

We see the only difference between the DDGIC method (2.3) and the SIPG method (2.2) is that we include the solution's second derivative jump term $[\![(u_h)_{xx}]\!]$ in the numerical flux $\widehat{(u_h)_x}$. In a word, the DDGIC method is only different to the SIPG method when high order P^k ($k \geq 2$) polynomials are applied. The DDGIC method degenerates to the SIPG method when piecewise constant and linear polynomials are considered.

2.3 Symmetric DDG Method [20]

Now we introduce test function numerical flux $\widetilde{v_x}$ and refine the interface correction terms to obtain a symmetric DDG method as,

$$\int_{I_j} (u_h)_t v dx - \widehat{(u_h)_x} v^- \big|_{j+\frac{1}{2}} + \widehat{(u_h)_x} v^+ \big|_{j-\frac{1}{2}} + \int_{I_j} (u_h)_x v_x dx \\ + \widetilde{v_x} [\![u_h]\!]_{j+\frac{1}{2}} + \widetilde{v_x} [\![u_h]\!]_{j-\frac{1}{2}} = 0, \tag{2.4}$$

with numerical flux,

$$\begin{cases} \widehat{(u_h)_x} = \beta_{0u} \frac{[\![u_h]\!]}{h} + \{\{(u_h)_x\}\} + \beta_1 h [\![(u_h)_{xx}]\!] \\ \widetilde{v_x} = \beta_{0v} \frac{[\![v]\!]}{h} + \{\{v_x\}\} + \beta_1 h [\![v_{xx}]\!] \end{cases}, \text{ at } x_{j\pm1/2}. \tag{2.5}$$

Notice that the test function v is taken to be non-zero only inside the cell I_j. When evaluating $\widetilde{v_x}$ at cell interfaces $x_{j\pm1/2}$, only the quantities from the side of I_j contribute to the calculation of $\widetilde{v_x}$. For example at $x_{j+\frac{1}{2}}$ we have,

$$\widetilde{v_x}|_{j+\frac{1}{2}} = \beta_{0v} \frac{-v^-}{h} + \frac{v_x^-}{2} + \beta_1 h(-v_{xx}^-).$$

Summing (2.4) over all computational cells I_j, we have the primal formulation of symmetric DDG method solving (1.1) as,

$$\int_0^{2\pi} (u_h)_t v dx + \mathbb{B}(u_h, v) = 0,$$

where the bilinear form is defined as,

$$\mathbb{B}(u_h, v) = \sum_{j=1}^{N} \int_{I_j} (u_h)_x v_x \, dx + \sum_{j=1}^{N} \left(\widehat{(u_h)_x} [\![v]\!] + \widetilde{v_x} [\![u_h]\!] \right)_{j+\frac{1}{2}}.$$

We see the bilinear form $\mathbb{B}(u_h, v) = \mathbb{B}(v, u_h)$ is symmetric for the symmetric DDG method.

For time dependent parabolic problems, we find out the DDGIC method [14] is the most efficient diffusion solver. For time independent elliptic type problems, the symmetric DDG method [10] is more efficient. Among the four DDG methods [13,14,20,21] the symmetric DDG method is the only one that gives the mass matrix being symmetric. The non-symmetric mass matrix of the DDGIC method may cause extra issues when fast solvers are applied. In [10] we observe symmetric DDG method also gives the smallest condition number on the mass matrix. When comparing to the SIPG method, the symmetric DDG method saves 7–10% CPU running time, especially for high order polynomial approximations. In [10] we also observe that the symmetric DDG method resolves the oscillatory wave much better than the DDGIC method.

