CHAPTER SEVEN

ON APPROXIMATE INFERENCE

In chapters 2 – 6 we developed and investigated real-valued (or lattice-valued) propositional and predicate calculi based on the notion of a (continuous) t-norm and its residuum. Now our understanding of these calculi is reasonably advanced: we have obtained several completeness theorems etc. But now we must ask: how does this relate to fuzzy logic (in the narrow sense, obviously)?

Let us again quote from Zadeh [221]:

"In a narrow sense, fuzzy logic (FLn) is a logical system which aims at a formalization of approximate reasoning. As such, it is rooted in multivalued logic but its agenda is quite different from that of traditional multivalued logical systems, e.g., Łukasiewicz logic. In this connection, it should be noted that many of the concepts which account for the effectiveness of fuzzy logic as a logic of approximate reasoning are not a part of traditional multivalued logical systems. Among these are the concepts of a linguistic variable, canonical form, fuzzy rule, fuzzy graph, fuzzy quantifiers and such modes of reasoning as interpolative reasoning, syllogistic reasoning and dispositional reasoning."

In this chapter and also in some further chapters we shall investigate some items from Zadeh's agenda of questions or topics typical of fuzzy logic in this meaning: without denying that they do not belong to the "classical" questions of many-valued logic we shall try to show that these topics admit a strictly logical analysis in the sense of formal logic. It will turn out that much of approximate inference may be presented as *deduction* in a suitably chosen logic, i.e. in a truth-preserving way. This does not mean that we claim to *reduce* fuzzy logic to many valued logic; but it does mean that the role of many-valued logic as a base (foundation) of fuzzy logic is much greater and more important than one would guess at the beginning. It goes without saying that fuzzy logic indeed has aspects that are not grasped by classical many-valued logic (as control aspects of fuzzy control); but, I repeat, the *deductive* aspect (with its corresponding semantical counterpart) is very important.

Section 1 analyses Zadeh's compositional rule of inference and two particular cases of it: *generalized modus ponens* and a (less known) *generalized conjunctive rule*. We also comment on some dangers in certain popular but questionable uses of fuzzy logic in expert systems. Section 2 analyzes logical aspects of fuzzy control. Section 3 presents an alternative formalization of the two generalized rules and the corresponding alternative formalization of fuzzy control.

P. Hájek, *Metamathematics of Fuzzy Logic*

7.1. THE COMPOSITIONAL RULE OF INFERENCE

7.1.1 We start with the notion of a *variate*.[37] A variate is given by its *name* X and its *domain* D. X is just a symbol; D is a non-empty set. Examples are: age with the domain of integers ≤ 120 (say), temperature (with some domain), etc. Fuzzy logic notoriously uses expressions of the form "X is A" where A is (the name of) a fuzzy subset of D, e. g. "the age is high". These expressions typically occur in *fuzzy rules*, to be analyzed later.

It is not automatically clear how this fits into our formalism of predicate calculus, and, in fact, we shall present two ways of doing it. In this and the next section, we shall proceed as follows: having n variates $(X_1, D_1), \cdots,$ $\cdots (X_n, D_n)$ we understand the D's as domains of a many-sorted structure interpreting a predicate language; fixed fuzzy subsets of a domain interpret some unary predicates. Besides we may have predicates of higher arity and their interpretation in our many-sorted structure. The most important thing comes now: the name of a variate is taken to be an *object constant*, interpreted in each situation as the *actual* value of the variate. The expression "X is A" becomes an atomic closed formula $A(X)$ (once again: A is a unary predicate, X is an object constant). A typical rule "IF X is A THEN Y is B" may be interpreted as $A(X) \rightarrow B(Y)$. (Caution: this is not the only possible reading of a "fuzzy IF – THEN rule", as we shall see later.)

Summarizing: We have understood an n-tuple of variates $(X_i, D_i)_{i=1}^n$ as determining a language $\mathcal{I}$ with sorts $s_1, \cdots, s_n$ and with object constants $X_1, \cdots, X_n$, X_i of the sort s_i. Besides our language may contain arbitrary predicates of any type and any other object constants. Any $\mathcal{I}$-structure

$$\langle (D_i)_i, , (r_P)_{P\, pred.}, m_{X_1}, \cdots, m_{X_n} \cdots \rangle$$

(where the dots stand for the interpretation of additional constants, if any) is understood as a fuzzy structure over the given variates, with X_i denoting the actual value of the i-th variate.

Given this, we may start our analysis of Zadeh's compositional rule of inference.[38]

7.1.2 The *compositional rule of inference* in its traditional formulation can be stated as follows:

From "X is A" *and* "(X, Y) is R" *infer* "Y is B" if for all $v \in D_Y$,

$$r_B(v) = \sup_{u \in D_X} (r_A(u) * r_R(u, v)).$$

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[37] Similarly to [86] we reserve the term "variable" for variables of a logical calculus (propositional variable, object variable) and use the term "variate" for what several people call variable.

[38] See [213].

where $*$ is a continuous t-norm. The relation r_B is sometimes called the *composition* of r_A and r_R, or the *image* of r_A under the relation r_R.

7.1.3 We ask: what does this mean? What is inferred? Is this any *deduction*?

First observe that the rule is *semantical*: given any structure $\mathbf{D}$ of domains of variates, it formulates a semantical condition on the fuzzy set r_B interpreting the predicate B in terms of r_A (fuzzy set) and r_R (fuzzy relation). Typically, no more explanation is given. The rule is stated more or less as self-evident.

But observe that in fact the definition of r_B in terms of r_A and r_R is expressible in BL$\forall$: The condition above just means that the formula $(\forall y)(B(y) \equiv (\exists x)(A(x) \,\&\, R(x,y))$ is 1-true in $\mathbf{D}$. Call the last formula $Comp$ (or $Comp(A,R,B)$, the composition); thus the rule demands (assumes) $\|Comp\|_{\mathbf{D}} = 1$.

Lemma 7.1.4 Under the present notation,

$$\mathrm{BL}\forall \vdash Comp \to ((A(X) \& R(X,Y)) \to B(Y)).$$

Consequently, for each structure $\mathbf{D}$ such that $\|Comp\|_{\mathbf{D}} = 1$,
$\|A(X) \& R(X,Y)\|_{\mathbf{D}} \leq \|B(Y)\|_{\mathbf{D}}$

Proof: By 5.1.14,
BL$\forall$ $\vdash (\forall y)([(\exists x)(A(x) \& R(x,y)) \to B(y)] \to (\forall x)[(A(x) \& R(x,y)) \to B(y)])$ and
BL$\forall$ $\vdash (\forall y)(\forall x)((A(x) \& R(x,y)) \to B(y)) \to [(A(X) \& R(X,Y)) \to B(Y)]$.
Thus
BL$\forall \vdash Comp \to ((A(X) \& R(X,Y)) \to B(Y))$
by transitivity of implication. The rest is evident. □

We also show that $Comp$ is the best condition making the above inference possible.

Lemma 7.1.5 Let B' be a unary predicate of the same sort as B. Then
BL$\forall$ $\vdash Comp \,\&\, (\forall y)(\forall x)((A(x) \& R(x,y)) \to B'(y)) \to (\forall y)(B(y) \to B'(y))$
(thus in each model $\mathbf{D}$, if $Comp$ is true then r_B is the smallest fuzzy subset of D_Y making the inference of the composition rule sound).

Proof: Evidently,
BL$\forall$ $\vdash$ $(\forall y)(\forall x)((A(x) \& R(x,y))$ $\to$ $B'(y))$ $\to$

$\rightarrow (\forall y)[(\exists x)(A(x)\&R(x,y)) \rightarrow B'(y)]$,
but assuming $Comp$,the formula $(\exists x)(A(x)\&R(x,y))$ is equivalent to $B(y)$, thus we get the desired implication.
Proof: □

We shall now discuss two important particular cases. For this purpose we replace the atomic formula $R(x,y)$ by an arbitrary formula $\varphi(x,y)$.

Corollary 7.1.6 (1) Let $Comp$ be the formula
$(\forall y)(B(y) \equiv (\exists x)(A(x) \,\&\, \varphi(x,y)))$. Then
$\mathrm{BL}\forall \vdash (Comp \,\&\, A(X) \,\&\, \varphi(X,Y)) \rightarrow B(Y)$.

Proof: Evident modifications of the above. □

7.1.7 Now we shall consider Zadeh's Generalized Modus Ponens as a particular case of the Compositional Inference rule. To this end let us slightly change notation: we replace A by A^*, B by B^* and then take $\varphi(x,y)$ to be $A(x) \rightarrow B(y)$ for some predicates A,B. Then 7.1.4 gives the following theorem.

Theorem 7.1.8 Let $Comp_{MP}$ be the formula

$$(\forall y)(B^*(y) \equiv (\exists x)(A^*(x) \,\&\, (A(x) \rightarrow B(y)))).$$

Then BL∀ proves

$$(Comp_{MP} \,\&\, A^*(X) \,\&\, (A(X) \rightarrow B(Y))) \rightarrow B^*(Y).$$

Remark 7.1.9 (1) This may be visualized as a deduction rule:

$$\frac{Comp_{MP}, A^*(X), A(X) \rightarrow B(Y)}{B^*(Y)}$$

The obvious reading is: if $Comp_{MP}, A^*(X), A(X) \rightarrow B(Y)$ are 1-true (in a given structure **D**) then $B^*(Y)$ is 1-true. But 7.1.8 gives more:

$$\|Comp_{MP} \,\&\, A^*(X) \,\&\, (A(X) \rightarrow B(Y))\|_{\mathbf{D}} \leq \|B^*(Y)\|_{\mathbf{D}},$$

in particular, if $Comp_{MP}$ is 1-true (B^* is defined as above), $A^*(X)$ is r-true and $A(X) \rightarrow B(Y)$ is s-true then $B^*(Y)$ is at least $(r * s)$ - true (where $*$ is the truth function of &).

