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Walter Krämer (Editor)

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Econometrics of Structural Change

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Preface

Econometric models are made up of assumptions which never exactly match reality. Among the most contested ones is the requirement that the coefficients of an econometric model remain stable over time. Recent years have therefore seen numerous attempts to test for it or to model possible structural change when it can no longer be ignored. This collection of papers from Empirical Economics mirrors part of this development.

The point of departure of most studies in this volume is the standard linear regression model

$$y_t = x_t'\beta_t + u_t \quad (t = 1, \ldots, T),$$

where notation is obvious and where the index t emphasises the fact that structural change is mostly discussed and encountered in a time series context. It is much less of a problem for cross section data, although many tests apply there as well.

The null hypothesis of most tests for structural change is that $\beta_t = \beta_0$ for all t, i.e. that the same regression applies to all time periods in the sample and that the disturbances u_t are well behaved. The well known Chow test for instance assumes that there is a single structural shift at a known point in time, i.e. that $\beta_t = \beta_0$ $(t < t^*)$, and $\beta_t = \beta_0 + \Delta\beta$ $(t \geq t^*)$, where t^* is known.

It can easily be generalized to multiple structural shifts, the timing of which must however still be known. Another generalisation, provided by Toyoda and Ohtani in this volume, is to different change points for individual coefficients. Under the usual alternative all coefficients change at once, but here it is shown in a demand for fuel application that change points for individual coefficients might well be different.

Pötzelberger and Polasek consider the standard Chow test from a Bayesian viewpoint. By varying the prior distribution of $\Delta\beta$, they determine whether or not the structural change is robust against the different choices for the prior distribution, the major point being that a structural change can be diagnosed with much more confidence if it is found substantial irrespective of the prior distribution.

Leybourne and McCabe consider regression coefficients which follow a random walk, i.e. where

$$\beta_t = \beta_{t-1} + v_t \quad (v_t \sim \text{i.i.d. } (0, \sigma_V^2)).$$

Here, the null hypothesis of structural stability is equivalent to $H_0: \sigma_V^2 = 0$. Alternatively, one can dismiss specific alternatives altogether and look for pure significance tests, as is done by King and Edwards. By suitable transformations of recursive (or other LUS) residuals, they reduce the problem to one of testing independently distributed uniform random variables. This is similar to the established CUSUM and CUSUM of squares tests, which likewise do not require any prior knowledge about the type and timing of structural shifts.

Another group of papers in this volume consider standard procedures in non-standard situations. MacKinnon modifies the Chow test such as to become robust to heteroskedasticity among the disturbances u_t of the model, and Ploberger et al. adapt the CUSUM test to dynamic models of the form

$$y_t = \gamma y_{t-1} + x_t'\beta + u_t,$$

which are ruled out in the classical analysis with nonstochastic regressors. The problem is that recursive residuals are then no longer nid $(0, \sigma^2)$ (given nid disturbances), and the standard assessment of their cumulative sums breaks down.

A different but perennial problem in all empirical work is adressed by Lütkepohl and Phillips/McCabe. This is the possible presence of several complications at time. Lütkepohl considers test of causality in vector autoregressions, and shows that the true significance level far exceeds the nominal one when there is structural change in the regression coefficients. This implies that many rejections of non-causality which have been reported in empirical work in recent years may well be due to structural change.

Phillips and McCabe suggest a sequential approach to testing for structural change to take care of such multiple violations of the assumptions of the model. It has become common practice in empirical econometrics (and a good one at that) to test a model for various misspecifications such as omitted variables, autocorrelated or heteroskedastic disturbances, incorrect functional form or structural change at a time. The obvious problem with this approach, which Phillips and McCabe at least partially resolve, is how to control the Type I error probability and how to draw conclusions from the results of the tests.

An annotated bibliography containing about 400 items by Hackl and Westlund of econometric and statistical work on structural change concludes this volume. It is a tribute to the dynamics of this literature that in the few months after the acceptance for publication of this bibliography, dozens of additional papers have appeared which deal with the testing and modelling of structural change. A huge literature, which is not touched upon here, has for instance evolved around the Kalman filter approach to parameter instability. New tests for structural change keep appearing at an increasing rate, and given the multitide of possible models and alternatives, this will continue for quite some time. I should be pleased if readers would judge this volume as a useful contribution to this fascinating field.

Dortmund, April 1989 Walter Krämer

Contents

A Modification of the CUSUM Test in the Linear Regression Model with Lagged Dependent Variables

By W. Ploberger, W. Krämer, and R. Alt[1]

Abstract: We consider testing for structural change in a dynamic linear regression model, and show that the well known CUSUM test, which has been initially devised only for the standard static model, can easily be modified such as to remain asymptotically valid also in this nonstandard situation.

1 Introduction

Consider the simple dynamic regression model

$$y_t = \gamma y_{t-1} + \beta_1 x_{t1} + \ldots + \beta_K x_{tK} + u_t \quad (t = 1, \ldots, T), \tag{1}$$

where the disturbances u_t are idd$(0, \sigma^2)$ (not necessarily normal), $|\gamma| < 1, u_t$ is independent of y_{t-j} $(j \geqq 1)$, and the pre-sample observation y_0 is some fixed number. This paper is concerned with testing whether the regression coefficients γ

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Preliminary versions of this paper were presented at the first meeting of the IIASA working group on "Statistical and Economic Identification of Structural Change" in Lodz, May 1985, at the Econometric Society European Meeting in Budapest, Sep. 1986, at the annual meeting of the econometrics section of the "Verein für Socialpolitik" in Gießen, March 1987, and in seminars at CORE, Manchester, Rotterdam and Amsterdam. We are grateful to G. Chamberlain, J. Drèze, S. Dutta, P. Hackl, A. Harvey, J. Kiviet, T. Kloek, H. Lütkepohl, B. McCabe, G. D. A. Phillips, R. Quandt, and in particular to Wang Liqun, for helpful criticism and comments.

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and β remain stable over time. In particular, we address the applicability of the wellknown Brown-Durbin-Evans (1975) CUSUM test, which has initially been devised only for nonstochastic regressors, to the above dynamic model.

Let $x_t = [x_{t1}, \ldots, x_{tK}]'$, $X = [x_1, \ldots, x_T]'$, and $X_i^+ = [0, \ldots, 0, x_1, x_2, \ldots, x_{T-i}]'$. We then impose the following additional assumptions: X is nonstochastic, with $\|x_t\| = 0(1)$, and there exist a finite vector c and finite matrices Q_0 (nonsingular) and Q_i such that, as $T \to \infty$,

$$\frac{1}{T} \sum_{t=1}^{T} x_t \to c, \tag{2}$$

$$\frac{1}{T} X'X \to Q_0, \quad \text{and} \tag{3}$$

$$\frac{1}{T} X'X_i^+ \to Q_i. \tag{4}$$

Assumptions (3) and (4) guarantee consistency and asymptotic normality of OLS in the model (1) (Theil 1971, p. 412), and (2) is implied by (3) whenever there is a constant in the regression.

Let $z_t = [y_{t-1}, x_t']$, $Z = [z_1, \ldots, z_T]'$, $y = [y_1, \ldots, y_T]'$, $u = [u_1, \ldots, u_T]'$, and $\delta = [\gamma, \beta_1, \ldots, \beta_K]'$. The model (1) can then be rewritten as

$$y = Z\delta + u, \tag{5}$$

where

$$\frac{1}{T} \sum_{t=1}^{T} z_t \xrightarrow{p} d = \begin{bmatrix} \beta' c/(1-\gamma) \\ c \end{bmatrix} \tag{6}$$

and $(1/T)Z'Z \to R$ for some finite matrix R.

Disregarding for the moment the stochastic nature of the first column of Z, the CUSUM test for the stability of δ is based on successive partial sums of recursive residuals w_r, which for $K + 2 \leqq r \leqq T$ are defined as

$$w_r = (y_r - z_r' \hat{\delta}^{(r-1)})/f_r, \tag{7}$$

where

$$f_r = (1 + z_r'(Z^{(r-1)\prime}Z^{(r-1)})^{-1}z_r)^{1/2}, \tag{8}$$

$Z^{(r-1)} = [z_1, \ldots, z_{r-1}]'$, and $\hat{\delta}^{(r-1)}$ is the OLS estimate for δ from the first $r-1$ observations (superscripts will in the sequel always signify that the respective quantity is based on observations with index no larger than the superscript). The test statistic is

$$S = \max_{K+1<r\leqq T} \left| \frac{W^{(r)}}{\sqrt{T-K-1}} \right| \Bigg/ \left(1+2\frac{r-K-1}{T-K-1}\right), \tag{9}$$

where

$$W^{(r)} = \frac{1}{\hat{\sigma}} \sum_{t=K+2}^{r} w_t \tag{10}$$

is the cumulated sum of the recursive residuals and where

$$\hat{\sigma} = \left(\frac{1}{T-K-2} \sum_{r=K+2}^{T} (w_r - \bar{w})^2\right)^{1/2}. \tag{11}$$

(see Harvey 1975 for a discussion of the appropriate estimate for σ).

Given some significance level α, Brown, Durbin and Evans determine the appropriate critical value a, rather heuristically, by viewing the $W^{(r)}$'s as discrete readings from continuous Brownian Motion, i.e. by solving the expression

$$\Pr\left(\max_{K+1<r\leqq T} \frac{\tilde{W}^{(r)}}{\sqrt{T-K-1}} \Bigg/ \left(1+2\frac{r-K-1}{T-K-1}\right) \geqq a\right) = \frac{\alpha}{2} \tag{12}$$

for a, where $\tilde{W}^{(r)}$ is a continuous Gaussian process with mean and covariance function

$$E\tilde{W}^{(r)} = 0, \quad E\tilde{W}^{(r)2} = r - K - 1$$

$$E(\tilde{W}^{(r)}\tilde{W}^{(s)}) = \min(r, s) - K - 1. \tag{13}$$

Sen (1982) shows that, in the static model, this is asymptotically correct, in the sense that

$$\lim_{T\to\infty} \Pr\left\{ \max_{K<r\leqq T} \frac{W^{(r)}}{\sqrt{T-K}} \Bigg/ \left(1+2\frac{r-K}{T-K}\right) \geqq a \right\} = \frac{a}{2}. \tag{14}$$

The probability that S is greater than a therefore tends to

$$a - \Pr\,(\tilde{W}^{(r)} \text{ crosses both lines}), \tag{15}$$

where the latter term is negligible for the usual values of a (i.e. from one to ten percent).

2 Asymptotic Null Distribution in the Dynamic Model

Now consider the dynamic case. One can of course disregard the dynamic character of the regression and proceed with the CUSUM test as described above. We call this the dynamic CUSUM test. However, there is prima facie little reason to believe that the true rejection probability of this procedure will continue to be approximated by the corresponding probability from a Gaussian process. Even if the disturbances were normal, the recursive residuals are now neither normal nor independent, due to the presence of common stochastic components.

Dufour (1982, p. 46) notes that if we knew the true value of γ, the model could be reduced to standard form via

$$y_t - \gamma y_{t-1} = \beta_1 x_{t1} + \ldots + \beta_K x_{tK} + u_t \quad (t = 1, \ldots, T). \tag{16}$$

One could then proceed as usual and recursively estimate the vector β. When γ is unknown, one can replace it by the OLS estimate $\hat{\gamma}$ from the full sample, and hope that the resulting recursive residuals and any tests based on them will have approximately the same properties as those based on the true γ. We show next that this is indeed the case. The resulting variant of the CUSUM test will be referred to as the Dufour test. (Since this procedure does not fare well in our power investigation below, it is however only fair to say that Dufour did not in any way advocate this test.)

Let $\overset{*}{y}_t = y_t - \hat{\gamma} y_{t-1}$ $(t = 1, \ldots, T)$. Similar to (7), (8) and (10), define for $K < r \leqq T$

$$\overset{*}{w}_r = (\overset{*}{y}_r - x_r' \hat{\beta}^{(r-1)})/g_r, \tag{17}$$

$$g_r = (1 + x_r'(X^{(r-1)\prime} X^{(r-1)})^{-1} x_r)^{1/2}, \quad \text{and} \tag{18}$$

$$\overset{*}{W}^{(r)} = \sum_{t=K+1}^{r} \overset{*}{w}_t / \hat{\sigma}. \tag{19}$$

We then have the following result:

Theorem 1: Let a be determined from (12), and assume that there is a constant in the regression (1). Then, under the conditions imposed at the beginning of this section,

$$\lim_{T \to \infty} \Pr \left\{ \max_{K < r \leqq T} \frac{\overset{*}{W}^{(r)}}{\sqrt{T-K}} \bigg/ \left(1 + 2\frac{r-K}{T-K}\right) \geqq a \right\} = \frac{a}{2}. \tag{20}$$

The proof of Theorem 1 is rather involved, since it appears impossible to avoid the theory of weak convergence of probability measures on metric spaces. The problem is: how can we derive the limiting probability in (20) from the corresponding probability of a suitably defined limit process? Unfortunately, finite dimensional distribution theory does not apply here, since the $\overset{*}{W}^{(r)}$ processes do not converge in distribution (in the ordinary sense) to anything. Therefore we have to view these sequences (properly standardized) as mappings from a probability space into something more general than finite dimensional Euclidean space. The most authoritative treatment of such issues, on which we will draw heavily in our proof below, is still Billingsley (1968). Breiman (1968) and Gänssler and Stute (1977) also provide useful introductions. A convenient summary of the state of the art is Serfling (1980, Chapter 1.11), and various special issues are discussed in depths in Hall and Heyde (1980, Chapter 4).

Proof of Theorem 1: Let $D[0, 1]$ be the set of all real valued functions on the $[0, 1]$-interval that are right continuous and have left limits, and let $\mathscr{D}$ denote the σ-field generated by the Skorohod metric on $D[0, 1]$ (see Billingsley 1968, Chapter 3). A mapping f from some probability space into $D[0, 1]$ measurable with respect to $\mathscr{D}$ is then called a random element. This generalizes the conventional notion of a random variable, i.e. a mapping from a probability space into Euclidean space, to infinite dimensions. A sequence $f^{(T)}$ of random elements is said to converge in distribution (or weakly) to f (in symbols: $f^{(T)} \xrightarrow{p} f$) if

$$\Pr(f^{(T)} \in M) \rightarrow \Pr(f \in M)$$

for all $M \in \mathscr{D}$ with boundary of f-measure zero. This again generalizes the usual convergence concept for probability distributions on Euclidean spaces.

Associated with each random element $f(\omega)$ (where ω is an element of the underlying probability space) is a stochastic process $\tilde{f}(z)$, $0 \leqq z \leqq 1$, via $\tilde{f}(z, \omega) = f(\omega)(z)$, where we often drop the explicit reference to ω. Conversely, for every stochastic process $\tilde{f}(z)$ whose trajectories are constant or constant on intervals, there exists exactly one such random element $f(\omega)$. Since we will only encounter processes of this type below, we will henceforth not distinguish between random elements and stochastic processes and drop the $\tilde{}$-superscript.

The following results are either well known or easily shown and will subsequently be used to establish weak convergence of certain random elements:

Lemma 1 (Billingsley 1968, Theorem 4.1): Let $f^{(T)}$ and $g^{(T)}$ be random elements in $D[0, 1]$ such that $f^{(T)} \xrightarrow{d} f$ as $T \rightarrow \infty$ and

$$\sup_{0 \leqq z \leqq 1} |f^{(T)}(z) - g^{(T)}(z)| \xrightarrow{p} 0. \tag{21}$$

Then $g^{(T)}$ converges also in distribution to f.

Lemma 2 (Ploberger and Krämer 1986): Let x_t be random variables such that

$$\frac{1}{T} \sum_{t=1}^{T} x_t \xrightarrow{p} c \tag{22}$$

for some constant c. Then

$$\sup_{0 \leqq z \leqq 1} \left| \frac{1}{T} \sum_{t=1}^{Tz} x_t - cz \right| \rightarrow 0 \quad \text{(a.s.)}. \tag{23}$$

Now, for the proof of the theorem, consider the random element

$$\overset{*}{W}^{(T)}(z) = \frac{1}{\hat{\sigma}\sqrt{T-K}} \sum_{r=K+1}^{\tau(z)} \overset{*}{w}_r = \frac{1}{\sqrt{T-K}} \overset{*}{W}^{\tau(z)}, \tag{24}$$

where $\tau(z) = [K + (T-K)z]$ is the largest integer less than or equal to $K + (T-K)z$. The trajectories of the process (24) are constant on the half open intervals $((n-1)/(T-K), n/(T-K)]$ $(n = 1, \ldots, T-K)$, so $\overset{*}{W}^{(T)}$ is indeed a random element in $D[0, 1]$. Moreover, the probability in (20) can now be expressed as

$$\Pr\left(\max_{K < r \leqq T} \frac{\overset{*}{W}^{(r)}}{\sqrt{T-K}} \bigg/ \left(1 + 2\frac{r-K}{T-K}\right) \geqq a \right)$$

$$= \Pr\left(\max_{0 \leqq r \leqq 1} \overset{*}{W}^{(T)}(z)/(1 + 2z) \geqq a \right). \tag{25}$$

Since the boundary of the event $\{ \sup_{0 \leqq z \leqq 1} W(z)/(1+2z) = a \}$ has W-measure zero, Theorem 1 therefore follows from

$$\overset{*}{W}^{(T)} \xrightarrow{d} W. \tag{26}$$

The hard part is to establish (26). To this purpose, consider the random elements

$$\tilde{W}^{(T)}(z) = \frac{1}{\hat{\sigma}\sqrt{T-K}} \sum_{r=K+1}^{\tau(z)} \tilde{w}_r, \tag{27}$$

where $\tilde{w}_r$ is defined similar to $\overset{*}{w}_r$, but with the true γ in place of $\hat{\gamma}$. The $\tilde{w}_r$ can be viewed as recursive residuals from the standard static model, so $\tilde{W}^{(T)} \xrightarrow{d} W$ in view of Sen (1982), and (26) follows from

$$\sup_{0 \leqq z \leqq 1} |\overset{*}{W}^{(T)}(z) - \tilde{W}^{(T)}(z)| \xrightarrow{p} 0 \tag{28}$$

and Lemma 1.

For proof of (28), keep z initially fixed, let $Q^{(t)}=X^{(t)\prime}X^{(t)}$, and consider

$$\overset{*}{W}{}^{(T)}(z)-\tilde{W}^{(T)}(z)=\frac{1}{\hat{\sigma}\sqrt{T-K}}\sum_{r=K+1}^{\tau(z)}(\overset{*}{w}_r-\tilde{w}_r)$$

$$=\frac{T}{\hat{\sigma}\sqrt{T-K}}(\hat{\gamma}-\gamma)\left\{\frac{1}{T}\sum_{t=K+1}^{\tau(z)}y_{t-1}\right.$$

$$\left.-\frac{1}{T}\sum_{t=K+1}^{\tau(z)}x_t'[Q^{(t-1)}]^{-1}\sum_{s=1}^{t-1}x_s y_{s-1}\right\}/g_t. \tag{29}$$

It is easily seen that

$$\frac{T}{\hat{\sigma}\sqrt{T-K}}(\hat{\gamma}-\gamma)=0_p(1). \tag{30}$$

We show next that the term in pointed brackets on the rightmost side of (29) tends to zero in probability (uniformly in z). This is done by considering the two sums separately. As to the first, we have $\left(\sum_{t=K+1}^{T}\frac{y_{t-1}}{g_t}\right)/T\xrightarrow{p}\beta'c/(1-\gamma)$, where $c=\lim\,(\Sigma\,x_t)/T$ is from (2), so

$$\frac{1}{T}\sum_{t=K+1}^{\tau(z)}\frac{y_{t-1}}{g_t}\xrightarrow{p}z\beta'c/(1-\gamma) \tag{31}$$

(uniformly in z) in view of Lemma 2. Along similar lines, we show now that the expression on the right of (31) is also the probability limit (uniformly in z) of the second sum.

From $t[Q^{(t-1)}]^{-1}\rightarrow Q_0^{-1}$ and

$$\frac{1}{t}\sum_{s=1}^{t}x_s y_{s-1}\rightarrow\left(\sum_{i=0}^{\infty}\gamma^i Q_{i+1}\right)\beta\quad\text{(a.s.)}, \tag{32}$$

we have

$$\frac{1}{T}\sum_{t=K+1}^{\tau(z)}\frac{x_t'}{\gamma_\tau}[Q^{(t-1)}]^{-1}\sum_{s=1}^{t}x_s y_{s-1}\xrightarrow{p}c'Q_0^{-1}\left(\sum_{i=0}^{\infty}\gamma^i Q_{i+1}\right)\beta z \tag{33}$$

uniformly in z, where the uniformity of the convergence again follows from Lemma 2. Now we need the assumption that there is a constant in the regression. Moreover, assume without loss of generality that the constant is the last regressor, and that $Q_0 = I_K$. This implies that the mean regressor c equals $c = [0, \ldots, 0, 1]'$, and that the Q_i $(i = 1, 2, \ldots)$ are block diagonal with unity in the K, K-position. Therefore, the limit on the right hand side of (33) equals the limit on the right hand side of (31), and the term in pointed brackets on the rightmost side of (29) tends to zero in probability, uniformly in z. Since the term in front of the pointed brackets remains stochastically bounded, this in turn establishes (28) and the theorem.

3 Finite Sample Null Distribution

In view of Theorem 1, it does not matter asymptotically whether γ is known or estimated, given the model is correct (no structural change). Krämer, Ploberger and Alt (1987) show in addition that the Dynamic CUSUM test is likewise valid in the model (1). Below we report briefly on some Monte Carlo experiments to explore which procedure approximates its nominal size better in finite samples. (Any choice between them must of course also rest on their relative power under alternatives, but this issue is outside the scope of the present paper.)

Most experiments below were based on the model

$$y_t = 0.5y_{t-1} + (-1)^t + 1 + u_t \quad (t = 1, \ldots, T), \tag{34}$$

where $y_0 = 0$, $u_t = \text{nid}(0, 1)$, and with T equal to 30, 60, 120 and 1,000. The particular x-series was chosen to ensure condition (2), and for ease of comparison with similar experiments in Ploberger, Kontrus, and Krämer (1986).

Table 1 reports the empirical rejection probabilities for nominal significance levels α equal to one, five and ten per cent, based on 1,000 independent replications (trials, runs). Under the heading of "static CUSUM test", we also give the results for the case where $\gamma = 0.5$ is assumed known. This obviously amounts to the ordinary CUSUM test in the nonstochastic linear model.

Table 1. Monte Carlo estimates of finite sample significance levels

T	significance level α (%, nominal)								
	static C·-test			Dufour test			dynamic C·-test		
	1.0	5.0	10.0	1.0	5.0	10.0	1.0	5.0	10.0
30	0.3	2.7	6.8	0.1	1.4	3.5	0.3	2.8	6.0
60	0.1	2.8	7.0	0.3	2.2	5.4	0.3	4.4	8.6
120	0.7	3.7	6.8	0.5	2.5	7.4	0.4	3.0	7.5
1000	0.8	3.9	7.9	1.0	4.6	9.5	1.1	5.8	9.7

Table 1 shows that the nominal size for all variants of the CUSUM test consistently overstates the true significance level, sometimes drastically so. Somewhat unexpectedly, the asymptotic approximation works better for the dynamic version than for the Dufour test. The gap between true and nominal size narrows as sample size increases, as predicted by our analytical results.

We also investigated the robustness of these results to changes in the experimental design. Table 2 for instance reports empirical rejection probabilities for various alternative values of γ, and for $T=120$ (remaining design unchanged). These experiments show that the true size is fairly robust to changes in γ in case of the Dynamic CUSUM test, but varies widely in case of the Dufour test, improving as $\gamma \to -1$ and being completely off the mark as $\gamma \to 1$. (There is no point in including the corresponding results for the Static CUSUM test, since its true rejection probability is the same for all γ.)

We found this volatility of the true size of the Dufour test also when varying the β parameters. As the proof of Theorem 1 shows, this results from the form of the test statistic, which equals the test statistic of the Static CUSUM test, plus a remainder term that vanishes as $T \to \infty$. The correlation between these components depends on the underlying δ vector. The size of the Dufour test is larger than the corresponding figure for the Static CUSUM test when this correlation is positive, and smaller when the correlation is negative. For some parameter combinations, the actual size of the Dufour test even surpassed the nominal size.

The paper therefore ends on a rather unhappy note. Although the Dufour test turned out to be asymptotically valid irrespective of γ (i.e. it is asymptotically both valid and similar), it exhibits extreme non-similarity and possible violations of size in finite samples.

Table 2. Empirical significance level for alternative γ's

α (nominal, %)	γ								
	-.95	-.9	-.6	-.3	0	.3	.6	.9	.95
				a) Dufour Test					
1.0	0.5	0.7	0.3	0.7	0.5	0.7	0.4	0.3	0.1
5.0	3.8	3.8	4.1	3.1	3.9	3.3	2.6	2.0	1.0
10.0	8.7	8.4	7.4	8.2	7.7	6.9	6.3	3.5	1.5
				b) dynamic CUSUM Test					
1.0	0.6	0.4	0.3	0.3	0.3	0.5	0.4	0.9	0.8
5.0	2.9	3.3	3.3	2.8	3.1	3.4	4.3	5.2	4.5
10.0	6.5	6.8	7.1	5.8	7.6	7.3	9.6	10.1	8.4

References

Billingsley B (1968) Convergence of probability measures. Wiley, New York

Breiman L (1968) Probability. Addison-Wesley, Reading, Mass.

Brown RL, Durbin J, Evans JM (1975) Techniques for testing the constancy of regression relationships over time. Journal of the Royal Statistical Society B 27:149–163

Dufour JM (1982) Recursive stability analysis of linear regression relationships. Journal of Econometrics 19:31–76

Gänssler P, Stute W (1977) Wahrscheinlichkeitstheorie. Springer, Berlin

Hall P, Heyde CC (1980) Martingale limit theory and its application. Academic Press, New York

Harvey A (1975) Comment on the paper by Brown, Durbin and Evans. Journal of the Royal Statistical Society B 37:179–180

Johnston J (1984) Econometric methods, 3rd ed. McGraw-Hill, New York

Krämer W, Ploberger W, Alt R (1987) Testing for structural change in dynamic models. To appear in Econometrica

Ploberger W, Krämer W (1986) The local power of the CUSUM and CUSUM of squares test. Mimeo

Ploberger W, Kontrus K, Krämer W (1986) A new test for structural stability in linear regression. To appear in the Journal of Econometrics

Sen PK (1982) Invariance principles for recursive residuals. The Annals of Statistics 10:307–312

Serfling RJ (1980) Approximation theorems of mathematical statistics. Wiley, New York

Theil H (1971) Principles of economerics. Wiley, New York

Heteroskedasticity-Robust Tests for Structural Change[1]

By J. G. MacKinnon[2]

Summary. It is remarkably easy to test for structural change, of the type that the classic *F* or "Chow" test is designed to detect, in a manner that is robust to heteroskedasticity of possibly unknown form. This paper first discusses how to test for structural change in nonlinear regression models by using a variant of the Gauss-Newton regression. It then shows how to make these tests robust to heteroskedasticity of unknown form, and discusses several related procedures for doing so. Finally, it presents the results of a number of Monte Carlo experiments designed to see how well the new tests perform in finite samples.

1 Introduction

A classic problem in economerics is testing whether the coefficients of a regression model are the same in two or more separate subsamples. In the case of time-series data, where the subsamples generally correspond to different economic environments, such as different exchange-rate or policy regimes, such tests are generally referred to as tests for structural change. They are equally applicable to cross-section data, where the subsamples might correspond to different groups of observations such as large firms and small firms, rich countries and poor countries, or men and women. Evidently there could well be more than two such groups of observations.

The classical *F* test for the equality of two sets of coefficients in linear regression models is commonly referred to by economists as the Chow test, after the early and influential paper by Chow (1960). Another exposition of this procedure is Fisher (1970). The classic approach is to partition the data into two parts, possibly after re-ordering. The n-vector y of observations on the dependent variable is

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divided into an n_1-vector y_1 and an n_2-vector y_2, and the $n \times k$ matrix X of observations on the regressors is divided into an $n_1 \times k$ matrix X_1 and an $n_2 \times k$ matrix X_2, with $n = n_1 + n_2$. Thus the maintained hypothesis may be written as

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad E(uu^T) = \sigma^2 I, \tag{1}$$

where β_1 and β_2 are each k-vectors of parameters to be estimated. The null hypothesis to be tested is that $\beta_1 = \beta_2 = \beta$. Under it, (1) reduces to

$$y \equiv \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \beta + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \equiv X\beta + u, \quad E(uu^T) = \sigma^2 I. \tag{2}$$

In the usual case where both n_1 and n_2 are greater than k, it is easy to construct a test of (2) against (1) by using an ordinary F test. The unrestricted sum of squared residuals from OLS estimation of (1) is

$$\text{USSR} \equiv \text{SSR}_1 + \text{SSR}_2 = y_1^T M_1 y_1 + y_2^T M_2 y_2, \tag{3}$$

where $M_i \equiv I - X_i(X_i^T X_i)^{-1} X_i^T$ for $i = 1, 2$ denotes the $n \times n$ matrix which projects orthogonally off the subspace spanned by the columns of the matrix X_i. The vectors $M_1 y_1$ and $M_2 y_2$ are the residuals from the regressions of y_1 on X_1 and y_2 on X_2 respectively. Thus USSR is simply the sum of the two sums of squared residuals.

The restricted sum of squared residuals, from OLS estimation of (2), is

$$\text{RSSR} = y^T M_x y, \tag{4}$$

where $M_x \equiv I - X(X^T X)^{-1} X^T$. Thus the ordinary F statistic is

$$\frac{(y^T M_x y - y_1^T M_1 y_1 - y_2^T M_2 y_2)/k}{(y_1^T M_1 y_1 + y_2^T M_2 y_2)/(n-2k)} = \frac{(\text{RSSR} - \text{SSR}_1 - \text{SSR}_2)/k}{(\text{SSR}_1 + \text{SSR}_2)/(n-2k)}. \tag{5}$$

This test statistic, which is what many applied econometricians refer to as the "Chow test", has k and $(n-2k)$ degrees of freedom because the unrestricted model has $2k$ parameters while the restricted model has only k. It will be exactly distributed as $F(k, n-2k)$ if the error turms $\boldsymbol{u}$ are normal and independent of the fixed regressors $\boldsymbol{X}$, and k times it will be asymptotically distributed as $\chi^2(k)$ under much weaker conditions.

