Int J Game Theory (2018) 47:543–555
https://doi.org/10.1007/s00182-018-0623-0

ORIGINAL PAPER

A note on the eternal dominating set problem

Stephen Finbow[1] · Serge Gaspers[2] · Margaret-Ellen Messinger[3] · Paul Ottaway[4]

Accepted: 30 April 2018 / Published online: 8 May 2018

Abstract We consider the "all guards move" model for the eternal dominating set problem. A set of guards form a dominating set on a graph and at the beginning of each round, a vertex not in the dominating set is attacked. To defend against the attack, the guards move (each guard either passes or moves to a neighboring vertex) to form a dominating set that includes the attacked vertex. The minimum number of guards required to defend against any sequence of attacks is the "eternal domination number" of the graph. In 2005, it was conjectured [Goddard et al. (J. Combin. Math. Combin. Comput. 52:169–180, 2005)] there would be no advantage to allow multiple guards to occupy the same vertex during a round. We show this is, in fact, false. We also describe algorithms to determine the eternal domination number for both models for eternal domination and examine the related combinatorial game, which makes use of the reduced canonical form of games.

✉ Margaret-Ellen Messinger
mmessinger@mta.ca

Stephen Finbow
sfinbow@stfx.ca

Serge Gaspers
sergeg@cse.unsw.edu.au

Paul Ottaway
paulottaway@capilanou.ca

1 St. Francis Xavier University, Antigonish, Nova Scotia, Canada

2 UNSW Sydney and Data61, CSIRO, Eveleigh, Australia

3 Mount Allison University, Sackville, Canada

4 Capilano University, North Vancouver, Canada

Keywords Graph protection · Graph domination · Eternal domination · Combinatorial game theory

Mathematics Subject Classification 05C69 · 05C57 · 68R15 · 91A46

1 Introduction

Surveillance cameras that monitor rooms in a building may form a dominating set, but the cameras are in fixed locations. Soldiers that guard intersections may form a dominating set, but they also may be required to move in order to respond to an attack at a neighbouring intersection. The former is an example of the *dominating set problem*, however, the latter is an example of a *graph protection* model. Since the concept of *graph protection* was first formalized by Arquilla and Fredricksen (1995) and further communicated by ReVelle (1997), Stewart (1999), and ReVelle and Rosing (2000), there have been many advances on the topic. We refer to a recent survey by Klostermeyer and Mynhardt (2016) for more background and the state of the art of graph protection. One model of graph protection is the eternal dominating set problem.

In the "all guards move" model for the *eternal dominating set problem*, a set of guards form a dominating set on a graph and then a vertex is attacked. In response, each guard may remain where he is or move to a neighbouring vertex with the collective goal of forming a dominating set containing the attacked vertex. In accomplishing this goal, the guards have *defended against the attack*. Thus, during each "round", a vertex is attacked and then the guards move to defend against the attack. The *eternal domination number* of a graph G, denoted $\gamma_{\text{all}}^{\infty}(G)$, is the minimum number of guards required to defend against any sequence of attacks. This parameter has also been denoted by $\sigma_m(G)$ (Goddard et al. 2005), $\gamma_m(G)$ (Klostermeyer and MacGillivray 2009), and $\gamma_m^{\infty}(G)$ (Klostermeyer and Mynhardt 2016). The $\gamma_{\text{all}}^{\infty}(G)$ notation, of which we make use in this note, has been used more recently by Beaton et al. (2013) and Finbow et al. (2015).

In 2005, Goddard et al. stated a series of open questions and conjectured that there would be no advantage to allow multiple guards to be located at the same vertex in any round in the "all-guards move" model. As a result, subsequent publications on the eternal dominating set problem have defined the problem in such a way that at most one guard can be located on a vertex in any round. A 2009 article by Klostermeyer and MacGillivray stated the above conjecture as an open problem. In Sect. 2 we show the conjecture to be false.