(2) We stress that we have shown *provability in* BL∀. Thus the above rule may be read for implication and conjunction in Ł∀, G∀, Π∀ and any other predicate calculus given by a continuous t-norm.

(3) Furthermore, observe that you may replace & by $\wedge$ in 7.1.8 and make the obvious modifications, e.g. if the (modified) $Comp_{MP}$ is 1-true, $A^*(X)$ is r-true and $A(X) \to B(Y)$ is s-true then $B^*(Y)$ is $\min(r, s)$-true.

(4) Let us mention that the use of A, A^*, B, B^* should suggest that A^* is similar to A in some sense - and then $Comp_{MP}$ should say that B^* is similar to B in some other sense.[39] But be careful: If A, B, A^* are interpreted by crisp (0, 1 - valued) subsets of the respective domains then the interpretation of B^* is also crisp and

(i) either $r_{A^*} \subseteq r_A$ and $r_{A^*} \neq \emptyset$ and $r_{B^*} = r_B$,

(ii) or $r_{A^*} \subseteq r_A$ and $r_{A^*} = \emptyset$ and $r_{B^*} = \emptyset$,

(iii) or r_{A^*} is not a subset of r_A and then $r_{B^*} = D_Y$ (the full set). (This can be formulated and proved in BL$\forall$ - exercise.) See Figure 7.1.

(5) In general, if $Comp_{MP}$ is defined as in 7.1.8 then
$Comp_{MP} \vdash (\forall y)[((\exists x)(A^*(x) \,\&\, \neg A(x)) \to B^*(y)]$.
Thus for each $v \in D_Y, r^*_B(v) \geq \sup_{u \in D_X}(r_{A^*}(u) * (-)r_A(u))$.

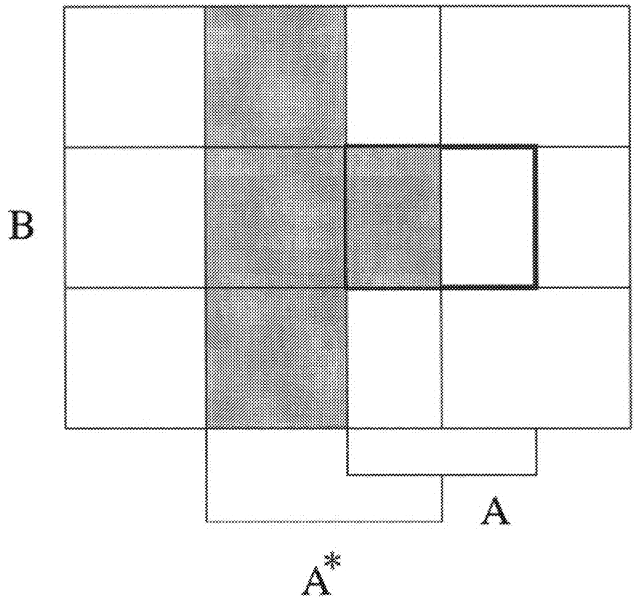

Figure 1. The grey domain is the set of pairs satisfying $A^*(x)\&(A(x) \to B(y))$.

(6) Observe that, under the present notation, and over BL$\forall$,
$Comp_{MP} \vdash (\exists x)A^*(x) \to (\forall y)(B(y) \to B^*(y))$.
Indeed, the following formulas are provable:

[39] The notorious example is: If the colour is red then the tomato is ripe; the colour is very red – what follows?

$B(y) \to (A(x) \to B(y))$,
$A^*(x) \,\&\, B(y)) \to (A^*(x) \,\&\, (A(x) \to B(y))$,
$(\exists x)(A^*(x) \,\&\, B(y)) \to (\exists x)(A^*(x) \,\&\, (A(x) \to B(y))$,
$(\exists x)(A^*(x) \,\&\, B(y)) \to B^*(y)$,
$[(\exists x)A^*(x) \,\&\, B(y)] \to B^*(y)$,
$(\exists x)A^*(x) \to (B(y) \to B^*(y))$ (by 5.1.18, (9)),
$(\exists x)A^*(x) \to (\forall y)(B(y) \to B^*(y))$.
Thus in particular, if for some $u \in D_X$, $r_{A^*}(u) = 1$, then for all $v \in D_Y$, $r_B(v) \leq r_{B^*}(v)$.

*

7.1.10 Now let us have predicates A, A^* (of the same sort), B, B^* (of the same sort) and take $A(x) \,\&\, B(y)$ for $\varphi(x, y)$. (Note that we could also take $A(x) \wedge B(y)$.) We automatically get a BL∀- provable tautology and hence a sound deduction rule. We shall ask what it means and how it relates to classical logic. In the next section we shall use our analysis of Generalized Modus Ponens and the present Generalized Conjunctive Rule to an analysis of the logical background of fuzzy controllers. As above, 7.1.4 gives the following

Theorem 7.1.11 Let $Comp_{CR}$ be

$$(\forall y)(B^*(y) \equiv (\exists x)(A^*(x) \,\&\, A(x) \,\&\, B(y))).$$

Then BL∀ proves

$$Comp_{CR} \,\&\, A^*(x) \,\&\, A(x) \,\&\, B(y) \to B^*(y)$$

Remark 7.1.12 (1) As above, we get a sound deduction rule

$$\frac{Comp_{CR}, A^*(X), A(X) \,\&\, B(Y)}{B^*(Y)},$$

Soundness means: for each domain structure **D**,

$$\|Comp_{CR}, A^*(X) \,\&\, A(X) \,\&\, B(Y)\|_{\mathbf{D}} \leq \|B^*(Y)\|_{\mathbf{D}}.$$

In the language of "fuzzy rules": From "X is A and Y is B" and "X is A^*" infer "Y is B^*". This is not a frequently used "fuzzy rule", but we get it naturally from the Compositional rule.

(2) There are at least two particular cases in the classical (Boolean) case. First, the usual rule for conjunction, second a rule of contradiction:

$$\frac{A(X) \,\&\, B(Y)}{B(Y)} \qquad \frac{\neg A(X), A(X) \,\&\, B(Y)}{\neg B(Y) \,\&\, B(Y)}$$

(3) In general, for A, B, A^* crisp, we get the picture for $A^*(x)$ & $A(x)$ & & $B(y)$ given in Fig. 7.2:

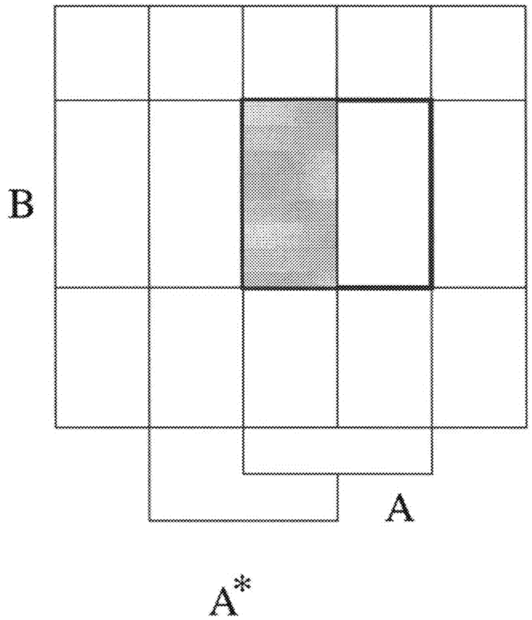

Figure 2.

(i) either $r_A \cup r_{A^*} \neq \emptyset$ (for some u, $r_A(u) = r_{A^*}(u) = 1$) and $r_B = r_{B^*}$
(ii) or $r_A \cup r_{A^*} = \emptyset$ and then $r_{B^*} = \emptyset$ ($r_{B^*}(v) = 0$ for all $v \in D_Y$).

(4) In full generality, the following is provable over BL∀: (for the present meaning of $Comp_{CR}$):

(i) $Comp_{CR} \vdash (\forall y)(B^*(y) \rightarrow B(y))$,

(ii) $Comp_{CR} \vdash (\forall y)(B^*(y) \rightarrow (\exists x)(A(x) \,\&\, (A^*(x)))$.

Indeed, assuming $Comp_{CR}$ we prove
$B^*(y) \rightarrow (\exists x)(A^*(x) \,\&\, A(x) \,\&\, B(y)) \rightarrow [(\exists x)(A^*(x) \,\&\, A(x)) \,\&\, B(y)]$ (by 5.1.18 (9)).

From (ii) we also get
$Comp_{CR} \vdash (\exists y)B^*(y) \rightarrow (\exists x)(A(x) \,\&\, A^*(x))$.

and hence
$Comp_{CR} \vdash \neg(\exists x)(A(x) \,\&\, A^*(x) \rightarrow \neg(\exists y)B^*(y)$.

(5) We again stress that all provabilities are over BL∀. These are valid over

Ł∀, G∀, Π∀ and any other t-norm logic. Moreover, the reader may check that everything remains valid if we replace & by $\wedge$ throughout.

Example 7.1.13 We shall present here a critical analysis of the use of the compositional rule of inference in some fuzzy expert systems like CADIAG-2 (developed by K.P. Adlassnig and his group; see [3, 4, 5, 117] or, for a short description , [115]; for a detailed analysis see [32]). We have three sets: M – the set of patients, S – the set of symptoms and D - the set of diagnoses. Further we have three fuzzy binary relations: $P : (M \times S) \to [0, 1]$ expressing for each pair $(p, s), p \in M, s \in S$, how much p has the symptoms s, further $R : (S \times D) \to [0, 1]$ expressing for each $s \in S$ and $d \in D$ how much s confirms d, and $P' : (M \times D) \to [0, 1]$ expressing how much a patient p has the diagnosis d. Thus we have a structure with three domains M, S, D and three binary relations. Let $Has, Conf, Has'$ be binary predicates naming P, R, P' respectively; thus $Has(x, s)$ says "x has the symptom s", similarly $Has'(x, d)$; and $Conf(s, d)$ says "s confirms d". Using the compositional rule (for an arbitrary but fixed x) one defines

$$Diag(x, d) \equiv (\exists s)(Has(x, s)\&Conf(s, d)).$$

This defines over Ł∀ or other logic a relation $C : (P \times D) \to [0, 1]$, expressing for each p and d how much d is confirmed for p.