The ordinary Chow test (5) has one obvious and very serious limitation. Like all conventional F tests, it is (in general) valid only under the rather strong assumption that $E(\boldsymbol{u}\boldsymbol{u}^T) = \sigma^2 \boldsymbol{I}$. This assumption may be particularly implausible when one is testing the equality of two sets of regression parameters, since if the parameter vector $\boldsymbol{\beta}$ differs between two regimes the variance σ^2 may well be different as well. A number of papers have addressed this issue, including Toyoda (1974), Jayatissa (1977), Schmidt and Sickles (1977), Watt (1979), Honda (1982), Phillips and McCabe (1983), Ohtani and Toyoda (1985), Toyoda and Ohtani (1986) and Weerahandi (1987). However, none of these papers proposes the very simple approach of using a test which is robust to heteroskedasticity of unknown form. The work of Eicker (1963) and White (1980) has made such tests available, and Davidson and MacKinnon (1985) have provided simple ways to calculate them using artificial regressions. In this paper *I* show how the results of the latter authors may be used to calculate several heteroskedasticity-robust variants of the Chow test.

The plan of the paper is as follows. In Section 2 I discuss how to test for structural change in nonlinear regression models by using a variant of the Gauss-Newtonn regression. In Section 3 I then discuss ways to make the tests discussed in Section 2 robust to heteroskedasticity of unknown form. Finally, in Section 4, I present the results of some Monte Carlo experiments designed to see how well the new tests perform in finite samples.

2 Testing for Structural Change in Nonlinear Regression Models

Nonlinear regression models may seem unnecessarily complicated, but studying them makes it easier to see how to make Chow-type tests robust to heteroskedasticity. Suppose that the null hypothesis is

$$H_0: \quad y_t = x_t(\boldsymbol{\beta}) + u_t, \quad E(\boldsymbol{u}\boldsymbol{u}^T) = \sigma^2 \boldsymbol{I}, \tag{6}$$

where the regression functions $x_t(\boldsymbol{\beta})$, which may depend on exogenous and/or lagged dependent variables and on a k-vector of parameters $\boldsymbol{\beta}$, are assumed to be twice continuously differentiable. The matrix $\boldsymbol{X}(\boldsymbol{\beta})$, with typical element

$$X_{ti}(\boldsymbol{\beta}) = \frac{\partial x_t(\boldsymbol{\beta})}{\partial \beta_i}, \tag{7}$$

will play a major role in the analysis. In the case of the linear regression model $\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{u}$, $\boldsymbol{X}(\boldsymbol{\beta})$ is simply the matrix $\boldsymbol{X}$. It is assumed that

$$\operatorname*{plim}_{n \to \infty} \; (n^{-1}\boldsymbol{X}^T(\boldsymbol{\beta})\boldsymbol{X}(\boldsymbol{\beta})) \tag{8}$$

exists and is a positive-definite matrix.

For simplicity I shall assume that the sample is to be divided into only two groups of observations; extensions to the many-group case are obvious. We first define a vector $\boldsymbol{\delta} \equiv [\delta_1 \ldots \delta_n]^T$, letting $\delta_t = 0$ if observation t belongs to group 1 and $\delta_t = 1$ if observation t belongs to group 2. Note that it would be possible to let δ_t take on values between zero and one for some observations, which might be useful if it were thought that the transition between regimes was gradual rather than abrupt. If the null hypothesis is (6) the alternative hypothesis may be written as

$$H_1: \quad y_t = x_t(\boldsymbol{\beta}_1(1 - \delta_t) + \boldsymbol{\beta}_2\delta_t) + u_t, \quad E(\boldsymbol{uu}^T) = \sigma^2\boldsymbol{I}. \tag{9}$$

Thus the regression function is $x_t(\boldsymbol{\beta}_1)$ if $\delta_t = 0$ and $x_t(\boldsymbol{\beta}_2)$ if $\delta_t = 1$.

The alternative hypothesis H_1 can be rewritten as

$$y_t = x_t(\boldsymbol{\beta}_1 + (\boldsymbol{\beta}_2 - \boldsymbol{\beta}_1)\delta_t) + u_t = x_t(\boldsymbol{\beta}_1 + \gamma\delta_t) + u_t, \tag{10}$$

where $\gamma \equiv \boldsymbol{\beta}_2 - \boldsymbol{\beta}_1$. This makes it clear that H_0 is equivalent to the null hypothesis that $\gamma = 0$. Since the latter is simply a set of zero restrictions on the parameters of a nonlinear regression function, we can use a Gauss-Newton regression to test it; see Engle (1982b) or Davidson and MacKinnon (1984). The Gauss-Newton regression, or GNR, for testing H_0 against H_1 is easily seen to be

$$y_t - x_t(\tilde{\boldsymbol{\beta}}) = \boldsymbol{X}_t(\tilde{\boldsymbol{\beta}})\boldsymbol{b} + \delta_t\boldsymbol{X}_t(\tilde{\boldsymbol{\beta}})\boldsymbol{c} + \text{errors}, \tag{11}$$

where $\tilde{\boldsymbol{\beta}}$ denotes the nonlinear least squares (NLS) estimates of $\boldsymbol{\beta}$ for the whole sample.

The GNR (11) may be written more compactly as

$$\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{X}}\boldsymbol{b} + \boldsymbol{\delta} * \tilde{\boldsymbol{X}}\boldsymbol{c} + \text{errors}, \tag{12}$$

where $\tilde{\boldsymbol{u}}$ is an n-vector with typical element $y_t - x_t(\tilde{\boldsymbol{\beta}})$ and $\tilde{\boldsymbol{X}}$ is an $n \times k$ matrix with typical row $\boldsymbol{X}_t(\tilde{\boldsymbol{\beta}})$. Here "$*$" denotes the direct product of two matrices, a typical element of $\boldsymbol{\delta} * \tilde{\boldsymbol{X}}$ being $\delta_t X_{ti}(\tilde{\boldsymbol{\beta}})$, so that $\boldsymbol{\delta} * \tilde{\boldsymbol{X}}_t$ equals $\tilde{\boldsymbol{X}}_t$ when $\delta_t = 1$ and 0 when $\delta_t = 0$. Thus we can perform the test by estimating the model using the entire sample and regressing the residuals on the matrix of derivatives $\tilde{\boldsymbol{X}}$ and on the matrix $\boldsymbol{\delta} * \tilde{\boldsymbol{X}}$, which is $\tilde{\boldsymbol{X}}$ with the rows which correspond to group 1 observations set to zero. There is no need to reorder the data. Several asymptotically valid test statistics can then be computed, including the ordinary F statistic for the null hypothesis that $\boldsymbol{c} = 0$. In the usual case where k is less than $\min(n_1, n_2)$, it will have k degrees of freedom in the numerator and $(n - 2k)$ degrees of freedom in the denominator.

Unlike the ordinary "Chow test" (5), this procedure is applicable even if $\min(n_1, n_2) < k$. Suppose, without loss of generality, that $n_2 < k$ and $n_1 > k$. Then the matrix $\boldsymbol{\delta} * \tilde{\boldsymbol{X}}$, which has k columns, will have $n_2 < k$ rows which are not just rows of zeros, and hence will have rank at most n_2. When equation (12) is estimated, at most n_2 elements of $\boldsymbol{c}$ will be identifiable, and the residuals corresponding to all observations which belong to group 2 will be zero. Thus the degrees of freedom for the numerator of the F statistic, which is equal to the rank of $[\tilde{\boldsymbol{X}}\ \boldsymbol{\delta} * \tilde{\boldsymbol{X}}]$ minus the rank of $\tilde{\boldsymbol{X}}$, must be at most n_2. The degrees of freedom for the denominator will normally be $n_1 - k$. Note that when $x_t(\boldsymbol{\beta}) = \boldsymbol{X}_t\boldsymbol{\beta}$ and $\min(n_1, n_2) > k$, the F test based on the GNR (12) is *numerically identical* to the "Chow test" (5). This follows from the fact that the sum of squared residuals from (12) will then be equal to $\text{SSR}_1 + \text{SSR}_2$, the sum of the SSR's from estimating the regression separately over the two groups of observations.

It may be of interest to test whether a subset of the parameters of a model, rather than all of the parameters, are the same over two (or more) subsamples. It is easy to modify the tests already discussed to deal with this case. The null and alternative hypotheses can now be written as

$$H_0\colon \quad y_t = x_t(\boldsymbol{\alpha}, \boldsymbol{\beta}) + u_t, \quad E(\boldsymbol{u}\boldsymbol{u}^T) = \sigma^2 \boldsymbol{I}, \tag{13}$$

and

$$H_0\colon \quad y_t = x_t(\boldsymbol{\alpha}, (1 - \delta_t)\boldsymbol{\beta}_1 + \delta_t\boldsymbol{\beta}_2) + u_t, \quad E(\boldsymbol{u}\boldsymbol{u}^T) = \sigma^2 \boldsymbol{I}, \tag{14}$$

where $\boldsymbol{\alpha}$ is an l-vector of parameters which are assumed to be the same over the two subsamples and $\boldsymbol{\beta}$ is an m-vector of parameters the constancy of which is to be tested. The Gauss-Newton regression is easily seen to be

$$\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{X}}_\alpha \boldsymbol{a} + \tilde{\boldsymbol{X}}_\beta \boldsymbol{b} + \boldsymbol{\delta} * \tilde{\boldsymbol{X}}_\beta \boldsymbol{c} + \text{errors}. \tag{15}$$

where $\tilde{\boldsymbol{X}}_\alpha$ is an $n \times l$ matrix with typical element $\partial x_t(\boldsymbol{\alpha}, \boldsymbol{\beta})/\partial \alpha_i$ and $\tilde{\boldsymbol{X}}_\beta$ is an $n \times m$ matrix with typical element $\partial x_t(\boldsymbol{\alpha}, \boldsymbol{\beta})/\partial \beta_j$, both evaluated at the estimates $(\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}})$ from (13). One would then use the F statistic for $\boldsymbol{c} = 0$, which if $m < \min(n_1, n_2)$ will have m and $(n\text{-}l\text{-}2m)$ degrees of freedom.

There are several asymptotically equivalent test statistics which may be calculated from the artificial regression (12). They all have the same numerator, which is the explained sum of squares from that regression. The denominator may be anything which consistently estimates σ^2, and if the statistic is to be compared to the $F(k, 2n-k)$ rather than the $\chi^2(k)$ distribution, it must first be multiplied by $(n-2k)/k$. If we let $\tilde{\boldsymbol{Z}}$ denote $\boldsymbol{\delta} * \tilde{\boldsymbol{X}}$, then the numerator of all the test statistics is

$$\tilde{\boldsymbol{u}}^T \tilde{\boldsymbol{M}}_x \tilde{\boldsymbol{Z}} (\tilde{\boldsymbol{Z}}^T \tilde{\boldsymbol{M}}_x \tilde{\boldsymbol{Z}})^{-1} \tilde{\boldsymbol{Z}}^T \tilde{\boldsymbol{M}}_x \tilde{\boldsymbol{u}}, \tag{16}$$

where $\tilde{\boldsymbol{M}}_x \equiv \boldsymbol{I} - \tilde{\boldsymbol{X}}(\tilde{\boldsymbol{X}}^T \tilde{\boldsymbol{X}})^{-1} \tilde{\boldsymbol{X}}^T$. What may be the best of the many valid test statistics is the ordinary F statistic for $\boldsymbol{c} = 0$ in (12), which is

$$\frac{\tilde{\boldsymbol{u}}^T \tilde{\boldsymbol{M}}_x \tilde{\boldsymbol{Z}} (\tilde{\boldsymbol{Z}}^T \tilde{\boldsymbol{M}}_x \tilde{\boldsymbol{Z}})^{-1} \tilde{\boldsymbol{Z}}^T \tilde{\boldsymbol{M}}_x \tilde{\boldsymbol{u}}/k}{\tilde{\boldsymbol{u}}^T \tilde{\boldsymbol{M}}_{x;z} \tilde{\boldsymbol{u}}/(n-2k)}, \tag{17}$$

where $\tilde{\boldsymbol{M}}_{x;z}$ is the matrix which projects orthogonally off the subspace spanned by $\tilde{\boldsymbol{X}}$ and $\tilde{\boldsymbol{Z}}$ jointly. Expression (17) is just $(n-2k)/k$ times the explained sum of squares from (12) divided by the sum of squared residuals from (12).

Rewriting expression (16) so that all factors are $O(1)$, we obtain

$$(n^{-1/2} \tilde{\boldsymbol{u}}^T \tilde{\boldsymbol{M}}_x \tilde{\boldsymbol{Z}})(n^{-1} \tilde{\boldsymbol{Z}}^T \tilde{\boldsymbol{M}}_x \tilde{\boldsymbol{Z}})^{-1} (n^{-1/2} \tilde{\boldsymbol{Z}}^T \tilde{\boldsymbol{M}}_x \tilde{\boldsymbol{u}}). \tag{18}$$

This expression is a quadratic form in the vector

$$n^{-1/2} \tilde{\boldsymbol{Z}}^T \tilde{\boldsymbol{M}}_x \tilde{\boldsymbol{u}}. \tag{19}$$

Standard asymptotic theory tells us that this vector is asymptotically normally distributed with mean vector zero and covariance matrix

$$\sigma^2 \operatorname*{plim}_{n\to\infty} (n^{-1}\tilde{\boldsymbol{Z}}^T\tilde{\boldsymbol{M}}_x\tilde{\boldsymbol{Z}}). \tag{20}$$

The middle matrix in (18), times anything which consistently estimates σ^2, provides a consistent estimate of (20). Thus (18), divided by anything which consistently estimates σ^2, must be asymptotically distributed as $\chi^2(k)$.

The key point which emerges from the above discussion is that every test statistic based on the GNR (12) is actually testing whether the k-vector (19) has mean zero asymptotically. Under relatively weak assumptions this vector will be asymptotically normal, since it is essentially a weighted sum of n independent random variables (the elements of the vector $\boldsymbol{u}$). Under the much stronger assumption of homoskedasticity, its asymptotic covariance matrix will be given by (20), which allows us to use tests based on the GNR. Without this assumption, we will still be able to compute test statistics as quadratic forms in $n^{-1/2}\tilde{\boldsymbol{Z}}^T\tilde{\boldsymbol{M}}_x\tilde{\boldsymbol{u}}$ and expect them to be asymptotically distributed as $\chi^2(k)$, provided that we can somehow obtain an estimate of the asymptotic covariance matrix of $n^{-1/2}\tilde{\boldsymbol{Z}}^T\tilde{\boldsymbol{M}}_x\tilde{\boldsymbol{u}}$ which is consistent in the presence of heteroskedasticity. How this may be done is discussed in the next section.

3 Heteroskedasticity-Robust Tests

We are now ready to drop the often implausible assumption that $E(\boldsymbol{u}\boldsymbol{u}^T)=\sigma^2\boldsymbol{I}$. Instead, we shall assume initially that

$$E(\boldsymbol{u}\boldsymbol{u}^T) = \boldsymbol{\Omega}, \quad \Omega_{tt} = \sigma_t^2, \quad \Omega_{ts} = 0 \text{ for } t \neq s, \quad 0 < \sigma_t < \sigma_{\max}. \tag{21}$$

Thus the covariance matrix of the error terms $\boldsymbol{u}$, $\boldsymbol{\Omega}$, is assumed to be an $n \times n$ diagonal matrix with σ_t^2 as its t-th diagonal element. Except that σ_t is assumed to be bounded from above by some possibly very large number $\sigma_{\max}$, we are not assuming that anything is known about the σ_t^2's. These assumptions admit virtually any interesting pattern of heteroskedasticity, including autoregressive conditional heteroskedasticity (ARCH errors; see Engle 1982a), since there is nothing which prevents σ_t^2 from depending on variables which affect $\boldsymbol{x}_t(\boldsymbol{\beta})$. They do however rule out serial correlation or any other sort of dependence across observations.

Under the assumptions (21), it is easy to see that the asymptotic covariance matrix of the vector (19) is

$$\operatorname*{plim}_{n\to\infty} (n^{-1}\tilde{\boldsymbol{Z}}^T\tilde{\boldsymbol{M}}_x\boldsymbol{\Omega}\tilde{\boldsymbol{M}}_x\tilde{\boldsymbol{Z}}). \tag{22}$$

It is in general not possible to estimate $\boldsymbol{\Omega}$, an $n\times n$ matrix which in this case has n non-zero elements, consistently. However, by a slight modification of the arguments used by White (1980), one can show that the matrix

$$n^{-1}\tilde{\boldsymbol{Z}}^T\tilde{\boldsymbol{M}}_x\acute{\boldsymbol{\Omega}}\,\tilde{\boldsymbol{M}}_x\tilde{\boldsymbol{Z}} \tag{23}$$

consistently estimates (22), where $\acute{\boldsymbol{\Omega}}$ is an $n\times n$ diagonal matrix with $\acute{\sigma}_t^2$ as the t-th diagonal element, and the diagonal elements $\acute{\sigma}_t^2$ have the property that

$$\acute{\sigma}_t^2 \to \sigma_t^2 + v_t \quad \text{as} \quad n\to\infty. \tag{24}$$

Here v_t is a random variable which asymptotically has mean zero and finite variance and is independent of $\tilde{\boldsymbol{X}}$ and $\tilde{\boldsymbol{Z}}$. There are many choices for $\acute{\sigma}_t^2$, of which the simplest is $\tilde{u}_t^2$, the square of the t-th residual from the initial NLS estimation of H_0.

Combining (19) and (23), we obtain the family of test statistics

$$(n^{-1/2}\tilde{\boldsymbol{u}}^T\tilde{\boldsymbol{M}}_x\tilde{\boldsymbol{Z}})(n^{-1}\tilde{\boldsymbol{Z}}^T\tilde{\boldsymbol{M}}_x\acute{\boldsymbol{\Omega}}\tilde{\boldsymbol{M}}_x\tilde{\boldsymbol{Z}})^{-1}(n^{-1/2}\tilde{\boldsymbol{Z}}^T\tilde{\boldsymbol{M}}_x\tilde{\boldsymbol{u}})$$

$$= \tilde{\boldsymbol{u}}^T\tilde{\boldsymbol{M}}_x\tilde{\boldsymbol{Z}}(\tilde{\boldsymbol{Z}}^T\tilde{\boldsymbol{M}}_x\acute{\boldsymbol{\Omega}}\tilde{\boldsymbol{M}}_x\tilde{\boldsymbol{Z}})^{-1}\tilde{\boldsymbol{Z}}^T\tilde{\boldsymbol{M}}_x\tilde{\boldsymbol{u}}. \tag{25}$$

Since $n^{-1/2}\tilde{Z}^T\tilde{\boldsymbol{M}}_x\tilde{\boldsymbol{u}}$ is asymptotically normal with covariance matrix (22) and the matrix (23) consistently estimates (22), it is clear that (25) will be asymptotically distributed as $\chi^2(k)$ under H_0. As shown by Davidson and MacKinnon (1985), variants of (25) can be computed by means of two different artificial regressions. The most generally applicable of these is

$$\tilde{u}_t/\acute{\sigma}_t = \acute{\sigma}_t(\tilde{\boldsymbol{M}}_x\tilde{\boldsymbol{Z}})_t\boldsymbol{c} + \text{error}. \tag{26}$$

The explained sum of squares from regression (26) is the test statistic (25). The inner product of the regressor matrix with itself is $\tilde{\boldsymbol{Z}}^T\tilde{\boldsymbol{M}}_x\acute{\boldsymbol{\Omega}}\tilde{\boldsymbol{M}}_x\tilde{\boldsymbol{Z}}$, while its inner product with the regressand is $\tilde{\boldsymbol{u}}^T\tilde{\boldsymbol{M}}_x\tilde{\boldsymbol{Z}}$. The latter expression does not involve the $\acute{\sigma}_t$'s because the $\acute{\sigma}_t$ which multiplies each of the regressors cancels with the $1/\acute{\sigma}_t$ which multiplies the regressand. For regression (26) to be computable, $\acute{\sigma}_t$ must never be exactly equal to zero, since if it were the regressand would be undefined; this problem can be avoided in practice by setting $\acute{\sigma}_t$ to a very small number whenever it should really be zero.

If $\tilde{u}_t^2$ is used for $\acute{\sigma}_t^2$, and it is probably the most natural choice, an even simpler artificial regression is available. It is

$$\boldsymbol{\iota} = \tilde{\boldsymbol{U}}\tilde{\boldsymbol{M}}_x\tilde{\boldsymbol{Z}}\boldsymbol{c} + \text{errors}, \tag{27}$$

where $\boldsymbol{\iota}$ is an n-vector of ones and $\tilde{\boldsymbol{U}}$ is an $n \times n$ diagonal matrix with $\tilde{u}_t$ as the t-th diagonal element. The explained sum of squares from (27) is

$$\boldsymbol{\iota}^T\tilde{\boldsymbol{U}}\tilde{\boldsymbol{M}}_x\tilde{\boldsymbol{Z}}(\tilde{\boldsymbol{Z}}^T\tilde{\boldsymbol{M}}_x\tilde{\boldsymbol{U}}^T\tilde{\boldsymbol{U}}\tilde{\boldsymbol{M}}_x\tilde{\boldsymbol{Z}})^{-1}\tilde{\boldsymbol{Z}}^T\tilde{\boldsymbol{M}}_x\tilde{\boldsymbol{U}}\boldsymbol{\iota}. \tag{28}$$

The vector $\boldsymbol{\iota}^T\tilde{\boldsymbol{U}}$ is simply $\tilde{\boldsymbol{u}}^T$, and the matrix $\tilde{\boldsymbol{U}}^T\tilde{\boldsymbol{U}}$ is simply $\acute{\boldsymbol{\Omega}}$ with $\tilde{u}_t^2$ being used for $\acute{\sigma}_t^2$, so that (28) is just a special case of (25). The artificial regression (27) is very easy to compute. The regressand is a vector of ones. Each of the regressors is the vector of residuals from a regression of $\tilde{\boldsymbol{Z}}$ on $\tilde{\boldsymbol{X}}$, each element of which has been multiplied by the appropriate element of $\tilde{\boldsymbol{u}}$ (to see this, observe that $\tilde{\boldsymbol{U}}\tilde{\boldsymbol{M}}_x\tilde{\boldsymbol{Z}} = \tilde{\boldsymbol{u}} * \tilde{\boldsymbol{M}}_x\tilde{\boldsymbol{Z}}$). Thus one simply has to perform $k+1$ linear regressions. Since k of them involve the same set of regressors (the matrix $\tilde{\boldsymbol{X}}$), the computational burden (given appropriate software) is only moderately greater than that of performing two linear regressions.

There are other choices for $\acute{\sigma}_t^2$ besides $\tilde{u}_t^2$. One that was proposed in the context of heteroskedasticity-consistent covariance matrix estimators (HCCME's) for linear regression models by MacKinnon and White (1985) is

$$\ddot{\sigma}_t^2 = \tilde{u}_t^2/(\tilde{\boldsymbol{M}}_x)_{tt}, \tag{29}$$

where $(\tilde{\boldsymbol{M}}_x)_{tt}$ denotes the t-th diagonal element of the matrix $\tilde{\boldsymbol{M}}_x$. The reason for using (29) is that in the case of a linear regression model with homoskedastic residuals, it provides an unbiased estimate of σ_t^2 $(=\sigma^2)$, correcting the tendency of squared residuals to be too small.

In the context of testing for structural change, assumptions (21) may seem more unrestrictive than is needed. What has traditionally worried econometricians about the ordinary F test is not the possibility that there may be heteroskedasticity of unknown form, but the possibility that the variance of the error terms may simply be different in the two sub-samples. It is easy to derive a version of (25) which allows only for this possibility. First, estimate the model over each of the two groups of observations, obtaining sums of squared residuals SSR_1 and SSR_2 respectively. Then make the definitions:

$$\hat{\sigma}_1 = \left(\frac{\mathrm{SSR}_1}{n_1 - k}\right)^{1/2} \quad \text{and} \quad \hat{\sigma}_2 = \left(\frac{\mathrm{SSR}_2}{n_2 - k}\right)^{1/2}, \tag{30}$$

and let $\acute{\sigma}_t = \hat{\sigma}_1$ for all observations where $\delta_t = 0$ and $\acute{\sigma}_t = \hat{\sigma}_2$ for all observations where $\delta_t = 1$. Now run regression (26) using the $\acute{\sigma}_t$'s so defined. The explained sum of squares from this regression will have the form of (25), and will clearly provide an asymptotically valid test statistic if in fact group 1 observations have variance σ_1^2 and group 2 observations have variance σ_1^2. Of course, if one is willing to make the assumption that the variance is constant over each of the sub-samples, various other procedures are available; see Jayatissa (1977), Weerahandi (1987), Watt (1979), Honda (1982) and Ohtani and Toyoda (1985), among others.

4 Finite-sample Properties of the Tests

The tests suggested in the previous section are valid only asymptotically. If they are to be useful in practice, their known asymptotic distributions must provide reasonably good approximations to their unknown finite-sample distributions. In this section I report the results of several Monte Carlo experiments designed to investigate whether this is so. For obvious reasons, attention is restricted to the case of linear regression models. Experiments were run for samples of sizes 50, 200 and 800, with n_1 equal to θn, θ being either 0.5 or 0.2, and with σ_1 variously equal to σ_2, four times σ_2 or one quarter of σ_2. In all experiments there were four regressors including a constant term. The X matrix was initially chosen for a sample of size 50 and replicated as many times as necessary as the sample size was increased, so as to ensure that the matrix $n^{-1}X^TX$ did not change. The regressors were a constant, the Canadian 90-day treasury bill rate, the quarterly percentage rate of change in real Canadian GNP, seasonally adjusted at annual rates, and the exchange rate between the Canadian and U.S. dollars, in Canadian dollars per U.S. dollar, all for the period 1971:3 to 1983:4.

Choosing the X matrix in this way makes it easy to see how the sample size affects the results. However, it may make the performance of the heteroskedasticity-robust (HR) tests appear to be unrealistically good in moderately large samples. As Chesher and Jewitt (1987) have shown, the values of the few smallest diagonal elements of $\boldsymbol{M}_x$ can have a very big impact on the finite-sample performance of HCCME's. Replicating the $\boldsymbol{X}$ matrix as the sample size is increased ensures that all diagonal elements of $\boldsymbol{M}_x$ approach one at a rate proportional to $1/n$, so that once n becomes large the HR tests are bound to perform reasonably well. With real data sets, one would certainly expect the smallest elements of $\boldsymbol{M}_x$ to approach one as n tends to infinity, but possibly at a rate much slower than $1/n$, thus implying that the HR tests might perform less well for larger samples than these experiments suggest. In the experiments, the smallest elements of $\boldsymbol{M}_x$ were 0.7965 for $n=50$, 0.9491 for $n=200$ and 0.9873 for $n=800$.

The four test statistics that were computed in the course of the experiments were the following:

1. The ordinary F test, expression (5), which is valid only under homoskedasticity. It will be denoted F.

2. The heteroskedasticity-robust test statistic (28), based on the artificial regression (27). It will be denoted HR_1.

3. A heteroskedasticity-robust test statistic like (25), in which $\ddot{\sigma}_t$ defined by (29) is used in place of $\tilde{u}_t^2$. This statistic, which will be denoted HR_2, is somewhat harder to compute that HR_1.

4. A test statistic with the form of (25), but where $\acute{\sigma}_t$ is either $\hat{\sigma}_1$ or $\hat{\sigma}_2$, where the latter were defined in (30). This statistic, which will be denoted $2V$ (for two variances) will be asymptotically valid under much less general assumptions than HR_1 and HR_2.

The results of the Monte Carlo experiments are presented in Tables 1 and 2. Table 1 contains results for 18 experiments where the null hypothesis that $\boldsymbol{\beta}_1=\boldsymbol{\beta}_2$ was correct. The percentage of the time that each test rejected the null hypothesis at the nominal 1%, 5% and 10% levels is shown in the table. These numbers should thus be very close to 1.0, 5.0 and 10.0 if the tests are behaving in finite samples as asymptotic theory says they should.

In the first group of experiments, the variance in the two subsamples was equal. The ordinary F test is thus completely valid, and, as we would expect, the rejection frequencies for the F test were indeed very close to what they should be. All the other tests performed reasonably well when $\sigma_1=\sigma_2$. However, HR_1 and HR_2 tended to under-reject, especially for $\theta=0.2$ (when n_1 was one-quarter the size of n_2), while $2V$ tended to over-reject somewhat. The performance of all tests improved sharply with the sample size, and one could feel confident about using any of them for $n\geqslant 200$.

In the second group of experiments, σ_2 was four times as large as σ_1. The F test was therefore no longer valid, but it continued to perform quite well for $\theta=0.5$. However, it rejected the null far too infrequently for $\theta=0.2$. The two HR tests

Table 1. Rejection Frequencies when the Null Hypothesis is True

n	σ_1/σ_2	θ	Test	Rejection Frequencies 1%	5%	10%	θ	Test	Rejection Frequencies 1%	5%	10%
50	1/1	.5	F	1.10	5.15	10.30	.2	F	0.70	4.65	9.85
			HR_1	0.45	5.00	10.30		HR_1	0.00†	0.70†	5.10†
			HR_2	0.25†	3.00†	8.05*		HR_2	0.00†	0.25†	2.40†
			2V	2.60†	7.30†	13.05†		2V	6.10†	12.90†	19.00†
200	1/1	.5	F	1.05	4.85	10.25	.2	F	1.35	5.90	9.95
			HR_1	0.55	4.50	9.65		HR_1	0.55	3.75	9.70
			HR_2	0.55	4.25	9.10		HR_2	0.55	3.55	8.90
			2V	1.30	5.30	11.00		2V	2.25†	7.55†	11.80*
800	1/1	.5	F	1.25	5.30	10.40	.2	F	1.25	5.30	10.30
			HR_1	1.15	5.45	10.05		HR_1	0.95	5.00	9.85
			HR_2	1.10	5.45	9.90		HR_2	0.95	4.85	9.60
			2V	1.35	5.40	10.50		2V	1.20	5.60	10.70
50	1/4	.5	F	2.65†	7.10†	11.80*	.2	F	0.00†	0.10†	0.15†
			HR_1	0.60	4 45	11.70		HR_1	0.00†	0.25†	0.80†
			HR_2	0.15†	2.70†	7.80†		HR_2	0.00†	0.10†	0.55†
			2V	2.50†	7.50†	13.80†		2V	3.70†	9.60†	14.75†
200	1/4	.5	F	2.35†	8.35†	12.60†	.2	F	0.00†	0.15†	0.30†
			HR_1	1.20	5.45	10.80		HR_1	0.25†	1.90†	5.65†
			HR_2	0.95	5.05	10.40		HR_2	0.20†	1.65†	5.30†
			2V	1.60*	6.60*	11.95*		2V	1.65*	6.05	11.55
800	1/4	.5	F	1.80†	5.35	10.10	.2	F	0.00†	0.05†	0.15†
			HR_1	1.00	4.85	10.10		HR_1	0.75	4.05	7.55†
			HR_2	1.00	4.70	10.00		HR_2	0.75	3.95	7.35†
			2V	1.20	5.25	9.90		2V	1.45	5.75	9.60
50	4/1	.5	F	2.45†	8.70†	13.90*	.2	F	47.45†	63.70†	70.50†
			HR_1	0.60	4 95	11.15		HR_1	0.90	7.40†	16.05†
			HR_2	0.20†	3.05†	7.75†		HR_2	0.40*	4.90	11.60
			2V	2.45†	7.45†	12.40†		2V	9.15†	16.15†	22.20†
200	4/1	.5	F	2.70†	8.05†	12.95†	.2	F	38.70†	56.70†	64.90†
			HR_1	1.10	4.90	10.30		HR_1	1.25	5.30	10.40
			HR_2	0.95	4.60	9.75		HR_2	0.90	4.50	9.70
			2V	1.50	5.90	11.35		2V	2.30†	6.65†	11.95*
800	4/1	.5	F	2.40†	7.60	12.60	.2	F	39.05†	55.60†	65.15†
			HR_1	0.65	4.35	10.25		HR_1	1.00	5.75	10.60
			HR_2	0.65	4.35	10.05		HR_2	0.95	5.45	10.50
			2V	0.80	4.65	9.90		2V	0.95	5.55	10.85

Notes: All results are based on 2,000 replications.