In the "1 guard moves" model of the eternal dominating set problem, only one guard may move in response to an attack during each round. In this model, the minimum number of guards required to defend against any sequence of attacks on a graph G is denoted $\gamma_1^{\infty}(G)$. This notation was used by Klostermeyer and Mynhardt and we use it to be consistent with the "all guards move" notation. [The notations $\sigma_1(G)$ (Goddard et al. 2005) and $\gamma_{\infty}(G)$ (Klostermeyer and MacGillivray 2009) have also been used.] Burger et al. (2004) showed that there is no advantage to allow multiple guards to be located at the same vertex in a given round in the "1 guard moves" model. However, Goddard et al. (2005) asked about the complexity of the associated recognition and decision problems for the "1 guard moves" model and the "all guards move" model. In Sect. 3, we describe algorithms to determine $\gamma_1^\infty(G)$ and $\gamma_{\text{all}}^\infty(G)$ in time $O^*(8^n)$. [1] We also describe an algorithm to determine the (all-guards move) eternal domination number for the variant when multiple guards are permitted to occupy the same vertex during a given round. In this case, the algorithm runs in time $O^*\left((\frac{81}{8}\sqrt{3})^n\right)$ (note $\frac{81}{8}\sqrt{3} \approx 17.54$). In Sect. 4, we describe a combinatorial game based on eternal domination, given a graph, a set of attackers, and a set of defenders. For many situations a player would like to ignore the infinitesimal values since they only determine the parity of the number of moves once the associated non-infinitesimal value has reached zero. Grossman and Siegel (2009) showed that the idea of a simplest game infinitely close to a given game, called the *reduced canonical game*, was well-defined. The reduced canonical form can be useful in determining optimal play in sums with other games. For the combinatorial game based on eternal domination, we determine the reduced canonical form for the game, given a graph, a set of attackers, and a set of defenders.

We conclude this section with formal definitions. Let $G = (V, E)$ be a graph. A *dominating set* of G is a subset of V whose closed neighbourhood is V. The smallest cardinality of a dominating set is denoted $\gamma(G)$ and is called the *domination number* of G. Let $\mathbb{D}_q(G)$ be the set of all dominating sets of G which have cardinality q. Let $D, D' \in \mathbb{D}_q(G)$. We will say D can be *transformed* to D' (or D *transforms* to D') if $D = \{v_1, v_2, \dots, v_q\}$, $D' = \{u_1, u_2, \dots, u_q\}$ and $u_i \in N[v_i]$ for $i = 1, 2, \dots, q$.

In the "eternal dominating set problem", a defender is given q guards to protect the graph from a series of attacks on vertices made by an attacker. An *eternal dominating family of* G is a subset $\mathcal{E} \subseteq \mathbb{D}_q(G)$ for some q so that for every $D \in \mathcal{E}$ and every possible attack $v \in V(G)$, there is a dominating set $D' \in \mathcal{E}$ so that $v \in D'$ and D transforms to D'. When the value of q in the above definition is known we will refer to this family as an eternal dominating family with q guards and the minimum such q is denoted $\gamma_{\text{all}}^\infty(G)$. A set $D \in \mathbb{D}_q(G)$ is an *eternal dominating set* if it is a member of some eternal dominating family. Note that the set of all eternal dominating sets of a particular cardinality is an eternal dominating family, provided the family is non-empty.

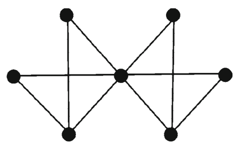
2 Allowing multiple guards on a vertex during a given round

In this section, we disprove the conjecture of Goddard et al. (2005) that there is never any advantage in allowing two or more guards to occupy the same vertex in the all-guards move model. To avoid confusion, let $\gamma_{\text{all}}^{\infty *}(G)$ denote the eternal domination number for the variant in which multiple guards are permitted to occupy the same vertices during each round. Clearly, the variant $\gamma_{\text{all}}^{\infty *}$ forms a lower bound for $\gamma_{\text{all}}^\infty$:

Observation 1 *For any graph G, $\gamma_{\text{all}}^{\infty *}(G) \leq \gamma_{\text{all}}^\infty(G)$.*

[1] The O^*-notation used here is similar to the O-notation but hides polynomial factors in the size of an input instance. See Woeginger (2001) and Fomin and Kratsch (2010) for more on the O^*-notation and exponential-time algorithms in general.

Fig. 1 Graph H



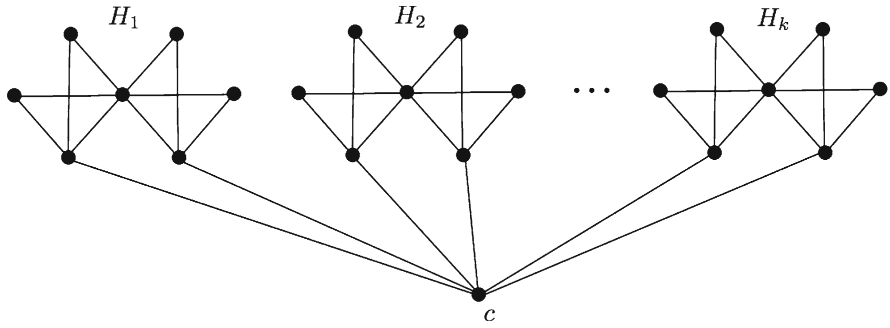


Fig. 2 Graph G_k

Theorem 2 *For every integer $\ell \geq 0$, there is a graph G_ℓ such that $\gamma_{\text{all}}^{\infty}(G_\ell) - \gamma_{\text{all}}^{\infty *}(G_\ell) \geq \ell$.*