So far so good; but *what does this mean*? Clearly, the answer depends on the definition of the meaning of $Conf$, i.e. of the relation R. Here our *warning comes:* In CADIAG and similar systems, one defines R from some data. $R(s, d)$ is taken to be the relative frequency $Fr(d|s)$ of presence of d among objects (patients) having d (for implicity, symptoms and diagnoses are assumed to be crisp). What does $Diag(x, d)$ mean in this case? This is difficult to say; but one thing is clear. Let p be a patient and let $s_1, \ldots, s_k$ be the symptoms he has. We would be interested in knowing, or at least estimating $Fr(d|s_1 \ldots s_k)$ – the relative frequency of d among object having $s_1 \ldots, s_k$ – as a possible estimate of the value of $Has'(x, d)$. But it must be clearly said that $Diag$ does *not* estimate this relative frequency; $Diag$ just defines $\max(Fr(d|s_1), \ldots, Fr(d(s_k))$. And observe that it can happen, for example, that $(k = 2)$, $Fr(d|s_1) = Fr(d|s_2) = 0.9$ but $Fr(d|s_1, s_2) = 0.2$. (See the following frequency table)

s_1	s_2	d	
1	1	1	1
1	0	1	44
0	1	1	44
0	0	1	5
1	1	0	4
1	0	0	1
0	1	0	1
0	0	0	3

Note that it does not help to allow S to contain conjunctions of symptoms – see [32] for details.

Let us offer one possible interpretation of $Conf$ with desirable properties (without any claim that this is the only right interpretation). Assume the relations P, P' interpreting Has, Has' to be fuzzy and let

$$Conf(s,d) \equiv (\forall x)(Has(x,s) \to Has'(x,d)).$$

Thus the truth degree of $Conf(s,d)$ is the minimum, over all patients x, of the truth degrees of the implication $Has(x,s) \to Has'(x,d)$. *Caution.* For example, in Ł∀, $\|Conf(s,d)\| = 0.9$ means that for each patient, $\|Has(x,s)\| \leq \min(1, Has'(x,d)\| + 0.1)$. (All truth values in the given structure.) Then for each patient p, the following formulas are true:

$(Has(p,s)\&Conf(s,d)) \to Has'(p,d)$, thus

$(\exists s)(Has(p,s)\&Conf(s,d)) \to Has'(p,d)$,

i.e. $Diag(p,d) \to Has'(p,d)$. Consequently, the truth degree of $Diag(p,d)$ is a lower bound of the degree in which p has the diagnosis d. This is surely pleasing; the question remains if e.g. the physician can supply knowledge necessary to evaluate $Conf$ in the present meaning.

7.1.14 In the rest of this section we shall discuss the question of what are "the present (actual) values of variables X, Y" and sketch a formalism for this. The present discussion will be used in an example in Ch. 8 and may be skipped now if desired.

We can imagine a new domain M of objects (situations, time elements, persons) interpreting a new sort of variables; denote the new variables by $z, z_1, \cdots$. We assume that for each object $m \in M$, each variate X takes a value $f_X(m) \in D_X$ (temperature, colour, etc. *of* m). Thus X becomes a *function symbol*; $X(z)$ is a term. Fixing an $m_0 \in M$ as a meaning of a new constant c for *actual object*, $X(c), Y(c)$ are terms for the values of the corresponding variates for the actual object.

Definition 7.1.15 Formally, this leads to a many-sorted many-valued calculus whose language consists of a sort of s_0 objects, sorts $s_1, \cdots, s_n$ for domains of variates,

- unary function symbols $X_1, \cdots, X_n$ for variates and an object constant c for "the actual object" (possibly other constants),
- variables for each sort,
- predicates of various types.

The language is called the *ground language*. A *structure* for this language has the form

$$\mathbf{M} = \langle M, D_{X_1}, \cdots, D_{X_n}, f_1, \cdots, f_n, (r_P)_{P\,predicate}, m_c, \cdots \rangle$$

(dots for possible meanings for other constants) where each f_i maps M into D_{X_i}. **M** is called a *ground structure*.

Terms are variables, constants and expressions $X_i(z)$ where X_i is the name of the i-th variate and z is a variable of the sort s_0. Everything else is as usual;

$$\|X_i(z)\|_{M,v} = f_i(v(z)).$$

7.1.16 We may now formulate axioms like $(\forall z)(A(X(z)) \to B(Y(z))$, saying "for each object z (situation etc.), if the value of the variate X (on the object z, in the situation z etc.) is A (big, etc) then the value of Y (on z) is B". Then e.g. the Generalized Modus Ponens with the old condition $Comp_{MP}$ saying $(\forall y)(B^*(y) \equiv (\exists x)(A^*(x)\,\&\,A(x) \to B(y))$ can be formulated as

$$Comp_{MP} \to (\forall z)((A^*(X(z))\,\&\,(A(X(z)) \to B(Y(z))) \to B^*(Y(z))$$

and shown to be provable in the corresponding obvious modification of BL$\forall$ (with a limited use of function symbols). You may derive various corollaries, e.g.

$$\frac{Comp_{MP}, (\forall z)(A(X(z)) \to B(Y(z)), A^*(X(c))}{B^*(Y(c))}$$

saying that if B^* is defined as $Comp_{MP}$ demands, if for all situations, [X is A implies Y is B] and if in the actual situation X is A^* then in the actual situation Y is B^*. Soundness of the rule guarantees e.g. that if $Comp_{MP}$ and [for all z, X is A implies Y is B] are true (i.e. 1-true) in a given ground structure and [X is A^*] is r-true in the actual situation then [Y is B^*] is at least r-time in this situation.

This appears to be the the way that the Generalized Modus Ponens is actually used.

7.2. Fuzzy Functions and Fuzzy Controllers

7.2.1 Fuzzy control is apparently the most broadly used application of fuzzy logic. Various books explaining fuzzy control, written by non-logicians, suffer by logical mismatch caused by the fact that "fuzzy IF-THEN rules" are presented as implications but then used to construct a fuzzy relation having little to do with any implication, at least at first glance (the relation is defined by a disjunction of conjunctions). Attempts to call e.g. the min-conjunction a "Mamdani implication" (see e.g. [35]) must be strictly rejected since we insist that the fuzzy truth function of a connective must behave classicaly for extremal values $0, 1$ – and this is not the case for minimum as implication. It has slowly become clear that fuzzy control deals with *approximation of functions* on the basis of pieces of fuzzy information of the kind "for arguments approximately equal c_i the image is approximately equal to d_i".[40]

It is illuminating to analyze the crisp situation. Assume we have two domains M_1, M_2 and a crisp, possibly partial, function f from M_1 to M_2. Moreover, let us have distinct elements $(u_1, v_1), \ldots, (u_n, v_n) \in M_1 \times M_2$ such that for each $i = 1, \ldots, n$, $f(u_i) = v_i$. Let us have a two-sorted language with equality (denoted $=$ for both domains) and a binary predicate F interpreted by f, let $\mathbf{M} = \langle M_1, M_2, f, =_1, =_2 \rangle$ where $=_i$ is identity on M_i, x-variables range on M_1, y-variables on M_2. The fact that f is a partial mapping is expresed by the sentence $(\forall x, y_1, y_2)((F(x, y_1) \wedge F(x, y_2)) \to y_1 = y_2)$. Let c_i be the constants for u_i, and d_i for v_i respectively.

(1) The formula

$$\bigwedge_i F(c_i, d_i)$$

just expresses the fact that $f(u_i) = v_i$; it is true in $\mathbf{M}$.

(2) The formula

$$\bigwedge_i ((x = c_i) \to (y = d_i))$$

defines a relation $r \subseteq M_1 \times M_2$ whose restriction to $\{u_1, \ldots, u_n\}$ coincides with the restriction of f to $\{u_1, \ldots, u_n\}$ and containing all pairs (u, v) where u is distinct from all $u_1, \ldots, u_n$ and $v \in M_2$; thus $f \subseteq r$.

(3) The formula

$$\bigvee_i (x = c_i \wedge y = d_i)$$

defines a relation $s \subseteq M_1 \times M_2$ which is the restriction of f to $\{u_1, \ldots, u_n\}$; i.e. no pair (u, v) with u distinct from all $u_1, \ldots, u_n$ belongs to s. Thus $s \subseteq f$.

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[40] For analyses of IF-THEN rules see [43, 118, 119, 155, 198]. Our presentation is a free elaboration of Kruse et al. [118] Sec. 4.4-4.5; and Godo and Hájek [61, 63, 62]; but our notion of a fuzzy function seems to be new.

Compare this, in the fuzzy case, with the deduction rules of the last section. We shall develop a general theory of fuzzy functions and "partial knowledge" on them and then apply it to describe (the logical aspect of) fuzzy control. We shall systematically develop the theory in BL$\forall$ (showing various statements to be provable); thus this will give, in particular, sound results for any t-norm logic $\mathcal{C}\forall$. To simplify matters, we shall deal only with unary functions (having one argument); a generalization to functions of several variables is easy.

After having discussed fuzzy functions we shall investigate the general logical structure of fuzzy controllers, not using fuzzy functions. The hurrying reader, not interested in fuzzy functions, may skip to 7.2.17.