* and † indicate that the quantity in question differs significantly at the 0.01 and 0.001 level respectively from what it should be if the test statistic were distributed as $\chi^2(4)$ or $F(4, n-8)$.

performed reasonably well for $\theta = 0.5$, but also grossly under-rejected for $\theta = 0.2$. Even for $n = 800$, they tended to reject too infrequently in the latter case. The $2V$ test over-rejected quite severely for $n = 50$ and moderately for $n = 200$, but performed very well for $n = 800$. The third group of experiments was similar to the second, except that σ_1 was now four times as large as σ_2. This changed many results dramatically. The F test continued to perform surprisingly well for $\theta = 0.5$, but rejected the null far too often for $\theta = 0.2$. The two HR tests generally performed well, although they over-rejected somewhat when $n = 50$. The $2V$ test continued to over-reject quite severely when $n = 50$ and moderately when $n = 200$.

From Table 1 two conclusions emerge. First, the two HR tests generally perform quite well, but usually tend to under-reject. There is thus no reason to prefer HR_2 to the simpler HR_1; the former simply under-rejects more severely in most cases. Nevertheless, there are evidently some cases where HR_1 can seriously over-reject, at least for small samples, so that routine use of this test as if it were an exact test is not justified. Secondly, the $2V$ test performs very well in medium and large samples but tends to over-reject in smaller ones. Its good performance in reasonably large samples makes sense, because it would be an exact test if $\hat{\sigma}_1$ and $\hat{\sigma}_2$ were replaced by σ_1 and σ_2. Provided that both n_1 and n_2 are reasonably large, $\hat{\sigma}_1$ and $\hat{\sigma}_2$ will provide good estimates of σ_1 and σ_2, and hence it is not surprising that the test performs well. Of course, in these circumstances the Wald test examined by Watt (1979), Honda (1982) and Ohtani and Toyoda (1985), which also uses the estimates $\hat{\sigma}_1$ and $\hat{\sigma}_2$, might well perform even better.

Table 2 presents results for 18 experiments where the null hypothesis was false. The parameters were chosen so that for the case where $\sigma_1 = \sigma_2$ and $\theta = 0.5$, the F test would reject the null roughly half the time. The difference between $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ was made proportional to $n^{-1/2}$ so that there would be no tendency for the rejection frequencies to increase with the sample size. What should happen under this scheme as $n \rightarrow \infty$ is that all tests which are asymptotically equivalent will tend to the same random variable, and thus reject the null the same fraction of the time. The results in Table 2 largely speak for themselves. Once again, the $2V$ test performs well. It performs quite similarly to HR_1 and HR_2 in most cases for $n = 800$, but generally rejects the null more frequently for smaller sample sizes.

The limited Monte Carlo experiments reported on here certainly do not provide a definitive study of heteroskedasticity-robust tests for structural change. For example, no attempt was made to study the effect of combining the ordinary F test with the $2V$ test by first doing a pretest of the hypothesis that $\sigma_1 = \sigma_2$ (see Phillips and McCabe 1983 or Toyoda and Ohtani 1986). Such a strategy seems appealing, and would presumably produce results somewhere between those for F and $2V$, depending on the significance level of the pretest. There was also no attempt to quantify the size-power tradeoffs of the various tests, although how useful such an exercise is when size is not known in practice is unclear.

The most substantial omission is that the undoubtedly very complex relationships between test performance, the number of regressors and the structure of the $\boldsymbol{X}$ matrix were not studied at all. To do so would be a major undertaking, because it seems unlikely that Monte Carlo evidence alone, without a strong theoretical framework based on work like that of Chesher and Jewitt (1987), would allow one

Table 2. Rejection Frequencies when the Null Hypothesis is False

n	σ_1/σ_2	θ	Test	Rejection Frequencies 1%	5%	10%	θ	Test	Rejection Frequencies 1%	5%	10%
50	1/1	.5	F	22.40	45.60	59.05	.2	F	16.70	38.65	52.35
			HR_1	10.65	39.00	56.70		HR_1	0.25	8.00	23.15
			HR_2	5.65	29.40	47.90		HR_2	0.15	3.60	13.75
			2V	30.45	52.80	63.70		2V	31.45	47.85	58.00
200	1/1	.5	F	26.00	50.20	62.20	.2	F	20.55	41.50	54.60
			HR_1	23.65	48.50	61.05		HR_1	9.80	31.85	47.60
			HR_2	22.45	46.55	59.70		HR_2	8.90	30.25	45.55
			2V	28.05	51.30	63.10		2V	23.65	43.90	56.30
800	1/1	.5	F	26.80	51.25	64.60	.2	F	21.60	42.55	55.05
			HR_1	26.40	51.20	64.15		HR_1	18.05	40.20	53.50
			HR_2	25.70	51.05	63.95		HR_2	17.40	39.75	53.20
			2V	27.25	51.35	64.80		2V	22.00	42.10	56.00
50	1/4	.5	F	23.50	43.90	56.55	.2	F	0.95	5.45	12.10
			HR_1	11.90	39.40	57.85		HR_1	0.00	3.20	19.15
			HR_2	6.40	29.85	48.95		HR_2	0.00	1.10	9.45
			2V	36.55	57.35	69.50		2V	52.05	72.45	80.55
200	1/4	.5	F	25.70	46.65	58.05	.2	F	0.85	5.95	12.90
			HR_1	24.95	51.05	64.95		HR_1	15.50	45.60	63.15
			HR_2	23.70	49.20	63.65		HR_2	14.60	42.80	61.10
			2V	31.90	56.15	67.40		2V	47.75	71.50	80.10
800	1/4	.5	F	24.80	44.90	57.45	.2	F	0.95	6.50	14.20
			HR_1	28.80	52.25	65.80		HR_1	37.10	65.30	76.95
			HR_2	28.50	51.90	65.50		HR_2	36.85	64.70	76.65
			2V	30.95	53.40	66.30		2V	47.50	71.10	80.40
50	4/1	.5	F	24.35	47.55	61.60	.2	F	78.30	88.70	92.45
			HR_1	19.95	55.50	73.35		HR_1	2.35	19.30	36.90
			HR_2	11.95	44.10	65.10		HR_2	1.05	11.60	27.50
			2V	47.95	69.85	80.25		2V	26.25	40.05	48.95
200	4/1	.5	F	26.20	51.70	63.25	.2	F	76.15	86.30	89.80
			HR_1	39.50	66.05	77.55		HR_1	9.25	29.20	42.60
			HR_2	37.50	64.35	76.25		HR_2	8.15	27.50	41.25
			2V	46.55	69.75	79.00		2V	17.10	33.25	44.60
800	4/1	.5	F	28.80	51.70	64.00	.2	F	75.75	85.75	90.15
			HR_1	44.55	69.00	79.75		HR_1	13.45	30.55	43.60
			HR_2	44.20	68.70	79.50		HR_2	13.15	30.30	43.05
			2V	45.55	69.70	79.95		2V	14.85	32.50	43.25

Note: All results are based on 2,000 replications.

to say anything interesting about those relationships. Nevertheless, a few fairly strong results do seem to emerge from the Monte Carlo experiments. These are:

1. There seems to be no reason to use HR_2 instead of the simpler HR_1.
2. Since HR_1 never seriously over-rejects at the 1% level, one should probably

view an HR_1 statistic which is significant at the 1% level as providing quite strong evidence against the null hypothesis.

3. The $2V$ test performs very well in medium and large samples, although it over-rejects somewhat in small samples. It generally has more power than the HR tests.

5 Conclusion

This paper has shown that it is remarkably easy to test for structural change in a fashion which is robust to heteroskedasticity of unknown form. The tests can also be modified so that they are robust only to a more structured form of heteroskedasticity in which the variance differs between the two subsamples, although since numerous other solutions to this simpler problem are available, this modification may be of limited interest. The new tests are asymptotically valid for both linear and nonlinear regression models. Monte Carlo evidence for the linear case suggests that, although the finite-sample performance of even the best tests is sometimes poor, the ordinary F test can be so misleading that it clearly makes no sense to ignore the possibility of heteroskedasticity when testing for structural change. At the very least one should double-check the results of the F test by using one of the tests discussed in this paper.

References

Chesher A, Jewitt I (1987) The bias of a heteroskedasticity consistent covariance matrix estimator. Econometrica 55:1217–1222

Chow GC (1960) Tests of equality between sets of coefficients in two linear regressions. Econometrica 28:591–605

Davidson R, MacKinnon JG (1984) Model specification tests based on artificial linear regressions. International Economic Review 25:485–502

Davidson R, MacKinnon JG (1985) Heteroskedasticity-robust tests in regression directions. Annales de l'INSÉE 59/60:183–218

Eicker F (1963) Asymptotic normality and consistency of the least squares estimators for families of linear regressions. Annals of Mathematical Statistics 34:447–456

Engle RF (1982a) Autoregressive conditional heteroskedasticity with estimates of the variance of United Kingdom inflation. Econometrica 50:987–1007

Engle RF (1982b) A general approach to Lagrange multiplier model diagnostics. Journal of Econometrics 20:83–104

Fisher FM (1970) Tests of equality between sets of coefficients in two linear regressions: an expository note. Econometrica 38:361–366

Honda Y (1982) On tests of equality between sets of coefficients in two linear regressions when disturbance variances are unequal. The Manchester School 49:116–125

Jayatissa WA (1977) Tests of equality between sets of coefficients in two linear regressions when disturbance variances are unequal. Econometrica 45:1291–1292

MacKinnon JG, White H (1985) Some heteroskedasticity consistent covariance matrix estimators with improved finite sample properties. Journal of Econometrics 29:305–325

Ohtani K, Toyoda T (1985) Small sample properties of tests of equality between sets of coefficients in two linear regressions under heteroskedasticity. International Economic Review 26:37–44

Phillips GDA, McCabe BP (1983) The independence of tests for structural change in regression models. Economics Letters 12:283–287

Schmidt P, Sickles R (1977) Some further evidence on the use of the Chow test under heteroskedasticity. Econometrica 45:1293–1298

Toyoda T (1974) Use of the Chow test under heteroskedasticity. Econometrica 42:601–608

Toyoda T, Ohtani K (1986) Testing equality between sets of coefficients after a preliminary test for equality of disturbance variances in two linear regressions. Journal of Econometrics 31:67–80

Watt PA (1979) Tests of equality between sets of coefficients in two linear regressions when disturbance variances are unequal: some small sample properties. The Manchester School 47:391–396

Weerahandi S (1987) Testing regression equality with unequal variances. Econometrica 55:1211–1215

White H (1980) A heteroskedasticity-consistent covariance matrix estimator and a direct test for heteroskedasticity. Econometrica 48:817–838

A Switching Regression Model with Different Change-Points for Individual Coefficients and its Application to the Energy Demand Equations for Japan[1]

By T. Toyoda[2] and K. Ohtani[3]

Abstract: In this paper, we set up a switching regression model in which individual coefficients are allowed to shift at different change-points. We also apply it to the energy demand equations and examine structural change in the demands for total fuel oil and for light oil and kerosene at the second oil crisis. It is shown that assuming the different change-points for individual coefficients yields more plausible results than assuming the same change-point for all coefficients.

1 Introduction

Since Quandt (1958) proposed a switching regression model, the model has often been used to detect a structural change-point in some economic equations. Based on the switching regression model, for example, Stern/Baum/Greene (1979) studied structural change in the aggregate import and export demand equations for the United States and Boughton (1981) studied structural change in the demand equation for money.

From the theoretical and practical viewpoints, the switching regression model has been extended to some directions. For example, Salazar/Breomeling/Chi (1981) and Ohtani (1982) considered the switching regression model when the error terms are autocorrelated. Also, Bacon/Watts (1971), Tsurumi (1980) and Katayama/Ohtani/Toyoda (1987) considered the switching regression model when the change in regression coefficients occurs gradually.

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Although the switching regression models studied so far assume that all coefficients shift at the same change-point, the change-point may be different among the regression coefficients in some practical situations. The first purpose of this paper is to set up a switching regression model in which individual coefficients are allowed to shift at the different change-points.

As an application of the gradual switching regression model, Ohtani/Katayama (1985) examined structural change at the first oil crisis in the energy demand equation for Japan which is explained both by the relative price and economic activity variables. Although Toyoda/Ohtani/Katayama (1987) examined structural change in the same-type energy demand equations for Japan both at the first and second oil crises, we used no formal methods to detect change-points. Namely, we selected some plausible change-points by conjecture, and conducted the Chow test proposed by Chow (1960) and the Wald test proposed by Watt (1979). In the process of our study in Toyoda/Ohtani/Katayama (1987), we found a strong evidence that the individual coefficients for the explanatory variables, i.e., the relative price and an economic activity variable, might shift at different time-points. This evidence has motivated us to our second purpose of this paper, i.e., to examine and estimate change-points of individual coefficients in some energy demand equations for Japan. It is shown that assuming the different change-points for individual coefficients yields more plausible results than assuming the same change-point for all coefficients.

2 The Different Change-Points Model

Consider a switching regression model

$$y_t = \sum_{i=1}^{k} (\beta_i + \lambda_{it}\delta_i)x_{it} + \varepsilon_t, \tag{1}$$

where, for $t = 1, 2, \ldots, T$, y_t is the t-th observation on the dependent variable, x_{it} is the t-th observation on the i-th independent variable, λ_{it} is the dummy variable defined as

$$\begin{aligned} \lambda_{it} &= 0 \quad \text{for} \quad t \leqq t_i^*, \\ \lambda_{it} &= 1 \quad \text{for} \quad t > t_i^*, \end{aligned} \tag{2}$$

and ε_t is the error term which is normally and independently distributed with zero mean and constant variance σ^2 (i.e., $\varepsilon_t \sim \text{NID}(0, \sigma^2)$). Although it may be possible to allow the error variance also shift between two regimes as in, e.g., Quandt (1958), we rather prefer simplicity to complexity as our first approach to the present new problem, i.e., we assume that it remains constant over the whole period.

Defining λ_{it} as in (2) means that the i-th coefficient shifts from β_i to $\beta_i + \delta_i$ at an unknown change-point t_i^* ($k \leqq t_i^* \leqq T-k$). If we assume that all t_i^*'s are the same (i.e., $t_1^* = t_2^* = \ldots = t_k^*$), the switching regression model defined in (1) and (2) (say, the different change-points model) reduces to the traditional switching regression model that all coefficients shift at the same change-point (say, the same change-point model). If we have prior knowledge that the j-th coefficient does not shift, the prior knowledge can be utilized by putting $\delta_i = 0$ with appropriate adjustment of the number of the independent variables.

Denoting

$$y^* = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{bmatrix}, \quad X^* = \begin{bmatrix} x_{11} \ldots x_{k1} & \lambda_{11}x_{11} \ldots \lambda_{k1}x_{k1} \\ x_{12} \ldots x_{k22} & \lambda_{12}x_{12} \ldots \lambda_{k2}x_{k2} \\ \vdots \quad \vdots & \vdots \quad \vdots \\ x_{1T} \ldots x_{kT} & \lambda_{1T}x_{1T} \ldots \lambda_{kT}x_{kT} \end{bmatrix},$$

$$\theta = (\beta_1, \beta_2, \ldots, \beta_k, \delta_1, \delta_2, \ldots, \delta_k)', \quad \varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_T)',$$

the model (1) and (2) can be rewritten in the matrix form as

$$y^* = X^*\theta + \varepsilon; \quad \varepsilon \sim N(0, \sigma^2 I_T). \tag{3}$$

Note that X^* depends on the vector of change-points, $t^* = (t_1^*, t_2^*, \ldots t_k^*)$, through the dummy variables λ_{it}'s ($i = 1, 2, \ldots, k$).

The log-likelihood function for (3) is

$$L(t^*, \theta, \sigma^2) = -(T/2)\log 2\pi - (T/2)\log \sigma^2 - (y^* - X^*\theta)'(y^* - X^*\theta)/2\sigma^2. \tag{4}$$

Differentiating (4) with respect to θ and σ^2, and equating the resultant equations to zero, we obtain the conditional maximum likelihood (ML) estimates of θ and σ^2 given t^*:

$$\hat{\theta}^* = (X^{*\prime}X^*)^{-1}X^{*\prime}y^*, \tag{5}$$

$$\hat{\sigma}^{*2} = (y^* - X^*\hat{\theta}^*)'(y^* - X^*\hat{\theta}^*)/T. \tag{6}$$

Substituting (5) and (6) into (4), we obtain the concentrated log-likelihood function:

$$L_{\max}(t^*) = -(T/2)(1 + \log 2\pi) - (T/2) \log \hat{\sigma}^{*2}. \tag{7}$$

Since $L_{\max}(t^*)$ depends on t^* only, the ML estimate of t^* can be obtained by a grid search over the region $k \leqq t_i^* \leqq T-k$ $(i=1,2,\ldots,k)$.

The likelihood ratio test for stability of coefficients cannot be conducted, since t_i^*'s are defined as integer values (e.g., Johnston 1984, p. 409). However, the change in the i-th coefficient (i.e., β_i) can be tested by conducting a conditional test for the null hypothesis, H_0: $\delta_i = 0$, given the ML estimates of t_i^*'s.

3 Structural Change in the Energy Demand Equations for Japan

Applying the different change-points model set up in the previous section, we examine structural change in the demand equations for total fuel oil and for light oil and kerosene in Japan before and after the second oil crisis. Note that a large component of total fuel oil is heavy oil and it is used mainly in the industrial sector (i.e., about 55% in 1985) while a considerable part of light oil and kerosene is consumed in the household sector (i.e., about 87% used in the non-industrial sector in 1985).

The model we adopt here is a partial adjustment demand equation, which is most popular in studies in this area. It is simple but valuable in allowing for instantaneous and non-instantaneous demand adjustments to price and income (or an economic activity level). Our model is specified as

$$\log E_t = \beta_1 + \beta_2 \log P_t + \beta_3 \log Y_t + \beta_4 \log E_{t-1} + \varepsilon_t, \tag{8}$$

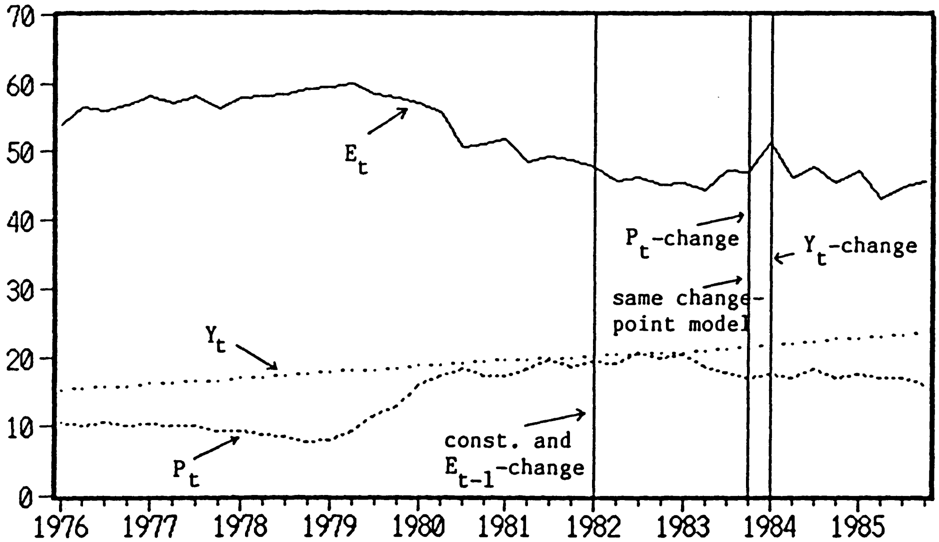

Fig. 1. Behaviours of the demand for total fuel oil (E_t), its relative price (P_t) and real GDP (Y_t)

where, at the time-point t, E_t is the energy demand, P_t is the relative price of energy to the general price, Y_t is the real income (real GDP), E_{t-1} is the energy demand lagged one period and ε_t is the error term distributed as NID$(0, \sigma^2)$. The estimates of coefficients β_2 and β_3 are the estimates of short-run (instantaneous) price and income elasticities, respectively. These estimates multiplied by $1/(1-\beta_4)$ are the long-run estimates of the same elasticities.

The data used in our study are seasonally adjusted quarterly data for Japan from the first quarter of 1976 (1976:Q1) to the fourth quarter of 1985 (1985:Q4). See Appendix for their sources and definition of the variables. Figures 1 and 2 show the behaviours of the variables used in this study. From the figures, it seems that the demand for total fuel oil has had a declining tendency after the second oil crisis (i.e., after around 1979–1980), but the demand for light oil and kerosene has had an increasing tendency except for the period 1979–1983 when the demand remained unchanged or rather slightly decreased. Converting the basic energy demand equation given in (8) into the different change-points model given in (1) and (2), we estimated the change-points and other parameters of the demand equations for total fuel oil and for light oil and kerosene. For comparison, we also estimated the change-points and other parameters of the energy demand equations based on the traditional same change-point model. The estimation results are shown in Tables 1 and 2. The estimates of the change-points are also shown by the vertical lines in Figs. 1 and 2. Note that if the coefficient δ_i for each independent variable is significantly different from zero, the coefficient β_i significantly shifts from β_i to $\beta_i + \delta_i$. Since the change-points in the different change-points model vary with the

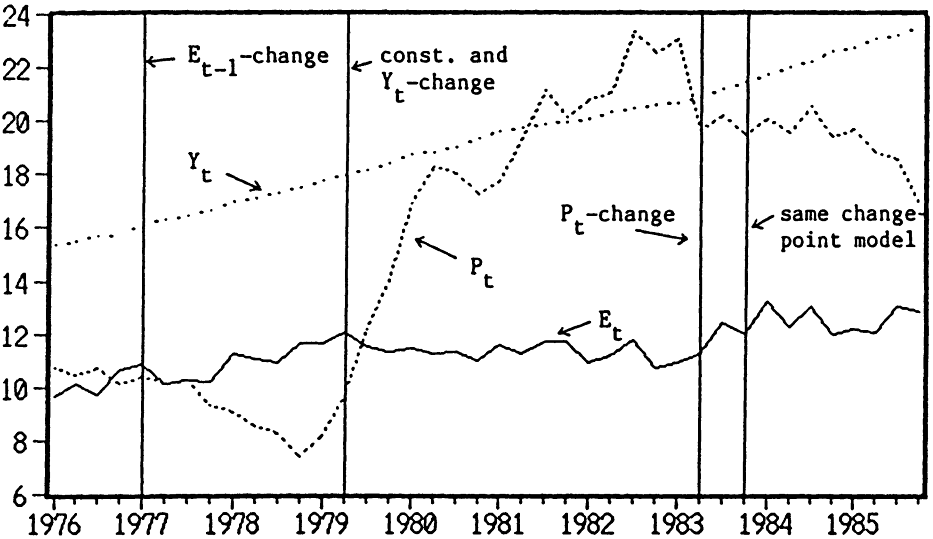

Fig. 2. Behaviours of the demand for light oil and kerosene (E_t), its relative price (P_t) and real GDP (Y_t)

Table 1. Estimation results for total fuel oil

Model		Const.	P_t	Y_t	E_{t-1}	$\bar{R}^2$	σ	D.W.
Same change-point	t*	1983:4	-	-	-	0.947	0.025	2.830
	β	3.764	-0.084	-0.045	0.706			
		(2.60)	(-3.08)	(-0.56)	(9.11)			
	δ	57.292	-0.635	-3.149	-1.686			
		(5.03)	(-1.74)	(-4.50)	(-5.69)			
	$\beta+\delta$	61.056	-0.719	-3.194	-0.980			
Different change-points	t*	1982:1	1983:4	1984:1	1982:1	0.945	0.025	2.583
	β	4.082	-0.080	-0.045	0.677			
		(2.63)	(-2.68)	(-0.51)	(5.97)			
	δ	7.669	0.202	-0.010	-0.716			
		(3.50)	(4.24)	(-4.50)	(-3.52)			
	$\beta+\delta$	11.751	0.122	-0.055	-0.039			

Notes: Values in parentheses are t-values. $\bar{R}^2$ is the coefficient of determination adjusted by degrees of freedom. D.W. is the value of the Durbin-Watson ratio.

Table 2. Estimation results for light oil and kerosene

Model		Const.	P_t	Y_t	E_{t-1}	R^2	σ	D.W.
Same change-point	t^*	1983:4	-	-	-	0.772	0.037	2.206
	β	-0.582	-0.110	0.666	0.200			
		(-0.40)	(-2.96)	(3.76)	(1.23)			
	δ	36.814	-0.654	-2.271	-0.889			
		(2.41)	(-1.31)	(-2.16)	(-2.08)			
	$\beta+\delta$	36.232	-0.764	-1.605	-0.689			
Different change-points	t^*	1979:2	1983:2	1979:2	1977:1	0.834	0.031	2.176
	β	-12.339	-0.070	1.904	-0.131			
		(-3.21)	(-1.34)	(5.35)	(-0.92)			
	δ	21.557	0.151	-1.783	-0.009			
		(4.44)	(3.89)	(-4.43)	(-2.74)			
	$\beta+\delta$	9.218	0.081	0.121	-0.140			

Notes: The same as in Table 1.

Table 3. Short- and long-run elasticities of price and income

			Short-run			Long-run	
Energy	Model	Period	P_t	Y_t	E_{t-1}	P_t	Y_t
Total fuel oil	Same change-point	1976:Q1-1983:Q4	-0.084	-0.045	0.706	-0.286	-0.153
		1984:Q1-1985:Q4	-0.719	-3.194	-0.980	-0.363	-1.613
	Different change-points	1976:Q1-1982:Q1	-0.080	-0.045	0.677	-0.248	-0.139
		1982:Q2-1983:Q4	-0.080	-0.045	-0.039	-0.077	-0.043
		1984:Q1-1984:Q1	0.122	-0.045	-0.039	0.117	-0.043
		1984:Q2-1985:Q4	0.122	-0.055	-0.039	0.117	-0.053
Light oil and kerosene	Same change-point	1976:Q1-1983:Q4	-0.110	0.666	0.200	-0.138	0.833
		1984:Q1-1985:Q4	-0.764	-1.605	-0.689	-0.452	-0.950
	Different change-points	1976:Q1-1977:Q1	-0.070	1.904	-0.131	-0.062	1.683
		1977:Q2-1979:Q2	-0.070	1.904	-0.140	-0.061	1.670
		1979:Q3-1983:Q2	-0.070	0.121	-0.140	-0.061	0.106
		1983:Q3-1985:Q4	0.081	0.121	-0.140	0.071	0.106

coefficient, the long-run elasticities both of price and income shift not only by the change in their short-run elasticities but also by the change in the adjustment parameter. Thus, to clarify the changes in long-run elasticities of price and income, we show in Table 3 the estimates of the short-run and long-run elasticities for the subperiods divided by the change-point of each coefficient.

4 Interpretation of the Estimation Results

First, as to structural change in the demand equation for total fuel oil, we see the following facts from Tables 1 and 3.

(1) Based on the same change-point model, the estimate of the change-point is 1983:Q4, which is considerably lagged from the period of the second oil crisis. The short-run and long-run price elasticities before and after the change are negative and their absolute values become larger after the change. Also, the short-run and long-run income elasticities before and after the change are negative. Although the negative income elasticities are not expected from the theory, the reason may be as follows. That is, total fuel oil is mainly used in the industrial sector, and the industrial sector introduced the oil-saving technology after the first oil crisis so that the demand for total fuel oil rather tends to decrease even if GDP increases. From Fig. 1, the behaviour of demand of total fuel oil before the change seems consistent with that of the price. Since the income elasticity is not highly significant before the change, the effect of the income on the demand for total fuel oil may be weak. However, since the change in the income elasticity is significant and its absolute value becomes considerably larger, the demand for total fuel oil after the change may tend to decrease by the larger negative income effect though the price is stable or rather tends to decrease.

(2) Based on the different change-points model, the price elasticity shifts at 1983:Q4, which is the same as the change-point based on the same change-point model. However, the price elasticity becomes positive after the change. The change-point of the income elasticity is 1984:Q1, which is slightly different from the change-point based on the same change-point model. The change-points of the adjustment parameter and the constant term are 1982:Q1, which is considerably different from the change-point based on the same change-point model. This result means that the change in adjustment occurred in an earlier stage than the changes in price and income elasticities. Although the absolute values of the price and income elasticities are smaller in the different change-points model than in the same change-point model, the behaviour of demand for total fuel oil before the change

seems to be equally explained by the behaviours of the price and income variables in both models. However, the absolute value of the income elasticity based on the same change-point model seems too large after the change. Specifically, Table 3 shows that the long-run income elasticity based on the same change-point model is negative and its absolute value becomes larger by more than 1.0 after the change. Although the signs of the estimates of the changed price and income elasticities based on the different change-points model are opposite to the ones expected from the demand theory after the change, their absolute values both in the short-run and long-run are much smaller than those based on the same change-point model. Since energy is the indispensable necessity particularly in the industrial sector, the smaller elasticities depicted in the different change-points model seem more plausible in the long-run.

Next, as to structural change in the demand for light oil and kerosene, we see the following facts from Tables 2 and 3.