Proof Let H be the graph given in Fig. 1 and let $k := \ell + 3$. Form the graph G_k as follows: the vertex set of G_k consists of the vertices of k copies of H (labeled $H_1, H_2, \ldots, H_k$) along with an additional vertex c . The edge set of G_k contains the edges from each of the k copies of H, along with an edge between vertex c and the degree 3 vertices in each copy of H. Graph G_k is illustrated in Fig. 2.

We first consider the traditional eternal dominating set problem on G_k, where each vertex is occupied by at most one guard in a given round. Our first claim is that at the end of each round (i.e. after the guards have moved), each copy of H must contain at least two guards. For a contradiction, suppose that at the end of round t, H_i contains only one guard g. In round $t + 1$, if a degree 2 vertex of H_i is attacked, then guard g must move to that attacked vertex during round $t + 1$. However, this leaves at least three vertices of degree 2 in H_i that are not dominated by g. Since all vertices must be dominated after the guards have moved during round $t + 1$, a guard must move from c to a vertex in H_i. However, regardless of which of the two vertices in H_i the guard from c moves to, there will remain at least one vertex of degree 2 in H_i that is not dominated. Therefore, at the end of every round, each copy of H must contain at least two guards. Furthermore, $2k$ guards are not sufficient: consider an attack at vertex c in G_k. Then a guard from H_j must move to c, leaving only one guard in H_j, which cannot occur. Therefore, $\gamma_{\text{all}}^{\infty}(G_k) \geq 2k + 1$.

Certainly $2k + 1$ guards suffice: initially, one guard occupies vertex c and for each $1 \leq i \leq k$, one guard occupies the degree 6 vertex of H_i and one guard occupies a different vertex of H_i. Certainly G_k is dominated by the vertices occupied by these $2k + 1$ guards. Since vertex c is occupied by a guard, at each round, a vertex in a copy of H is attacked. If a vertex in H_j is attacked, one guard moves from the degree 6 vertex to the attacked vertex, and the other guard moves to the degree 6 vertex. Thus, $2k+1$ guards suffice to defend any sequence of attacks and $\gamma_{\text{all}}^{\infty}(G_k) = 2k+1$.

Suppose now that multiple guards are permitted to occupy the same vertices at the end of a round. We first show $\gamma_{\text{all}}^{\infty *}(G_k) \leq k+4$: initially place two guards at c, one guard at each of the two vertices of degree four in H_1, one guard on any other vertex of H_1 and one guard at the vertex of degree 6 in each H_i $(2 \leq i \leq k)$; such a degree 6 vertex is called the *central* vertex of H_i. Clearly, each vertex of G_k is dominated. Suppose a vertex in H_i is attacked, $i \neq 1$. The guard g at the central vertex of H_i moves to the attacked vertex and two guards move from c to each of the two vertices (of degree four) of H_i that are adjacent to c. The two guards in H_1 adjacent to c move to c (leaving two guards at c) and the remaining guard on H_1 moves to the central vertex of H_1. No other guards move. We note the guards have responded to the attack and, up to symmetry, have remained in the same position as the beginning of the round.

Instead of an attack in H_i, we consider an attack in H_1. Let g be the guard in H_1 that does not occupy a vertex adjacent to c. Let u and v be the two vertices in H_1 adjacent to c. If the attacked vertex is the central vertex, g moves to the central vertex and no other guard moves. Otherwise, the attacked vertex is adjacent to exactly one of u, v, suppose v. The guard at v moves to the attacked vertex. If the vertex occupied by g is adjacent to v, then g moves to v and no other guard moves. Otherwise, the vertex occupied by g is adjacent to u, but not v. In this case, g moves to u, the guard at u moves to c, and a guard at c moves to v; no other guard moves. In H_1, the new configuration of guards has again 2 guards on the neighbors of c and one guard on another vertex of H_1. The guards have responded to the attack and, up to symmetry, have remained in the same position. We conclude that $k+4$ guards can respond to any attack and, up to symmetry, have remained in the same position.