Up to now, we have taken the method of lines approach and have left time variable t continuous. For time discretization, TVD Runge–Kutta method [16,17] is used to solve the ODE to match the accuracy in space,

$$(u_h)_t = L(u_h). \tag{2.6}$$

Specifically the third-order TVD Runge–Kutta method that we use in this paper is given by,

$$\begin{aligned} u_h^{(1)} &= u_h^n + \Delta t L(u_h^n), \\ u_h^{(2)} &= \frac{3}{4} u_h^n + \frac{1}{4} \left(u_h^{(1)} + \Delta t L(u_h^{(1)}) \right), \\ u_h^{n+1} &= \frac{1}{3} u_h^n + \frac{2}{3} \left(u_h^{(2)} + \Delta t L(u_h^{(2)}) \right). \end{aligned}$$

3 Fourier Analysis for the Moment Errors

In this section we write out the SIPG method, the DDGIC method and the symmetric DDG method in details and lay out the three DG methods as finite difference schemes. We need the assumption of uniform mesh and still zero Neumann boundary condition is considered. Treated as finite difference methods, we then perform the standard Von Neumann Fourier analysis and symbolically calculate the moment errors for each method and finally compare the analytical errors with the numerical ones.

We use the SIPG method (2.2) to demonstrate the procedure of the Fourier analysis. After picking a local basis for the solution space $\mathbb{V}_h^h$ and inverting a local $(k+1) \times (k+1)$ mass matrix (which could be done by hand), the DG method of (2.2) can be rewritten out as:

$$\frac{d}{dt} \vec{u}_j = \frac{1}{h^2} \left(A \vec{u}_{j-1} + B \vec{u}_j + C \vec{u}_{j+1} \right), \tag{3.1}$$

where $\vec{u}_j$ is a small vector of size $k+1$ containing the coefficients of the DG solution $u_h(\cdot,t)$ in the local basis inside cell I_j and A, B and C are $(k+1)\times(k+1)$ constant matrices. If we choose the point values of the solution $u_h(\cdot,t)$ inside cell I_j as the degree of freedom, denoted by

$$u_{j+\frac{2i-k}{2(k+1)}}, \qquad i=0,\ldots,k,$$

at the $k+1$ equally spaced points, then the DG method, rewritten in terms of these degrees of freedom, can be considered as a finite difference scheme on a globally uniform mesh (with mesh size $h/(k+1)$); however they are not standard finite difference schemes because each point in the group of $k+1$ points belonging to the cell I_j obeys a different form of finite difference scheme. Since we focus on P^2 quadratic polynomial approximations, let us discuss the procedure in detail with $k=2$. The degree of freedom are now the point values at the $3N$ uniformly spaced points,

$$u_{j-\frac{1}{3}},\, u_j,\, u_{j+\frac{1}{3}}, \quad j=1,\ldots,N.$$

The DG polynomial solution inside cell I_j is then represented by,

$$u_h(x,t)=u_{j-\frac{1}{3}}(t)\phi_{j-\frac{1}{3}}(x)+u_j(t)\phi_j(x)+u_{j+\frac{1}{3}}(t)\phi_{j+\frac{1}{3}}(x), \tag{3.2}$$

where $\phi_{j-\frac{1}{3}}(x), \phi_j(x), \phi_{j+\frac{1}{3}}(x)$ are the Lagrange interpolation polynomials at points $x_{j-\frac{1}{3}}, x_j$ and $x_{j+\frac{1}{3}}$. Now we obtain the finite difference representation of the SIPG method (2.2) as in (3.1) with the numerical solution vector defined as,

$$\vec{u}_j=\begin{pmatrix} u_{j-\frac{1}{3}}(t)\\ u_j(t)\\ u_{j+\frac{1}{3}}(t)\end{pmatrix}, \quad \text{on cell } I_j. \tag{3.3}$$