(1) Based on the same change-point model, the estimate of the change-point is 1983:Q4, which is the same as the result for total fuel oil. The income elasticity is positive before the change, but it becomes negative after the change. Also, the absolute value of the price elasticity becomes larger after the change. Although the price elasticity is negative and highly significant before the change, the effect of price hike between 1978:Q4 and 1983:Q1 does not seem to be fully reflected in the demand for light oil and kerosene since Fig. 2 shows that the decrease of the demand is very slight during that period.

(2) Based on the different change-points model, the income elasticity and the constant term shift at 1979:Q2, which is just around the second oil crisis. Also, the price elasticity shifts at 1983:Q2. However, the price elasticity becomes positive after the change though its absolute value is small. It is interesting that the change in price elasticity occurs around the period when the price begins to decrease. The adjustment parameter shifts at 1977:Q1 though the magnitude of change is very small.

(3) Based on the results in the different change-points model, the effects of the price on the demand for light oil and kerosene seem weak before 1979:Q2 since the absolute value of the price elasticity is small and also the price variable is not highly significant. However, since the income elasticity is large and highly significant before 1979:Q2, the demand for light oil and kerosene seems to increase by the income effects. Since the absolute value of the price and income elasticities are small for the period between 1979:Q3 and 1983:Q2, the decrease in the demand for light oil and kerosene might be slight in that period. Specifically, since the price variable is not highly significant before 1983:Q2, the price hike might not affect the demand for light oil and kerosene. Comparing the results based on the same

change-point model with the one based on the different change-points model, the behaviour of the demand for light oil and kerosene during the period of the price hike (i.e., 1978:Q4–1983:Q1) seems to be explained better by the latter model than the former. Since the price elasticity is positive after 1983:Q3 and the price tends to decrease after 1983:Q1, it is expected that the demand for light oil and kerosene decreases after 1983:Q3. On the other hand, the income elasticity is positive and the growth rate of GDP seems slightly higher after 1983:Q1 than before 1982:Q4. Thus, the price and income effects might be offset, so that the demand for light oil and kerosene might be rather stable after 1983:Q3.

Appendix: Data Sources and Definition of Variables

Energy demand was drawn from various issues of *Yearbook of Coal, Petroleum and Coke Statistics* compiled by the Research and Statistics Department, Ministry of International Trade and Industry. The units of total fuel oil and of light oil and kerosene are kilocalories and they are measured in logarithms in Figs. 1 and 2.

The ratios of the domestic wholesale prices of total fuel oil and of light oil and kerosene to the GDP deflator are used as their relative prices. The domestic wholesale prices were drawn from *Price Indexes Annual,* Bank of Japan (1975 = 100.0) and the GDP deflator (1975 = 100.0) and GDP were drawn from *Annual Report of National Account,* Economic Planning Agency. The relative prices and GDP are measured in logarithms in Figs. 1 and 2, and GDP is re-scaled so as to match with the units of other variables.

References

Bacon DW, Watts DG (1971) Estimating the transition between two intersecting straight lines. Biometrika 58:525–534

Boughton JM (1981) Recent instability of the demand for money: an international perspective. Southern Economic Journal 47:579–597

Chow GC (1960) Tests of equality between sets of coefficients in two linear regressions. Econometrica 28:591–605

Johnston J (1984) Econometric methods, 3rd ed. McGraw-Hill, New York

Katayama S, Ohtani K, Toyoda T (1987) Estimation of structural change in the import and export equations: an international comparison. The Economic Studies Quarterly 38:148–158

Ohtani K (1982) Bayesian estimation of the switching regression model with autocorrelated errors. Journal of Econometrics 18:251–261

Ohtani K, Katayama S (1985) An alternative gradual switching regression model and its application. The Economic Studies Quarterly 36:148–153

Quand RE (1958) The estimation of the parameters of a linear regression system obeying two separate regimes. Journal of the American Statistical Association 53:873–880

Salazar D, Broemeling L, Chi A (1981) Parameter changes in a regression model with autocorrelated errors. Communications in Statistics-Theory and Methods A10:1751–1758

Stern RM, Baum CF, Greene MN (1979) Evidence on structural change in the demand for aggregate U.S. imports and exports. Journal of Political Economy 87:179–192

Toyoda T, Ohtani K, Katayama S (1987) Structural change in oil consumption in Japan: an econometric analysis of effects of the two oil crises. Kobe University Economic Review 33:33–47

Tsurumi H (1980) A Bayesian estimation of structural shifts by gradual switching regressions with an application to the US gasoline market. In: Zellner A (ed) Bayesian analysis in econometrics and statistics: essays in honor of Harrold Jeffreys. North-Holland, Amsterdam, pp 213–240

Watt PA (1979) Tests of equality between sets of coefficients in two linear regressions when disturbance variances are unequal: some small sample properties. The Manchester School of Economic and Social Studies 47:391–396

Testing for Coefficient Constancy in Random Walk Models with Particular Reference to the Initial Value Problem

By S. J. Leybourne[1] and B. P. M. McCabe[2]

Summary: This article is concerned with Locally Best Invariant tests for coefficient stability in a univariate random walk coefficient regression model. In particular, we explore the effects that different assumptions about the initial value of the random walk process have on the form and asymptotic distribution of the resulting test statistics. When this initial value is allowed to be random, it is shown that the test statistics are either exactly the same, or possess the same asymptotic distributions, as when the initial value is fixed.

Key words: Brownian Motion, Brownian Bridge, Invariance, Locally Best Invariant Test, Mixing, Random Walk, Weak Convergence.

1.0 Introduction and Summary

This article explores the effect of different assumptions made about the initial value β_0 on the Locally Best Invariant test of $\omega^2 = 0$ in the model

$$y_t = x_t \beta_t + \varepsilon_t \qquad \varepsilon_t \sim \mathrm{IN}(0, \sigma^2) \tag{1}$$

$$\beta_t = \beta_{t-1} + \eta_t \qquad \eta_t \sim \mathrm{IN}(0, \sigma^2 \omega^2) \tag{2}$$

$$t = 1, \ldots, T.$$

We assume that σ^2 is an unknown nuisance parameter and that x_t is a known exogenous variable. When $\omega^2 = 0$ is true then β_t is constant and its value depends on

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what is assumed about the initial value of the sequence. The initial value of β_0 may be considered to be either fixed or random. When it is considered to be fixed, it is either assumed to be known (i.e. zero without loss of generality) or unknown and equal to β, say. In the case where it is random it is conventionally assumed to be $N(0, \sigma^2\omega^2\xi^2)$, where ξ^2 is a known and possibly large number. Most generally, one may assume that β_0 is distributed as $N(\beta, \sigma^2\omega^2\xi^2)$ with β and ξ^2 unknown. Of course, the distribution of β_0 is assumed independent of those of ε_t and η_t.

The fixed β_0 case has been studied by, for example, Garbade (1977). However, the situation where β_0 is random does not seem to have been studied before and this article derives the Locally Best Invariant test of $\omega^2=0$, showing that the test statistics are either exactly or asymptotically the same as in the case when β_0 is fixed. Section 4 summarises the asymptotic distribution theory required for implementing the test in the absence of normality (under normality, the method of Imhof (1961) could be used to determine exact distributions). This is done under standard mixing conditions.

2.0 The Likelihood Function of the Observables

The above model can be cast in an alternative but equivalent form as follows. By repeated back substitution of (2) into (1)

$$y_t = x_t \sum_{i=1}^{t} \eta_i + x_t\beta_0 + \varepsilon_t$$

from which it is easily established that

$$E(y_t) = \boldsymbol{x}_t E(\beta_0) = \boldsymbol{x}_t \beta$$

$$\begin{aligned} V(y_t y_{t-k}) &= \sigma^2(t\omega^2\boldsymbol{x}_t^2 + 1 + \omega^2\xi^2\boldsymbol{x}_t^2) \quad k=0 \\ &= \sigma^2((t-k)\omega^2\boldsymbol{x}_t\boldsymbol{x}_{t-k} + \omega^2\xi^2\boldsymbol{x}_t\boldsymbol{x}_{t-k}) \quad t>k>0. \end{aligned}$$

Of course, when β_0 is fixed, then $\xi^2=0$. In a vector notation we may write, for $\boldsymbol{y}=(y_1, y_2, \ldots, y_T)'$,

$$\boldsymbol{y} \sim N(\boldsymbol{x}\beta, \sigma^2 \boldsymbol{Q}(\omega^2, \xi^2)), \tag{3}$$

$$\boldsymbol{Q}(\omega^2, \xi^2) = \omega^2 \boldsymbol{XVX} + \boldsymbol{I}_T + \omega^2 \xi^2 \boldsymbol{Xii'X}$$

where $\boldsymbol{X}$ is a $T \times T$ diagonal matrix with t-th diagonal element equal to x_t, $\boldsymbol{x}$ is a $T \times 1$ vector of the x_t's and $\boldsymbol{V}$ is a $T \times T$ symmetric positive definite matrix whose (i,j)-th element is equal to $\min(i,j)$. The $T \times 1$ vector $\boldsymbol{i}$ consists of a column of ones.

3.0 Locally Best Invariant Tests

From (3) we see that σ^2 is always a nuisance parameter and so too are β and ξ^2 if they are unknown. It is clear that the role of β is the same irrespective of whether ξ^2 is zero or not i.e. whether β_0 is random or not. A great advantage of testing problems being invariant to certain transformations is that the distributions of maximal invariants often depends on a smaller number of parameters, thus eliminating the effect of other parameters. For example, irrespective of the status of β and ξ^2, testing for $\omega^2 = 0$ is invariant under

$$\boldsymbol{y} \rightarrow \alpha \boldsymbol{y}, \quad \alpha > 0, \tag{4}$$

and a maximal invariant is given by

$$\boldsymbol{\varepsilon}/(\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon})^{1/2}$$

where $\boldsymbol{\varepsilon} = \boldsymbol{y} - \boldsymbol{x}\beta$ and it is distributed free of σ^2. If β and ξ^2 are known, the Locally Best Invariant test is given by

$$\boldsymbol{\varepsilon}'\boldsymbol{A}\boldsymbol{\varepsilon}/\boldsymbol{\varepsilon}'\boldsymbol{\varepsilon}, \quad \boldsymbol{A} = \partial \boldsymbol{Q}(\omega^2, \xi^2)/\partial\omega^2 \,|\, \omega^2 = 0. \tag{5}$$

For further details see King and Hillier (1985). If β is unknown and ξ^2 is known or unknown then under the transformation

$$\boldsymbol{y} \rightarrow \alpha \boldsymbol{y} + \boldsymbol{x}\delta, \tag{6}$$

where a is a positive scalar and δ is an arbitrary scalar, a maximal invariant is $\boldsymbol{w}=\boldsymbol{P}\hat{\boldsymbol{\varepsilon}}/(\hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}})^{1/2}$, where $\boldsymbol{P}$ is a $(T-1)\times T$ dimensional matrix which satisfies $\boldsymbol{PP}'=\boldsymbol{I}_{T-1}$ and $\boldsymbol{P}'\boldsymbol{P}=\boldsymbol{M}=\boldsymbol{I}-\boldsymbol{xx}'/(\boldsymbol{x}'\boldsymbol{x})$. The density of this maximal invariant is proportional to

$$|\boldsymbol{PQP}'|^{-1/2}(\boldsymbol{w}'(\boldsymbol{PQP}')^{-1}\boldsymbol{w})^{-1/2(T-1)}. \tag{7}$$

Evaluating $\boldsymbol{PQP}'$, we see

$$\begin{aligned}\boldsymbol{P}(\omega^2\boldsymbol{XVX}+\boldsymbol{I}_T+\omega^2\xi^2\boldsymbol{Xii}'\boldsymbol{X})\boldsymbol{P}' &= \omega^2\boldsymbol{PXVXP}'+\boldsymbol{I}+\omega^2\xi^2\boldsymbol{PXii}'\boldsymbol{XP}'\\ &= \omega^2\boldsymbol{PXVXP}'+\boldsymbol{I}\end{aligned}$$

since $\boldsymbol{i}'\boldsymbol{X}=\boldsymbol{x}'$ and $\boldsymbol{i}'\boldsymbol{XP}'=\boldsymbol{x}'\boldsymbol{P}'=0$. Hence, the distribution of this maximal invariant does not depend, interestingly enough, on ξ^2 i.e. the location and scale invariance rule automatically eliminates the covariance parameter ξ^2 in addition to β and σ^2. The Locally Best Invariant test is

$$\hat{\boldsymbol{\varepsilon}}'\boldsymbol{A}\hat{\boldsymbol{\varepsilon}}/\hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}}, \quad \boldsymbol{A}=\partial\boldsymbol{Q}(\omega^2,0)/\partial\omega^2|\omega^2=0, \tag{8}$$

where $\hat{\boldsymbol{\varepsilon}}=\boldsymbol{y}-\boldsymbol{x}\hat{\beta}$ and $\hat{\beta}$ is the estimated value of β from the regression of $\boldsymbol{y}$ on $\boldsymbol{x}$.

Further insight into this phenomenon may be obtained in the case where $x_t=1$ for all t and σ^2 is assumed to be known. Then, under the transformation

$$\boldsymbol{y}\rightarrow\boldsymbol{y}+\delta\boldsymbol{i}$$

where δ is an arbitrary constant, maximal invariants include $\{y_t-y_k, t=1,\ldots,T, t\neq k\}$ (for any value of k) and $\{y_t-\bar{y}, t=1,\ldots,T\}$. By writing

$$y_t=\sum^{t}\eta_i+\beta_0+\varepsilon_t$$

it is clear that these maximal invariants do not involve any distributional characteristics of β_0 whatsoever. It is immaterial whether β_0 has a diffuse prior distribution or, indeed, what value of k is chosen.

It is perhaps of interest to note that invariance rule (6) is not appealing in the case where $\beta_0 \sim N(0, \sigma^2\omega^2\xi^2)$ since the family of distributions given in (3) is not closed under location invariance in this case.

3.1 The Locally Best Invariant Test when β_0 is Fixed

When β_0 is known we see that y is distributed as in (3) with $\beta = \xi^2 = 0$. The problem of testing $\omega^2 = 0$ is seen to be invariant under the transformation (4) and so the Locally Best Invariant test follows from (5) and is given by

$$y'XVXy/y'y. \tag{9}$$

Under normality, the exact distribution of (9) may be calculated via Imhof's method. Section 4 gives the asymptotic distribution when y is allowed to be α-mixing.

When β_0 is unknown, and hence is a nuisance parameter as well, we note that the testing problem is invariant under (6) and the Locally Best Invariant test

$$\hat{\varepsilon}'(XVX)\hat{\varepsilon}/\hat{\varepsilon}'\hat{\varepsilon}$$

follows from (8). The distribution of this statistic may be calculated, as before, using Imhof's method under normality. The asymptotic distribution is also given in Section 4.

3.2 The Locally Best Invariant Test when β_0 is Random

We first consider the case when β_0 is distributed as $N(0, \sigma^2\omega^2\xi^2)$ and ξ^2 is known. From (5), the Locally Best Invariant test, under transformation (4), is given by

$$y'X(V + \xi^2 ii')Xy/y'y = y'XVXy/y'y + \xi^2 y'Xii'Xy/y'y. \tag{10}$$

As ξ^2 is known, under the assumption of normality we may determine the exact distribution of this statistic via Imhof's method. We note that the statistic is increasing in ξ^2 and, as it increases, the power of the test will approach one. Thus, whilst one may assume that ξ^2 is large in order to simulate the effect of a noninformative prior for β_0, it is clear that ξ^2 is very informative about the distribution of y under the alternative.

When β_0 is distributed as $N(\beta, \sigma^2\omega^2\xi^2)$ then under transformation (6), the Locally Best Invariant test is

$$\hat{\varepsilon}'(XVX)\hat{\varepsilon}/\hat{\varepsilon}'\hat{\varepsilon}$$

as follows from (8). This is, of course, the same test as was obtained when β_0 was fixed but unknown.

4.0 Aysmptotic Distributions of the Tests

Whilst we have used the normality assumption to derive the Locally Best Invariant test we need only assume that $\{\varepsilon_t\}$ forms an α-mixing sequence under the null in order to derive its asymptotic distribution. This allows $\{\varepsilon_t\}$ to be, subject to mild regularity conditions, nonstationary, heteroscedastic and serially correlated. Accordingly, we make the following assumptions for any specified sequence $\{\xi_t\}$

Assumption 1: The sequence $\{\xi_t\}$ satisfies

1) $E(\xi_t)=0$ for all t,

2) $\sup_t E|\xi_t|^{\beta+\varepsilon}<\infty$ for some $\beta>2$ and $\varepsilon>0$,

3) $T^{-1}V(\Sigma\,\xi_t)\rightarrow\sigma_\xi^2$ as $T\rightarrow\infty$, $0<\sigma_\xi<\infty$

4) $\{\xi_t\}$ is α-mixing with coefficients a_m which satisfy

$$\sum_{m=1}^{\infty} a_m^{1-2/\beta}<\infty.$$

We define the partial sum process for a sequence ξ_t as a function on $D[0 \quad 1]$ by

$$W_T(r)=T^{-1/2}\sigma_\xi^{-1}\sum_{t=1}^{i}\xi_t \quad i/T\leqslant r<(i+1)/T,\ i=0,\ldots,T.$$

It follows from Herrndorf (1984) that $W_T(r) \Rightarrow W(r)$ where "$\Rightarrow$" means converges weakly and $W(r)$ is a Brownian motion on $C[0 \quad 1]$. Since the limit processes considered here are all in $C[0 \quad 1]$ use of the sup norm will suffice as a metric. Further details on the results presented below (and possible generalisations) are given in Leybourne and McCabe (1989).

Lemma 1: If the sequence $\{x_t y_t\}$ satisfies Assumption 1, then

$$\sigma_y^2 \sigma_{xy}^{-2} T^{-1} \boldsymbol{y}' \boldsymbol{X} \boldsymbol{V} \boldsymbol{X} \boldsymbol{y} / \boldsymbol{y}' \boldsymbol{y} \Rightarrow W^2 \equiv \int_0^1 W(r)^2 dr$$

where $\sigma_y^2 = Lt\, V(\Sigma^T y_t)/T$, $\sigma_{xy}^2 = Lt\, V(\Sigma^T x_t y_t)/T$ and $W(r)$ is a Brownian motion process.

Now define

$$x_T(r) \equiv \sum_{t=1}^{i} x_j^2 / \sum_{t=1}^{T} x_t^2 \quad i/T \leqslant r < (i+1)/T.$$

Lemma 2: Under Assumption 1 for the sequence $\{x_t \varepsilon_t\}$ and the condition that $x_T(r) \rightarrow r$,

$$\sigma_\varepsilon^2 \sigma_{x\varepsilon}^{-2} T^{-1} \hat{\boldsymbol{\varepsilon}}'(\boldsymbol{X}\boldsymbol{V}\boldsymbol{X})\hat{\boldsymbol{\varepsilon}} / \hat{\boldsymbol{\varepsilon}}'\hat{\boldsymbol{\varepsilon}} \Rightarrow B^2 \equiv \int_0^1 B(r)^2 dr$$

where $\sigma_\varepsilon^2 = Lt\, V(\Sigma^T \varepsilon_t)/T$, $\sigma_{x\varepsilon}^2 = Lt\, V(\Sigma^T x_t \varepsilon_t)/T$ and $B(r)$ is a Brownian bridge process.

The proof is similar to Lemma 1 and is also omitted. Note that if $\Sigma^T x_t^2/T$ converges to a constant then $x_T(r) \rightarrow r$ in $C[0 \quad 1]$.

Lemma 3: Under Assumption 1 for the sequence $\{x_t y_t\}$,

$$\sigma_y^2 \sigma_{xy}^{-2} T^{-1} \boldsymbol{y}' \boldsymbol{X}(\boldsymbol{V} + \xi^2 \boldsymbol{i}\boldsymbol{i}')\boldsymbol{X}\boldsymbol{y} / \boldsymbol{y}'\boldsymbol{y}$$

is asymptotically equivalent to $\sigma_y^2 \sigma_{xy}^{-2} T^{-1} \boldsymbol{y}' \boldsymbol{X} \boldsymbol{V} \boldsymbol{X} \boldsymbol{y} / \boldsymbol{y}' \boldsymbol{y}$ and its asymptotic distribution is given by Lemma 1.

Proof: From (10)

$$y'X(V+\xi^2 ii')Xy/y'y = y'XVXy/y'y + \xi^2 y'Xii'Xy/y'y$$

and the Lemma follows if T^{-1} times the latter term converges to zero. Since $i'Xy/T = \Sigma^T x_t y_t/T$ converges in probability to zero under Assumption 1, the second term converges to zero and the result holds.

Hence, asymptotically, the test statistic (10) does not depend on the value of ξ^2, and thus it is not necessary that ξ^2 be known. It follows that there is no difference, asymptotically, between the assumption that $\beta_0 = 0$ and that of $\beta \sim N(0, \sigma^2\omega^2\xi^2)$ in the sense that the same test statistic arises in both cases.

References

Garbade K (1977) Two methods for examining the stability of regression coefficients. Journal of the American Statistical Society 72:54–63

Herrndorf N (1984) A functional central limit theorem for weakly dependent sequences of random variables. Annals of Probability 12:141–153

Imhof JP (1961) Computing the distribution of quadratic forms in normal variables. Biometrika 48:419–426

King ML, Hillier GH (1985) Locally best invariant tests of the error covariance matrix of the linear regression model. Journal of the Royal Statistical Society B 47:98–102

Leybourne SJ, McCabe BPM (1989) On the distribution of some test statistics for coefficient constancy. Biometrika 76:1

Transformations for an Exact Goodness-of-Fit Test of Structural Change in the Linear Regression Model[1]

By M. L. King and P. M. Edwards[2]

Abstract: This paper considers testing for structural change of unknown form in the linear regression model as a problem of testing for goodness-of-fit. Transformations of recursive (or other LUS) residuals that reduce the problem to one of testing independently distributed uniform variables are presented. Exact empirical distribution function tests can then be applied without having to estimate unknown parameters. The tests are illustrated by their application to a money demand model.

1 Introduction

In many applications, the standard assumptions required for the classical linear regression model are somewhat questionable. This is particularly true in econometric applications, where for example, it is often difficult to find convincing arguments as to why the regression relationship is constant over time. In fact, the main point of the Lucas (1976) critique of quantitative economic policy analysis is that policy changes can cause parameter changes in economic relationships over time. Of course, if these changes are of a minor nature, then it may well be that the standard linear regression model provides a useful and meaningful approximation. It would be silly to build a complicated model when a simple one will do. It is therefore important to be able to test the adequacy of a fitted linear regression model. Typically, little may be known about how and when the regression relationship might change so that the test will need to cast a wide net. One possible approach is to apply a goodness-of-fit test to the linear regression.

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The first such test that usually springs to mind is the well-known χ^2 test. This is less than ideal for, as Stephens (1974) observed, it has long been known that for goodness-of-fit problems in which the distribution function is continuous and completely specified, tests based on the empirical distribution function (EDF) are more powerful than the χ^2 test. A disadvantage of EDF based tests is that when unknown parameters in the distribution function are replaced by their estimates, the distributions of the test statistics under the null hypothesis change. Stephens gives some approximate critical values of various statistics for a random sample from the normal distribution with zero mean and unknown variance as well as unknown mean and variance.

The Cusum of squares test for structural change proposed by Brown, Durbin and Evans (1975) can be viewed as an approximate Kolmogorov-Smirnov EDF test applied to recursive residuals that have undergone a secondary nonlinear transformation. To see this, let

$$y = X\beta + u, \quad u \sim N(0, \sigma^2 I_n) \tag{1}$$

denote the standard linear regression model where y is $n \times 1$, X is an $n \times k$ nonstochastic matrix of rank $k < n$, β is a $k \times 1$ vector of unknown parameters and σ is an unknown scale parameter. Also let $\hat{u}_j$, $j = k+1, \ldots, n$ denote the recursive residuals from (1). (For a definition of recursive residuals see, for example, Phillips and Harvey 1974, Brown, Durbin and Evans 1975 or Farebrother 1976b.) The Cusum of squares test is based on whether, for $r = k+1, \ldots, n$,

$$s_r = \left(\sum_{j=k+1}^{r} \hat{u}_j^2 \right) \Bigg/ \left(\sum_{j=k+1}^{n} \hat{u}_j^2 \right)$$

is always in the range

$$\pm c_0 + (r-k)/m, \tag{2}$$

where c_0 is an appropriately chosen value and $m = n - k$. Because $s_n = 1$, this acceptance region is equivalent to

$$\max_{i=1,\ldots,m-1} \{s_{k+i} - i/m\} < c_0$$

and

$$\max_{i=1,\ldots,m-1} \{i/m - s_{k+i}\} < c_0$$

which is of the form of the modified Kolmogorov-Smirnov test provided $s_{k+1}, \ldots, s_n$ is an ordered sample of independent observations from the uniform (0, 1) distribution.

For the case when m is even, Brown, Durbin and Evans noted that the joint distribution of

$$s_{k+2}, s_{k+4}, \ldots, s_{n-2} \tag{3}$$

is identical to that of an ordered sample of independent observations from the uniform (0, 1) distribution. If the test is based on these $(m/2) - 1$ statistics then Durbin's (1969) table of significance points for the modified Kolmogorov-Smirnov EDF test can be used to determine c_0. Brown, Durbin and Evans suggested using this value, or a linearly interpolated value if m is odd, for c_0 in (2). They reported that Monte Carlo evidence indicated that this choice of c_0 value yields true significance levels slightly above nominal levels. An exact EDF test when m is even, could have been based on the $(m/2) - 1$ statistics given by (3) with an obvious reduction in power.

In this paper we propose alternative transformations of recursive and other residuals which allow exact EDF tests to be applied to a full set of observations. Invariance arguments are used to reduce the goodness-of-fit testing problem to one of testing independent variables from the uniform (0, 1) distribution so that the standard EDF tests such as the Kolmogorov-Smirnov, Cramer-von Mises, Kuiper, Watson and Anderson-Darling tests can be used. A similar approach has been suggested by Csörgö, Seshadri and Yalovsky (1973) (also see Mardia 1980) for the special case of a random sample from the normal distribution with unknown mean and variance.

The proposed transformations are discussed in the next section and the results of an application of the proposed testing procedure to an annual model of the demand for money in the USA are presented in Section 3.

2 The Transformation

Our goodness-of-fit problem is one of testing

$$H_0 : y \sim N(X\beta, \sigma^2 I_n)$$

against

$$H_a : y \not\sim N(X\beta, \sigma^2 I_n)$$

where both β and σ^2 are unknown. Observe that if H_a is true then at least one of either

(i) $E(y) \neq X\beta$,

(ii) $\mathrm{Var}(y) \neq \sigma^2 I_n$,

(iii) y is non-normal,

is true so we are indeed casting a wide net. While it is obvious how a structural change might result in (i) or (ii) being true, note that (iii) will occur in a regression whose errors switch distribution at some point in time.

This testing problem is invariant to transformations of the form

$$y^* = \gamma_0 y + X\gamma, \tag{4}$$

where γ_0 is a scalar and γ is a $k \times 1$ vector. This is because if H_0 holds then

$$y^* \sim N(X(\gamma_0\beta + \gamma), \gamma_0^2 \sigma^2 I_n)$$

which means that H_0 also holds for y^*. Furthermore, if H_a is true because of at least one of (i), (ii) or (iii) holding then the same will also be true of y^* given the form of (4).

As King (1980) notes, the $m \times 1$ vector

$$v = P_1 z / (z' P_1' P_1 z)^{1/2}$$

is a maximal invariant under the group of transformations defined by (4) where $z = My$ is the vector of ordinary least squares residuals, $M = I_n - X(X'X)^{-1}X'$, and P_1 is an $m \times n$ matrix such that $M = P_1'P_1$ and $P_1P_1' = I_m$.

Under H_0, v is uniformly distributed over the surface of the unit m-sphere. Because of this, when v is transformed to polar coordinates, $\theta_j \in [0, \pi]$, $j = 1, 2, \ldots, m-2, \theta_{m-1} \in [0, 2\pi]$, via

$$v_1 = \cos\theta_1,$$

$$v_j = \left(\prod_{i=1}^{j-1} \sin\theta_i\right)\cos\theta_j \quad 2 \le j \le m-1,$$

$$v_m = \prod_{i=1}^{m-1} \sin\theta_i,$$

it follows (see Goldman 1976) that $\theta_1, \ldots, \theta_{m-1}$ are independent random variables under H_0 with probability density functions:

$$P_{\theta_j}(\theta_j) = \Gamma\{(m-j+1)/2\}\pi^{-1/2}[\Gamma\{(m-j)/2\}]^{-1}\sin^{m-1-j}\theta_j,$$

$$\theta_j \in [0, \pi], \quad j = 1, 2, \ldots, m-2,$$

$$P_{\theta_{m-1}}(\theta_{m-1}) = 1/(2\pi), \quad \theta_{m-1} \in [0, 2\pi].$$

Observe that if e_i is the $m \times 1$ vector of zeros with the i-th element being unity, then θ_1 is the angle between e_1 and v, and θ_j is the angle between e_j and the projection of v onto the manifold spanned by $e_j, e_{j+1}, \ldots, e_m$ for $j = 2, \ldots, m-1$.

Given the independence of $\theta_1, \ldots, \theta_{m-1}$ under H_0, the transformations

$$w_j = \int_0^{\theta_j} P_{\theta_j}(x)dx, \quad j = 1, \ldots, m-1,$$

result in independently distributed uniform variables on the interval (0, 1) under H_0. These transformations can be performed using the following formulae:

$$w_{m-1} = \theta_{m-1}/(2\pi).$$

For $1 \leq j \leq m-2$ and $m-j$ odd, let $q=(m-1-j)/2$. Then

$$w_j=\Gamma(q+1)\pi^{-1/2}\left[\Gamma\left(q+\frac{1}{2}\right)\right]^{-1}\left[2^{-2q}\binom{2q}{q}\theta_j\right.$$

$$\left.+(-1)^q 2^{-(2q-1)}\sum_{k=0}^{q-1}(-1)^k\binom{2q}{k}\{\sin(2q-2k)\theta_j\}/\{2q-2k\}\right].$$

For $1 \leq j \leq m-2$ and $m-j$ even, let $q=(m-2-j)/2$. Then

$$w_j=\Gamma(q+3/2)\pi^{-1/2}\{\Gamma(q+1)\}^{-1}$$

$$\left[2^{-2q}(-1)^{q+1}\sum_{k=0}^{q}(-1)^k\binom{2q+1}{k}(\cos\{(2q+1-2k)\theta_j\}-1)/(2q+1-2k)\right]$$

The resultant w_j, $j=1, \ldots, m-1$, after having been sorted into ascending order

$$w_j^{(1)} \leq w_j^{(2)} \leq \ldots \leq w_j^{(m-1)},$$

can be used to calculate standard test statistics based on the EDF as follows:

(i) The Kolmogorov-Smirnov statistics D, D^+, D^-:

$$D^+=\max_{1\leq i\leq m-1}\{i/(m-1)-w_j^{(i)}\},\quad D^-=\max_{1\leq i\leq m-1}[w_j^{(i)}-\{(i-1)/(m-1)\}]$$

and $D=\max(D^+, D^-)$.