Finally, we show $\gamma_{\text{all}}^{\infty *}(G) \geq k+4$. Suppose that H_i contains exactly one guard g at the end of round t. To dominate the vertices of degree 2 in H_i, g must be located at the central vertex of H_i. If a vertex of degree 2 in H_i is attacked at the beginning of round $t+1$, g must move to the attacked vertex, then there are three vertices of degree 2 that are not dominated by g. So for H_i to be dominated at the end of round $t+1$, two guards g_1, g_2 must move from c to the two vertices of degree 4 in H_i (since each of g_1, g_2 dominates at most 2 vertices of degree 2 in H_i). Let H_j be a copy of H that contains only one guard g' at the end of round $t+1$. To dominate the vertices of degree 2 in H_j, g' must be located at the central vertex of H_j. If a vertex of degree 2 in H_j is attacked at the beginning of round $t+2$, g' must move to the attacked vertex. However, for H_j to be dominated at the end of round $t+2$, two guards g_3, g_4 must move from c to the two vertices of degree 4 in H_j. Observe that g_1, g_2, g_3, g_4 are all distinct guards (since at the beginning of round $t+2$, g_1, g_2 are located in H_i). Further, each copy of H apart from H_i, H_j) must contain at least one guard, to dominate the degree 2 vertices of that copy of H. Coupled with the upper bound, $\gamma_{\text{all}}^{\infty *}(G) = k+4$. □

The graph described by Fig. 2 illustrates that the difference between $\gamma_{\text{all}}^{\infty}$ and $\gamma_{\text{all}}^{\infty *}$ can be arbitrarily large. Observe that the described graph has a cut-vertex. As a result, we leave the following as an open problem: does there exist a 2-connected graph for which the difference between $\gamma_{\text{all}}^{\infty}$ and $\gamma_{\text{all}}^{\infty *}$ is arbitrarily large?

3 Algorithms

We describe algorithms to determine $\gamma_1^\infty(G)$ and $\gamma_{\text{all}}^\infty(G)$ in time $O^*(8^n)$ and $\gamma_{\text{all}}^{\infty*}(G)$ in time $O^*((\frac{81}{8}\sqrt{3})^n)$. The general framework for these algorithms is as follows. We would like to determine if the graph parameter we consider is at most k for each $k \in \{0, \ldots, n\}$. We may restrict our attention to connected graphs as these parameters can easily be derived from those of all connected components of G. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and consider the graph parameter $p \in \{\gamma_1^\infty, \gamma_{\text{all}}^\infty, \gamma_{\text{all}}^{\infty*}\}$.

The configuration graph is a useful and powerful tool in such configuration-space type problem. It has been used in the context of a variety of problems, including vertex colouring, minimum spanning trees, independent sets, and domination. Fricke et al. (2011) used a configuration graph (called the γ-graph) to study minimum dominating sets: sets forming the nodes of the configuration graph with two nodes adjacent if the dominating sets differ in exactly one place.

For the graph parameter p, we now construct the *configuration graph* $\mathcal{C}_p = (C_p, T_p)$. A node or configuration $c \in C_p$ is a vector $c = (c(v_1), c(v_2), \ldots, c(v_n))$ where $c(v_i)$ is the number of guards placed at vertex v_i, $(1 \leq i \leq n)$ such that $\sum_{1 \leq i \leq n} c(v_i) = k$. For the parameters γ_1^∞ and $\gamma_{\text{all}}^\infty$, we require additionally that for $1 \leq i \leq n$, $c(v_i) \leq 1$. A transition $t \in T_p$ is a set $\{c_1, c_2\}$ of two configurations in C_p such that one can be obtained from the other by moving at most 1 (for $\gamma_1^\infty(G)$) or a subset of the guards (for $\gamma_{\text{all}}^\infty(G)$) along an edge of G to a neighbouring vertex.

Lemma 1 *The configuration graph can be constructed in time* $|C_p|^2 \cdot n^{O(1)}$.

Proof To construct the configuration graph $\mathcal{C}_p = (C_p, T_p)$ of G, start with $C_p = T_p = \emptyset$. First, enumerate all possible configurations and add them to C_p. This can easily be done with polynomial delay by a simple branching algorithm. Now, determine for every two configurations $c_1, c_2 \in C_p$ whether $\{c_1, c_2\} \in T_p$ in polynomial time as follows.

If at most one guard is allowed to move, then $\{c_1, c_2\} \in T_p$ if and only if c_1 differs from c_2 in exactly two positions and the corresponding vertices are adjacent. This test can obviously be done in polynomial time.