As a finite difference method, we perform the following Von Neumann Fourier analysis. The analysis depends on the assumption of uniform mesh and periodic boundary condition. We make an ansatz of the solution with the form,

$$u_j(t)=\hat{u}_j(t)e^{ix_j}, \tag{3.4}$$

and substitute the ansatz of (3.4) into the SIPG scheme (3.1) to find the coefficient vector evolving in time as,

$$\frac{d}{dt}\begin{pmatrix} \hat{u}_{j-\frac{1}{3}}(t)\\ \hat{u}_j(t)\\ \hat{u}_{j+\frac{1}{3}}(t)\end{pmatrix}=G(h)\begin{pmatrix} \hat{u}_{j-\frac{1}{3}}(t)\\ \hat{u}_j(t)\\ \hat{u}_{j+\frac{1}{3}}(t)\end{pmatrix}. \tag{3.5}$$

The amplification matrix $G(h)$ is given by,

$$G(h)=\frac{1}{h^2}\left(A\,e^{-ih}+\,B\,+\,C\,e^{ih}\right). \tag{3.6}$$

The matrices A, B, C are from (3.1) which is the rewritten of the SIPG scheme (2.2) with Lagrange interpolation polynomials as basis functions.

The general solution of the ODE system (3.5) is given by,

$$\begin{pmatrix} \hat{u}_{j-\frac{1}{3}}(t)\\ \hat{u}_j(t)\\ \hat{u}_{j+\frac{1}{3}}(t)\end{pmatrix}=a_1\,e^{\lambda_1 t}\,V_1\,+a_2\,e^{\lambda_2 t}\,V_2+a_3\,e^{\lambda_3 t}\,V_3, \tag{3.7}$$

where $\lambda_1, \lambda_2, \lambda_3$, and V_1, V_2, V_3 are the eigenvalues and the corresponding eigenvectors of the amplification matrix $G(h)$ respectively.

To fit the initial condition $u(x, 0) = \cos(x)$, we set,

$$u_{j-\frac{1}{3}}(0) = e^{i\,x_{j-\frac{1}{3}}},\ u_j(0) = e^{i\,x_j},\ u_{j+\frac{1}{3}}(0) = e^{i\,x_{j+\frac{1}{3}}},$$

whose real part is the given initial condition of the model Heat equation (1.1). Thus we require, at $t = 0$,

$$\begin{pmatrix} \hat{u}_{j-\frac{1}{3}}(0) \\ \hat{u}_j(0) \\ \hat{u}_{j+\frac{1}{3}}(0) \end{pmatrix} = \begin{pmatrix} e^{-i\,\frac{h}{3}} \\ 1 \\ e^{i\,\frac{h}{3}} \end{pmatrix} e^{i\,x_j}.$$

Fitting the initial condition determines the coefficients a_1, a_2, and a_3 in the general solution (3.7). Thus we can explicitly write out the solution expression (3.2) for the SIPG method. With the exact solution spatial derivative given as $u_x(\cdot, t) = -e^{-t}\sin(x)$, we can symbolically calculate the moment errors of (1.3). Through the simple Taylor expansions, now we are able to find out the leading error terms in the moment errors expression. In the following sections we compare the predicted errors by Fourier analysis to the errors numerically calculated from the SIPG, DDGIC and symmetric DDG methods. We should mention this is not a standard error estimate technique, yet the Fourier type error analysis is a very powerful tool and can carry out error estimate for problems that can be not analyzed through standard finite element technique.

In this paper we study the super convergence property of the DDGIC and the symmetric methods under the moment errors (1.3). Specifically the test function on cell I_j is taken as $v = (\frac{x-x_j}{h/2})^m$ with $m = 0, 1, \ldots, 2k-1$ as the polynomial power. The moment error of (1.3) is laid out in detail as the following:

$$ME_m\,[u_x - (u_h)_x] = \max_{1\le j\le N} \left| \frac{\int_{I_j}((u_h)_x(\cdot, t) - u_x(\cdot, t))v(x)\,dx}{\|v\|_{L_1}} \right|,$$
$$v = \left(\frac{x - x_j}{h/2}\right)^m. \tag{3.8}$$

Here the exact solution is given with $u(x, t) = e^{-t}\cos(x)$.