(ii) The Cramer-von Mises statistic W^2:

$$W^2=\sum_{i=1}^{m-1}[w_j^{(i)}-\{(2i-1)/(2m-2)\}]^2+1/\{12(m-1)\}.$$

(iii) The Kuiper statistic V:

$$V = D^+ + D^-.$$

(iv) The Watson statistic U^2:

$$U^2 = W^2 - (m-1)(\bar{w} - 0.5)^2$$

$$\text{where } \bar{w} = \left(\sum_{i=1}^{m-1} w_i\right)/(m-1).$$

(v) The Anderson-Darling statistic A^2:

$$A^2 = -\sum_{i=1}^{m-1} [(2i-1)\{\log w_j^{(i)} + \log(1 - w_j^{(m-i)})\}/(m-1)] - (m-1).$$

Stephens (1974) presents tables for finding the critical values of each of the statistics. (Also see Pearson and Hartley 1972.)

How should one compute v? Observe that Var $(P_1 z) = \sigma^2 P_1 M P_1' = \sigma^2 I_m$ so that $P_1 z \sim N(0, \sigma^2 I_m)$. This implies that v can be regarded as a linear unbiased with scalar covariance matrix (LUS) residual vector divided by its norm. For any given regression model there are an infinite number of LUS residual vectors. Some of the best known are Theil's (1965, 1968) BLUS residuals and recursive residuals. These and other LUS residuals are reviewed by King (1987).

When testing for structural change, we recommend the use of recursive residuals. They can be calculated recursively either forwards in time or backwards in time. If one suspects that a change may have occurred late in the estimation period then tests based on backward recursive residuals are likely to have better power. Because BLUS residuals are "best" estimates of m of the unknown disturbances they may be preferable when testing specifically for non-normality. Algorithms for computing BLUS and recursive residuals may be found in Farebrother (1976a, 1976b).

Table 1. Values of the EDF test statistics for Klein's demand for money model; 1879–1974

Test Statistics	Forward recursive residuals	Backward recursive residuals
D^+	0.2038	0.1328
D^-	0.2120	0.2469
D	0.2120	0.2469
W^2	1.6114	1.2790
V	0.4159	0.3797
U^2	1.6114	1.1222
A^2	9.3343	7.5476

3 An Example

This section considers the application of the above exact EDF tests to an annual regression model of the demand for money in the USA suggested by Klein (1977). This model was used by Krämer and Sonnberger (1986) to illustrate the use of diagnostic testing in practice. Using Klein's notation, the model is

$$\log M = a_0 + a_1 \log y_p + a_2 r_S + a_3 r_L + a_4 r_M + a_5 \log S(\dot{P}/P) + u \tag{5}$$

where M is the quantity of money ($M2$), y_p is real permanent income, r_S is a short term interest rate, r_M is the rate of return on money, $S(\dot{P}/P)$ is a measure of variability of the rate of price changes and u is the disturbance term. Annual observations of these variables for 1879–1974 are given by Krämer and Sonnberger (1986, Table A.1).

Farebrother's (1976b) algorithm was used to calculate recursive residuals forwards in time and backwards in time. Both sets of residuals, calculated using the full data set (1879–1974), were transformed as outlined above and the resultant w_j, $j = 1, \ldots, 89$, were sorted into ascending order. The calculated values of each of the EDF test statistics are given in Table 1. With one exception, all tests reject H_0 at the one per cent significance level. The one exception is the D^+ test based on backwards recursive residuals which is significant at the five per cent level. There is ample evidence that the classical linear regression based on (5) does not fit the data well.

References

Brown RL, Durbin J, Evans JM (1975) Techniques for testing the constancy of regression relationships over time. Journal of the Royal Statistical Society B 37:149–163

Csörgö M, Seshadri V, Yalovsky M (1973) Some exact tests for normality in the presence of unknown parameters. Journal of the Royal Statistical Society B 35:507–522

Durbin J (1969) Tests for serial correlation in regression analysis based on the periodogram of least squares residuals. Biometrika 56:1–15

Farebrother RW (1976a) Algorithm AS104: BLUS residuals. Applied Statistics 25:317–322

Farebrother RW (1976b) Recursive residuals - a remark on Algorithm AS75: Basic procedures for large, sparse or weighted linear least squares problems. Applied Statistics 25:323–324

Goldman J (1976) Detection in the presence of spherically symmetric random vectors. IEEE Transactions on Information Theory IT-22:52–59

King ML (1980) Robust tests for spherical symmetry and their application to least squares regression. The Annals of Statistics 8:1265–1271

King ML (1987) Testing for autocorrelation in linear regression models: a survey. In: King ML, Giles DEA (eds) Specification analysis in the linear model. Routledge and Kegan Paul, London

Klein B (1977) The demand for quality-adjusted cash balances: Price uncertainty in the US demand for money function. Journal of Political Economy 85:691–715

Krämer W, Sonnberger H (1986) The linear regression model under test. Physica-Verlag, Heidelberg

Lucas RE (1976) Econometric policy evaluation: a critique. Carnegie-Rochester Conferences in Public Policy 1:19–46

Mardia KV (1980) Tests for univariate and multivariate normality. In: Krishnaiah PR (ed) Handbook of statistics 1: Analysis of variance. North-Holland, Amsterdam

Pearson ES, Hartley HO (1972) Biometrika tables for statisticians 2. Cambridge University Press, Cambridge

Phillips GDA, Harvey AC (1974) A simple test for serial correlation in regression analysis. Journal of the American Statistical Association 69:935–939

Stephens MA (1974) EDF statistics for goodness of fit and some comparisons. Journal of the American Statistical Association 69:730–737

Theil H (1965) The analysis of disturbances in regression analysis. Journal of the American Statistical Association 60:1067–1079

Theil H (1968) A simplification of the BLUS procedure for analyzing regression disturbances. Journal of the American Statistical Association 63:242–251

Robust Bayesian Analysis of a Parameter Change in Linear Regression

By K. Pötzelberger and W. Polasek[1]

Summary: Robust Bayesian analyses in a conjugate normal framework have been developed by Leamer (1978) and Polasek and Pötzelberger (1987). Fixing the prior mean and varying the prior covariance matrix yields a so-called feasible ellipsoid for the posterior mean and robust HPD regions, also called HiFi-regions. This paper considers the application of this approach to gain robust Bayesian inference in case of a parameter change in regression models.

1 Introduction

The estimation and detection of a parameter shift in a linear regression model has gained increasing attention in the econometric literature. Many adhoc models have been proposed from a classical point of view, but only a few Bayesian treatments are known. Tsurumi (1977), Tsurumi and Sheflin (1984), and Ilmakunnas and Tsurumi (1984) have used Bayesian highest posterior density (HPD) intervals to test for shifts in the parameters of a model in the presence of heteroscedastic and autocorrelated errors. These methods assume a known switching point, whereas Smith (1977), Salazar, Broemeling and Chi (1981), and Ohtani (1981) are searching for the unknown join point.

In this paper we follow a slightly different route for the linear model with switching regimes and known join point. We assume a partial prior specification for the amount of the shift in the coefficients and then we find via a Bayesian robustness analysis as to how sensitive the posterior distribution reacts to changes in the strength of the prior distribution. The results are presented by the so-called feasible ellipsoid (Leamer 1978) and the HiFi-region (Polasek and Pötzelberger 1987), a robust version of the well known HPD-intervalls. The assumption of a shift

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has a robust Bayesian justification, if the size in the parameter change can be judged large enough from different a prior views.

The next section introduces the basic linear model with two regimes and section 3 derives the feasible ellipsoid and the so-called HiFi-region for this model. In a concluding section we summarize our results. The appendix gives details of the calculation of the posterior mean.

2 The Basic Model

Let $\boldsymbol{y}=(y_1,\ldots,y_T)'$ be a $T\times 1$ dependent variable and $\boldsymbol{X}=(x_1,\ldots,x_T)'$ a $T\times K$ matrix vector of independent variables. We assume a linear regression of the form $\boldsymbol{y}=\boldsymbol{X}\beta+\boldsymbol{u}$, where $\boldsymbol{u}$ is the error term. Furthermore, consider a change in the parameters after time n resulting in a regression model in two regimes.

$$y_t=x_t'\beta+u_{t1} \quad t=1,\ldots,n; \tag{2.1}$$

$$y_t=x_t'(\beta+\delta)+u_{t2} \quad t=n+1,\ldots,T.$$

The residuals are assumed to be i.i.d. with mean 0 and precision σ_1 and σ_2, respectively:

$$u_{t1}\sim N(0,\sigma_1^{-1}) \quad \text{and} \quad u_{t2}\sim N(0,\sigma_2^{-1}). \tag{2.2}$$

Let $\boldsymbol{y}_1$ be the dependent variable and $\boldsymbol{X}_1$ the independent variables in the first regime and $\boldsymbol{y}_2$ and $\boldsymbol{X}_2$ in the second regime. Then we can write the model (2.1) in the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} X_1 & 0 \\ X_2 & X_2 \end{pmatrix} \begin{pmatrix} \beta \\ \delta \end{pmatrix} + \boldsymbol{u} \quad \text{with} \quad \boldsymbol{u}\sim N\left(0, \begin{pmatrix} \sigma_1^{-1}\boldsymbol{I}_n & 0 \\ 0 & \sigma_1^{-1}\boldsymbol{I}_{T-n} \end{pmatrix}\right), \tag{2.3}$$

where δ is the change in the parameters at point n.

The likelihood function for this model (2.3) is given by

$$1(\beta,\delta)\propto e^{-1/2\{\sigma_1(y_1-X_1\beta)'(y_1-X_1\beta)+\sigma_2(y_2-X_2(\beta+\delta))'(y_2-X_2(\beta+\delta))\}}. \tag{2.4}$$

With prior information about β and δ given as block-normal distribution

$$\begin{pmatrix}\beta\\ \delta\end{pmatrix} \sim N\left(\begin{pmatrix}\beta^*\\ \delta^*\end{pmatrix}, \begin{pmatrix}P^{*-1} & 0\\ 0 & Q^{*-1}\end{pmatrix}\right), \tag{2.5}$$

we find after some algebra (see appendix) that the posterior distribution of δ after seeing the data $y=(y_1,y_2)'$ is normal with mean δ^{**} and variance-covariance matrix Q^{**-1} given by

$$\delta|y \sim N(\delta^{**}, Q^{**-1}), \tag{2.6}$$

$$\delta^{**} = (Q^* + X_2'\Psi X_2)^{-2}(Q^*\delta^* + X_2'\Psi(y_2 - \zeta)), \tag{2.7}$$

$$Q^{**} = Q^* + X_2'\Psi X_2. \tag{2.8}$$

Ψ in (2.7) and (2.8) is the metric of the log-likelihood function of $y_2 - X_2\delta$ given by

$$\Psi = \sigma_2 I_K - \sigma_2^2 X_2 P^{**-1} X_2'. \tag{2.9}$$

ζ in (2.7) is the mode of the likelihood of $y_2 - X_2\delta$:

$$\zeta = \Psi^{-1}\sigma_2 X_2 Q^{**-1}(P^*\beta^* + \sigma_1 X_1' y_1), \tag{2.10}$$

and P^{**} is the metric of the posterior density of β given δ:

$$P^{**} = \sigma_1 X_1' X_1 + \sigma_2 X_2' X_2 + P^*. \tag{2.11}$$

The posterior mean (2.7) of the shift parameter δ can be written as a matrix weighted average of the prior location δ^* and the diffuse parameter-location δ^{non}, a posterior mean one would obtain if the prior knowledge for δ would be noninformative (but not necessarily about β, because Ψ depends on P^{**} and therefore on P^*).

$$\delta^{**} = (Q^* + X_2'\Psi X_2)^{-1}(Q^*\delta^* + X_2'\Psi X_2\delta^{\text{non}})). \tag{2.12}$$

δ^{non} can be expressed as

$$\delta^{\text{non}} = \hat{\delta} - (\boldsymbol{P}^* + \sigma_1 \boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{P}^*\beta^*, \tag{2.13}$$

where $\hat{\delta}$ is the ML-estimate of the difference between the regression estimates of the first and the second regime:

$$\hat{\delta} = (\widehat{\beta + \delta}) - \hat{\beta}, \tag{2.14}$$

with the OLS-estimates

$$\hat{\beta} = (\boldsymbol{X}_1'\boldsymbol{X}_1)^{-1}\boldsymbol{X}_1'\boldsymbol{y}_1 \quad \text{and} \quad (\widehat{\beta + \delta}) = (\boldsymbol{X}_2'\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\boldsymbol{y}_2. \tag{2.15}$$

The case of a diffuse prior for β in (2.5) (i.e. the classical ML- or OLS-estimate) is included in the formulas (2.7) to (2.11) by setting the precision matrix $\boldsymbol{P}^*$ to zero. The noninformative estimate δ^{non} reduces then to the ML-location $\hat{\delta}$ in (2.13). The estimates of the residual variances are

$$\sigma_1^2 = (y_1 - \boldsymbol{X}_1\beta)'(y_1 - \boldsymbol{X}_1\beta)/n, \tag{2.16}$$

$$\sigma_2^2 = (y_2 - \boldsymbol{X}_2(\beta + \delta))'(y_2 - \boldsymbol{X}_2(\beta + \delta))/(\boldsymbol{T} - \boldsymbol{n}).$$

3 Feasible Ellipsoids and HiFi-Regions

3.1 The Feasible Ellipsoid

The first result of conjugate Bayesian robustness was derived in Chamberlain and Leamer (1976) for the normal linear regression model with a full rank $T \times p$ data matrix $\boldsymbol{X}$ and the full rank precision matrices R and Σ, i.e.

$$y \sim N(\boldsymbol{X}\beta, \sigma^2 \boldsymbol{R}^{-1}), \quad \beta \sim N(\boldsymbol{b}^*, \Sigma^{-1}); \tag{3.1}$$

They showed that the posterior mean

$$b_{\Sigma}=(X'RX+\Sigma)^{-1}(X'RY+)b^{*}). \tag{3.2}$$

is constrained to lie in the ellipsoid

$$(\beta-h)'X'RX(\beta-h)\leq c, \tag{3.3}$$

where $h=(b+b^{*})/2$, $b=(X'RX)^{-1}X'RY$ is the OLS-estimate of β, and $c=(b^{*}-b)'X'RX(b^{*}-b)/4$ is a constant. This ellipsoid can be written also in the form

$$F=\text{closure}\,\{b_{\Sigma}|\Sigma\,\text{pos. def. and symmetric}\}=\text{ELL}(b^{*},b,X'RX), \tag{3.4}$$

where ELL(*,*,*) describes an ellipse with diameter b^{*} to b, and metric $X'RX$, as in (3.3).

3.2 Extreme Bound Analysis (EBA)

Reporting of ellipsoids can be done graphically only in two dimensions and for higher dimensions one would like to have simpler tools available. Projecting this ellipsoid onto the coordiante axes yields the so-called extreme bounds:

$$\text{EBA}^{u}_{l}(i)=h_i+z(c[X'RX]^{ii})^{1/2},\quad i=1,\ldots,p, \tag{3.5}$$

where $[A]^{ii}$ stands for the i-th diagonal element of A^{-1} and c is given as in (3.3). z is 1 for the upper bound EBA^{u} and -1 for the lower bound EBA_{l}.

3.3 HiFi-Regions

A HPD region of size a for the normal linear regression model (3.1) is given by

$$\text{HPD}_{a}(\Sigma)=\{\beta|(\beta-b_{\Sigma})'(\Sigma+X'RX)(\beta-b_{\Sigma}\}\leq\chi^{2}(p,a)\}, \tag{3.6}$$

where $\chi^2(p,a)$ denotes the a-quantile of the chi2-distribution with p degrees of freedom. The closure of the union of all HPD region of fixed size a is denoted by HiFi_a:

$$\mathrm{HiFi}_a = \text{closure} \bigcup_{\Sigma \in \mathbb{M}^+} \mathrm{HPD}_a(\Sigma), \tag{3.7}$$

where $\mathbb{M}^+$ is the set of all positive definite symmetric matrices. To each ellipsoid F we can construct a HiFi-region with $0.5 \leq a \leq 1$. By $\mathrm{HiFi}_i^u(i,a)$ we denote as before the lower und the upper bound for the i-th coefficient of the HiFi-region of size a.

3.4 Robust Shift Analysis for Lower Bounded Prior Variances

In the two-regime model (2.3) we apply the Bayesian bounded robustness ideas given in Leamer (1982) or Polasek (1984) for the precision matrix $\boldsymbol{Q}$ of the shift parameter δ. By bounding the prior precision matrices $\boldsymbol{Q}$ from above (i.e. the variance from below) we derive special robustness results in form of smaller feasible ellipsoids than (3.3). Bounding the prior precision matrices $\boldsymbol{Q}$ means excluding orthodox priors, where the mean of the shift parameter has really a degenerate (one point) distribution.

For fixed prior knowledge for the shift parameter δ^* and fixed $\boldsymbol{Q}_0$ the set of δ^{**} with $\boldsymbol{Q}$ being any precision matrix such that $\boldsymbol{Q}_0 - \boldsymbol{Q}$ is a positive definite and symmetric matrix is given by the ellipsoid

$$(\delta^{**} - \boldsymbol{m})'\boldsymbol{H}(\delta^{**} - \boldsymbol{m}) \leq c^*, \tag{3.8}$$

$$\boldsymbol{H} = \boldsymbol{M} + \boldsymbol{M}\boldsymbol{Q}_0^{-1}\boldsymbol{M} \quad \text{with} \quad \boldsymbol{M} = \boldsymbol{X}_2'\Psi\boldsymbol{X}_2, \tag{3.9a}$$

$$\boldsymbol{m} = (\delta^{\mathrm{non}} + \delta_{\mathrm{II}})/2, \tag{3.9b}$$

$$c^* = (\delta^{\mathrm{non}} - \delta_{\mathrm{II}})'\boldsymbol{H}(\delta^{\mathrm{non}} - \delta_{\mathrm{II}})/4, \tag{3.9c}$$

with δ^{non} being the noninformative part of (2.12). The feasible ellipsoid (3.8) is now determined by the parameters $F^u = \text{ELL}(\delta^{\text{non}}, \delta_{\text{II}}, \boldsymbol{H})$, where δ_{II} is given by

$$\delta_{\text{II}} = \boldsymbol{H}^{-1}(\boldsymbol{M}\delta^* + \boldsymbol{M}\boldsymbol{Q}_0^{-1}\boldsymbol{M}\delta^{\text{non}}). \tag{3.10}$$

This ellipsoid can be also obtained if we make the following consideration: After a break has occured in a time series, the variance of the error process might be different than before. If the second variance is larger then the original ellipsoid (3.1) can be used. If it is smaller than in the first regime then this kind of uncertainty changes the set of posterior means only by changing the upper bound matrix $\boldsymbol{Q}_0$ to $\boldsymbol{Q}_0\sigma_1/\sigma_2^*$ where we allow σ_2 to vary in the interval $0 \leq \sigma_2 \leq \sigma_2^*$.

4 Examples

This section demonstrates the approach with 2 examples: The first one checks whether the simple Keynesian consumption function in Austria has changed after the oil-shock depression in 1975. The second example checks whether the Swiss consumption function has changed after 1975 as well.

Example 4.1: Consumption Function in Austria

As prior information for the consumption function and the shift parameter we assume a conjugate normal distribution for β:

$$\beta \sim N\left[\begin{pmatrix} 0 \\ 0.8 \end{pmatrix}, \begin{pmatrix} 100 & 0 \\ 0 & 0.25^2 \end{pmatrix} = \begin{pmatrix} 0.01 & 0 \\ 0 & 16 \end{pmatrix}^{-1}\right], \quad \delta \sim N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Sigma\right]. \tag{4.1}$$

For the shift parameter δ we assume a priori no shift-effects, implying a prior mean $\delta^* = 0$.

Then the lower bound of the variance of the shift parameter is determined by excluding the set of prior distributions which are considered as too informative, i.e. too sharp around the prior mode. We suggest a simple approach for our example: Expecting in general no shift effects imply a prior mode (expectation in a conjugate

Table 4.1 a. Regression estimates for Austria: Classical summary

Variable Name	Coefficient	Dependent Variable: C% Std. Err. Estimate	t Statistic	Prob > t
Constant	3.772	1.171	3.220	0.003
GNP%	0.193	0.224	0.859	0.397
Dummy	-3.209	1.373	-2.337	0.026
Dum*GNP	0.599	0.344	1.742	0.092

Table 4.1.b. EBA and HiFi-regions for the shift parameter

sets / bounds	Constant lower	Constant upper	slope lower	slope upper
F	-7.10	0.1	-0.38	1.74
F^u	-7.00	-3.08	0.40	1.37
HPD(α =.9)	-11.31	-2.68	0.09	2.62
HiFi(α =.9)	-11.40	1.25	-1.06	2.92

framework) $\delta^* = 0$, but we want to exclude all prior distributions which have roughly 50% (exact 46.2%) of their mass inside the bivariate $\pm\sigma$ square region $(-1, 1) \times (-1, 1)$. This approach gives an upper bound precision matrix $Q_0 = I_2$, the identity matrix.

The classical summary of our regression shift model is given in Table 4.1.a, and the associated scatterplot and regression lines are shown in Fig. 4.1. The robust Bayesian summaries are listed in Table 4.1 b and shown graphically in Fig. 4.2. All HPD and HiFi-regions are given for $\alpha = 90\%$. The parameters of the upper bounded ellipsoid F^u are $\delta^{\text{non}} = (-7.0, 1.36)$ and $\delta_{\text{II}} = (-3.08, 0.41)$. They are marked by a square and a triangle in Fig. 4.2. Recall that δ^{non} is the location for the shift parameter δ where we are diffuse about δ but informative with prior (4.1) about β. δ_{II} is the (limiting) location parameter if we incorporate the precision bound Q_0, i.e. if we exclude all orthodox priors beyond this precision bound. Q_0 denotes the upper bound ellipsoid between these two points. Note that it is relatively thin. The HPD-region is centered around δ^{non} and denotes the diffuse 90% credibility region for the shift parameters δ where the prior information for β included. Note that all HiFi-regions for δ have to be larger than this HPD-region.

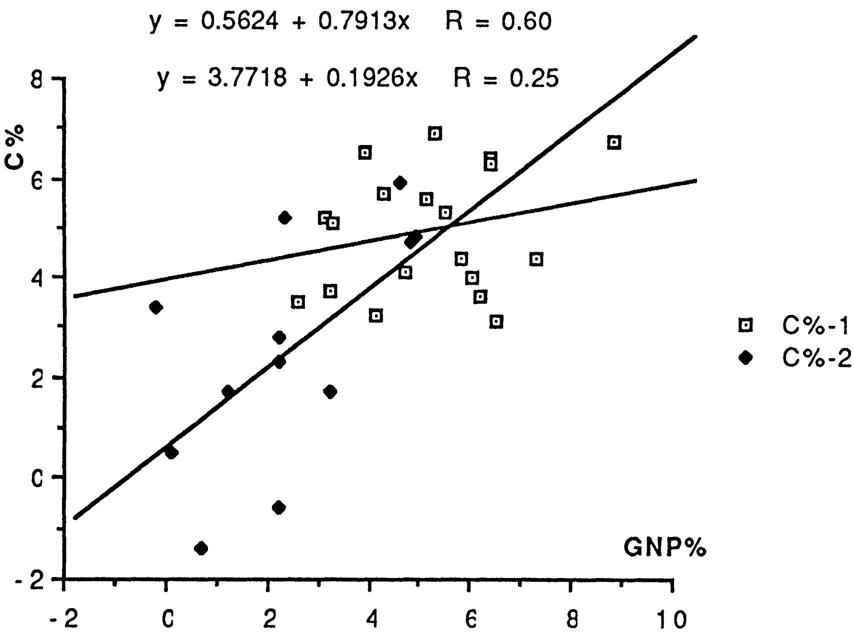

Fig. 4.1. The Austrian consumption function before and after the oil shock

The HiFi-region is shown for the unbounded ellipsoid F, which passes through δ^{non} and $\delta^* = \mathbf{0}$ (marked by a*).

To draw robust Bayesian inferences we have to take into account that the extreme bounds of the simple ellipsoid F always cover the origin, since the prior location for δ^* was chosen that way. Only by excluding orthodox priors we can bound away the smaller ellipsoid F^u from the coordinate axis. It is interesting to observe that the mass of the HiFi-region is in the NW-orthant of the parameter space. The HiFi^u-region which corresponds to the F^u ellipsoid (not shown in the Figure for technical reasons) is only slightly larger than the HPD region.

The HiFi-region shows that even with very "weird" prior precisions (from the point of view of the data), a Bayesian analysis of this data set leads to conclusions which are in the neighborhoud of the diffuse HPD region. This implies that the data are very conclusive for a parameter change in the consumption function, even if we take into account very dissentive prior views about the shift parameters.

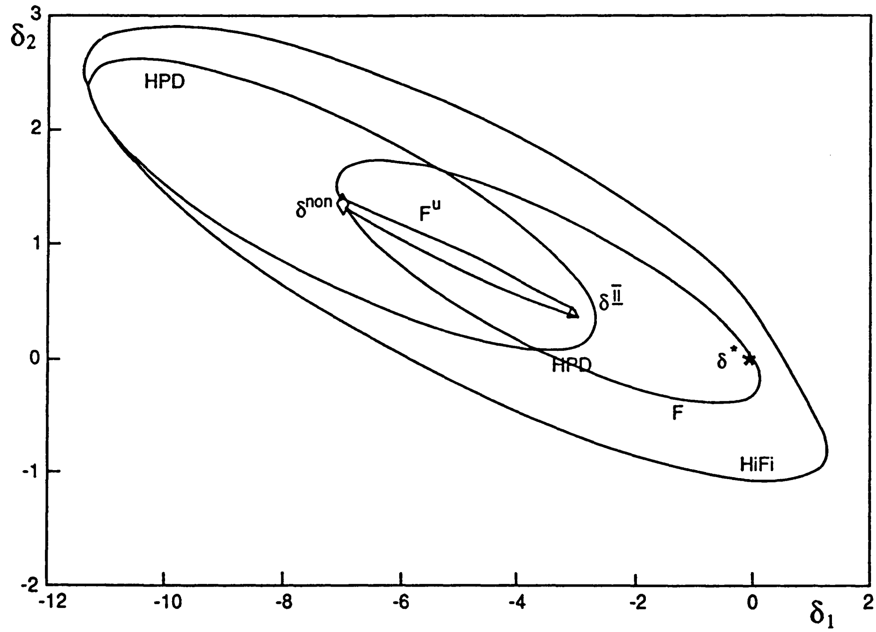

Fig. 4.2. Robust Bayesian summaries for structural change in Austria. ◇ ... δ^{non} ... (left) point on the F-ellipsoid; △ ... δ_{II} (right) point on the F-ellipsoid; * ... prior location $\delta^* = (0,0)$; $F^u = \text{ELL}(\delta^{\text{non}}, \delta_{\text{II}}, \boldsymbol{H})$ upper bound ellipsoid, $F = \text{ELL}(\delta^{\text{non}}, \delta^*, \boldsymbol{X'RX})$ feasible ellipsoid, H_0 ... diffuse 90% HPD-intervall (classical confidence region)

Example 4.2: Consumption Function in Switzerland

As in example 4.1 we want to find out about the effects of the oil shock for the consumption pattern in Switzerland. In particular we are interested if the data are consistent with the hypothesis that there was no oil shock effect. As before we set the prior means of the shift parameters to zero, but exclude all orthodox priors, i.e. we bound the prior precision (covariance) matrices away from the zero location. The upper bound precision matrix (lower bound variance matrix) is again set to the identity matrix: $\boldsymbol{Q}_0 = \boldsymbol{I}_2$. This means we exclude all priors which are to sharp around the prior location, i.e. assign at least 46.5% to the unit $t\sigma$-square $(-1, 1) \times (-1, 1)$.