Definition 7.2.2 Let T be a theory with a binary predicate F of a type (t_1, t_2), let $\approx_i$ be a similarity predicate in T for the sort t_i. (We shall write $\approx$ both for $\approx_1$ and $\approx_2$ without any danger of confusion.) We say that F *defines a (partial) fuzzy function* in T *with respect to* $\approx$ if T proves the following:

$$(x \approx x' \,\&\, y \approx y') \to (F(x,y) \equiv F(x',y')),$$

$$(F(x,y) \,\&\, F(x,y')) \to y \approx y'.$$

The first formula is the congruence axiom (cf. 5.6.5); the second says that any two images of x are similar.

Lemma 7.2.3 Let F define a partial fuzzy function in T w.r.t. $\approx$. Let c, d be constants such that $T \vdash F(c,d)$.

(1) Then $T \vdash (x \approx c \,\&\, F(x,y)) \to y \approx d$.

(2) Moreover, if $A(x)$ is the formula $x \approx c$ and $B(y)$ is the formula given by the condition $Comp$ of the compositional rule of inference from F and A (cf. 7.1.6 (2)), i. e. $B(y)$ is $(\exists x)(x \approx c \,\&\, F(x,y))$ then $T \vdash (B(y) \equiv y \approx d)$. (Thus the compositional rule transforms $x \approx c$ and $F(x,y)$ to $y \approx d$.)

Proof: (1) In T, $x \approx c \,\&\, F(x,y)$ implies $F(c,y)$ which, due to the provability of $F(c,d)$ gives $y \approx d$.

(2) Clearly, $T \vdash y \approx d \to [c \approx c \,\&\, F(c,y)] \to [(\exists x)(x \approx c \,\&\, F(x,y))]$. On the other hand, $T \vdash (\exists x)(x \approx c \,\&\, F(x,y)) \to y \approx d$ follows from (1). □

Definition 7.2.4 Let r_i be a similarity on M_i ($i = 1, 2$). A fuzzy relation $s : (M_1 \times M_2) \to [0,1]$ is a *fuzzy mapping* from M_1 into M_2 *w.r.t.* r_1, r_2 if s is extensional, i. e. for all $x, x' \in M_1, y, y' \in M_2$,

$$r_1(x,x') * r_2(y,y') * s(x,y) \leq s(x',y')$$

and functional, i. e.

$$s(x,y) * s(x,y') \leq r_2(y,y')$$

Lemma 7.2.5 Let F define a function w.r.t. $\approx$ in T. If $\langle M_1, M_2, r_1, r_2, s\rangle$ is a model of T then s is a fuzzy mapping from M_1 into M_2 w.r.t. r_1, r_2.

Proof: Obvious. □

Example 7.2.6 (1) Assume that s is a fuzzy mapping from M_1 into M_2 w.r.t. r_1, r_2 and let r_1, r_2, s be crisp (0, 1-valued). Then r_i is an equivalence on M_i and if we factorize, i. e. put $M_i' = M_i/r_i$ (in more details: $[u]_1 = \{u' \in M_1 | r_1(u,u') = 1\}$, analogously $[v]_2$ then putting $f([u]_1) = [v]_2$ iff $s(u,v) = 1$ we get a crisp mapping from M_1' into M_2' – see Fig. 7.3.

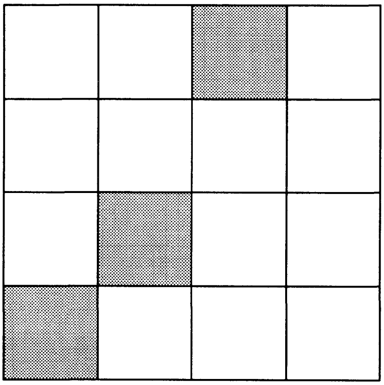

Figure 3.

(2) Now let r_i be similarities on M_i $(i = 1, 2)$ and let f be a crisp partial mapping from M_1 into M_2. Take the least extensional relation s containing f (cf. 5.6.13). Our question is under which conditions s is a fuzzy mapping from M_1 into M_2. The condition is that f respects the similarities r_1, r_2, i. e., for all $x_1, x_2 \in M_1$,

$$r_1(x_1, x_2) \leq r_2(f(x_1), f(x_2)).$$

Indeed, consider $\mathbf{M} = \langle M_1, M_2, r_1, r_2, f, s\rangle$ and let $\approx_1, \approx_2, F, \hat{F}$ be names of r_1, r_2, f, s; thus the formula

$$(\forall \cdots)(\hat{F}(x,y) \equiv (\exists x', y')(x \approx x' \,\&\, y \approx y' \,\&\, F(x',y'))\qquad (*)$$

is 1-true in **M**. Our condition reads

$$(\forall\cdots)((x_1 \approx x_2 \,\&\, F(x_1, y_1) \,\&\, F(x_2, y_2)) \to y_1 \approx y_2) \qquad (**)$$

Clearly, if the axioms of a fuzzy function are 1-true for $\approx_i$, $\hat{F}$ in **M** then (**) is 1-true. Conversely, assume (*) 1-true; we have to show that the formula

$$(\forall\cdots)((\hat{F}(x, y_1) \,\&\, \hat{F}(x, y_2)) \to y_1 \approx y_2)$$

is 1-true. Let T contain axioms of similarity for $\approx_i$ and (*).

$T \vdash [x \approx x' \,\&\, y_1 \approx y' \,\&\, x \approx x'' \,\&\, y_2 \approx y'' \,\&\, F(x', y') \,\&\, F(x'', y'')] \to y_1 \approx y_2$ (since the left-hand side implies $x' \approx x'' \,\&\, F(x', y') \,\&\, F(x'', y'') \,\&\, y_1 \approx y' \,\&\, y_2 \approx y''$, which in turn implies $y' \approx y'' \,\&\, y_1 \approx y' \,\&\, y_2 \approx y''$ (by (**)), and this implies $y_1 \approx y_2$). Thus

$$T \vdash (\exists x', y', x'', y'')[\cdots] \to y_1 \approx y_2$$

(cf. 5.1.14 (2))),
$T \vdash [(\exists x', y')(x \approx x' \,\&\, y_1 \approx y' \,\&\, F(x', y')) \,\&\, (\exists x'', y'')(x \approx x'' \,\&\, y_2 \approx y'' \,\&\, F(x'', y''))] \to y_1 \approx y_2$ (cf. 5.1.18 (9)), which gives

$$T \vdash (\hat{F}(x, y_1) \,\&\, \hat{F}(x, y_2)) \to y_1 \approx y_2.$$

See Fig. 7.4.

(3) We give an example of the meaning of the previous condition for Łukasiewicz logic. Assume $M_i = [a_i, b_i]$ are real intervals; let for $u, v \in m_i$, $r_i(u, v) = \max(0, 1 - c_i|u - v|)$. Then a sufficient condition for a mapping f of M_1 into M_2 is to satisfy the Lipschitz condition

$$|f(x_1) - f(x_2)| \leq \varepsilon|x_1 - x_2|$$

for an $\varepsilon \leq c_2/c_1$.

*

Our next task is to investigate the situation described as follows: there is a fuzzy mapping s from M_1 into M_2 (w.r.t. r_1, r_2), that is not at our disposal as a whole; but we know finitely many examples u_i, v_i ($i = 1, \ldots, n$) such that $s(u_i, v_i) = 1$, i. e. if F names s, c_i name u_i and d_i name v_i then $F(c_i, d_i)$ is 1-true in $\mathbf{M} = \langle M_1, M_2, r_1, r_2, s, u_i, v_i \rangle$. It follows immediately that each formula

$$x \approx c_i \,\&\, F(x, y) \to y \approx d_i$$

is 1-true; and this resembles an "IF- THEN rule"

IF x is similar to c_i THEN y is similar to d_i.

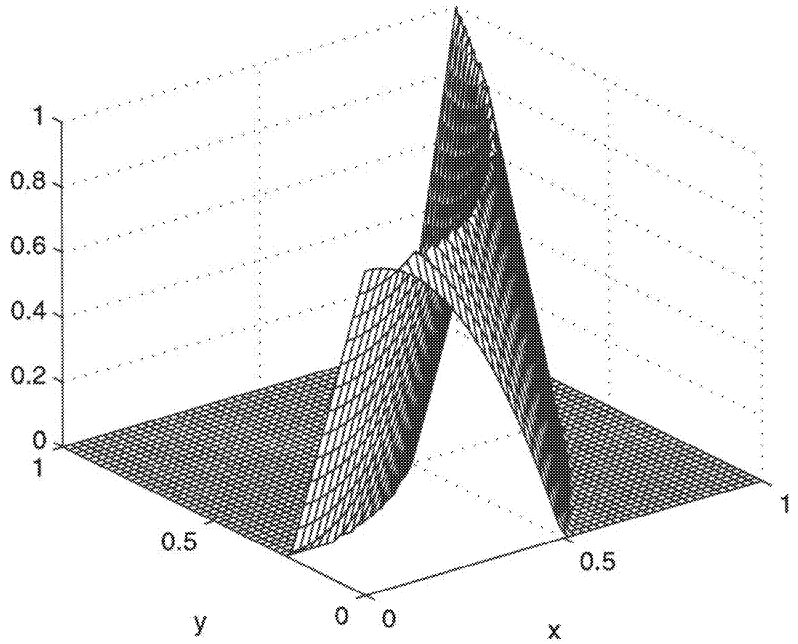

Figure 4. A fuzzy mapping given by the crisp function $y = x^2$.

What more can we say?

Definition 7.2.7 We say that, in a theory T, F *defines a* $\approx$*-function with examples* (c_i, d_i) $(i = 1, \ldots n)$ if F defines a fuzzy function in T w.r.t. a similarity $\approx$ and for $i = 1, \ldots, n$, T proves $F(c_i, d_i)$.