If all guards are allowed to move, construct a bipartite graph H with bipartition (A, B) such that for each vertex $v_i \in V(G)$, A contains $c_1(v_i)$ copies of v_i and B contains $c_2(v_i)$ copies of v_i. Two vertices $a \in A$ and $b \in B$ are adjacent in H if they are copies of one and the same vertex in G or if they are copies of two adjacent vertices in G. Now, $\{c_1, c_2\} \in T_p$ if and only if H has a perfect matching. Each edge in a perfect matching corresponds to the movement of a guard. Clearly, the size of H is polynomial in n and a maximum matching of H can be found in polynomial time. □

We say that a configuration is 0-*step p-dominating* if the vertices occupied by guards form a dominating set in G; that is, if $\{v_i \in V(G) : c(v_i) \geq 1\}$ dominates G. A configuration c is *k-step p-dominating* if for each vertex $v_i \in V(G)$, there exists a configuration $c' \in N_{\mathcal{C}_p}[c]$ such that $c'(v_i) \geq 1$ and c' is $(k-1)$-step p-dominating.

Lemma 2 *A configuration c is $|C_p|$-step p-dominating if and only if c is ∞-step p-dominating.*

Proof For the sake of contradiction, assume a configuration c_0 is k-step p-dominating but not $(k+1)$-step p-dominating for some $k \geq |C_p|$. Then c_0 has a neighboring configuration c' in $\mathcal{C}_p$ that is $(k-1)$-step p-dominating but not k-step p-dominating. Repeating this argument eventually finds a directed path in $\mathcal{C}_p$ starting at c_0 and ending at a configuration that is 0-step p-dominating but not 1-step p-dominating. But this path has $k+1 > |C_p|$ vertices, a contradiction. □

We next show that for any graph G on n vertices and an integer k, it can be decided in time $n^{3k} \cdot n^{O(1)}$ whether there exists a configuration that is ∞-step p dominating. To do this, the algorithms proceeds in rounds numbered from 1 to $|C_p|$. In round ℓ, $1 \leq \ell \leq |C_p|$, it determines which configurations are ℓ-step p-dominating by checking each configuration according to the definition that precedes Lemma 2. The computation for one round can be done in time $(|C_p|+|T_p|)\cdot n^{O(1)}$ as each configuration is checked once and each edge traversed twice.

As we may stop the computation after $|C_p|$ rounds, the total running time of the algorithm is $|C_p| \cdot (|C_p| + |T_p|) \cdot n^{O(1)} \leq |C_p|^3 \cdot n^{O(1)}$.

As for any fixed number of guards k, $|C_p| \leq n^k$, the running time of the algorithm is polynomial if the number of guards is a constant.

Theorem 3 *For any graph G on n vertices and an integer k, it can be decided in time $n^{3k} \cdot n^{O(1)}$ whether $\gamma_1^{\infty}(G) \leq k$, whether $\gamma_{\text{all}}^{\infty}(G) \leq k$ or whether $\gamma_{\text{all}}^{\infty *}(G) \leq k$.*

More generally, as $|C_p| \leq 2^n$ if at most one guard is allowed to be placed on each vertex, Theorem 4 follows.

Theorem 4 *For any graph G on n vertices, $\gamma_1^{\infty}(G)$ and $\gamma_{\text{all}}^{\infty}(G)$ can be determined in time $O^*(8^n)$.*

If multiple guards are allowed to be placed on the vertices, the number of configurations equals the number of weak compositions of k into n parts, which is $\binom{k+n-1}{k}$. Since $k \leq n$, $|C_{\gamma_{\text{all}}^{\infty *}}| \leq 4^n$, which leads to an algorithm with running time $O^*(64^n)$. Using Klostermeyer and Mynhardt's result that $\gamma_{\text{all}}^{\infty}(G) \leq \lceil n/2 \rceil$ when G is connected, we can significantly improve this running time. Namely, since $\gamma_{\text{all}}^{\infty *}(G) \leq \gamma_{\text{all}}^{\infty}(G) \leq \lceil n/2 \rceil$, the number of configurations can be bounded above by $\binom{3n/2}{\lceil n/2 \rceil}$.

Theorem 5 *For any graph G on n vertices, $\gamma_{\text{all}}^{\infty *}(G)$ can be determined in time $O^*\left((\frac{81}{8}\sqrt{3})^n\right)$.*

Proof We assume G is connected. Otherwise, process each connected component independently. We have that $|C_{\text{all}}^{\infty *}|$ is at most $\binom{3n/2}{\lceil n/2 \rceil}$ and the running time of the algorithm is upper bounded by $|C_p|^3 \cdot n^{O(1)}$. Using Stirling's formula, the result follows. □

4 The combinatorial game

This section will draw upon material from combinatorial game theory and for terminology and background, we refer the reader to Albert et al. (2007), Berlekamp et al. (2001), and Siegel (2013). In particular, we consider the eternal dominating set problem as a combinatorial game; that is, a two-player game with perfect information and no chance devices, where players alternate moves until the player whose turn it is has no legal moves available. The players are *the attackers* and *the defenders* and we label the attackers as the Left player and the defenders as the Right player.