As one can see from Table 4.2.a, the classical data evidence is not strong about the shifting slope parameter, but from is more conclusive for the intercept. This weak data evidence transforms in Fig. 4.3 into a large 90% HPD interval which intersects the coordiante axis. With the bounded prior information we can conclude that there was a downward shift in the (simple) consumption function for the posterior means of the shift parameters because the defining parameters,

Table 4.2.a. Regression estimates for Switzerland: Classical summary

	Dependent Variable: Consumption			
Variable Name	Coefficient	Std. Err. Estimate	t Statistic	Prob > t
Constant	1.882	0.456	4.128	0.000
GNP	0.428	0.090	4.734	0.000
Dummy1	-1.001	0.600	-1.667	0.104
Dum*GNP	0.006	0.153	0.039	0.969

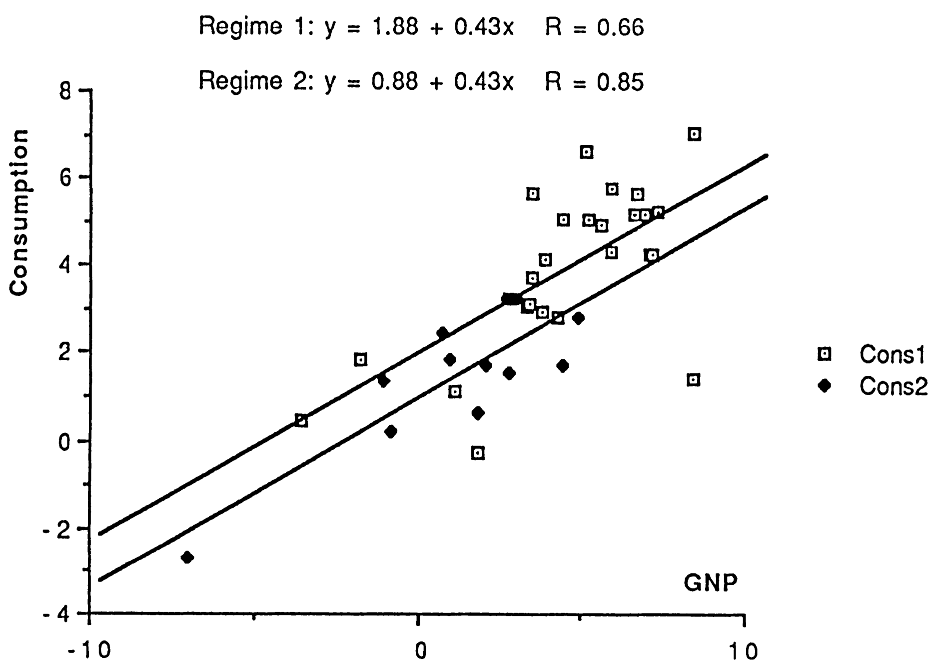

Fig. 4.3. Consumption function in Switzerland before and after the oil shock 1975

$\delta^{\text{non}} = (-1.58, 0.14)$ and $\delta_{\text{II}} = (-1.24, 0.09)$, lie close together. But this small F^u ellipsoid is deceptive, because the corresponding HiFi region remains large. Even if we exclude these strong prior views, then the upper bounded HiFi^u-region still intersects the coordinate axes, because the HiFi region has to be larger than the HPD region. This means that the robust Bayesian inference is quite fragile for a shift in the consumption function. Therefore we conclude that there exists prior views which can show that there was a shift in the consumption function and on the

Table 4.2.b. EBA and HiFi-regions for the shift parameters

bounds / sets	Constant lower	Constant upper	slope lower	slope upper
F	-1.61	0.03	-0.11	0.26
F^u	-1.58	-1.24	0.09	0.15
HPD(α =.9)	-3.61	0.45	-.32	0.60
HiFi(α =.9)	-3.63	0.88	0.44	0.67

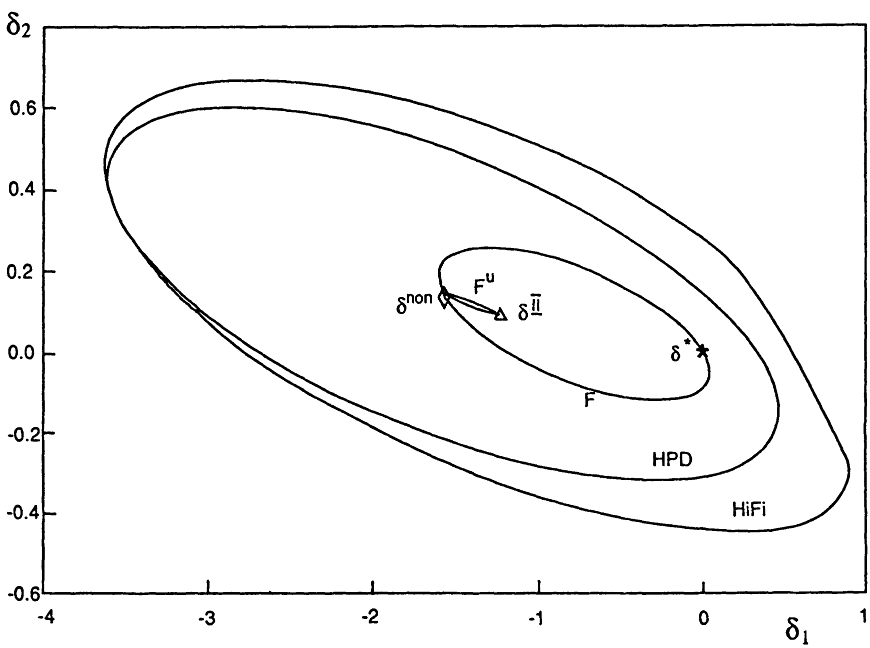

Fig. 4.4. Robust Bayesian summaries for structural change in Switzerland. ◇ ... δ^{non} ... (left) point on the F-ellipsoid; △ ... δ_{II} (right) point on the F-ellipsoid; * ... prior location $\delta^* = (0, 0)$; $F^u = \text{ELL}(\delta^{\text{non}}, \delta_{\text{II}}, H)$ upper bound ellipsoid, $F = \text{ELL}(\delta^{\text{non}}, \delta^*, X'RX)$ feasible ellipsoid, H_0 ... diffuse 90% HPD-intervall (classical confidence region)

other side there might be other prior views which lead to the conclusions that there was no change in the consumption function at all.

Note that judging a possible shift in the regression by HiFi regions can be viewed as an approximate robust Bayes test. This is similar to the usual procedure that HPD (or confidence) intervals represents sets of hypotheses which cannot be rejected. A full Bayesian test treatment would have to take into account prior probabilities for the hypotheses.

The extreme bounds for the robust Bayesian summaries are given in Table 4.2.b. They convey in general much less information than the graphical summary in Fig. 4.4.

4 Conclusions

The approach has shown that the ordinary Bayes analysis of the linear model can be extended to the case of a change in the regime during the observation period. The robust analysis allows to judge the change parameters from different priori views. As the examples for the simple consumtion function in Switzerland and Austria show, both countries react quite differently to the oil shock in 1975. By excluding orthodox or too sharp prior densities we find conclusive robust Bayesian evidence that the oil shock has shifted the Austrian cosumption function, but not necessarily the Swiss one.

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Appendix: Derivation of the Posterior Mean of δ:

By multiplying the joint likelihood function $l(\beta, \delta | y)$ in (2.4) with the prior density of β we find for the joint density $p(\beta, \delta, y)$ a normal kernel:

$$l(\beta, \delta | y) p(\beta) \propto \exp(-g/2) \tag{A1}$$

with g in the exponent given as the sum of three quadratic forms in β:

$$g = \sigma_1 y_1' y_1 - 2\sigma_1 y_1' X_1 \beta + \beta' X_1' X_1 \beta \sigma_1 + \sigma_2 (y_2 - X_2 \delta)'(y_2 - X_2 \delta) \tag{A2}$$

$$-2\sigma_2 (y_2 - X_2 \delta)' X_2 \beta + \sigma_2 \beta' X_2' X_2 \sigma_2 \beta + \beta' \boldsymbol{P}^* \beta - 2\beta' \boldsymbol{P}^* \beta^* + \beta^{*\prime} \boldsymbol{P}^* \beta^* .$$

Completing the quadratic form in β we find for g the expression

$$g=(\beta-\beta^{**})'\boldsymbol{P}^{**}(\beta-\beta^{**})+c, \tag{A3}$$

where the posteriori parameters are given by the posteriori precision P^{**}:

$$\boldsymbol{P}^{**}=\sigma_1\boldsymbol{X}_1'\boldsymbol{X}_1+\boldsymbol{P}^*+\sigma_2\boldsymbol{X}_2'\boldsymbol{X}_2 \tag{A4a}$$

and the posterior mean β^{**}:

$$\beta^{**}=\boldsymbol{P}^{**-1}(\sigma_1\boldsymbol{X}_1'\boldsymbol{y}_1+\sigma_2\boldsymbol{X}_2'(\boldsymbol{y}_2-\boldsymbol{X}_2\delta)+\boldsymbol{P}^*\beta^*) \tag{A4b}$$

and the constant c depends on the shift parameter δ:

$$c=-\beta^{**\prime}\boldsymbol{P}^{**}\beta^{**}+\sigma_1\boldsymbol{y}_1'\boldsymbol{y}_1+\sigma_2(\boldsymbol{y}_2-\boldsymbol{X}_2\delta)'(\boldsymbol{y}_2-\boldsymbol{X}_2\delta)+\beta^{*\prime}\boldsymbol{P}^*\beta^*. \tag{A4c}$$

Integrating out β in (A1) we find now the likelihood function of the shift parameter δ as:

$$l(\delta)\propto\exp\left(-\frac{1}{2}(\delta-\phi)'\Psi(\delta-\phi)\right). \tag{A5}$$

Now Ψ is given in (2.9) and ϕ by

$$\phi=(\boldsymbol{X}_2'\Psi\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2'\Psi(\boldsymbol{y}_2-\phi). \tag{A6}$$

Since the marginal likelihood function of δ has the form of the usual normal density we can multiply it with the normal prior density of δ given in (2.5) and following the usual algebra we finally get the result (2.7) and (2.8).

References

Becker RA, Chambers JM (1984) An interactive environment for data analysis and graphics. Wadswoth, Belmont CA

Berger JO (1984) The robust Bayesian viewpont. In: Kadane J (ed) Robustness in Bayesian statistics. North-Holland, pp 63–144

Broemling LD, Tsurumi H (1987) Econometrics and structural change. Marcel Dekker, NY

Chamberlain G, Leamer EE (1976) Matrix weighted averages and posterior bounds. J Roy Stat Soc B 38:73–84

Ilmakunnas P, Tsurumi H (1984) Testing for parameter shifts in a regression model with two regimes of autocorrelated errors. Economic Studies Quarterly 35:46–54

Leamer EE (1978) Specification searches. Wiley, NY

Leamer EE (1982) Sets of posterior means with bounded variance priors. Econometrica 50:725–736

Poetzelberger K (1987) HPD-regions for the linear model. In: Viertl R (ed) Probability and Bayesian statistics. Plenum, NY, pp 395–402

Polasek W (1984) Multivariate regression systems: estimation and sensitivity analysis for two-dimensional data. In: Kadane J (ed) Robustness in Bayesian statistics. North-Holland, pp 229–309

Polasek W, Poetzelberger K (1987) Robust HPD-regions in Bayesian regression models. Inst. f. Statistics and Econometrics, Basel, mimeo

Salazar D, Broemling LD, Chi A (1981) Parameter changes in a regression model with autocorrelated errors. Comm in Statistics A10:1751–1758

Smith AFM (1977) A Bayesian analysis of some time-varying model. In: Barra JK et al (eds) Recent developments in statistics. North-Holland, Amsterdam

Tsurumi H, Sheflin N (1984) Bayesian tests of a parameter shift under heteroscedasdicity: weighted-*t* vs. double-*t* approaches. Communications in Statistics A13:1003–1013

The Stability Assumption in Tests of Causality Between Money and Income

By H. Lütkepohl[1]

Abstract: This note argues that structural stability is an important condition for tests of Granger-causality. Despite this fact the standard causality tests are sometimes applied to data for which structural stability cannot be assumed a priori. Therefore the stability of GNP/M1 systems of the U.S., Canada, and West Germany in the aftermath of the 1973/74 oil crisis is analyzed using formal statistical tests. Prediction tests are particularly useful for that purpose. The stability of the model for Canadian data is rejected whereas stability is not rejected for the U.S. and West Germany.

1 Introduction

In recent years the Granger-causal relationship between money and income has been discussed in a large number of articles for various periods and countries (e.g., Sims 1972; Williams, Goodhart, and Gowland 1976; Ciccolo 1978; Hsiao 1979a, b, 1981; DeReyes, Starleaf and Wang 1980; Thornton and Batten 1985 to list just a few). The results of the various tests and the conclusions drawn for the money-income relationship differ considerably in some of the studies. Therefore the limitations of the tests and the methodology on which the tests are based have been investigated. For example, measurement errors (Newbold 1978; Schwert 1979), seasonal adjustment (Geweke 1982), data transformations (Feige and Pearce 1979), omitted variables (Lütkepohl 1982), and lag length selection (Thornton and Batten 1985) may have an impact on the outcome of the tests. A further problem will be considered in the following.

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One of the basic assumptions on which many of the tests rely is the stationarity of the money-income system. The stationarity assumption excludes trends, certain seasonal components and structural instabilities in the sample period. While trends and seasonal terms are usually taken care of by data transformations or inclusion of seasonal dummies and/or time trends in the model, the possibility of structural instability has not been allowed for by some authors. On the other hand, recent studies suggest that in economic systems the stability assumption may be problematic for the period after World War II. In particular, there is some evidence that the oil price shock in 1973/74 has caused substantial turbulence in some economies (e.g., Darby 1982; Hamilton 1983; Burbidge and Harrison 1984). In several studies data from that period have been used in testing for Granger-causality, without precautions for structural instabilities. Therefore the reliability of these tests may be questionable if indeed structural instabilities can be detected in the data series used. The purpose of this study is to look into the structural stability of some time series used in causality tests without adjustments for structural change.

We acknowledge that there are studies where structural change is allowed for. Moreover, in some investigations only data prior to the 1973/74 oil crisis have been used. The focus in this study is on series that implicitly have been assumed stationary although they cover periods before and after the 1973 oil price shock.

The structure of the remainder of the paper is as follows. In the next section some aspects of the concept of Granger-causality will be reviewed briefly and the stationarity tests will be explained. They are based on predictions and are therefore in line with Granger's causality concept. In Section 3 the stationarity of some time series that have been used in causality analyses will be investigated. It turns out that stationarity of some of the series is indeed rejected by the tests. Conclusions are presented in the last section.

2 Granger-Causality and Stability Tests

Granger-causality between two variables y_t and x_t is often considered in a bivariate system with autoregressive (AR) reduced form

$$y_t = \nu_1 + a_{11}(L)y_{t-1} + a_{12}(L)x_{t-1} + u_{1t} \tag{1a}$$

$$x_t = \nu_2 + a_{21}(L)y_{t-1} + a_{22}(L)x_{t-1} + u_{2t} \tag{1b}$$

where v_1 and v_2 are intercept terms, $\underline{u}_t = (u_{1t}, u_{2t})'$ is bivariate white noise with covariance matrix $E(\underline{u}_t \underline{u}_t') = \Sigma_{\underline{u}}$ and $\underline{u}_t$ is independent of $\underline{u}_s$ for $s \neq t$. Furthermore, the

$$a_{ij}(L) = \sum_{n=0}^{\infty} a_{ij,n} L^n$$

are polynomials in the lag operator L of possibly infinite order and the lag operator is defined such that $L^n y_t = y_{t-n}$.

If the system (1) is stationary and contains all relevant information y_t is not Granger-caused by x_t if and only if $a_{12}(L) \equiv 0$ and x_t is not Granger-caused by y_t if and only if $a_{21}(L) \equiv 0$. Various tests of these restrictions have been proposed in the literature. They are based on the assumption that the system (1) is stationary.

As mentioned in the introduction, stationarity of (1) requires that there are no trends, nonstationary seasonal cycles or structural changes in the series x_t and y_t. To remove trends and seasonal components initial data transformations such as seasonal adjustment and differencing are sometimes used. Alternatively time trends and/or seasonal dummies may be included in the system (1). We will focus on structural instabilities in the following.

To demonstrate that such instabilities may indeed have a substantial impact on the outcome of causality tests we have conducted a small Monte Carlo experiment. We have generated 1,000 realizations of the bivariate Gaussian AR(1) process

$$y_t = v_1 + 0.5 y_{t-1} + 0.5 x_{t-1} + u_{1t}, \tag{2a}$$

$$\Sigma_{\underline{u}} = I_2,$$

$$x_t = v_2 + 0 y_{t-1} + 0.4 x_{t-1} + u_{2t}, \tag{2b}$$

with $v_1 = v_2 = 0$ for $t = 0, 1, \ldots, 100$. The equation errors u_{1t} and u_{2t} are independent standard normal variates generated by a NAG library subroutine. We have fitted unrestricted vector AR(1) models to the system (2) by LS estimation for each separate equation. The first value for each variable ($t = 0$) was used as presample value in the estimation. Note that (2) is a system with Granger-causality from x to y and no causality from y to x. A test for Granger-noncausality from y to x in this simple system may be based on the t-ratio of the coefficient of y_{t-1} in (2b). In the 1,000 replications of the experiment the absolute value of this ratio exceeded 1.96 (the critical value of an asymptotic 5% level test) in 58 cases. Thus, noncausality from y to x is rejected in 5.8% of the replications under the present ideal conditions with no structural change. This reflects that the size of the test for this process and sample size is about right.

We have repeated the experiment with $v_1 = v_2 = 0$ for $t = 0, \ldots, 50$ and $v_1 = v_2 = 1$ for $t = 51, \ldots, 100$. Thus, now there is a structural change after period $t = 50$. In this case noncausality from y to x was rejected in 719 replications. In other words, causality from y to x is incorrectly accepted in almost 72% of the cases. Consequently the structural change has a remarkable impact on the test.

Ashley, Granger, and Schmalensee (1980) emphasize that Granger's concept of causality is connected with out-of-sample prediction. Therefore it makes sense to base the stationarity tests on out-of-sample predictions. For that purpose the original sample is partitioned. The first part is used for estimating a time series model which is then used for predicting the second part of the sample. If the predictions deviate considerably from the actually observed values the stationarity hypothesis is rejected. In other words, the data in the two subsamples are assumed to be generated by different processes, if the model for the first subsample cannot predict the second subsample with the expected precision.

To explain the idea behind the tests used below we denote the optimal forecast of a K-dimensional stationary process $\underline{y}_t$, h periods into the future, by $\underline{y}_t(h)$ and the corresponding vector of forecast errors by $\underline{e}(h) = \underline{y}_{t+h} - \underline{y}_t(h)$. If $\underline{y}_t$ is Gaussian (normally distributed) $\underline{e}(h)$ is also normally distributed with mean (vector) zero and the variance-covariance matrix is the forecast mean square error (MSE) matrix, say $\underline{\Sigma}(h)$. In other words, $\underline{e}(h) \sim N(0, \underline{\Sigma}(h))$ and consequently $t(h) = \underline{e}(h)' \underline{\Sigma}(h)^{-1} \underline{e}(h)$ has a central χ^2 distribution with K degrees of freedom if the null hypothesis of no structural change is true. This way, a sequence of statistics is obtained for forecast horizons $h = 1, 2, \ldots$ that can be used to test whether the forecast error is in agreement with the stationarity hypothesis.

Alternatively the $(Kh \times 1)$ vector of forecast errors $\underline{f}(h) = (\underline{e}(1)', \ldots, \underline{e}(h)')'$ may be considered. Under the aforementioned assumptions this vector has a multivariate normal distribution with zero mean vector and covariance or MSE matrix $\underline{\Sigma}(h) = E[\underline{f}(h)\underline{f}(h)']$, say. Thus, $\lambda(h) = \underline{f}(h)' \underline{\Sigma}(h)^{-1} \underline{f}(h)$ has a central χ^2 distribution with Kh degrees of freedom. The statistic $\lambda(h)$ can be used to check whether the observed values for h postsample periods are in agreement with the stationarity assumption. For $h = 1$ the tests based on $t(1)$ and on $\lambda(1)$ are equivalent. However, for $h > 1$ using both tests is useful because they have different power against different alternatives. A more detailed discussion of this topic can be found in Lütkepohl (1989).

It may be worth noting that these tests are in particular sensitive to increases in the variability (heteroskedasticity) of the underlying process. For the present purpose this is a valuable property since homoskedasticity is a prerequisite of stationarity and is assumed in causality studies.

The stationarity of the system (1) implies stationarity of the individual series y_t and x_t and the existence of individual AR representations, say

$$y_t = \eta_1 + \beta_1(L)y_{t-1} + e_{1t}, \tag{3a}$$

$$x_t = \eta_2 + \beta_2(L)x_{t-1} + e_{2t}, \tag{3b}$$

where the η_i are intercept terms,

$$\beta_i(L) = \sum_{n=0}^{\infty} \beta_{i,n}L^n, \quad i = 1, 2,$$

and the e_{it} are univariate white noise processes (Lütkepohl 1987). Note, however, that $\underset{\sim}{e}_t = (e_{1t}, e_{2t})'$ will not be bivariate white noise in general. If any of the two univariate processes in (3) is nonstationary the same will hold for the bivariate system (1). Thus, a stationarity test of (1) may be conducted either by applying the aforementioned tests to the bivariate system or by testing the stationarity of the individual series (3a/b). If stationarity is rejected for one of the univariate processes, stationarity of (1) is also rejected. We will apply the prediction tests based on $t(h)$ and $\lambda(h)$ to bivariate ($K=2$) and univariate ($K=1$) series since univariate and multivariate tests have different power against different alternatives (see Lütkepohl 1989 for details).

Of course, in practice the forecasts and hence the forecast errors and MSEs are based on estimated processes. In the following section only finite order AR processes will be fitted and the tests will be based on AR models chosen by the three model selection criteria AIC, HQ, and SC (see Judge et al. 1985, Sections 7.5.2 and 16.6.1a). These criteria have been used in various studies and some other criteria are very similar. The SC criterion is the most parsimonious criterion and always chooses the smallest order whereas AIC chooses the greatest order and HQ an order in between. Lütkepohl (1988) has shown for the univariate case that using such a procedure is justified even if the actual data generation process is not a finite order AR process, provided the $t(h)$ and $\lambda(h)$ statistics are appropriately modified and used in conjunction with critical values from F rather than χ^2 distributions. As suggested by Lütkepohl (1989) the statistics $t(h)$ and $\lambda(h)$ will be multiplied by factors $T/(T+Kp+1)K$ and $T/(T+Kp+1)hK$, respectively, where p is the order of the AR model used for forecasting and T is the sample size used for estimation. These correction factors follow from asymptotic approximations of the forecast MSEs that take into account that estimated rather than known processes are used.

Table 1. Results of Stability Tests for Quarterly, Seasonally Adjusted U.S. Data: Estimation Period 1960.I–1973.II

		forecast	univariate models				bivariate models	
		horizon	$\Delta \ell n$GNP			$\Delta \ell n$M1		
test	quarter	h	AR(0) (SC)	AR(1) (HQ)	AR(7) (AIC)	AR(1) (AIC,HQ,SC)	AR(0) (HQ,SC)	AR(6) (AIC)
t	1973.III	1	.28	.28	.07	3.07	1.44	1.74
	IV	2	1.40	1.34	.07	.72	.85	.02
	1974.I	3	.69	.84	1.46	.10	.78	.49
	II	4	.30	.24	1.38	.42	.59	1.29
	III	5	.16	.12	.09	.56	.57	1.50
	IV	6	.06	.10	1.27	.03	.10	.47
	1975.I	7	3.24	3.63	3.39	.17	1.78	1.47
	II	8	1.85	1.75	1.28	1.94	1.35	.31
	III	9	7.94**	7.90**	3.86	.25	5.99**	3.33*
	IV	10	.19	.14	.05	.36	.45	.21
λ	1973.III	1	.28	.28	.07	3.07	1.44	1.74
	IV	2	.84	.68	.06	2.65	1.15	1.05
	1974.I	3	.79	1.22	.69	1.77	1.02	.85
	II	4	.67	1.14	1.09	1.49	.92	1.12
	III	5	.57	.92	.92	1.25	.85	1.52
	IV	6	.48	.81	1.02	1.08	.72	1.41
	1975.I	7	.87	1.23	1.22	.96	.87	1.48
	II	8	1.00	1.75	1.25	1.17	.93	1.37
	III	9	1.77	2.24*	2.19*	1.16	1.49	1.90*
	IV	10	1.61	2.09*	1.97	1.07	1.39	1.77

* Significant at 5% level.
** Significant at 1% level.

3 Empirical Results

This section discusses the stability of money and income models of the U.S., Canada, and West Germany after the first oil crisis in late 1973. For all three countries data for the period 1970–1975 have been used in previous causality tests without precautions for possible structural changes.

3.1 Results for the U.S.

Examples of studies for the U.S. in which causality tests have been based on data covering the time of the first oil crisis include Thornton and Batten (1985), DyReyes, Starleaf, and Wang (1980) (DSW) and Hsiao (1979a). They use quarterly data for GNP and M1 for 1962.II–1982.III, 1950.I–1975.III, and 1947.I–1977.III, respectively. While seasonally adjusted data were used in the last two studies, Thornton and Batten are not precise about the data used. We will use quarterly, seasonally adjusted, nominal GNP and M1 data for the period 1960.I–1975.IV as published by the OECD (Historical Statistics 1960–1979). The tests are applied to first differences of the logarithms of the original data. The estimation and specification period is 1960.I–1973.II and forecasts are computed for 1973.III–1975.IV. The maximum lag length used in the AR model specification procedure is eight. The resulting values of the test statistics are given in Table 1. Note that the $t(h)$ and $\lambda(h)$ statistics in the table have approximate F distributions under the null hypothesis of no structural change with $[K, 52-(K+1)p]$ and $[Kh, 52-(K+1)p]$ degrees of freedom, respectively. Here $K=1$ for the univariate tests and $K=2$ for tests based on the bivariate models. Of course, the test values in the table are not independent.

Significant test values are obtained only for the second half of 1975. In other words, the stability hypothesis is rejected only for the end of 1975 and not for the period immediately following the oil price shock in late 1973. The instability seems to arise from an unusual value of $\Delta \ln$ GNP in 1975.III. Of course, such a result may occur by chance, that is, the rejection of the null hypothesis may be a type I error. Therefore the overall conclusion is that the tests do not strongly support the hypothesis of a structural change caused by the 1973/74 oil crisis in the money-income system of the U.S.

Table 2. Results of Stability Tests for Quarterly, Seasonally Adjusted Canadian Data: Estimation Period 1955.I–1973.II

		forecast	univariate models			bivariate model
		horizon	$\Delta \ell n$GNP		$\Delta \ell n$M1	
test	quarter	h	AR(0) (HQ,SC)	AR(3) (AIC)	AR(1) (AIC,HQ,SC)	AR(1) (AIC,HQ,SC)
t	1973.III	1	1.07	.00	.21	.28
	IV	2	10.05**	4.55*	.27	5.00**
	1974.I	3	5.51*	3.24	.92	2.49
	II	4	2.12	.66	3.52	2.13
	III	5	.87	.17	3.68	2.76
	IV	6	.02	.03	.33	.21
	1975.I	7	.08	.00	6.82*	3.35*
	II	8	.30	.07	1.35	.69
	III	9	2.71	1.82	2.42	2.00
	IV	10	.59	.30	7.25**	3.53*
λ	1973.III	1	1.07	.00	.21	.28
	IV	2	5.56**	2.29	.43	2.61*
	1974.I	3	5.54**	2.54	.92	2.62*
	II	4	4.69**	1.94	1.34	2.27*
	III	5	3.92**	1.64	3.20*	3.04**
	IV	6	3.27**	1.52	2.70*	2.56**
	1975.I	7	2.82*	1.33	3.90**	2.96**
	II	8	2.50*	1.17	3.41**	2.59**
	III	9	2.53*	1.30	3.18**	2.45**
	IV	10	2.33*	1.19	3.35**	2.41**

* Significant at 5% level.
** Significant at 1% level.

3.2 Results for Canada

Studies using Canadian M1 and GNP data for 1973/74 in tests for causality include DSW and Hsiao (1979b, 1981). The data used in this section are seasonally adjusted, quarterly figures from 1955.I–1975.IV as published in the Appendix of Hsiao (1979b). Again first differences of the logarithms of the original data are used. The estimation period is 1955.I–1973.II and the test values are computed for 1973.III–1975.IV as in the U.S. case. The maximum AR order used in the AR order selection procedures is 14. Here we have used a higher maximum AR order than in the previous section because more data are available. The results of the stability tests are shown in Table 2. The degrees of freedom of the F distributions corresponding to the $t(h)$ and $\lambda(h)$ tests are $[K, 72-(K+1)p]$ and $[hK, 72-(K+1)p]$ respectively. Obviously the stability hypothesis is quite clearly rejected in this case by the univariate as well as the bivariate tests.

Since one purpose of the study is do determine whether a structural instability may have had in impact on the causality tests we have performed such tests for the period 1955.I–1973.II and 1955.I–1975.IV. The tests are standard F tests of the null hypotheses $\alpha_{12}(L)\equiv 0$ (M1 does not cause GNP) and $\alpha_{21}(L)\equiv 0$ (GNP does not cause M1). Since AIC, HQ, and SC have all chosen a bivariate AR(1) for the period 1955.I–1973.II we have based the tests on AR(1) models. For the period 1955.I–1973.II we get

$$\underset{}{\Delta \ln \mathrm{GNP}_t =} \underset{(5.38)}{0.016} + \underset{(0.61)}{0.072}\, \Delta \ln \mathrm{GNP}_{t-1} + \underset{(2.12)}{0.207}\, \Delta \ln \mathrm{M1}_{t-1} + \hat{u}_{1t} \qquad (4a)$$

$$\Delta \ln \mathrm{M1}_t = \underset{(1.40)}{0.004} + \underset{(1.32)}{0.172}\, \Delta \ln \mathrm{GNP}_{t-1} + \underset{(4.31)}{0.462}\, \Delta \ln \mathrm{M1}_{t-1} + \hat{u}_{2t} \qquad (4b)$$

and for 1955.I–1975.IV we get

$$\Delta \ln \mathrm{GNP}_t = \underset{(4.87)}{0.014} + \underset{(2.02)}{0.218}\, \Delta \ln \mathrm{GNP}_{t-1} + \underset{(2.44)}{0.215}\, \Delta \ln \mathrm{M1}_{t-1} + \hat{u}_{1t} \qquad (5a)$$

$$\Delta \ln \mathrm{M1}_t = \underset{(1.39)}{0.005} + \underset{(1.83)}{0.237}\, \Delta \ln \mathrm{GNP}_{t-1} + \underset{(3.81)}{0.405}\, \Delta \ln \mathrm{M1}_{t-1} + \hat{u}_{2t} \qquad (5b)$$

Here the numbers in parentheses are asymptotic t statistics. t tests are equivalent to F tests for the present AR(1) models. Hence, noncausality from GNP to M1 can be rejected at a 10% level of significance in (5) whereas the same is not true in (4). Thus, applying the test to the data from the period prior to 1973.II one would clearly conclude that GNP is not likely to be causal for M1 while the same conclusion is not reached from a 10% level test based on data up to 1975. In line with the simulations reported in Section 2, this example demonstrates that not taking into account possible structural changes may indeed have a significant effect on the conclusions drawn from causality tests. Note that other causality tests may lead to different results. However, if the foregoing strategy is used, different conclusions may be obtained for the two periods.

3.3 Results for West Germany

West German data were also considered by DSW. We use quarterly, seasonally adjusted, nominal GNP and M1 for the period 1960.I–1975.IV as published by the Deutsche Bundesbank. Again first differences of logarithms are used. As for the U.S. the estimation period is 1960.I–1973.II and test values are computed for 1973.III–1975.IV. Using a maximum AR order of eight in the search procedure all three criteria AIC, HQ, and SC choose $p=0$ as optimal AR order for the bivariate system as well as the univariate series. The resulting test values are given in Table 3. In this case the degrees of freedom of the F distributions corresponding to the t and λ tests are $(K, 52)$ and $(hK, 52)$, respectively. None of the test values is significant at the 1% level and those significant at the 5% level may be spurious. This view is supported by the results of Lütkepohl (1988) where it was found that, for the univariate case, the tests tend to reject the null hypothesis, when it is true, more often than is indicated by the significance level chosen. Consequently, there is no overwhelming evidence supporting the hypothesis of structural change.