Repeating once more, the definitions say that the following formulas are provable in T: Similarity axioms for $\approx$,
extensionality of F: $(F(x,y) \,\&\, x \approx x' \,\&\, y \approx y') \to F(x', y')$,
functionality of F: $(F(x,y) \,\&\, F(x,y')) \to y \approx y')$,
examples of F: $\bigwedge_{i=1}^{n} F(c_i, d_i)$.
Let us agree that in the sequel $A_i(x)$ will stand for $x \approx c_i$ and $B_i(y)$ for $y \approx d_i$, unless stated otherwise.

Theorem 7.2.8 Let T be a theory over BL$\forall$ and assume that in T, F defines a $\approx$-function with examples (c_i, d_i) $(i = 1, \cdots, n)$. Then T proves the following formulas:

$$F(x,y) \to \bigwedge_i (A_i(x) \to B_i(y)),$$

$$\bigvee_i (A_i(x) \,\&\, B_i(y)) \to F(x,y).$$

Proof: (1) $T \vdash (F(x,y) \,\&\, x \approx c_i) \to F(c_i,y)$ (from extensionality),
$T \vdash F(c_i,y) \to y \approx d_i$ (from functionality and $T \vdash F(c_i,d_i)$).
Thus $T \vdash (F(x,y) \,\&\, A_i(x)) \to B_i(y)$,
$T \vdash F(x,y) \to (A_i(x) \to B_i(y))$,
$T \vdash F(x,y) \to \bigwedge_i (A_i(x) \to B_i(y))$. (cf. 7.2.3 (1).)
(2) $T \vdash (x \approx c_i \,\&\, y \approx d_i) \to F(x,y)$ from extensionality, thus
$T \vdash (A_i(x) \,\&\, B_i(y)) \to F(x,y)$,
$T \vdash \bigvee_i (A_i(x) \,\&\, B_i(y)) \to F(x,y)$. □

Remark 7.2.9 Given predicates A_i, B_i, we let $RULES(x,y)$ stand for the formula

$$\bigwedge_i (A_i(x) \to B_i(y))$$

and $MAMD(x,y)$ (resembling the name Mamdani, see his [131, 132, 133]) for the formula

$$\bigvee_i (A_i(x) \& B_i(y))$$

We shall prove various results on the relation of these two formulas. In particular, Theorem 7.2.8 says that under the assumptions made,

$$T \vdash MAMD(x,y) \to F(x,y) \to RULES(x,y).$$

Lemma 7.2.10 Let $T, F, \approx, c_i, d_i, A_i, B_i$ be as above and let $T \vdash$ $\vdash MAMD(x,y) \equiv \bigvee_i (A_i(x) \,\&\, B_i(y))$. Then $MAMD$ defines in T a $\approx$-function with examples (c_i, d_i).

Proof: *Extensionality*:
$T \vdash (x \approx c_i \,\&\, y \approx d_i) \to ((x' \approx x \,\&\, y' \approx y) \to (x' \approx c_i \,\&\, y' \approx d_i))$,
$T \vdash (A_i(x) \,\&\, B_i(y)) \to ((x' \approx x \,\&\, y' \approx y) \to (A_i(x') \,\&\, B_i(y')))$,
$T \vdash (A_i(x) \,\&\, B_i(y)) \to ((x' \approx x \,\&\, y' \approx y) \to MAMD(x',y'))$,
$T \vdash \bigvee_i (A_i(x) \,\&\, B_i(y)) \to ((x' \approx x \,\&\, y' \approx y) \to MAMD(x',y'))$,
$T \vdash (MAMD(x,y) \,\&\, x' \approx x \,\&\, y' \approx y) \to MAMD(x',y')$.
Functionality:
$T \vdash MAMD(x,y) \to F(x,y)$; thus
$T \vdash (MAMD(x,y) \,\&\, MAMD(x,y')) \to (F(x,y) \,\&\, F(x,y'))$, hence
$T \vdash MAMD(x,y) \,\&\, MAMD(x,y')) \to y \approx y'$, by the functionality of F.
Examples: Clearly, $T \vdash A_i(c_i) \,\&\, B_i(d_i)$,hence $T \vdash MAMD(c_i,d_i)$. □

Remark 7.2.11 Thus the formula $MAMD(x,y)$, i. e. $\bigvee_i(A_i(x) \,\&\, B_i(y))$ defines in T the least $\approx$-function with examples (c_i, d_i). *Caution*: The formula $RULES(x,y)$, i. e. $\bigwedge_i(A_i(x) \rightarrow B_i(y))$ (with our fixed assumptions, $A_i(x)$ is $x = c_i$ etc.) need not define a $\approx$-function! This can be seen already in the crisp case: for x non-equivalent to any of $c_1, \dots, c_n$, our formula gives no restriction to the value of y. In more details, if T is as above and $T \vdash RULES(x,y) \equiv \bigwedge_i(A_i(x) \rightarrow B_i(y))$ then
$T \vdash (\bigwedge_i \neg(A_i(x)) \rightarrow RULES(x,y)$ (since $T \vdash \neg A_i(x) \rightarrow (A_i(x) \rightarrow B_i(y)))$. see Fig.7.5.

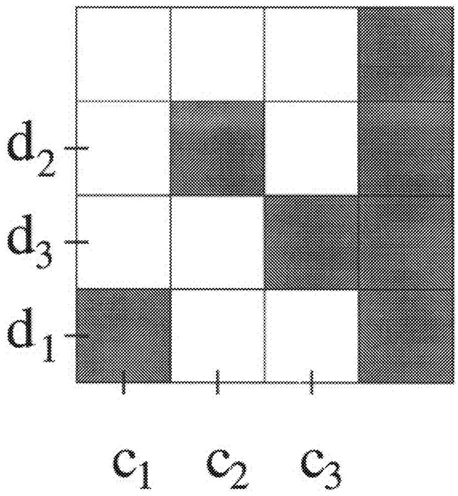

Figure 5.

Thus keeping our assumptions on T we may ask under which conditions the two formulas, $RULES(x,y)$ and $MAMD(x,y)$ are equivalent. The following lemma gives the answer:

Lemma 7.2.12 Let $T, F, \approx, c_i, d_i, A_i, B_i$ be as above, let $MAMD(x,y)$ stand for $\bigvee_i(A_i(x) \,\&\, B_i(y))$ (i. e. for $\bigvee_i(x \approx c_i \,\&\, y \approx d_i)$) and let $RULES(x,y)$ stand for $\bigwedge_i(A_i(x) \rightarrow B_i(y))$ i. e. for $\bigwedge_i(x \approx c_i \rightarrow y \approx d_i)$. Then

$$T \vdash (\bigvee_i A_i^2(x)) \rightarrow (MAMD(x,y) \equiv RULES(x,y)).$$

Proof: $T \vdash (A_i(x) \rightarrow B_i(y)) \rightarrow [(A_i(x) \,\&\, A_i(x)) \rightarrow (A_i(x) \,\&\, B_i(y))]$,
$T \vdash A_i^2(x) \rightarrow [(A_i(x) \rightarrow B_i(y)) \rightarrow (A_i(x) \,\&\, B_i(y))]$,

$T \vdash A_i^2(x) \to [\bigwedge_i (A_i(x) \to B_i(y)) \to \bigvee_i (A_i(x) \,\&\, B_i(y))]$,
$T \vdash \bigvee_i A_i^2(x) \to [MAMD(x,y) \equiv RULES(x,y)]$ (by 7.2.9; recall that in propositional calculus, $p \equiv q$ is equivalent to $(p \to q) \wedge (q \to p)$, cf. 2.2.16 (29)). □

Corollary 7.2.13 Under the present notation,
$T \cup \{(\forall x)(\bigvee_i A_i(x))\} \vdash (\forall x, y)(MAMD(x,y) \equiv RULES(x,y))$.
(Note that if **M** is a model of the theory in question then for each $u \in M$ there is an i such that if c_i denotes u_i in **M**, u is similar to u_i in degree 1.)

Proof: This follows from facts on the propositional calculus: we know that $\bigvee_i p_i \vdash (\bigvee_i p_i)^2$ (since $q \to (q \to q^2)$ is BL-provable); and $(\bigvee_i p_i)^2 \vdash \bigvee_i (p_i^2)$ by 2.2.24. □

What we have done up to now may be described (or interpreted) as follows: We have two domains M_1, M_2 (you could write D_X, D_Y instead), similarities r_1, r_2 on M_1, M_2 respectively and a partial fuzzy mapping s from M_1 to M_2, thus a model $\mathbf{M} = \langle M_1, M_2, r_1, r_2, s\rangle$. We introduce the language $\approx_1, \approx_2, F$ and assume we have n examples (u_i, v_i) named (c_i, d_i) such that $F(c_i, d_i)$ is 1-true in **M** (i. e. $s(u_i, v_i) = 1$). This can be expressed by saying "F sends c_i to d_i", or, "F sends (x similar to c_i) to (y similar to d_i)", or "IF x is similar to c_i (and $F(x,y)$) THEN y is similar to d_i". We know that in each model **M** as above the formula $(\forall x, y)(F(x,y) \to \bigwedge_i ((x \approx c_i) \to (y \approx d_i)))$ is 1-true, thus s is a subrelation of the fuzzy relation defined by $\bigwedge_i (x \approx c_i \to y \approx d_i)$ (which itself need not be a $\approx$ mapping); on the other hand, the formula $\bigvee_i (x \approx c_i) \,\&\, y \approx d_i)$ defines in **M** a $\approx$-fuzzy mapping h which is a subrelation of s and satisfies $h(u_i, v_i) = 1$.

Lemma 7.2.12 says that for each $u \in M_1$, $v \in M_2$, the degree in which u satisfies $\bigvee_i (x \approx c_i)^2$ (i. e. which u is *very similar* to an u_i, $i = 1, \ldots, n$) is a lower bound for the degree in which (u, v) satisfies $MAMD(x,y) \equiv RULES(x,y)$.