We first consider to apply Fourier analysis to symbolically calculate the solution derivative moment errors (3.8) with different moment powers for the SIPG, DDGIC and symmetric DDG methods. Then we numerically compute the moment errors (3.8) with the three DG methods and carry out comparisons to the estimated errors through Fourier analysis. Notice that numerical solutions are obtained with third order TVD Runge–Kutta method (2.6) for time discretization. We choose smaller time step size to make sure spatial errors are dominant. For the average error corresponding to $m = 0$ in (3.8), an expected 2rd order convergence is obtained for the SIPG method with P^2 quadratic polynomial approximations. On the other hand we obtain 4th order super convergence for the average error with the DDGIC method and the symmetric DDG method. For the moment error of (3.8) with $m = 1$, we obtain 5th order super convergence with the DDGIC and the symmetric DDG methods. All super convergence orders are verified through analytical error estimate and numerical simulations.

3.1 SIDG Method of (2.2)

In this section we perform Fourier analysis on the SIPG method (2.2) with the numerical flux taken in the form of,

$$\widehat{(u_h)_x} = \beta_0 \frac{[\![u_h]\!]}{h} + \{\!\{(u_h)_x\}\!\}.$$

The matrices A, B, C in (3.1) for the SIPG method are:

$$A = \frac{1}{48}\begin{pmatrix} 3(-119+23\beta_0) & 2(549-115\beta_0) & 3(-319+115\beta_0) \\ -27(-9+\beta_0) & 18(-43+5\beta_0) & -27(-33+5\beta_0) \\ 3(1-\beta_0) & 2(-3+5\beta_0) & 3(-7-5\beta_0) \end{pmatrix},$$

$$B = \frac{1}{24}\begin{pmatrix} -9(-17+19\beta_0) & 2(-93+55\beta_0) & -9(-17+3\beta_0) \\ 27(-1+3\beta_0) & -18(17+5\beta_0) & 27(-1+3\beta_0) \\ -9(-17+3\beta_0) & 2(-93+55\beta_0) & -9(-17+19\beta_0) \end{pmatrix}, \quad (3.9)$$

$$C = \frac{1}{48}\begin{pmatrix} 3(-7-5\beta_0) & 2(-3+5\beta_0) & 3(1-\beta_0) \\ -27(-33+5\beta_0) & 18(-43+5\beta_0) & -27(-9+\beta_0) \\ 3(-319+115\beta_0) & 2(549-115\beta_0) & 3(-119+23\beta_0) \end{pmatrix}.$$

We take $\beta_0 = 4$ for the SIPG method and the same coefficient value $\beta_0 = 4$ is applied in the following Sect. 3.2 for the the DDGIC method and Sect. 3.3 for the symmetric DDG method. In Table 1 we list the analytically calculated moment errors (3.8) with the SIPG method through Fourier technique. For the average error [(3.8) with $m = 0$] we see the leading order is of 2nd order which is an expected order for the SIPG method with quadratic polynomial approximations. No super convergence phenomena is observed in this case. In

Table 1 Analytically calculated moment errors (3.8) with the SIPG method (3.1) and (3.9)

	$\beta_0 = 4$
$m = 0$	$\left(-\frac{1}{36}e^{-t}\sin(x_j)\right)h^2 + O(h^4)$
$m = 1$	$\left(\frac{1}{72}e^{-t}\sin(x_j)\right)h^3 + O(h^5)$
$m = 2$	$\left(-\frac{11}{540}e^{-t}\sin(x_j)\right)h^2 + O(h^4)$
$m = 3$	$\left(\frac{37}{4200}e^{-t}\sin(x_j)\right)h^3 + O(h^5)$

Table 2 SIPG method moment errors (3.8) comparison: numerical solution moment errors from scheme (2.2) and estimated errors from Table 1