4 Conclusions

This note has pointed out that structural stability of the system under investigation is a crucial prerequisite for Granger-causality tests. Since the oil crisis in 1973/74 has been blamed for some turbulence in major industrialized economies we have tested the structural stability of GNP/M1 systems for the U.S., Canada, and West Germany. For all three countries data covering the critical 1973/74 period have been used in causality tests by some authors without taking into account possible structural changes. For the U.S. and West Germany structural stability is not clearly

Table 3. Results of Stability Tests for Quarterly, Seasonally Adjusted West German Data: Estimation Period 1960.I–1973.II

test	quarter	forecast horizon h	univariate models: Δℓn GNP AR(0) (AIC,HQ,SC)	univariate models: Δℓn M1 AR(0) (AIC,HQ,SC)	bivariate model AR(0) (AIC,HQ,SC)
t	1973.III	1	.01	4.95*	2.79
	IV	2	.06	.64	.32
	1974.I	3	.13	.27	.15
	II	4	.18	.13	.25
	III	5	.05	.03	.03
	IV	6	1.13	3.33	3.47*
	1975.I	7	3.24	.03	2.04
	II	8	.06	1.98	1.04
	III	9	.18	3.24	2.34
	IV	10	.01	.07	.03
λ	1973.III	1	.01	4.95*	2.79
	IV	2	.03	2.79	1.56
	1974.I	3	.07	1.95	1.09
	II	4	.09	1.50	.88
	III	5	.09	1.20	.71
	IV	6	.26	1.56	1.17
	1975.I	7	.69	1.34	1.29
	II	8	.61	1.42	1.26
	III	9	.56	1.62	1.38
	IV	10	.50	1.47	1.25

* Significant at 5% level.

rejected so that this potential source of error in a causality test may not be a serious one. On the other hand, stability is rejected for Canada. It is shown that not taking into account the instability may give rise to misleading conclusions regarding the causal structure of the system. As a consequence for applied work we suggest that stability tests be conducted routinely prior to causality investigations if the structural stability is in doubt.

References

Ashley R, Granger CWJ, Schmalensee R (1980) Advertising and aggregate consumption: an analysis of causality. Econometrica 48:1149–1167

Burbidge J, Harrison A (1984) Testing for the effects of oil-price rises using vector autoregressions. International Economic Review 25:459–484

Ciccolo JH Jr (1978) Money, equity values, and income. Journal of Money, Credit and Banking 10:46–64

Darby MR (1982) The price of oil and world inflation and recession. American Economic Review 72:738–751

DyReyes FR Jr, Starleaf DR, Wang GH (1980) Tests of the direction of causation between money and income in six countries. Southern Economic Journal 47:477–487

Feige EL, Pearce DK (1979) The casual causal relationship between money and income: some caveats for time series analysis. The Review of Economics and Statistics 61:521–533

Geweke J (1982) Causality, exogeneity and inference. In: Hildenbrand W (ed) Advances in econometrics. Cambridge University Press, New York, pp 209–235

Hamilton JD (1983) Oil and the macroeconomy since World War II. Journal of Political Economy 91:228–248

Hsiao C (1979a) Causality tests in econometrics. Journal of Economic Dynamics and Control 1:321–346

Hsiao C (1979b) Autoregressive modeling of Canadian money and income data. Journal of the American Statistical Association 74:553–560

Hsiao C (1981) Autoregressive modelling and money-income causality detection. Journal of Monetary Economics 7:85–106

Judge GG, Griffiths WE, Hill RC, Lütkepohl H, Lee T-C (1985) The theory and practice of econometrics, second edition. John Wiley, New York

Lütkepohl H (1982) Non-causality due to omitted variables. Journal of Econometrics 19:367–378

Lütkepohl H (1987) Forecasting aggregated vector ARMA processes. Springer, Berlin

Lütkepohl H (1988) Prediction tests for structural stability. Journal of Econometrics 39:267–296

Lütkepohl H (1989) Prediction tests for structural stability of multiple time series. Journal of Business & Economic Statistics (forthcoming)

Newbold P (1978) Feedback induced by measurement errors. International Economic Review 19:787–791

Schwert GW (1979) Tests of causality: the message in the innovations. In: Brunner K, Meltzer AH (eds) Three aspects of policymaking: knowledge, data and institutions, Carnegie Rochester Conference Series on Public Policy, Vol 10. North-Holland, Amsterdam, pp 55–96

Sims CA (1972) Money, income and causality. American Economic Review 62:540–555

Thornton DL, Batten DS (1985) Lag-length selection and tests of Granger causality between money and income. Journal of Money, Credit and Banking 17:164–178

Williams D, Goodhart CAE, Gowland DH (1976) Money, income and causality: the U.K. experience. American Economic Review 66:417–423

A Sequential Approach to Testing for Structural Change in Econometric Models

By G. D. A. Phillips[1] and B. P. M. McCabe[2]

Summary. The paper shows that the sequential approach to testing econometric models, particularly testing for structural change, is both feasible and potentially very useful. In fact, this paper makes clear the possibility of using the sequential approach as suggested by Dhrymes et al. (1972) and shows that the statistical dependence between successive tests can be overcome in some cases.

1 Introduction

Modern econometric practice advocates that a given specification should be subject to a rigorous testing procedure and it is now becoming routine to test for misspecifications such as omitted variables, serially correlated disturbances, structural change, heteroscedasticity and incorrect functional form. This kind of intensive misspecification testing leads to problems of distortions in the inference procedures but leading econometricians believe that the importance of carrying out such tests overrides these problems.

While it is important to test econometric models rigorously it is also important to seek to structure the testing procedure in such a way that problems of data mining are minimised. In particular, we seek test procedures to test for the presence of, possibly, several misspecifications simultaneously in such a way that: (a) the overall Type 1 error probability is controlled within acceptable limits, and (b) the test procedure while having good power properties provides some opportunity for detecting individual types of misspecification.

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This paper shows that in some cases these aims may be at least partially achieved when the misspecifications are tested sequentially.

In a well known paper, Dhrymes et al. (1972, p. 299) drew attention to the desirability of using a sequential approach to test for, inter alia, structural change but it was acknowledged that no easy solution to this problem had been identified, a principal stumbling block involving the problem of statistical dependence between successive hypothesis tests. Here we consider a sequential approach to testing for misspecification and we focus, particularly, on the problem of testing for structural change when either serial correlation or heteroscedasticity, or both, may be present. We show that a sequence of independent tests may be based upon well known test statistics for these misspecifications.

2 A Sequential Approach to Testing for Misspecification

In the recent econometric literature, see especially, Mizon (1977), there has been much concern to develop an appropriate strategy for model selection. The practice of selecting models after applying numerous conventional tests of significance has well-recognised deficiencies and to overcome these problems, a search process has been advocated in which tests of specification are conducted on hypotheses within an overall maintained hypothesis which is carefully chosen to be the most general hypothesis likely to be relevant. If a composite hypothesis representing the most restricted model, is tested against the maintained and not rejected, then the position is straighforward but when the restricted model is rejected, one does not know which of the constituent hypotheses are responsible. However, if the hypotheses are nested and uniquely ordered, then when any hypothesis is true all preceding hypotheses in the nest must be true and if any hypothesis is false all succeeding ones must be false. This has the advantage of allowing a composite hypothesis to be tested using a sequential procedure which can determine the hypotheses responsible for the rejection. Mizon notes that the sequential approach has certain optimal power properties in the class of procedures that fix the probabilities of accepting a less restricted hypothesis than the true one. Also, this approach, which is outlined in Anderson (1971), may be extended to non-linear models. An important characteristic of the approach is that the asymptotic distribution of the statistic for testing any hypothesis in the ordered sequence against the less restricted hypothesis immediately preceding it, depends on the validity of all less restricted hypotheses in the sequence but not on that of more restricted hypotheses, and each of these test statistics is asymptotically independent. Thus control over the overall Type 1 error probability is possible. If the significance level for each test is chosen at α_1, then the significance level of the implicit test of the r-th hypothesis is $1-\prod_{i=1}^{r}(1-a_i)$ which is a monotonically non decreasing function of r.

Our concern is with multiple misspecification testing and it is worthwhile to examine a sequential approach particularly if robust procedures are required. In some cases there will be an ordering of nested hypotheses which will permit the development of mutually independent sequential tests and which will ensure that the distribution of the statistic for testing any hypothesis in the ordered sequence against the less restricted hypothesis immediately preceding it depends on the validity of all the less restricted hypotheses in the sequence but not on that of more restricted hypotheses.

An important result in developing independent tests is the Independence Theorem due to Basu (1955) which is noted in Hogg (1961). Broadly, the theorem states that if in a regular estimation problem there exists a boundedly complete set of joint sufficient statistics for m unknown parameters, a necessary and sufficient condition that a statistic Q be stochastically independent of the joint sufficient statistics is that the distribution of Q be free of the unknown parameters, see also Hogg and Craig (1956, p. 219).

Some applications of the Theorem in an econometric context are discussed in Phillips and McCabe (1988).

3 Sequential Testing: Useful Results

In an earlier paper Phillips and McCabe (1983) examined a sequential approach to testing for structural change in a linear regression model where the composite hypothesis includes changes in both the regression coefficients and the disturbance variance. In this case although there is no unique ordering of the constituent hypotheses it is possible to partition the composite hypothesis so that independent test statistics are available for the resulting tests. To see this we assume a linear regression model

$$y=X\beta+\varepsilon \tag{3.1}$$

where y is a $T\times 1$ vector of observations, X is a $T\times k$ matrix of rank k containing observations on k non-stochastic regressor variables, β is a $k\times 1$ vector of unknown parameters and $\varepsilon\sim N(0,\sigma^2 I_T)$. Rewriting (3.1) in the form

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_1 \end{pmatrix} \tag{3.2}$$

where $\varepsilon_i \sim N(0, \sigma_i^2 I_{T_i})$, $i = 1, 2$, and $T_1, T_2 > k$ with $T_1 + T_2 = T$, the structural change hypothesis may be written in the following sequence:

$$H_2 : \sigma_1^2 \neq \sigma_2^2,\ \beta_1 \neq \beta_2$$

$$H_1 : \sigma_1^2 = \sigma_2^2,\ \beta_1 \neq \beta_2$$

$$H_0 : \sigma_1^2 = \sigma_2^2,\ \beta_1 = \beta_2.$$

This ordering is not unique but it has the property that the distribution of the statistic for testing any hypothesis in the ordered sequence against the less restricted hypothesis immediately preceding it does not depend on the validity of more restricted hypotheses and the test statistics are independent under H_0. As a result the overall type 1 error probability can be controlled and the interpretation of a significant test result is straightforward. In fact, in this case, the procedure has the desirable property of yielding a uniformly most powerful invariant test of H_2 against H_0 as noted by Anderson and Mizon (1984).

In practice we shall often wish to combine a structural change test with other misspecification tests, particularly a test for serial correlation, and to extend the above analysis we shall write the model as

$$y = X\beta_1 + Z\Delta\beta + \varepsilon$$

$$\text{where} \quad Z = \begin{pmatrix} 0 \\ X_2 \end{pmatrix} \quad \text{and} \quad \Delta\beta = (\beta_2 - \beta_1). \tag{3.3}$$

It is well known that the Analysis of Covariance (AOC) test for structural change is identical to an F significance test of the coefficients of Z in (3.3). However, it is of interest to note that Basu's Independence Theorem may be invoked to deduce that the AOC test statistic is distributed independently of any misspecification test which is free of β_1, $\Delta\beta$ and σ^2 e.g. LM tests, the Durbin Watson test and various heteroscedasticity tests, when there are no misspecifications.

Here we are particularly concerned to examine a test for serial correlation and to do this we shall put $\beta^* = \begin{pmatrix} \beta_1 \\ \Delta\beta \end{pmatrix}$ and $X^* = (X:Z)$. The least squares estimator of β^* is then given by

$$\hat{\beta}^* = \begin{pmatrix} \hat{\beta}_1 \\ \Delta\hat{\beta} \end{pmatrix} = (X'^* X^*)^{-1} X'^* y$$

$$= \begin{pmatrix} X_1'X_1 + X_2'X_2, & X_2'X_2 \\ X_2'X_2 & X_2'X_2 \end{pmatrix}^{-1} \begin{pmatrix} X_1'y_1 + X_2'y_2 \\ X_2'y_2 \end{pmatrix}.$$

On inverting the above matrix using a well known theorem for inverting a partitioned matrix, we have

$$\begin{pmatrix} \hat{\beta}_1 \\ \Delta\hat{\beta} \end{pmatrix} = \begin{pmatrix} (X_1'X_1)^{-1}X_1'y_1 \\ (X_2'X_2)^{-1}X_2'y_2 - (X_1'X_1)^{-1}X_1'y_1 \end{pmatrix}$$

$$= \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 - \hat{\beta}_1 \end{pmatrix}.$$

It is easy to see that the residual vector is given by

$$\hat{\varepsilon} = \begin{pmatrix} y_1 - X_1\hat{\beta}_1 \\ y_2 - X_2\hat{\beta}_2 \end{pmatrix} = \begin{pmatrix} \hat{\varepsilon}_1 \\ \hat{\varepsilon}_2 \end{pmatrix}$$

where the sub-vectors are those which would be obtained when the two regressions are performed separately.

A test for serial correlation may be performed based upon the residual vector $\hat{\varepsilon}$ which has all the usual properties of a least squares residual vector. The bounds test statistic is

$$d = \frac{\hat{\varepsilon}' A \hat{\varepsilon}}{\hat{\varepsilon}'\hat{\varepsilon}}$$

where A is $T \times T$ and the number of degrees of freedom is $T-2k$. The AOC test is based upon

$$F = \frac{(\hat{\beta}_2 - \hat{\beta}_1)'((X_2'X_2)^{-1} + (X_1'X_1)^{-1})^{-1}(\hat{\beta}_2 - \hat{\beta}_1)/k}{\hat{\varepsilon}'\hat{\varepsilon}/(T-2k)} \tag{3.4}$$

and, given this form of the statistic, it is easy to deduce the independence of F and d under the null hypothesis of serial independence of the disturbance either by invoking the Independence Theorem or by noting that each term of the F ratio is, separately, distributed independently of d.

It is clear that if the following sequence is considered

$$H_2 : \rho \neq 0, \beta_1 \neq \beta_2$$

$$H_1 : \rho = 0, \beta_1 \neq \beta_2$$

$$H_0 : \rho = 0, \beta_1 = \beta_2$$

and the above test statistics are used, the tests are independent under H_0 and the overall type 1 error probability may be controlled exactly. Notice too that, in testing for serial correlation, one does not need to assume that $\beta_1 = \beta_2$.

Suppose now that a test for heteroscedasticity is required. Difficulties arise, though, when tests for serial correlation and heteroscedasticity are included in the same sequence since the null distribution of the usual test statistic for one misspecification is affected by the presence of the other misspecification. In addition, even when neither misspecification is present, the test statistics commonly employed are not independent in small samples.

However, in certain cases it is possible to modify the usual test statistics so that the null distribution of the test for serial correlation may be unaffected by the presence of heteroscedasticity and the test statistics are independent when neither misspecification is present. For example, the heteroscedasticity hypothesis of interest in the context of testing for structural change is given by

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = X\beta + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} \tag{3.5}$$

where $\varepsilon_i - N(0, \sigma_i^2 I_{T_i})$, $i = 1, 2$, with $\sigma_1^2 \neq \sigma_2^2$ and $T_1, T_2 > k$. The null hypothesis is chosen as $H_0 : \sigma_1^2 = \sigma_2^2$ and the appropriate test is the Variance Ratio (VR) test based on

$$F_1 = \frac{RSS_2/(T_2 - k)}{RSS_1/(T_1 - k)} = \frac{\hat{\varepsilon}'_2\hat{\varepsilon}_2/(T_2 - k)}{\hat{\varepsilon}'_1\hat{\varepsilon}_1/(T_1 - k)} \tag{3.6}$$

where $RSS_i = \hat{\varepsilon}'_i\hat{\varepsilon}_i$ is the residual sum of squares from a regression carried out on the corresponding T_i observations, $i = 1, 2$. Note that no reordering of the observations is involved and under H_0, $F_1 \sim F(T_2 - k, T_1 - k)$.

To find an appropriate test for the serial correlation hypothesis that the disturbances are generated by the first order autoregressive process

$$u_t = \rho u_{t-1} + \varepsilon_t, \quad |\rho| < 1, \; t = 1, 2, \ldots, T.$$

we shall consider the statistics

$$d_i = \frac{\hat{\varepsilon}'_i A_i \hat{\varepsilon}_i}{\hat{\varepsilon}'_i \hat{\varepsilon}_i}, \quad i = 1, 2. \tag{3.7}$$

When $\rho = 0$, d_1 and d_2 are each distributed as a Durbin-Watson ratio test and their distributions do not depend on the σ_i^2, $i = 1, 2$. In addition they are both distributed independently of the VR statistic under the overall null hypothesis, i.e. when neither misspecification is present.

It follows, therefore, that we can find a sequential test procedure having the desired characteristics provided that the test for serial correlation is performed first and is based on the d_i, $i = 1, 2$. One possibility is to pool the results of the separate tests and reject the hypothesis of no serial correlation if either test rejects. If this procedure is followed and each test is carried out at the $2^1/_2\%$ level, the overall test size is controlled at 5%. An alternative approach which yields a more powerful test, is to base the test on the LM type statistic

$$d_3 = \frac{T_1}{T} d_1 + \frac{T_2}{T} d_2. \tag{3.8}$$

The null distribution of d_3 is unknown but it can be approximated by the distribution of a β variate with the same mean and variance so that a test based on d_3 may be close to being exact. To examine this a set of Monte Carlo experiments was carried out and a simple version of (3.5) was simulated which included one regressor variable and a constant term. The regressor variable data was generated lognormally and T was chosen as 30 with $T_1 = T_2 = 15$. The experiments are discussed in section 5 but it is appropriate to note now that in several independent runs of 1,000 replications, the estimated rejection probability for a serial correlation test based on (3.8) was always close to the nominal significance level. Thus for the Monte Carlo experiments we may treat the test as being, essentially, exact.

It is of interest that a test of the structural change hypothesis

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} \tag{3.9}$$

based upon the AOC test, may be added to the sequential testing procedure.

The distributions of (3.6) and (3.8) are not affected by the structural change hypothesis and, furthermore, the three test statistics are mutually independent under the overall null hypothesis. A proof of this independence is given in the Appendix.

The sequence of nested hypotheses then takes the form

H_3: $\rho \neq 0$, $\sigma_1^2 \neq \sigma_2^2$, $\beta_1 \neq \beta_2$,

H_2: $\rho = 0$, $\sigma_1^2 \neq \sigma_2^2$, $\beta_1 \neq \beta_2$,

H_1: $\rho = 0$, $\sigma_1^2 = \sigma_2^2$, $\beta_1 \neq \beta_2$,

H_0: $\rho = 0$, $\sigma_1^2 = \sigma_2^2$, $\beta_1 = \beta_2$.

Note that when the individual tests are based on (3.6), (3.8) and the AOC test in (3.4), the distribution of the statistic for testing any hypothesis in the ordered sequence against the less restricted hypothesis immediately preceding it, does not depend on the validity of more restricted hypotheses. The hypotheses are tested in turn until one either accepts H_i, $i = 1, 2, 3$, or one rejects all hypotheses and arrives at H_0. If H_i, $i = 1, 2, 3$, is accepted, it is assumed that a misspecification has been found and the procedure stops. As a consequence, if H_2 is accepted, the structural change hypothesis regarding β is, essentially, untested. However, if $\sigma_1^2 \neq \sigma_2^2$ is regarded as an alternative hypothesis which is of intrinsic rather than merely instrumental interest, the structural change hypothesis could be tested using a Wald test, assuming that $\sigma_1^2 \neq \sigma_2^2$.

4 Non-Sequential Testing for Structural Change

In this section we consider the traditional non-sequential approach to testing for structural change. We suppose that, as in the case discussed in Section 3, tests for serial correlation and heteroscedasticity will also be carried out. The approach is to choose an optimal test for the particular case of interest. Thus the test for serial correlation is based upon the Durbin-Watson statistic using the residuals from the full regression in (3.1) i.e. $e = y - X\hat{\beta}$. The test statistic used is

$$\mathrm{VNR} = \frac{e'Ae}{e'e} \tag{4.1}$$

where A is a $T \times T$ tridiagonal matrix of well-known form. The distribution of VNR is approximated by a β-distribution with the same mean and variance.

A test for heteroscedasticity is based upon the LM statistic proposed by Harrison and McCabe (1977). For the particular type of heteroscedasticity hypothesis under consideration their test is locally best invariant. The test statistic used is:

$$H = \frac{e'Be}{e'e} \tag{4.2}$$

where B is an appropriate selector matrix of order $T \times T$ with T_2 ones and T_1 zeros on its principal diagonal, and zeros elsewhere. Notice that H is a ratio of the last T_2 squared residuals to the sum of squared residuals. Again the β-approximation to the distribution of H is used to determine its critical values.

Finally, a test for changes in the regression coefficients will be based upon the Analysis of Covariance (AOC) test. This is the test which is widely used in practice.

Our non-sequential testing procedure is to conduct all three tests at a nominal 1.7% level. Although the three test statistics are not mutually independent under the overall null i.e. when neither problem is present, the overall test size is close to 5%. In 1,000 replications of a Monte-Carlo experiment discussed in Section 5 the estimated size was 4.7%.

5 Sequential and Non-Sequential Testing for Structural Change: Some Monte Carlo Results

In this section we consider the results of a set of Monte Carlo experiments designed to compare a sequential approach to testing for structural change with the traditional non-sequential approach.

A simple linear regression model of the form

$$y_t = a + \beta x_t + \varepsilon_t, \quad t = 1, 2, \ldots, T,$$

was simulated where data for the explanatory variable x_t was generated lognormally from a distribution in which $\exp x \sim N(1, 1.31)$, and, in addition, $\varepsilon_t \sim N(0, 1)$. For simplicity, the parameters were chosen as $a = \beta = 0$. Serial correlation was introduced into the disturbance term by forming $\varepsilon_t = \rho\varepsilon_{t-1} + u_t$ where ρ was chosen as 0.3, 0.5 or 0.8. Heteroscedasticy was created by choosing $E(\varepsilon_t^2) = 1$, $t = 1, \ldots, 15$, and $E(\varepsilon_t^2) = 2$, $t = 16, \ldots, 30$. Finally, structural change in the regression parameter was introduced by putting $\beta = 0$, $t = 1, \ldots, 15$, and $\beta = 1$, $t = 16, \ldots, 30$.

One thousand replications were employed in each experiment and used to estimate the probability of rejecting the model in the presence of different combinations of misspecifications, for both sequential and nonsequential test procedures.

The results of the study are given in Table 1 in four sections. However, the reader should note some difficulty in comparing these results. The sequential test procedure stops whenever a significant test result is obtained since a respecification of the model is indicated. To continue would involve testing in the presence of a misspecification other than the one to be tested for. Because of this, not all possible misspecifications are tested. The data shown for sequential tests indicate the estimated probability that a rejection will occur at a particular stage of the sequential procedure and the estimated probability that the specification will be rejected at some stage is obtained by lateral summation to yield the column headed $P_r(R)$. Notice that, by its nature, the sequential procedure cannot detect more than one mispecification. On the other hand, with the nonsequential approach, all the mispecification tests are carried out and the estimated probabilities shown refer to individual tests which, in nearly all cases, are conducted in the context of more than one misspecification as indicated by the first three columns. The probability that a specification is rejected following a significant result in at least one of the tests, is given in the final column headed $P_r(R)$. This is not obtained as the lateral summation of the rejection probabilities for the individual tests, however.

Each individual test was carried out at a nominal 1.7% significance level. In the case of the sequential procedure, where the test statistics are independent under the

Table 1. Estimated Probabilities of Rejection in Sequential and Non-sequential Tests: $T = 30$

$T_1 = T_2 = 15$ 1000 replications				Nominal significance level is 1.7% for each test							
				Sequential Tests d_3 → VR → AOC				Non-sequential Tests			
	ρ	$\Delta\sigma^2$	$\Delta\beta$	d_3	VR	AOC	$P_r(R)$	VNR	H	AOC	$P_r(R)$
	0.0	0	0	.016	.014	.017	.047	.016	.014	.017	.047
	0.3	0	0	.171	.014	.081	.266	.245	.016	.092	.287
1	0.5	0	0	.510	.011	.115	.636	.626	.022	.217	.650
	0.8	0	0	.808	.003	.127	.938	.940	.032	.551	.947
	0.0	0	1	.011	.019	.833	.863	.344	.001	.857	.868
	0.3	0	1	.208	.012	.702	.922	.747	.004	.901	.928
2	0.5	0	1	.504	.012	.469	.985	.929	.008	.946	.978
	0.8	0	1	.802	.005	.193	1.00	.998	.010	1.000	1.000
	0.0	1	0	.025	.461	.011	.503	.022	.580	.016	.596
	0.3	1	0	.230	.344	.057	.631	.278	.551	.100	.676
3	0.5	1	0	.495	.203	.074	.772	.598	.488	.197	.801
	0.8	1	0	.796	.063	.088	.947	.934	.386	.512	.958
	0.0	1	1	.034	.455	.237	.726	.139	.302	.397	.647
	0.3	1	1	.230	.348	.240	.818	.486	.281	.520	.783
4	0.5	1	1	.478	.201	.255	.934	.774	.223	.691	.905
	0.8	1	1	.769	.072	.158	.999	.986	.089	.964	.998

Notes

1. ρ is the serial correlation parameter which is *fixed* for each experiment. $\Delta\sigma^2$ is the incremental change in the disturbance variance over the last T_2 observations. In fact $\Delta\sigma^2 = 1$ means that the disturbance variance doubles. $\Delta\beta$ is the incremental change in the parameter β over the last T_2 observations.
2. d_3 refers to a test for serial correlation based on (3.8) where its distribution is approximated by a β variate with the same mean and variance. VNR is the DW test again employing the β approximation. VR and AOC refer to the Variance Ratio and Analysis of Covariance tests, respectively, while H is the Harrison-McCabe LM test for heteroscedasticity.
3. $Pr(R)$ is the probability of rejecting the specification. In the case of non-sequential tests, this is the probability that at least one of the tests rejects.
4. For the sequential tests, the probabilities shown refer to the rejection at that stage of the sequential test procedure. The procedure terminates once a rejection occurs.
5. The nominal overall Type I error probability is $1-(0.983)^3 = 0.05$. This holds to a close approximation in both procedures.

overall null, the Type 1 error probability is $1-(0.983)^3 = 0.05$. In the non-sequential case, the test statistic used to test for serial correlation will not be independent of the H and AOC test statistics under the overall null but the dependence appears to be weak. Consequently, the overall Type 1 error probability of the non-sequential test procedure closely approximates that of the sequential procedure and, for practical purposes, we may assume that they are equal.

The Monte Carlo results are presented in Table 1. The four experiments are intended to examine the robustness or otherwise of individual tests to the presence of more than one misspecification, and to provide some indication of the comparative performance of the sequential and non-sequential test procedures.

The VNR and H tests are known to be more powerful than their counterparts in the sequential procedures, d_3 and VR, and this is demonstrated in experiments 1 and 3. The H test is seen to be non-robust to serial correlation in experiment 3 and to structural change in experiment 4 – compare the first rows of experiments 3 and 4.

In experiments 1 and 2 it seen that the VNR test is very non-robust to structural change. Indeed the VNR test has size 0.344 in the presence of structural change and the AOC test has size 0.551 when $\rho = 0.8$. However the rejection probabilities of both the VNR test in the context of structural change and the AOC test in the presence of serial correlation, are greatly increased.

The problems of distinguishing between serial correlation and structural change are largely avoided in the sequential approach. When serial correlation is the problem and not structural change, or when structural change is the problem and not serial correlation, there is a relatively high probability of detecting the misspecification.

A further problem of non-robustness of the AOC test is seen when heteroscedasticity also occurs – compare the first rows of experiments 2 and 4 where it is seen that the AOC test power falls sharply when heteroscedasticity is introduced.

The overall rejection probabilities for the sequential and non-sequential approaches are interesting. If either serial correlation is the sole misspecification or it occurs with structural change, there is little difference between the rejection probabilities in the two procedures. If serial correlation occurs with heteroscedasticity and not structural change, the non-sequential approach has the higher rejection probabilities. However, when both heteroscedasticity and structural change occur, with and without serial correlation, it is seen in experiment 4 that the sequential approach yields the highest rejection probabilities.

It seems therefore that when tests are structured to take account of more than one misspecification, there may be a gain in overall power if those misspecifications occur but if allowed for misspecifications fare not all present, this may lead to some loss of overall power.

Perhaps the single most important finding in this study is the support given to the sequential approach when testing for serial correlation and structural change.

Appendix

The Independence of the Test Statistics

Our proof is rather more general than required and we shall first show the mutual independence of $\frac{\varepsilon_1' A_1 \varepsilon_1}{\varepsilon_1' \varepsilon_1}, \frac{\varepsilon_2' A_2 \varepsilon_2}{\varepsilon_2' \varepsilon_2}$ $\varepsilon_1' \varepsilon_1$ and $\varepsilon_2' \varepsilon_2$ where A_1 and A_2 are *arbitrary* conformable matrices and $\varepsilon_i \sim N(0, \sigma^2 I_{T_i - k})$, $i = 1, 2$.

The joint characteristic function of these statistics is

$$\phi(t_1, t_2, t_3, t_4) = K \int_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \exp\left\{ t_1 \cdot \frac{\varepsilon_1' A_1 \varepsilon_1}{\varepsilon_1' \varepsilon_1} + t_2 \cdot \frac{\varepsilon_2' A_2 \varepsilon_2}{\varepsilon_2' \varepsilon_2} + t_3 \cdot \varepsilon_1' \varepsilon_1 + t_4 \cdot \varepsilon_2' \varepsilon_2 \right.$$

$$\left. - \frac{1}{2} (\varepsilon_1' \varepsilon_1 + \varepsilon_2' \varepsilon_2) \right\} \prod_{i=1}^{T_1 - k} d\varepsilon_{i1} \cdot \prod_{j=1}^{T_2 - k} d\varepsilon_{j2},$$

where $\varepsilon_1 \sim N(0, \sigma^2 I_{T_1 - k})$ and $\varepsilon_2 \sim N(0, \sigma^2 I_{T_2 - k})$ are independent.