Let $H = (V, E)$ be a graph, D be an ordered multi-set of guards (the elements of D being vertices in the set V), and $A \subset V$ be a set of attackers. To avoid a game that is a trivial win for the guards, we require $|D| < |V|$. Fix n to be the number of time steps allowed before we declare the defenders the winner (then the game does not continue indefinitely). Let t be the number of time steps remaining until the defenders have been declared the winner (i.e. the number of time steps remaining until a total of n time steps have passed). We define the game G by (H, D, A, t, n). To compute values for G, we assign values for the ending positions and describe the legal moves allowed to each player.

We first define the position $G = (H, D, A, 0, n) = 0$ which describes the situation when the defenders have protected all the vertices from the attackers for n time steps. In particular, this means that the underlying graph H can be defended successfully for n steps given the initial configuration of attackers and defenders. If this holds as we let n go to infinity, we would say the initial position is defender-win. Otherwise, it is attacker-win.

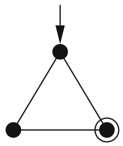
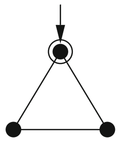
A legal move for Right is given by $(H, D, A, t, n) \to (H, D', A, t-1, n)$ where $|D'| = |D|$, $A \subset D'$ and for each $d_i \in D$, we require that $d_i' \in N[d_i]$ for $d_i' \in D'$. This represents the defenders moving each guard at most one step along an edge of the graph. Note that a move is legal only if the defenders can defend all of the vertices that are currently under attack.

If $A \subset D$ then Left has the legal move given by $(H, D, A, t, n) \to (H, D, A', t, n)$ where $|A'| = |A|$ and $A' \not\subset D$. This represents the fact that if all of the attackers are currently covered by defenders, she must move them to new vertices so that at least one unguarded vertex is being threatened. By definition, if it is Left's turn and $A \not\subset D$ then her only legal move is to 0. This represents the attackers starting a turn on a vertex which isn't being defended and therefore they win.

We denote by G^L, a Left option of G (i.e. a position Left can move to from G). This symbol is used even when a player has more than one option or none at all, so that the symbol G^L need not have a unique value. Then G^{LL} denotes a Left option of G^L (i.e. a position Left can move to from G^L). (G^R and G^{RR} are defined similarly.)

For an arbitrary game of eternal domination the canonical form would be infeasible to compute. However, we will show that the reduced canonical form of a game G, denoted $rcf(G)$ only takes on particular values and is therefore helpful in determining optimal play in sums with other games. The reduced canonical form of a game can found based on a game's left and right *stops* denoted $LS(G)$ and $RS(G)$, respectively.

Table 1 The graph C_3 with one attacker and one defender

Position	Canonical form for $t = 1$	Canonical form for $t = 2$	Canonical form for $t = 3$	Reduced canonical form for large t
	$*$	$*$	$*$	0
	$\downarrow$	$\downarrow_{[2]}$	$\downarrow_{[3]}$	0

They are defined in a mutually recursive fashion:

$$LS(G) = \begin{cases} G & \text{if } G \text{ is a number} \\ \max(RS(G^L)) & \text{if } G \text{ is not a number;} \end{cases}$$

$$RS(G) = \begin{cases} G & \text{if } G \text{ is a number} \\ \min(LS(G^R)) & \text{if } G \text{ is not a number.} \end{cases}$$

The *reduced canonical form* of a game G is then:

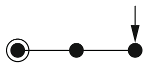
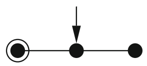
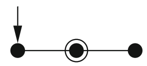
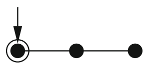
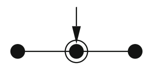
$$rcf(G) = \begin{cases} G & \text{if } G \text{ is a number} \\ RS(G) & \text{if } G \text{ is not a number and } RS(G) = LS(G) \\ \{LS(G)|RS(G)\} & \text{otherwise.} \end{cases}$$

The reduced canonical form of a game can equivalently be defined as the simplest game infinitely close to a given game and was shown by Grossman and Siegel (2009) to be well-defined.

We make use of left stops, right stops and reduced canonical forms in the following result (see the recent book by Siegel (2013) for a formal introduction to the concept of reduced canonical form). Tables 1 and 2 illustrate some particular small eternal domination games with their canonical forms and reduced canonical forms shown. In Tables 1 and 2, for each graph H, the arrows indicate the vertex is in the set A; the circled vertices indicate the vertex is in the set D.