	Numerical solutions		Predicted by analysis	
	Error	Order	Error	Order
	$m = 0$			
$N = 10$	0.65153E−02	–	0.66514E−02	–
$N = 20$	0.16343E−02	1.99	0.16424E−02	2.00
$N = 40$	0.41392E−03	1.98	0.41443E−03	2.00
$N = 80$	0.10382E−03	1.99	0.10385E−03	2.00
	$m = 1$			
$N = 10$	0.40371E−02	–	0.41792E−02	–
$N = 20$	0.51798E−03	2.96	0.51596E−03	3.00
$N = 40$	0.65162E−04	2.99	0.65098E−04	3.00
$N = 80$	0.81581E−05	2.99	0.81561E−05	3.00

Final time t = 0.5

Table 2 we list the moment errors numerically computed with the SIPG method (2.2) and the ones calculated from Fourier analysis, namely the leading error terms in Table 1. The two groups of errors match very well.

3.2 DDGIC Method of (2.3)

In this section we perform Fourier type error analysis on the DDGIC method (2.3) with quadratic P^2 polynomial approximations. The matrix vector format of the DDGIC method is given as (3.1) with the matrices A, B, C laid out as the following:

$$A = \frac{1}{48}\begin{pmatrix} 3(-119+23\beta_0+552\beta_1) & 2(549-115\beta_0-1656\beta_1) & 3(-319+115\beta_0+552\beta_1) \\ -27(-9+\beta_0+24\beta_1) & 18(-43+5\beta_0+72\beta_1) & -27(-33+5\beta_0+24\beta_1) \\ 3(1-\beta_0-24\beta_1) & 2(-3+5\beta_0+72\beta_1) & 3(-7-5\beta_0-24\beta_1) \end{pmatrix},$$
$$B = \frac{1}{24}\begin{pmatrix} -9(-17+19\beta_0+88\beta_1) & 2(-93+55\beta_0+792\beta_1) & -9(-17+3\beta_0+88\beta_1) \\ 27(-1+3\beta_0+24\beta_1) & -18(17+5\beta_0+72\beta_1) & 27(-1+3\beta_0+24\beta_1) \\ -9(-17+3\beta_0+88\beta_1) & 2(-93+55\beta_0+792\beta_1) & -9(-17+19\beta_0+88\beta_1) \end{pmatrix},$$
$$C = \frac{1}{48}\begin{pmatrix} 3(-7-5\beta_0-24\beta_1) & 2(-3+5\beta_0+72\beta_1) & 3(1-\beta_0-24\beta_1) \\ -27(-33+5\beta_0+24\beta_1) & 18(-43+5\beta_0+72\beta_1) & -27(-9+\beta_0+24\beta_1) \\ 3(-319+115\beta_0+552\beta_1) & 2(549-115\beta_0-1656\beta_1) & 3(-119+23\beta_0+552\beta_1) \end{pmatrix}. \tag{3.10}$$

We study the super convergence property of the DDGIC method with different choices of (β_0, β_1) coefficient in the numerical flux $\widehat{(u_h)_x}$ (2.3). The proper choice of β_1 coefficient leads to the super convergence on its approximation to the solution's derivative measured under the moment errors format of (3.8). Specifically we study two cases with $(\beta_0, \beta_1) = (4, \frac{1}{8})$ and $(\beta_0, \beta_1) = (4, \frac{1}{12})$. We symbolically calculate the moment errors (3.8) through Fourier analysis tool and we list the leading errors in Table 3.

With $\beta_1 = \frac{1}{2k(k+1)} = \frac{1}{12}$ chosen in the numerical flux $\widehat{(u_h)_x}$ of the DDGIC method (2.3), we see the average error ($m = 0$ case) is super convergent with $(k + 2) = 4$th order (the leading error term in Table 3). The moment error of (3.8) with $m = 1$ is super convergent with $(k+3) = 5$th order. Through finite element technique, authors in [3] prove that on Gaussian points $(k + 1)th$ order super convergence can be obtained with $\beta_1 = \frac{1}{2k(k+1)}$. Same β_1 coefficient formula works in our study on the moment errors. Different to [3], we do not need special initialization to guarantee super convergence results. Numerical experiments show Taylor expansion initialization, L^2 initialization and Lagrange interpolation initialization all lead to the observed super convergence orders. We should also mention that for higher order moment errors, for example $m = 2$ of (3.8), no super convergence is observed.