We should ask the following: What if we just have M_i, similarities r_i and (potential) examples (u_i, v_i)? What must be assumed to be sure that there is a fuzzy mapping s (w.r.t. r_i) such that $s(u_i, v_i) = 1$? The following lemma gives the answer.

Lemma 7.2.14 Let T be a theory with two sorts and similarity predicates $\approx_1, \approx_2$ of the respective sorts; let $c_1, \ldots, c_n$ be constants of the first sort and $d_1, \ldots, d_n$ constants of the second sort. If $T \vdash c_i \approx c_j \to d_i \approx d_j$ for each i, j (indices at $\approx$ deleted) and $T \vdash MAMD(x,y) \equiv \bigvee_i (x \approx c_i \,\&\, y \approx d_i)$

then $MAMD$ defines a $\approx$-function in T and $T \vdash MAMD(c_i, d_i)$ for $i = 1, \ldots, n$.

Proof: *Extensionality* as above.
Functionality: T proves the following chain of implications.
$[MAMD(x, y_1) \,\&\, MAMD(x, y_2)] \rightarrow$
$[(\bigvee_i x \approx c_i \,\&\, y_1 \approx d_i) \,\&\, \bigvee_j (x \approx c_j \,\&\, y_2 \approx d_j)] \rightarrow$
$[\bigvee_{i,j} (x \approx c_i \,\&\, x \approx c_j \,\&\, y_1 \approx d_i \,\&\, y_2 \approx d_j)] \rightarrow$
$[\bigvee_{i,j} (c_i \approx c_j \,\&\, y_1 \approx d_i \,\&\, y_2 \approx d_j)] \rightarrow$
$[\bigvee_{i,j} (d_i \approx d_j \,\&\, y_1 \approx d_i \,\&\, y_2 \approx d_j)] \rightarrow y_1 \approx y_2$.
Examples: Obviously, $T \vdash (c_i \approx c_i \,\&\, d_i \approx d_i)$, thus $T \vdash MAMD(c_i, d_i)$. □

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Proof: Let us be still more modest: let us have M_1, M_2 and fuzzy subsets r_{A_i} of M_1, r_{B_i} of M_2. We ask under which conditions we may assume

- similarities s_1 on M_1 and s_2 on M_2 with respect to which r_{A_i}, r_{B_i} are extensional,
- elements $u_1, \ldots, u_n \in M_1$, $v_1, \ldots, v_n \in M_2$ such that such that r_{A_i} are "fuzzy singletons given by u_i with respect to s_1" and similarly for r_{B_i}, v_i, s_2,
- a s_1, s_2-fuzzy mapping r_F "sending u_i to v_i".

We shall answer these questions.

Lemma 7.2.15 [41] Let T be a theory, A_i unary predicates of the same sort $(i = 1, \ldots, n)$.

(1) Define a binary predicate $\approx$ as follows:

$$(\forall x, x')(x \approx x' \equiv \bigwedge_i (A_i(x) \equiv A_i(x'))).$$

The resulting extension T' of T is conservative, $\approx$ is a similarity in T' and T' proves all A_i to be extensional.

(2) Add new constants c_i and axioms $(\forall x)(A_i(x) \equiv x \approx c_i))$. The resulting theory T'' is a conservative extension of T' iff T' proves all formulas

$$(\exists x) A_i(x),$$

$$(\exists x)(A_i(x) \,\&\, A_j(x)) \rightarrow (\forall x)(A_i(x) \equiv A_j(x)).$$

[41] See [118] 4.13.

Proof: (1) T' is a conservative extension of T by 5.2.15; the proof that in T' $\approx$ is a similarity making all A_i extensional is an easy variant of 5.6.14.

(2) First assume T'' to be a conservative extension of T'; then it suffices to prove the above formulas in T''.

$T'' \vdash c_i \approx c_i$ thus $T'' \vdash (\exists x)(x \approx c_i)$ and $T'' \vdash (\exists x)A_i(x)$. Furthermore, $T'' \vdash (\exists x)(A_i(x) \,\&\, A_j(x)) \equiv (c_i \approx c_j)$ (since $T'' \vdash c_i \approx c_j \to (c_i \approx c_i \,\&\, c_i \approx c_j)$ and $T'' \vdash (x \approx c_i \,\&\, x \approx c_j) \to c_i \approx c_j$); and $T'' \vdash c_i \approx c_j \to (x \approx c_i \equiv x \approx c_j)$. Thus $T'' \vdash (\exists x)(A_i(x)\&A_j(x)) \to (\forall x)(A_i(x) \equiv A_j(x))$.

Conversely, assume that T' proves the above formulas. Then we may extend T' conservatively by all axioms $A_i(c_i)$; call the resulting theory T'''. $T''' \vdash x \approx c_i \to A_i(x)$ immediately from the definition of $\approx$; on the other hand, $T''' \vdash A_i(x) \to (A_i(c_i) \equiv A_i(x))$ (since $A_i(c_i)$ is provable),
$T''' \vdash (A_i(x) \,\&\, A_j(x)) \to (\forall z)(A_i(z) \equiv A_j(z))$,
$T''' \vdash A_i(x) \to (A_j(x) \to (A_i(c_i) \equiv A_j(c_i)))$,
$T''' \vdash A_i(x) \to (A_j(x) \to A_j(c_i))$ (since $T''' \vdash A_i(c_i)$)
and also
$T''' \vdash (A_i(c_i)\&A_j(c_i) \to (A_i(x) \equiv A_j(x))$
$T''' \vdash A_j(c_i) \to (A_i(x) \to A_j(x))$
thus all together,
$T''' \vdash A_i(x) \to \bigwedge_j(A_j(x) \equiv A_j(c_i))$, hence
$T''' \vdash A_i(x) \to x \approx c_i$.
Thus T''' is stronger than T'' (in fact equivalent to T'') and hence T'' is a conservative extension of T'. □

Theorem 7.2.16 Let T be a theory, A_i unary predicates of one sort, B_i unary predicates of another sort. Assume

$$T \vdash (\exists x)A_i(x), \qquad T \vdash (\exists y)B_i(y),$$

$$T \vdash (\exists x)(A_i(x)\&A_j(x)) \to (\forall x)(A_i(x) \equiv A_j(x)),$$

$$T \vdash (\exists y)(B_i(y)\&B_j(y)) \to (\forall y)(B_i(y) \equiv B_j(y)).$$

Add definitions $x_1 \approx x_2 \equiv \bigwedge_i(A_i(x_1) \equiv A_i(x_2))$, $y_1 \approx y_2 \equiv \bigwedge_i(B_i(y_1) \equiv B_i(y_2))$, new constants c_i, d_i and axioms
$A_i(x) \equiv x \approx c_i, \qquad B_i(y) \equiv y \approx d_i$.
Finally add the definition

$$MAMD(x,y) \equiv \bigvee_i A_i(x)\&B_i(y).$$

The resulting theory T^M is a conservative extension of T and $\approx_1, \approx_2$ are similarities.

$MAMD$ defines in T^M a fuzzy mapping w.r.t. $\approx_1, \approx_2$ with the examples (c_i, d_i) iff

$$T \vdash (\exists x)(A_i(x) \& A_j(x)) \rightarrow (\exists y)(B_i(y) \& B_j(y)).$$

Proof: We only put things together. Adding $\approx_i, c_i, d_i$ and the axioms concerning them to T is conservative and the extension proves similarity axioms by the preceding lemma. Also recall that $c_i \approx c_j$ is equivalent to $(\exists x)(A_i(x) \& A_j(x))$ and similarly for $d_i \approx d_j$. Thus the fact that $MAMD(x, y)$ defines a fuzzy mapping follows by 7.2.14. □

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7.2.17 Afteı having discussed fuzzy functions at large, let us ask what we can say about the (logical) principles of fuzzy control in general, without relating it to the notion of similarity. (We again restrict ourselves to just one "input" variate X, a generalization to more input variates being easy.) The heartof the matter is as follows: We have n rules "IF X is A_i THEN Y is B_i" $r_i = 1, \cdots, n$, A_i unary predicates of the same sort, B_i predicates, all of the same sort (possibly different from the former sort). Let us write the rules as $A_i(x) \rightarrow B_i(y)$ cf. 7.1.1). We use A_i and B_i to define a binary predicate $MAMD$ by

$$(\forall x, y)(MAMD(x, y) \equiv \bigvee_i (A_i(x) \& B_i(y))) \qquad (Mamd)$$

and, given another unary predicate A^* of the first sort, define a B^* from A^*, via the compositional rule of inference, i. e.

$$(\forall y)(B^*(y) \equiv (\exists x)(A^*(x) \& MAMD(x, y))). \qquad (B*)$$

Given a model $\mathbf{M} = \langle D_X, D_Y, r_{A_i}, r_{B_i} \rangle$ this defines a functional associating to each fuzzy subset r_{A^*} of D_X the corresponding fuzzy subset r_{B^*} of D_Y. (Note that in fuzzy control this is used to define a crisp mapping of D_X into D_Y: one first uses a *fuzzification* operation, associating to each $u \in D_X$ a fuzzy set r_{A^*} ("approximately u"), then applies the functional to get r_{B^*} and finally applies a *defuzzification* procedure converting the fuzzy set r_{B^*} into a crisp output v. We shall disregard the operations of fuzzification and defuzzification.)

Our question now reads: *is there any logic here*? Let us try a positive answer, as general as possible. To this end we shall make the above formulas axioms of a theory of fuzzy control:

Definition 7.2.18 FC is a two-sorted theory having unary predicates $A_1, \ldots,$ $\ldots A_n, A^*$ of sort 1, unary predicates $B_1, \ldots, B_n, B^*$ of sort 2 a binary predicate $MAMD$ of the type $\langle 1, 2\rangle$. The axioms are the formulas (Mamd), (B*) above (defining $MAMD$ from A_i, B_i and defining B^* from $A^*, MAMD$). In addition, FC has two constants: X of sort 1 and Y of sort 2.