Lemma 3 *For any game G where $t > 0$, either $G^L = 0$ or $G^{LL} = 0$.*

Table 2 The graph P_3 with one attacker and one defender

Position	Canonical form for $t=1$	Canonical form for $t=2$	Canonical form for $t=3$	Reduced canonical form for large t
	1	1	1	1
	$*$	$*$	1	1
	$*$	1	1	1
	$\{1 \mid 0\}$	$1{+}_{1}$	$\{1 \mid 1{+}_{1}\}$	1
	$\downarrow$	$\{1 \mid \downarrow\}$	$1{+}_{1\uparrow}$	1

Proof There are two cases to consider. First, we assume that $A \not\subset D$. In this case, $G^L = 0$ since this is the only option the attackers have from a position where the defenders have not defended all vertices under attack. The second case is when $A \subset D$. We know that Left has made a legal move from this position and it must be to a position where $A \not\subset D$. This is the first case we considered, so we conclude that $G^{LL} = 0$. □

Lemma 4 *The game G will terminate after a finite number of moves.*

Proof Given the parameters t and n, we know that the defenders can make at most n moves in the game before we reach a value of 0. Also, if the attackers make two consecutive moves in a row at any point, the game is over with value 0 due to the previous lemma. Therefore, we achieve the maximum number of moves with alternate play and can have at most $2n$ moves taken in total. □

Lemma 5 *For a game G where there exists $v \in A$ but $v \notin N[D]$, then $G = 1$.*

Proof If Left plays first from this game, it must be to 0 since $A \not\subset D$. If Right plays first from this game, we find that he has no legal move. Since $v \notin N[D]$, there is no $d_i \in D$ which can move to v so that $A \subset D'$. Therefore, $G = \{0 \mid \} = 1$. □

Corollary 1 *For a game G where D is not a dominating set, there exists $G^L = 1$.*

Proof Left's move from such a game would be to choose a A' such that it contained a vertex $v \notin N[D]$. The previous lemma gives this left option the value 1. □

Theorem 6 *For any game G in the starting position, if the attackers can win the game by playing first, LS(G) = 1. Otherwise, LS(G) = 0. Likewise, if the defenders can win the game playing first, RS(G)=0 otherwise RS(G)=1.*

Proof We proceed by induction on t and denote a game by G_t for clarity. First note that if $t = 1$ we can have two cases.

The first case is when $A \subset D$. After Left makes a move, if Right cannot respond, we know the game has value 1. If Right has a legal move after Left's move, it must be to 0 since we will have $t = 0$. We can also note that if Left were to make two moves in a row, it would be to 0 as well. Finally, we note that if Right were to play first he has the legal move defined by $D' = D$ and it would have value 0 since t would become 0. So, if Left can win playing first we get $G_1 = \{1 \mid 0\}$ in which case LS(G_1) = 1. If Left cannot win playing first we get $G_1 = \{\{0 \mid 0\} \mid 0\} = \{* \mid 0\}$ and we note that LS(G_1) = 0. Either way, we note that Right could win playing first and that RS(G_1) = 0.

For the second case, we assume $A \not\subset D$ and note that such a position can only be reached after a Left move. If Left plays first, her only option is to 0. If Right plays first, we must consider whether or not he has a legal move. If he does, it must be to 0 since t would become 0. In that case, $G_1 = *$ and we have that RS(G_1) = 0. Otherwise, if Right does not have a legal move from G_1 then $G_1 = \{0 \mid \} = 1$ and LS(G_1) = RS(G_1) = 1.

In all but one case, the Left and Right stops were 0 if the defenders could win and 1 if the attackers could win. The case where the Left stop was 0 after the attackers had won was due to the fact that Left required two consecutive moves to end the game. Therefore, it is a position which cannot be reached with alternate play.

Now we will consider the game G_t. We again have two cases and begin by assuming $A \subset D$.

We start by letting Left play first. If Right cannot respond, we know the game has value 1 and we would find that LS(G_t) = 1. If Right has a legal move after Left's move, it must be to some game G_{t-1} where we again have $A \subset D$. So, LS(G_t) = LS(G_{t-1}) which must be 1 or 0 by induction. Now we consider Right's options from G_t. We know Right has a legal move to some game G_{t-1} by letting $D' = D$ so by induction RS(G_t) = LS(G_{t-1}) which must be 0 or 1.

Now we examine the case where $A \not\subset D$. If Left plays first, it must be to 0. If Right must play first, we must decide if he has a legal move. If he has no legal move, then $G_t = 1$ and LS(G_t) = RS(G_t) = 1. If Right does have a legal move, it must be to some game G_{t-1} where $A \subset D$. Therefore, RS(G_t) = LS(G_{t-1}) = 0 or 1 by induction.