We apply DDGIC scheme (2.3) to numerically solve the Heat equation and calculate the moment errors (3.8). In Table 4 we compare the numerical solution moment errors and the errors predicted by Fourier analysis. The numerical errors match very well with the analytical ones.

3.3 Symmetric DDG Method

In this section we perform Fourier analysis on the symmetric DDG method (2.4) with numerical flux (2.5). Here we have the notation of $\beta_0 = \beta_{0u} + \beta_{0v}$. Considering the rewritten of (2.4) in terms of matrix vector format (3.1), we have the matrices of A, B, C for the symmetric DDG method listed below as:

Table 3 Analytically calculated moment errors (3.8) with the DDGIC method (3.1) and (3.10)

	$\beta_1 = \frac{1}{8}$	$\beta_1 = \frac{1}{12}$
$m = 0$	$\left(\frac{1}{72} e^{-t} \sin(x_j)\right) h^2 + O(h^4)$	$\left(\frac{24t-7}{17280} e^{-t} \sin(x_j)\right) h^4 + O(h^6)$
$m = 1$	$-\left(\frac{1}{144} e^{-t} \sin(x_j)\right) h^3 + O(h^5)$	$-\left(\frac{667+840t}{3628800} e^{-t} \sin(x_j)\right) h^5 + O(h^7)$
$m = 2$	$\left(\frac{7}{1080} e^{-t} \sin(x_j)\right) h^2 + O(h^4)$	$-\left(\frac{1}{90} e^{-t} \sin(x_j)\right) h^2 + O(h^4)$
$m = 3$	$-\left(\frac{31}{8400} e^{-t} \sin(x_j)\right) h^3 + O(h^5)$	$\left(\frac{1}{2100} e^{-t} \sin(x_j)\right) h^3 + O(h^5)$

Table 4 DDGIC method moment errors (3.8) with $\beta_1 = 1/8$ or $\beta_1 = 1/12$ in $\widehat{(u_h)_x}$

	$\beta_1 = \frac{1}{8}$				$\beta_1 = \frac{1}{12}$			
	Numerical solutions		Predicted by analysis		Numerical solutions		Predicted by analysis	
	Error	Order	Error	Order	Error	Order	Error	Order
	$m = 0$							
$N = 10$	3.13e−03	–	3.32e−03	–	2.65e−05	–	2.73e−05	–
$N = 20$	8.09e−04	1.95	8.21e−04	2.00	1.67e−06	3.98	1.68e−06	4.00
$N = 40$	2.06e−04	1.97	2.07e−04	2.00	1.06e−07	3.98	1.06e−07	4.00
$N = 80$	5.18e−05	1.99	5.19e−05	2.00	6.67e−09	3.99	6.67e−09	4.00
	$m = 1$							
$N = 10$	1.97e−03	–	2.08e−03	–	3.39e−05	–	3.55e−05	–
$N = 20$	2.57e−04	2.93	2.57e−04	3.00	1.09e−06	4.94	1.09e−06	5.00
$N = 40$	3.25e−05	2.98	3.25e−05	3.00	3.46e−08	4.98	3.46e−08	5.00
$N = 80$	4.07e−06	2.99	4.07e−06	3.00	1.08e−09	4.99	1.08e−09	5.00