Theorem 7.2.19 FC proves the following (over BL∀):

$$[\bigwedge_i (A_i(X) \to B_i(Y)) \& \bigvee_i (A_i)^2(X)] \to (A^*(X) \to B^*(Y)):$$

7.2.20 Before we prove the theorem let us discuss its meaning. It says that, under the asumptions as how B^* is obtained, if the current value of the variate X (denoted by the constant X) satisfies, together with the current value the variate Y, all the rules $A_i(X) \to B_i(Y)$ *and* (sharp and) X satisfies $\bigvee_i {A_i}^2(X)$ then $A^*(X)$ implies $B^*(Y)$. In particular, assume $[\cdots]$ to be 1-true in **M**. Then $\|A_i(X)\|_M \leq \|B_i(Y)\|_M$ for all i and $\|A_i(X)\|_M = 1$ for at least one i. The conclusion is $\|A^*(X)\|_M \leq \|B^*(Y)\|_M$.

But this is not all. Assume the value of the antecedent $[\cdots]$ to be $\geq r$, i.e. the rules are *sufficiently* true and X *sufficiently* satisfies one of A_i's. The conclusion is that $\|B^*(Y)\|_M$ is *not much less* than $\|A^*(X)\|_M$. For example, if the rules are 1-true then $\|B^*(Y)\|_M \geq \|A^*(X)\|_M * \|\bigvee A_i^2(X))\|_M$ ($*$ being the interpretation of &). We shall come back to this discussion in a moment.

7.2.21 *Proof of* 7.2.19.
FC⊢ $(A_i(X) \& (A_i(X) \to B_i(Y))) \to B_i(Y)$, thus
FC⊢ $(A_i^2(X) \& \bigwedge_i (A_i(X) \to B_i(Y))) \to (A_i(X) \& B_i(Y))$, thus
FC⊢ $[A_i^2(X) \& \bigwedge_i (A_i(X) \to B_i(Y)) \& A^*(X)] \to A^*(X) \& MAMD(X, Y)$.
Consequently,
FC⊢ $(A_i^2(X) \& RULES(X, Y)) \to (A^*(X) \to$
$\to (\exists x)(A^*(x) \& MAMD(x, Y)))$,
which gives the result by the definition of B^*.

Now let us see what happens if we assume A^* to be equivalent to A_i:

Theorem 7.2.22 FC proves (over BL∀) the following:

$$[(\forall x)(A^*(x) \equiv A_i(x)) \& (\exists x) A_i^2(x)] \to (\forall y)(B_i(y) \to B^*(y)),$$

$$[(\forall x)(A^*(x) \equiv A_i(x)) \& (\forall x)(\bigwedge_{i \neq j} \neg(A_i(x) \& A_j(x)))] \to$$

$$\to (\forall y)(B^*(y) \to B_i(y)).$$

Proof: (i)
FC$\vdash (A_i^2(x)\&(A_i(x) \equiv A^*(x))\&B_i(y)) \to A^*(x)\&A_i(x)\&B_i(y)$,
FC$\vdash [(\exists x)(A_i^2(x)\&(A_i(x) \equiv A^*(x))\&B_i(y)] \to$
$\to (\exists x)(A^*(x)\& \bigvee_j(A_j(x)\&B_j(y)))$,
FC$\vdash [(\exists x)A_i^2(x)\&(\forall x)(A_i(x) \equiv A^*(x)] \to (\exists x)(A_i^2(x)\&(A_i(x) \equiv A^*(x)))$,
FC$\vdash [(\exists x)A_i^2(x)\&(\forall x)(A_i(x) \equiv A^*(x))\&B_i(y)] \to$
$\to (\exists x)(A^*(x)\&MAMD(x,y))$,
FC$\vdash [(\exists x)A_i^2(x)\&(\forall x)(A_i(x) \equiv A^*(x))] \to (B_i(y) \to B^*(y))$.
Now we prove (ii).
Write $Dsjnt(A_i)$ for $(\forall x) \bigwedge_{j\neq i} \neg(A_i(x)\&A_j(x))$,
$Equiv(A_i, A^*)$ for $(\forall x)(A^*(x) \equiv A_i(x))$.
FC$\vdash (B^*(y)\&Equiv(A^*, A_i)) \to (\exists x)(A_i(x)\& \bigvee_j(A_j(x)\&B_j(y)))$,
FC$\vdash Dsjnt(A_i) \to [(A_i(x)\& \bigvee_j(A_j(x)\&B_j(y)) \to A_i^2(x)\&B_i(y)]$
(since $A_i(x)\&A_j(x)\&B_j(y)$ implies $\bar{0}$ for $i \neq j$), i. e.
FC$\vdash [Dsjnt(A_i)\&Equiv(A^*, A)\&B^*(y)] \to (\exists x)(A_i^2(x)\&B_i(y))$,
FC$\vdash [Dsjnt(A_i)\&Equiv(A^*, A)] \to (B^*(y) \to B_i(y))$,
which gives the result by generalizing (by $(\forall y)$) and moving $(\forall y)$. □

Remark 7.2.23 (1) Again read the formulas as true in a model – first with the antecedent 1-true and then with the antecedent *sufficiently true.* We see that

(i) if $A^*(x)$ is sufficiently near to A_i and A_i is (sufficiently) non-empty then B_i is sufficiently included in B^*;

(ii) if A_i is sufficiently disjoint from all the other A_j's and A^* is sufficiently near to A_i then B^* is sufficiently included in B_i. Obviously, these are fuzzy readings; the precise meaning is given by the formulas proved and may be expressed in greater detail again as an exercise.

(2) Let us repeat once more that instead of antecedent of the form $A_i(X)$ we could investigate $A_{i1}(X_1)\&\ldots\&A_{ik}(X_k)$ or $A_{i1}(X_1) \wedge \ldots \wedge A_{ik}(X_k)$; this brings no problems but is more cumbersome.

(3) On the other hand, replacing & by $\wedge$ in the definition on $MAMD$ does bring additional problems (unless your logic is $G\forall$ – Gödel). We shall not go into them here.

7.3. AN ALTERNATIVE APPROACH TO FUZZY RULES

7.3.1 Up to now, we have worked with two variates X, Y with domains D_X, D_Y respectively; syntactically, we had just two sorts, predicates A_i, A^*

of the first sort and B_i, B^* of the second. X and Y were understood as *object constants* (for the actual value of the respective variate) and we had *rules* of the form $A_i(X) \to B_i(Y)$ saying "if the actual value of X is A_i then the actual value of Y is B_i". X and Y thus denoted possibly unknown but crisp elements of the respective domains.

Let us now try to be still more fuzzy and let X, Y denote *fuzzy subsets* of the respective domains, giving some vague information on the actual values of our variates. Syntactically this means that X and Y become *unary predicates* of the respective sorts. Then it is natural to formalize the assertion "X is A_i" to be just the formula $(\forall x)(X(x) \to A_i(x))$ (briefly, $X \subseteq A_i$). Indeed, the formula is 1-true if for each element m of D_X, the degree in which m satisfies X is a lower bound of the degree in which m satisfies A_i. (The reader may state in words the meaning of $X \subseteq A_i$ if this formula is r-true.) We shall reconsider the generalized modus ponens and the inference in fuzzy control in this new setting.[42]

Definition 7.3.2 (1) If X, A are unary predicates of the same sort then $X \subseteq A$ stands for $(\forall x)(X(x) \to A(x))$. Similarly, if φ, ψ are formulas with exactly one free variable x then $\varphi \subseteq \psi$ stands for $(\forall x)(\varphi(x) \to \psi(x))$

(2) Given A^*, A, B, B^{**} of the obvious sort, $Comp_{MPA}$ (alternative composition for generalized modus ponens) stands for the formula

$$(\forall y)(B^{**}(y) \equiv [(\forall x)(A^*(x) \to A(x)) \to B(y)]))$$

or, briefly,

$$B^{**}(y) \equiv [(A^* \subseteq A) \to B(y)].$$

Remark 7.3.3 Recall $Comp_{MP}$, i. e. the formula

$$(\forall y)(B^*(y) \equiv (\exists x)(A^*(x)\&(A(x) \to B(y)))$$

and observe that, over BL$\forall$, $Comp_{MP}, Comp_{MPA} \vdash B^* \subseteq B^{**}$. Indeed, assuming B^*, B^{**} to be defined by $Comp_{MP}, Comp_{MPA}$ respectively, we can prove the following:

$$B^{**}(y) \equiv ((\forall x)(A^*(x) \to A(x)) \to B(y)),$$

$$B^{**}(y) \equiv (\exists x)((A^*(x) \to A(x)) \to B(y)),$$

$$B^*(y) \equiv (\exists x)(A^*(x)\&(A(x) \to B(y))),$$

$$[A^*(x)\&(A(x) \to B(y))] \to [(A^*(x) \to A(x)) \to B(y)]$$

[42] Cf. [63, 62].

(note that the last formula is equivalent to the provable formula

$$[A^*(x)\&(A^*(x) \to A(x))\&(A(x) \to B(y))] \to B(y)\,),$$

$$(\exists x)(A^*(x)\&(A(x) \to B(y)) \to (\exists x)((A^*(x) \to A(x)) \to B(y)),$$

$$B^*(y) \to B^{**}(y).$$

Theorem 7.3.4 Let $Comp_{MPA}$ be as in 7.3.2, i. e. $(\forall y)(B^{**}(y) \equiv (A^* \subseteq A) \to B(y)))$. Then BL$\forall$ proves

$$[Comp_{MPA}\&(X \subseteq A^*)\&((X \subseteq A) \to (Y \subseteq B))] \to Y \subseteq B^{**}$$

Proof: Observe that it suffices to prove, in BL$\forall$, the formula

$$[(X \subseteq A^*)\&((X \subseteq A) \to (Y \subseteq B))] \to Y \subseteq [(A^* \subseteq A) \to B] \qquad (1)$$

Indeed, having (1) we get
$[(X \subseteq A^*)\&((X \subseteq A) \to (Y \subseteq B))\&Comp_{MPA} \to (Y \subseteq [(A^* \subseteq A) \to B]\&[(A^* \subseteq A) \to B] \subseteq B^{**}$ which gives the result by (provable) transitivity of $\subseteq$. Thus we prove the formula (1).