Again we find that except in one case, the left and right stops of G are 1 if the attackers can win and 0 if the defenders can win. The exceptional case occurs only when Left can make two consecutive moves and therefore is not reachable in alternate play from a position with $A \subset D$. □

Since the left stops and rights stops are 0 or 1, considering the four possible combinations of left and right stops yields the following corollary.

Corollary 2 *For any game* $G = (H, D, A, t, n)$, $Rcf(G)$ *is* 1, 0, *or* $\{1 \mid 0\}$.

Considering the original definition of eternal domination (in previous sections), there is no time limit for which the defenders need to protect vertices of the graph. To connect the results of this section to the rest of the paper, we consider the more natural analogous definition, $G = (H, D, A)$.

Theorem 7 *Given the game* $G = (H, D, A)$*, the reduced canonical form of* G *is* 1*,* 0*, or* $\{1 \mid 0\}$.

Proof For any fixed H, D, and A, there are only finitely many positions for the game. Therefore we can choose an n sufficiently large so that if the attackers can win, they can do so in at most n steps. Therefore, it is equivalent to the game (H, D, A, n, n). □

Tables 1 and 2 show the reduced canonical forms for some small games, in particular (with one attacker and one guard), P_3 is attacker-win and its reduced canonical form is 1 whereas C_3 is defender-win and its reduced canonical form is 0.

Acknowledgements S. Finbow acknowledges research support from the Natural Sciences and Engineering Research Council of Canada (Grant Application 2014-06571). Serge Gaspers is the recipient of an Australian Research Council (ARC) Future Fellowship (Project Number FT140100048) and he also acknowledges support under the ARC's Discovery Projects Funding Scheme (DP150101134). M.E. Messinger acknowledges research support from the Natural Sciences and Engineering Research Council of Canada (Grant Application 356119-2011). The authors acknowledge support from the Games and Graphs Collaborative Research Group of the Atlantic Association for Research in Mathematical Sciences.

References

Albert MH, Nowakowski RJ, Wolfe D (2007) Lessons in play. A K Peters Ltd, Massachusetts

Arquilla J, Fredricksen H (1995) "Graphing" an optimal grand strategy. Mil Oper Res 1(3):3–17

Beaton I, Finbow S, MacDonald JA (2013) Eternal domination numbers of $4 \times n$ grid graphs. J Combin Math Combin Comput 85:33–48

Berlekamp ER, Conway JH, Guy RK (2001) Winning ways for your mathematical plays, vol 1, 2nd edn. A K Peters Ltd, Massachusetts

Burger AP, Cockayne EJ, Gründlingh WR, Mynhardt CM, van Vuuren JH, Winterbach W (2004) Infinite order domination in graphs. J Combin Math Combin Comput 50:179–194

Finbow S, Messinger ME, van Bommel M (2015) Eternal domination of $3 \times n$ grid graphs. Aust J Combin 61(2):156–174

Fomin FV, Kratsch D (2010) Exact exponential algorithms, texts in theoretical computer science. Springer, Berlin

Fricke GH, Hedetniemi SM, Hedetniemi ST (2011) γ-graphs of graphs. Disc Math Graph Theory 31:517–531

Goddard W, Hedetniemi SM, Hedetniemi ST (2005) Eternal security in graphs. J Combin Math Combin Comput 52:169–180

Grossman JP, Siegel A (2009) Reductions of partizan games. In: Albert MH, Nowakowski RJ (eds) Games of no chance 3. Cambridge University Press, New York, pp 427–445

Klostermeyer WF, MacGillivray G (2009) Eternal dominating sets in graphs. J Combin Math Combin Comput 68:97–111

Klostermeyer WF, Mynhardt CM (2016) Protecting a graph with mobile guards. Appl Anal Disc Math 10:1–29

McCuaig W, Shepherd B (1989) Domination in graphs with minimum degree two. J Gr Theory 13:749–762

ReVelle CS (1997) Can you protect the Roman empire? John Hopkins Mag 50(2):40

ReVelle CS, Rosing KE (2000) Defendens Imperium Romanum: a classical problem in military strategy. Am Math Mon 107(7):585–594

Siegel AN (2013) Combinatorial game theory. American Mathematical Society, Providence

Stewart I (1999) Defend the Roman empire!. Sci Am 281:136–138

Woeginger GJ (2001) Exact algorithms for NP-hard problems: a survey. In: Proc. of the 5th international workshop on combinatorial optimization, Springer LNCS 2570, Berlin, pp 185–208