Numerical errors from scheme (2.3) and analytical errors from Table 3. Final time t = 0.5

$$
A = \frac{1}{48}\begin{pmatrix} 3(-119+23\beta_0+612\beta_1) & 2(549-115\beta_0-1956\beta_1) & 3(-319+115\beta_0+852\beta_1) \\ -27(-9+\beta_0+44\beta_1) & 18(-43+5\beta_0+172\beta_1) & -27(-33+5\beta_0+124\beta_1) \\ 3(1-\beta_0+36\beta_1) & 2(-3+5\beta_0-228\beta_1) & 3(-7-5\beta_0+276\beta_1) \end{pmatrix},
$$
$$
B = \frac{1}{24}\begin{pmatrix} -9(-17+19\beta_0+148\beta_1) & 2(-93+55\beta_0+1092\beta_1) & -9(-17+3\beta_0+148\beta_1) \\ 27(-1+3\beta_0+84\beta_1) & -18(17+5\beta_0+172\beta_1) & 27(-1+3\beta_0+84\beta_1) \\ -9(-17+3\beta_0+148\beta_1) & 2(-93+55\beta_0+1092\beta_1) & -9(-17+19\beta_0+148\beta_1) \end{pmatrix}
$$
$$
C = \frac{1}{48}\begin{pmatrix} 3(-7-5\beta_0+276\beta_1) & 2(-3+5\beta_0-228\beta_1) & 3(1-\beta_0+36\beta_1) \\ -27(-33+5\beta_0+124\beta_1) & 18(-43+5\beta_0+172\beta_1) & -27(-9+\beta_0+44\beta_1) \\ 3(-319+115\beta_0+852\beta_1) & 2(549-115\beta_0-1956\beta_1) & 3(-119+23\beta_0+612\beta_1) \end{pmatrix}. \tag{3.11}
$$

Similar to the super convergence study on the DDGIC method, we consider two cases with $(\beta_0, \beta_1) = (4, \frac{1}{8})$ and $(\beta_0, \beta_1) = (4, \frac{1}{12})$ chosen in the numerical fluxes (2.5). Notice that we have $\beta_0 = \beta_{0u} + \beta_{0v}$ for the symmetric DDG method.

Again we symbolically calculate the moment errors (3.8) and list the analytical results in Table 5. With $\beta_1 = \frac{1}{12}$ case, we have 4th order super convergence on the average error ($m = 0$ case) and 5th order super convergence on the moment error of (3.8) with $m = 1$. In Table 6 we list the numerical solution moment errors and the moment errors calculated from Fourier analysis. The two groups of errors match very well.

Table 5 Analytically calculated moment errors (3.8) with the symmetric DDG method (3.1) and (3.11)

	$\beta_1 = \frac{1}{8}$	$\beta_1 = \frac{1}{12}$
$m = 0$	$-\frac{1}{72}e^{-t}\sin(x_j)h^2 + O(h^4)$	$\frac{7-24t}{17280}e^{-t}\sin(x_j)h^4 + O(h^6)$
$m = 1$	$\frac{1}{288}e^{-t}\sin(x_j)h^3 + O(h^5)$	$\frac{-173+840t}{3628800}e^{-t}\sin(x_j)h^5 + O(h^7)$
$m = 2$	$\frac{7}{1080}e^{-t}\sin(x_j)h^2 + O(h^4)$	$\frac{1}{90}e^{-t}\sin(x_j)h^2 + O(h^4)$
$m = 3$	$\frac{9}{5600}e^{-t}\sin(x_j)h^3 + O(h^5)$	$-\frac{1}{2100}e^{-t}\sin(x_j)h^3 + O(h^5)$

Table 6 Symmetric DDG method moment errors (3.8) with $\beta_1 = 1/8$ and $\beta_1 = 1/12$ in $\widehat{(u_h)_x}$ and $\widetilde{v_x}$