First observe that, by our axiom on quantifiers, the following chain of implications is provable:

$$[(\forall x)(X(x) \to A(x)) \to (\forall y)(Y(y) \to B(y))] \to$$
$$\to \quad (\forall y)[(\forall x)(X(x) \to A(x)) \to (Y(y) \to B(y))] \to$$
$$\to (\forall y)[Y(y) \to ((\forall x)(X(x) \to A(x)) \to B(y))]. \qquad (2)$$

on the other hand, by the (provable) properties of implication,

$$(X(x) \to A^*(x)) \to [((X(x) \to A(x)) \to B(y)) \to$$
$$\to ((A^*(x) \to A(x)) \to B(y))],$$

thus

$$X \subseteq A^* \to [(\exists x)((X(x) \to A(x)) \to B(y)) \to$$
$$\to (\exists x)((A^*(x) \to A(x)) \to B(y))],$$

and

$$X \subseteq A^* \to [((\forall x)(X(x) \to A(x)) \to B(y)) \to$$
$$\to ((\forall x)(A^*(x) \to A(x)) \to B(y))];$$

in short,

$$X \subseteq A^* \to ([(X \subseteq A) \to B] \subseteq [(A^* \subseteq A) \to B]) \qquad (3)$$

The implications in (2) prove

$$((X \subseteq A) \to (Y \subseteq B)) \to (Y \subseteq [(X \subseteq A) \to B]); \qquad (4)$$

and (3) and (4) give

$$[(X \subseteq A \to Y \subseteq B) \,\&\, (X \subseteq A^*)] \to$$
$$\to (Y \subseteq [X \subseteq A) \to B] \,\&\, ([(X \subseteq A) \to B] \subseteq [(A^* \subseteq A) \to B]);$$

by transitivity of $\subseteq$ we get our formula (1); this completes the proof of the theorem. □

Remark 7.3.5 (1) This remark is analogous to 7.1.9: we may visualize the result as a rule

$$\frac{Comp_{MPA}, X \subseteq A^*, (X \subseteq A) \to (Y \subseteq B)}{Y \subseteq B^{**}}$$

Thus if the assumptions are 1-true in a structure **M** then so is the conclusion. But again , let us stress that 7.3.4 gives more:

$$\|Comp_{MPA}\&(X \subseteq A^*)\&((X \subseteq A) \to (Y \subseteq B))\|_{\mathbf{M}} \leq \|Y \subseteq B^{**}\|_{\mathbf{M}}$$

(2) Observe that taking A for A^* $Comp_{MPA}$ becomes equivalent to $B \subseteq B^{**}$, thus we get the trivial rule (modus ponens)

$$\frac{X \subseteq A \to Y \subseteq B, X \subseteq A}{Y \subseteq B}$$

as a particular case.

(3) More generally, assume **M** to be a model; iff $\|A^* \subseteq A\|_{\mathbf{M}} = 1$ then in $\mathbf{M}, Comp_{MPA}$ is equivalent to $B \subseteq B^{**}$, if $\|A^* \subseteq A\|_{\mathbf{M}} = r < 1$ then $\|Comp_{MPA}\|_{\mathbf{M}} = 1$ iff for each m from the common domain of Y, B, B^{**}, $r_{B^{**}}(m) \geq \min(1, r_B(m) + 1 - r)$.

(4) Furthermore, we show that the rule in (1) becomes ill (non-sound) if we replace $Comp_{MPA}$ (the alternative composition for modus ponens) by $Comp_{MP}$ (see 7.3.3) and B^{**} by B^*. We exhibit the simple example of a structure in which $Comp_{MP}, X \subseteq A^*, X \subseteq A \to Y \subseteq B$ are 1-true but $Y \subseteq B^*$ is not. Let $D_X = \{x_0, x_1\}, D_Y = \{y_0\}$. (The example works in any of the logics $L\forall, G\forall, \Pi\forall$.)

The following tables give the interpretation of X, A, A^* and Y, B, B^* :

	X	A	A^*
x_0	1	1	1
x_1	$\frac{1}{2}$	0	$\frac{1}{2}$

	Y	B	B^*
y_0	1	$\frac{1}{2}$	$\frac{1}{2}$

One trivially verifies $\|Comp_{MP}\| = \|X \subseteq A^*\| = 1$, $\|X \subseteq A\| = \|Y \subseteq B\| = \frac{1}{2}$, thus $\|(X \subseteq A) \to (Y \subseteq B)\| = 1$, but $\|Y \subseteq B^*\| = \frac{1}{2}$. Note also $\|(\exists x)X(x)\| = 1$.

Thus the rule

$$\frac{Comp_{MP}, (\exists x)X(x), (X \subseteq A^*), (X \subseteq A) \to (Y \subseteq B)}{Y \subseteq B^*}$$

is not sound (in Ł$\forall$, $G\forall$, $\Pi\forall$). The reader may show as an exercise that the last rule *is* sound in the Boolean logic $Bool\forall$.

(5) Finally, observe that if $\approx$ a similarity predicate in T and T proves A, B, A^*, B^* to be extensional (i.e. proves congruence axioms for them) and for some constants c, d, the theory T proves $(\forall x)(X(x) \equiv x \approx c)$ and $(\forall y)(Y(y) \equiv y \approx d)$ i.e. X, Y define fuzzy singletons) then $T \vdash (X \subseteq A) \equiv A(c)$, $T \vdash (X \subseteq A^*) \to A^*(c)$, $T \vdash (Y \subseteq B) \equiv Y(d)$ etc. and $T \vdash (Comp_{MPA} \& (X \subseteq A^*) \& ((X \subseteq A) \to Y \subseteq B))] \to (Y \subseteq B^*)$. Thus we get, in this particular case, again the result of 7.1.8.

*

Let us now turn to fuzzy control. We just restrict ourselves to one result (relating the Mamdani formula to the rules in our new sense).

Theorem 7.3.6 Let A_i, A^*, X be unary predicates of one sort, B_i, B^*, Y unary predicates of another sort. Let $MAMD(x, y)$ stand for the formula $\bigvee_i (A_i(x) \& B_i(y))$ (as above) and let $Comp_{MAMD}$ be the formula

$$(\forall y)(B^*(y) \equiv (\exists x)(A^*(x) \& MAMD(x, y))$$

Then BL$\forall$ proves

$$[Comp_{MAMD} \& X \subseteq A^* \& \bigwedge_i ((X \subseteq A_i) \to Y \subseteq B_i)) \&$$

$$\& (\exists x)X^2(x) \& \bigvee_i (X \subseteq A_i)^2] \to Y \subseteq B^*.$$

Remark 7.3.7 Before we prove the theorem let us comment on its meaning. The inference pattern of the Mamdani-like fuzzy control mechanism can be formulated as

"if X is A^*, B^* is defined from A_i, B_i using the Mamdani formula, and Y corresponds to X then Y is B^*".

We have analyzed this pattern as a sound deduction rule (or, better, as a provable implication) in the preceding section. The question is what it means that Y corresponds to X. In the theorem this is understood as the assumption that the pair (X, Y) satisfies all the rules $(\bigwedge_i((X \subseteq A_i) \rightarrow (Y \subseteq B_i))$. Alternatively we replace this assumption by the assumption that (X, Y) satisfies an analogon of the Mamdani formula $\bigvee_i(x \subseteq A_i \& Y \subseteq B_i)$. We shall do this in the next lemma and then show how this gives our Theorem.

Lemma 7.3.8 Under the assumptions of 7.3.6, BL$\forall$ proves

$$[Comp_{MAMD} \& X \subseteq A^* \& (\exists x)\, X^2(x) \& \bigvee_i (X \subseteq A_i \& Y \subseteq B_i)] \rightarrow Y \subseteq B^*$$

Proof: BL$\forall$ proves
$[(\exists x)(X(x) \& X(x) \& (X \subseteq A^*) \& (X \subseteq A_i)] \rightarrow (\exists x)(A^*(x) \& A_i(x))$, thus
$(X_i \subseteq A_i \& Y \subseteq B_i) \rightarrow$
$\rightarrow [(X \subseteq A^* \& (\exists x(X^* 2(x) \& Y(y)) \rightarrow (\exists x)(A^*(x) \& A_i(x) \& B_i(y))]$, and
$(\exists x)(A^*(x) \& A_i(x) \& B_i(y)) \rightarrow (\exists x)(A^*(x) \& \bigvee_i(A_i(x) \& B_i(y))$, thus
$\bigvee_i((X \subseteq A_i) \rightarrow [((X \subseteq A^*) \& (\exists x) X^2(x) \& Y(y)) \rightarrow$
$\rightarrow (\exists x)(A^*(x) \& MAMD(x, y))]$ thus
$[Comp_{MAMD} \& X \subseteq A^* \& (\exists x) X^2(x) \& \bigvee_i(X \subseteq A_i \& Y \subseteq B_i)] \rightarrow$
$\rightarrow (Y(y) \rightarrow B^*(y))$,
which gives the result. □

7.3.9 *Proof of* 7.3.6. The theorem follows from our lemma by observing the provability of

$$[\bigvee_i (X \subseteq A_i)^2 \& \bigwedge_i (X \subseteq A_i \rightarrow Y \subseteq B_i)] \rightarrow \bigvee_i ((X \subseteq A_i) \& (Y \subseteq B_i))$$

(which is an easy exercise in propositional calculus).