	$\beta_1 = \frac{1}{8}$				$\beta_1 = \frac{1}{12}$			
	Numerical solutions		Predicted by analysis		Numerical solutions		Predicted by analysis	
	Error	Order	Error	Order	Error	Order	Error	Order
	$m = 0$							
$N = 10$	3.14e−03	–	3.32e−03	–	2.78e−05	–	2.73e−05	–
$N = 20$	8.09e−04	1.96	8.21e−04	2.00	1.69e−06	4.04	1.68e−06	4.00
$N = 40$	2.06e−04	1.97	2.07e−04	2.00	1.06e−07	3.99	1.06e−07	4.00
$N = 80$	5.18e−05	1.99	5.19e−05	2.00	6.67e−09	3.99	6.67e−09	4.00
	$m = 1$							
$N = 10$	9.40e−04	–	1.04e−03	–	7.27e−06	–	8.08e−06	–
$N = 20$	1.27e−04	2.88	1.28e−04	3.00	2.46e−07	4.88	2.49e−07	5.00
$N = 40$	1.62e−05	2.97	1.62e−05	3.00	7.84e−09	4.97	7.87e−09	5.00
$N = 80$	2.03e−06	2.99	2.03e−06	3.00	2.46e−10	4.99	2.46e−10	5.00

Numerical solution errors from scheme (2.4) and estimated errors from Table 5. Final time t = 0.5

3.4 Piecewise Linear Approximations

As we mentioned previously in Sect. 2 that our diffusion solver the DDGIC and symmetric DDG methods degenerate to the SIPG method with low order (P^k with $k \leq 1$) polynomial approximations. With piecewise linear polynomial approximations it turns out that the SIPG method is also super convergent with $(k+1) = 2$ a second order for the average error [$m = 0$ in moment errors (3.8)].

We carry out Fourier analysis moment error estimate with piecewise linear approximations with SIPG method and list the first two moments errors below:

$$\begin{cases} ME_{m=0}\,[u_x - (u_h)_x] = \frac{1}{96}e^{-t}\sin(x_j)(7+8t)h^2 + O(h^4), \\ ME_{m=1}\,[u_x - (u_h)_x] = \frac{1}{6}e^{-t}\cos(x_j)h + O(h^3). \end{cases} \tag{3.12}$$

We also compute the moment errors (3.8) numerically with SIPG method (2.2) and P^1 approximations. We list the comparison between numerical errors and the estimated ones in Table 7 for the average error ($m = 0$ case). We see the SIPG method on its approximation to the solution's spatial derivative is also super convergent with $(k+1) = $ 2nd order. Based on the super convergence studies, the SIPG method seems not 'complete' and our diffusion solver the DDGIC and symmetric DDG methods complements the SIPG method in some way.

Table 7 Piecewise linear approximations moment errors (3.8) comparison: numerical errors with SIPG method (2.2) and analytical errors from estimate (3.12)

	Numerical solutions		Predicted by analysis	
	Error	Order	Error	Order
	$m = 0$			
$N = 10$	2.42e−02	–	2.74e−02	–
$N = 20$	6.56e−03	1.89	6.77e−03	2.00
$N = 40$	1.69e−03	1.95	1.70e−03	2.00
$N = 80$	4.27e−04	1.98	4.28e−04	2.00

Final time t = 0.5

4 Conclusion

We consider to solve the model Heat equation (1.1) with the classical SIPG method [1], the DDGIC method [14] and the symmetric DDG method [20]. Specifically we study the approximation to the solution's spatial derivative, namely $u_x(\cdot, t)$ of (1.1), measured under the moment errors (3.8). With kth degree polynomial applied, the error measured under L^2 and L^∞ strong norms is of kth order on its approximation to the solution's derivative. Measured under the weak sense or in the moment format, the error is super convergent with $(k+2)$th and $(k+3)$th order for its first two moments with the DDGIC and the symmetric DDG methods with even order approximations. No super convergence is observed with the classical SIPG method, even measured under weak sense. The observed super convergence orders are theoretically verified with Fourier type error estimate when P^2 quadratic polynomials approximations is considered. The analytically predicted errors match well with the numerical results.

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