Making the substitutions

$$y_1 = (1 - 2t_3)^{1/2} \varepsilon_1, \quad y_2 = (1 - 2t_4)^{1/2} \varepsilon_2,$$

we have

$$\phi(t_1, t_2, t_3, t_4) = (1 - 2t_3)^{-\frac{(T_1 - k)}{2}} (1 - 2t_4)^{-\frac{(T_2 - k)}{2}}$$

$$K \int_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \exp\left\{ t_1 \cdot \frac{\varepsilon_1' A_1 \varepsilon_1}{\varepsilon_1' \varepsilon_1} + t_2 \cdot \frac{\varepsilon_2' A_2 \varepsilon_2}{\varepsilon_2' \varepsilon_2} \right.$$

$$\left. - \frac{1}{2} (y_1' y_1 + y_2' y_2) \prod_{i=1}^{T_1 - k} dy_{i1} \prod_{j=1}^{T_2} dy_{j2} \cdot \right.$$

$$= chf^n \chi^2_{(T_1 - k)} \cdot chf^n \chi^2_{(T_2 - k)} \cdot chf^n \left(\frac{\varepsilon_1' A_1 \varepsilon_1}{\varepsilon_1' \varepsilon_1} \cdot \frac{\varepsilon_2' A_2 \varepsilon_2}{\varepsilon_2' \varepsilon_2} \right) \bigg\}.$$

Obviously

$$chf^n\left(\frac{\varepsilon_1'A_1\varepsilon_1}{\varepsilon_1'\varepsilon_1}, \frac{\varepsilon_2'A_2\varepsilon_2}{\varepsilon_2'\varepsilon_2}\right) = chf^n\left(\frac{\varepsilon_1'A_1\varepsilon_1}{\varepsilon_1'\varepsilon_1}\right)\cdot chf^n\left(\frac{\varepsilon_2'A_2\varepsilon_2}{\varepsilon_2'\varepsilon_2}\right).$$

and so $\phi(t_1, t_2, t_3, 3t_4)$ is simply a product of the characteristic functions. Hence $\frac{\varepsilon_1'A_1\varepsilon_1}{\varepsilon_1'\varepsilon_1}$, $\frac{\varepsilon_2'A_2\varepsilon_2}{\varepsilon_2'\varepsilon_2}$, $\varepsilon_1'\varepsilon_1$ and $\varepsilon_2'\varepsilon_2$ are mutually independent.

An immediate consequence of this result is the mutual independence of the $\frac{\varepsilon_1'A_i\varepsilon_i}{\varepsilon_i'\varepsilon_i}$, $i=1,2$, and any function of $\varepsilon_1'\varepsilon_1$ and $\varepsilon_2'\varepsilon_2$. In addition, it is well-known that $\frac{\varepsilon_2'\varepsilon_2}{\varepsilon_1'\varepsilon_1}$ and $\varepsilon_1'\varepsilon_1 + \varepsilon_2'\varepsilon_2$ are independent. It follows that the $\frac{\varepsilon_i A_i\varepsilon_i}{\varepsilon_i'\varepsilon_i}$, $i=1,2$, $\frac{\varepsilon_2'\varepsilon_2}{\varepsilon_1'\varepsilon_1}$ and $\varepsilon_1'\varepsilon_1 + \varepsilon_2'\varepsilon_2$ are mutually independent.

It is of interest that the statistics in (3.6) and (3.8) can be written in terms of recursive residuals which have the same properties as the ε_i, $i=1,2$, which appear in the foregoing analysis. Thus (3.6) and (3.8) may be written respectively as

$$F_1 = \frac{\varepsilon_2'\varepsilon_2/(T_2-k)}{\varepsilon_1'\varepsilon_1/(T_1-k)} \quad \text{and} \quad d_3 = \frac{T_1}{T}\frac{\varepsilon_1'A_1^*\varepsilon_1}{\varepsilon_1'\varepsilon_1} + \frac{T_2}{T}\frac{\varepsilon_2'A_2^*\varepsilon_2}{\varepsilon_2'\varepsilon_2}$$

where the ε_i, $i=1,2$, are vectors of recursive residuals and the A_i^* $i=1,2$, are suitably chosen conformable matrices.

Finally we need the following result which is, essentially, proved in Harvey and Phillips (1977).

Lemma: The analysis of covariance test statistic given in (3.4) can be written in the form

$$F = \frac{\varepsilon_3'\varepsilon_3/k}{(\varepsilon_1'\varepsilon_1 + \varepsilon_2'\varepsilon_2)/(T-2k)}$$

where the ε_i, $i=1,2,3$ are mutually independent normal random vectors of recursive residuals with zero means and common scalar covariance matrices.

Now it is clear that the $\frac{\varepsilon_i' A_i^* \varepsilon_i}{\varepsilon_i' \varepsilon_i}$, $i=1,2$, $\frac{\varepsilon_2' \varepsilon_2}{\varepsilon_1' \varepsilon_1}$, $\varepsilon_1' \varepsilon_1 + \varepsilon_1' \varepsilon_2$ and $\varepsilon_3' \varepsilon_3$ are all mutually independent. It follows, trivially, that F_1, d_3 and F are mutually independent also.

References

Anderson GJ, Mizon GE (1984) Parameter constancy tests: old and new. Discussion paper No 8325, University of Southampton

Anderson TW (1971) The statistical analysis of time series. Wiley, New York

Basu D (1955) On statistics independent of a complete sufficient statistic. Sankhya 15:377–380

Dhrymes PJ, Howrey EP, Hymans SH, Kmenta J, Leamer EE, Quandt RE, Ramsey JB, Shapiro HT, Zarnowitz V (1972) Criteria for the evaluation of econometric models. Annals of Economic and Social Measurement 1/3:291–324

Graybill F (1975) Theory and application of the linear model. Duxbury Press

Harrison MJ, McCabe BPM (1977) A test for heteroscedasticity based on ordinary least squares residuals. Journal of the American Statistical Association 74:494–499

Harvey AC, Phillips GDA (1977) Testing for structural change in simultaneous equation models. University of Kent, QSS Discussion Paper No 40

Hogg RB (1961) On the resolution of statistical hypotheses. Journal of the American Statistical Association 56:978–989

Hogg RV, Craig AT (1956) Sufficient statistics in elementary distribution theory. Sankhya 17:209–216

Mizon GE (1977) Inferential procedures in non-linear models: an application in a UK industrial cross section study of factor substitution and returns to scale. Econometrica 45:1221–1242

Phillips GDA, McCabe BP (1983) The independence of tests for structural change in regression models. Economic Letters 12:283–287

Phillips GDA, McCabe BP (1988) Some applications for Basu's independence theorem in testing econometric models. Statistica Neerlandica 42:37–46

Statistical Analysis of "Structural Change": An Annotated Bibliography

By P. Hackl[1] and A. H. Westlund[2]

1 Introduction

The typical "structural change" situation is – from the point of view of a statistician – as follows: To cope with a particular economic phenomenon a model is specified, and it is suspected that for different periods of time, or for different spatial regions, different sets of parameter values are needed in order to describe the reality adequately; the "change point" which separates these periods, or regions, is unknown. Questions that arise in this context include: Is it necessary to assume that the parameters are changing? When, or where, does a change occur or – if it takes place over a certain period of time – what is its onset and duration? How much do parameters before and after the change differ? What type of model is appropriate in a particular situation (e.g., two-phase regression, stochastic parameter models)?

Non-constancy of the parameters is an essential element of "structural change". This nonconstancy of the parameters can appear as an inadequacy of the model which is specified to represent the phenomenon in question; diagnostic checking methods can be applied to identify such nonconstancies. On the other hand, parameter variability can be incorporated in the model.

References included in this bibliography concentrate on two topics:

1. Detection of non-constancy of parameters in regression and time-series models.

2. Statistical analysis of models with time-varying parameters.

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The first group of references deals with the change point problem in the context of regression models. Constancy of a sequence of random variables is related to the analysis of residuals which might be performed in order to detect non-constancy of the regression parameters; therefore, papers are also included which discuss the analysis of parameter constancy of (time-ordered) sequences of random variables. Several papers discuss the analysis of constancy of parameters of time series models.

The second group of references is concerned with estimation procedures for regression models with time-varying parameters. These papers are of interest because time-varying parameter models might be appropriate for model specification in the presence of non-constancy. Also, such parameterizations can be used to detect instability in the coefficients. Some papers are included which discuss forecasting problems in the situation on non-constant parameters. No or nearly no weight is given to some topics which are related to those mentioned above, viz., continuous sampling inspection, heteroscedasticity, analysis of non-constancy of time-series parameters in the frequency domain, and disequilibrium models. The reason for these limitations lie partly in the subjects, partly in the fact that our efforts had to be restricted.

The close connection of questions of model stability with economic problems leads us to discuss briefly what is known under "structural change" among economists. In economics this notion is not clearly defined. However, a notion related to "structural change" which, in the context of a linear dynamic model, is clearly defined, is the concept of stability. It refers to the dominant root of the characteristic equation of the system: The system is stable if the dominant root lies within the unit circle (cf. Theil and Boot 1962; Oberhofer and Kmenta 1973). This concept, however, is of little help for defining "structural change" if it is accepted that structural change implies non-constant relations between elements (variables) of the system. Economists speak about structural change not only in this rather concrete sense but also if there are substantial changes in certain characteristics, e.g., the mean, of the endogenous variables of the system. Consequently, the borderline between structural change and stability is not strict, the notion "structural change" is not well-defined, and questions concerning the theoretical motivation of structural change, its measurability, and others, cannot be discussed properly. We hope that this bibliography contributes to a more commonly accepted use of the notion "structural change".

This paper resulted as a part of the activities of a IIASA (International Institute for Applied Systems Analysis, Laxenburg/Austria) Working Group on "Statistical Analysis and Forecasting of Economic Structural Change", a group of statisticians and econometricians which held meetings in 1985 and 1986. At that time no comprehensive basis in book-form was available on this subject, but four bibliographies: Hinkley et al. (1980), Johnson (1977, 1980), and Shaban (1980). A unified and updated compilation based on these papers (Hackl and Westlund 1985) is a forerunner of the bibliography in hand.

In the meanwhile, two books, viz. Broemeling and Tsurumi (1987) and Schulze (1987), appeared which treat the regression aspects of the subject from a Bayesian's and a frequentist's point of view, respectively. Furthermore, a number of recent monographies include special chapters related to the subject: Chow (1984), Judge et al. (1985), Nicholls and Pagan (1985). In a few months, Hackl (1988) will present results of the above-mentioned IIASA Working Group, including some specially invited papers, in form of a multi-author volume: Both surveying articles and specialized research papers give a comprehensive view of the subject, of related statistical and mathematical methods and problems, and of future directions.

Most references included in this bibliography were published in methodological (statistical and econometric) journals. Our work is partially based on the four above mentioned bibliographies which delivered about 50% of the references cited here. Most of the remaining papers appeared after these bibliographies were published, a fact that indicates the still growing interest in this subject. Papers which mainly deal with applications were not incorporated, except papers which were published in methodological journals. Of course, we do not claim that this bibliography is complete.

2 The Subject-Matter Codes

The entries in the list of papers (Chapter 3) are annotated according to their subject-matter. The corresponding codes consist of two digits which are separated by a period, indicating the following areas of statistical methodology:

0. General
0.1 Bibliography, survey.

1. Analysis of Constancy in a Sequence of Random Variables Ordered by Time
1.1 Tests for a change in the expectation. The change can be sudden or can continue over a certain period of time; the variance can be known or unknown.
1.2 Sequential test procedures for nonconstancy.
1.3 Tests for a change of parameters other than the mean or for a change of the whole distribution.
1.4 Estimation concerning the change point; estimation of the distribution parameters; sample theoretic approach.

1.5 Bayesian inference concerning the change point and/or the distribution parameters.
1.6 Estimation procedures concerning other parameters than the expectation in the presence of nonconstancy.

2. Analysis of Constany in Regression Models
2.1 Test procedures for nonconstancy of regression coefficients of linear regression models. The disturbance variance can be constant or can change in time.
2.2 Sequential test procedures for the detection of nonconstancy.
2.3 Inference concerning the linear regression model in the presence of nonconstancy; sample theoretic approach. Methods for estimating the unknown change point, distributional properties of such an estimate, and inference on the regression model parameters may be treated.
2.4 Bayesian inference in linear regression models in the presence of nonconstancy.
2.5 Special switching mechanisms.
2.6 Regression models with time-varying parameters. The mechanism of variation is assumed to be in action during the whole time of observation and may be deterministic or stochastic.
2.7 Inference concerning nonconstancy of non-linear regression models.
2.8 Methods of inference for models based on spline functions.
2.9 Forecasting under nonconstancy.

3. Estimation of Regression Models with Time-Varying Parameters
3.1 Ordinary least-squares estimation.
3.2 Generalized least-squares estimation, including the Hildreth-Houck and Swamy procedures.
3.3 Filtering and smoothing procedures.
3.4 Maximum likelihood estimation.
3.5 The varying parameter (VPR) procedure.
3.6 Bayesian estimation.
3.7 Adaptive estimation (AEP) procedures.
3.8 Other procedures.
3.9 Forecasting procedures in the presence of nonconstant parameters.

4. Analysis of Constancy in Time Series Models
4.1 Test procedures for nonconstancy of the mean and/or variance in ARIMA models.
4.2 Sequential test procedures for the detection of nonconstancy of an ARIMA model.
4.3 Test procedures for nonconstancy of parameters different from mean and variance in ARIMA models.

4.4 Estimation of parameters of an ARIMA model in the presence of nonconstancy; sample theoretic approach.
4.5 Bayesian inference concerning the parameters of an ARIMA model in the presence of nonconstancy.
4.6 Inference for models different from ARIMA models.
4.7 Forecasting under nonconstancy.
4.8 Inference concerning time dependence of (partially) known structure. Test and parameter estimation procedures; the nonconstancy is assumed to have a known onset and/or form (cf. intervention analysis).

In addition, the following code letters are used to qualify the subject-matter in more detail:

A Asymptotic Properties
B Bayesian Methods
C Comparison of Procedures
E Examples, Numerical Illustrations
M Multivariate Procedures
N Non-Parametric Methods
P Parametric Methods
R Robustness
S (Monte Carlo) Simulation Results
T Tables, Charts
U Univariate Procedures
V Computational Methods
X Non-Bayesian Methods

3 References

Abraham B (1980) [4.8; M] Intervention analysis and multiple time series. Biomtrka 67:73–78
Abraham B, Minder CE (1982) [3.8] A time series model with random coefficients. CommStA 11:1381–1391
Abraham B, Wei WWS (1979) [4.5] Inferences in a switching time series. ASAProBuEc, pp 354–358
Abraham B, Wei WWS (1984) [4.4; BE] Inferences about the parameters of a time series model with changing variance. Metrika 31:183–194
Akkina KR (1974) [3.1; C] Application of random coefficient regression models to the aggregation problem. Econmtca 42:369–375
Akman VE, Raftery AE (1986) [1.1, 1.6; AES] Asymptotic inferences for a change-point Poisson process. AnlsStat:14:1583–1590

Ali MM, Giaccotto C (1982) [1.1; NE] The identical distribution hypothesis for stock market prices. Location- and scale-shift alternatives. JASA 77:19–28

Ali MM, Silver JL (1985) [2.1; CV] Tests for equality between sets of coefficients in two linear regressions under heteroscedasticity. JASA 80:730–735

Amemiya T (1978) [2.6] A note on a random coefficients model. IntEconR 19:793–796

Andel J (1976) [4.4, 4.6] Autoregressive series with random parameters. MaOpfStS 7:735–741

Anderson GJ, Mizon GE (1983) [2.1; ACMU] Parameter constancy tests: Old and new. Disc. Papers Econ. Econometrics, No. 8325, Univ. Southampton

Anderson RL, Nelson LA (1975) A family of models involving intersecting straight lines and concomitant experimental designs useful in evaluating response to fertilizer nutrients. Biomtrcs 31:303–318

Anderson TW (1978) [4.3; MA] Repeated measurements on autoregressive processes. JASA 73:371–378

Andrews DWK, Fair RC (1987) Inference in models with structural change. Yale University, Cowles Foundation, New Haven

Aroian LA, Robison DE (1966) [1.2] Sequential life tests for the exponential distribution with changing parameter. Technmcs 8:217–227

Arora SS (1976) [3.2; CE] Alternative estimators of the determinants of inter-regional migration. ASA, Proc 1976 Annual Meeting, pp 217–219

Asatryan D, Safaryan I (1986) [1.4; ACE] Nonparametric methods for detecting changes in the properties of random sequences. In: Telksnys L (ed) Detection of changes in random processes. Springer, New York, pp 1–13

Ashley R (1984) [2.1; CES] A simple test for regression parameter stability. EconInquiry 22:253–268

Athans M (1974) [3.3] The importance of Kalman filtering methods for economic systems. AnnEcSoMt 3:49–64

Bacon DW, Watts DG (1971) [2.4, 2.5; A] Estimating the transition between two intersecting straight lines. Biomtrka 58:525–534

Bagshaw M, Johnson RA (1975) [1.1, 1.2, 4.1; CS] The effect of serial correlation on the performance of CUSUM tests II. Technmcs 17:73–80

Bagshaw M, Johnson RA (1975) [1.1, 1.2] Sequential detection of a drift change in a Wiener process. CommStA 4:787–796

Bagshaw M, Johnson RA (1977) [4.3; E] Sequential procedures for detecting parameter changes in a time-series model. JASA 72:593–597

Balmer DW (1976) [4.6] On a quickest detection problem with variable monitoring. JAppProb 13:760–767

Bandemer H, Schulze U (1985) [2.3, 2.4, 2.5; BE] On estimation in multiphase regression models with several regressors using prior knowledge. Statistics 16:3–13

Bansal RK, Papantoni-Kazakos P (1986) [1.2, 1.3; A] An algorithm for detecting a change in a stochastic process. IEEEInfo, IT-32:227–235

Barnard GA (1959) [1.2] Control charts and stochastic processes (with discussion). JRSS-B 21:239–271

Barone P, Roy R (1983) [4.1; ACUM] On a stability test for estimated multivariate autoregressive-moving average models. ASAProBuEc, pp 650–653

Barten AP, Bronsard LS (1970) [2.3; AM] Two-stage least-squares estimation with shifts in the structural form. Econmtca 38:938–941

Bass FM, Wittink DR (1975) [3.2; E] Pooling issues and methods in regression analysis with examples in marketing research. JMarRsr 12:414–425

Basseville M, Benveniste A (1983) [4.2; ACE] Sequential detection of abrupt changes in spectral characteristics of digital signals. IEEEInfo IT-29:709–724

Basseville M, Benveniste A (eds) (1986) Detection of abrupt changes in signals and dynamical systems. Springer, New York

Basseville M, Benveniste A, Moustakides GV (1986) [4.3; AEMS] Detection and diagnosis of abrupt changes in modal characteristics of nonstationary digital signals. IEEEInfo, IT-32:412–417

Batarshin R (1986) [1.1, 4.1] On the detection of the "disruption" of dynamic systems. In: Telksnys L (ed) Detection of changes in random processes. Springer, New York, pp 13–24

Bather JA (1967) [1.1] On a quickest detection problem. AnnMathStat 38:711–724

Baudin A, Westlund A (1985) [2.1; E] Structural instability analysis: The case of newsprint consumption in the United States. Forest Science 31:990–994

Baudin A, Nadeau S, Westlund A (1984) [3.3, 3.7; CE] Estimation and prediction under structural instability: The case of the US pulp and paper market. J. Forecasting 3:63–78

Bauer P, Hackl P (1978) [1.1, 1.2; EC] The use of MOSUMs for quality control. Technmcs 20:431–436

Bauer P, Hackl P (1980) [1.1, 1.2; ET] An extension of the MOSUM technique for quality control. Technmcs 22:1–7

Bauer P, Hackl P (1985) [2.1; V] The application of Hunter's inequality in simultaneous testing. BiomtrcJ 27:25–38

Beckmann RJ, Cook RD (1979) [2.1; AS] Testing for two-phase regressions. Technmcs 21:65–69

Bellman R, Roth R (1969) [2.3; VE] Curve fitting by segmented straight lines. JASA 64:1079–1084

Belsley DA (1973) On the determination of systematic parameter variation in the linear regression model. AnnEcSoMt 2:487–494

Belsley DA (1973) A test for systematic variation in regression coefficients. AnnEcSoMt 2:495–499

Belsley DA (1973) [3.3; C] The applicability of the Kalman filter in the determination of systematic parameter variation. AnnEcSoMt 2:531–533

Belsley DA, Kuh E (1973) [3.3; C] Time-varying parameter structures: An overview. AnnEcSoMt 2:375–379

Bennett RJ (1977) [3.8; CS] Consistent estimation of nonstationary parameters for small sample situations. A Monte Carlo study. IntEconR 18:489–502

Benveniste A, Basseville M, Moustakides GV (1987) [4.2, 4.3; A] The asymptotic local approach to change detection and model validation. IEEEAuCn 32:583–592

Bhattacharya PK (1978) [1.4, A] Estimation of change-point in the distribution of random variables. Unpublished manuscript

Bhattacharya PK, Brockwell PJ (1976) [1.4] The minimum of an additive process with applications to signal estimation and storage theory. ZeitWahr 37:51–75

Bhattacharya PK, Frierson D Jr. (1981) [1.1; NA] A non-parametric control chart for detecting small disorders. AnlsStat 9:544–554

Bhattacharyya GK (1984) [1.1, 1.2, 1.3; NA] Tests of randomness against trend or serial correlation. In: Krishnaiah PR, Sen PK (eds), Handbook of statistics, vol 4. North-Holland, Amsterdam, pp 89–111

Bhattacharyya GK, Johnson RA (1968) [1.1; NAS] Nonparametric tests for shift at unknown time point. AnnMathStat 39:1731–1743

Bhattacharyya MN, Andersen AP (1976) [4.1; AFUM] A post sample diagnostic test for a time series model. Australian J. Managemt 1:33–56

Bierens HJ (1984) [2.1; AS] Testing parameter constancy of linear regressions. In: Dijkstra TK (ed) Misspecification analysis. Springer, Berlin, pp 104–117

Blakemore AE, Schlagenhauf DE (1983) [2.5; E] A test for structural instability in the secular growth rate of productivity. Econ. Letters 13:153–159

Bookstein FL (1975) [2.3; V] On a form of piecewise linear regression. AmerStat 29:116–117

Booth NB, Smith AFM (1982) [4.2, 4.5; UM] A Bayesian approach to retrospective identification of change-points. JEconmcs 19:7–22

Borjas GJ (1982) [3.2; AC] On regressing regression coefficients. JStPlInf 7:131–137

Bowman HW, LaPorte AM (1972) [3.6; CE] Stochastic optimization in recursive equation systems with random parameters with an application to control of the money supply. AnnEcSoMt 1:419–435

Box GEP, Tiao GC (1965) [4.8; B] A change in level of a non-stationary time series. Biomtrka 52:181–192

Box GEP, Tiao GC (1975) [4.8; E] Intervention analysis with applications to economic and environmental problems. JASA 70:70–79

Box GEP, Tiao GC (1976) [4.1, 4.3; E] Comparison of forecasts and actuality. ApplStat 25:195–200

Brännäs K, Uhlin S (1984) [2.3; EC] Improper use of the ordinary least squares estimator in the switching regression model. CommStA 13:1781–1791

Brannigan M (1981) [1.1, 1.3; E] An adaptive piecewise polynomial curve fitting procedure for data analysis. CommStA 10:1823–1848

Bretschneider SI, Carbone R, Longini RL (1979) [3.7; E] An adaptive approach to time-series forecasting. DecisnSc 10:232–244

Bretschneider SI, Carbone R, Longini RL (1982) [3.7, 3.9; E] An adaptive multivariate approach to time series forecasting. DecisnSc 13:668–680

Bretschneider SI, Gorr WL (1983) [2.1; E] Ad hoc model boilding using time-varying parameter models. DecisnSc 14:221–239

Breusch TS (1986) [2.3] Hypothesis testing in unidentified models. REcon&St 53:635–651

Breusch TS, Pagan AR (1979) A simple test for heteroscedasticity and random coefficient variation. Econmtca 47:1287–1294

Broemeling LD (1972) [1.5] Bayesian procedures for detecting a change in a sequence of random variables. Metron 30:1–14

Broemeling LD (1974) [1.5] Bayesian inference about a changing sequence of random variables. CommStA 3:243–255

Broemeling LD (1977) [3.9; B] Forecasting future values of a changing sequence. CommStA 6:87–102

Broemeling LD, Chin Choy JH (1981) [2.4] Detecting structural change in linear models. CommStA 10:2551–2561

Broemeling LD, Tsurumi H (1987) [0.1, 2.1; B] Econometrics and structural change. Marcel Dekker, New York

Brown RL, Durbin J (1968) [2.1, 2.2] Methods of investigating whether a regression relationship is constant over time. In: Selected statistical papers I. Mathematisch Centrum, Amsterdam, pp 37–45

Brown RL, Durbin J, Evans JM (1975) [2.1, 2.3, 2.2; V] Techniques for testing the constancy of regression relationships over time (with discussion). JRSS-B 37:149–192

Bunke H, Schulze U (1985) [2.1] Approximation of change points in regression models. In: Pukkila T, Puntanen S (eds) Proceedings of the first international Tampere seminar on linear statistical models and their applications. University of Tampere, pp 161–177

Burnett TD, Guthrie D (1970) [3.8; S] Estimation of stationary stochastic regression parameters. JASA 65:1547–1553

Burobin N, Motte V, Muchnik I (1986) [4.1; EV] An algorithm of the instant of multiple change of properties of a random process based on the dynamic programming method. In: Telksnys L (ed) Detection of changes in random processes. Springer, New York, pp 38–46

Buse A, Lim L (1977) [2.8; E] Cubic splines as a special case of restricted least squares. JASA 72:64–68

Cadsby CB, Stengos T (1986) [2.1; E] Testing for parameter stability in a regression model with AR (1) errors. Econ. Letters 20:29–32

Campillo F (1986) [2.1] Testing for a change-point in linear systems with incomplete observation. In: Florens J-P et al. (eds) Asymptotic theory for non i.i.d. processes. Publications des Facultes Universitaires Saint-Louis, Bruxelles, pp 225–242

Carbone R, Longini RL (1977) [3.7; E] A feedback model for automated real estate assessment. Manag. Science 24:241–248

Carbone R, Makridakis S (1986) [4.7; CE] Forecasting when pattern changes occur beyond the historical data. Manag. Science 32:257–271

Carter RL, Blight BJN (1981) [2.4; E] A Bayesian change-point problem with an application to the prediction and detection of ovulation in women. Biomtrcs 37:743–751

Chan KS, Tong H (1986) [4.6] On estimating thresholds in autoregressive models. JTimSrAn 7:179–190

Chernoff H, Zacks S (1964) [1.4, 1.1; BXCS] Estimating the current mean of a normal distribution which is subjected to changes in time. AnnMathStat 35:999–1018

Chin Choy JH, Broemeling LD (1980) [2.4; E] Some Bayesian inferences for a changing linear model. Technmcs 22:71–78

Chow GC (1960) [2.1; E] Tests of equality between sets of coefficients in two linear regressions. Econmtca 28:591–605

Chow GC (1984) [0.1, 2.1, 2.3, 2.6; UM] Random and changing coefficient models. In: Griliches Z, Intriligator MD (eds) Handbook of econometrics, vol 2. North-Holland, Amsterdam, pp 1213–1245

Cobb GW (1978) [1.4; CBE] The problem of the Nile: Conditional solution to a changepoint problem. Biomtrka 65:243–251

Cooley TF (1975) [3.5; C] A comparison of robust and varying parameter estimates of a macro-econometric model. AnnEcSoMt 4:373–388

Cooley TF (1977) [3.2] Generalized least squares applied to time varying parameter models: A comment. AnnEcSoMt 6:313–314

Cooley TF, DeCanio SJ (1977) [3.5; E] Rational expectations in American agriculture 1867–1914. REcon&St 59:9–17

Cooley TF, Prescott EG (1973) [2.6] An adaptive regression model. IntEconR 14:364–371

Cooley TF, Prescott EC (1973) [3.5; E] Varying parameter regression: A theory and some applications. AnnEcSoMt 2:463–473

Cooley TF, Prescott EC (1976) [3.5; A] Estimation in the presence of stochastic parameter variation. Econmtca 44:167–184

Cooley TF, Rosenberg B, Wall KD (1977) [3.3] A note on optimal smoothing for time varying coefficient problems. AnnEcSoMt 6:453–456

Cooper JP (1973) [2.6, 3.8] Time-varying regression coefficients: A mixed estimation approach and operational limitations of the general Markov structure. AnnEcSoMt 2:525–530

Consigliere I (1981) The Chow test with serially correlated errors. Rivista Internazionale di Scienze Sociali 89:125–137

Corsi P, Pollack RE, Prakken JL (1982) [2.1; ES] The Chow test in the presence of serially correlated errors. In: Chow GC, Corsi P (eds) Evaluating the reliability of macro-economic models. John Wiley, New York, pp 163–192

Cox DD (1983) [2.8; ANR] Asymptotic form-type smoothing splines. AnlsStat 11:530–551

Crosier RB (1988) [1.2; MCE] Multivariate generalizations of cumulative sum quality-control schemes. Technmcs 30:291–303

Curnow RN (1973) [2.3, 2.8; E] A smooth population response curve based on an abrupt threshold and plateau model for individuals. Biomtrcs 29:1–10

Darkhovskii BS (1976) [1.1; N] A nonparametric method for the a posteriori detection of the "disorder" time of a sequence of independent random variables. ThProbAp 21:178–183

Darkhovskii BS (1984) [1.4; N] On two estimation problems for times of change of the probabilistic characteristics of a random sequence. ThProbAp 29:478–487

Davis WW (1979) [4.1; RS] Robust methods for detection of shifts of the innovation variance of a time serie. Technmcs 21:313–320

Dent WT, Hildreth C (1977) [3.4; CS] Maximum likelihood estimation in random coefficient models. JASA 72:69–72

Deshayes J, Picard D (1982) [2.1; A] Tests of disorder of regression: Asymptotic comparison. ThProbAp 27:100–115

Deshayes J, Picard D (1984) [1.1, 1.3, 1.4, 1.6; ABC) Lois asymptotiques des tests et estimateurs de rupture dans un modèle statistique classique. Ann.Inst.H.Poincaré, Prob. Statist. 20:309–328

DeJong DV, Collins DW (1985) [3.1; E] Explanations for the instability of equity beta: Risk-free rate changes and leverage effects. J. Financial and Quantitative Analysis 20:73–94

Diaz J (1982) [1.5; E] Bayesian detection of a change of scale parameter in sequences of independent gamma random variables. JEconmtcs 19:23–29

Diderrich GT (1985) [3.3; BC] The Kalman filter from the perspective of Goldberger-Theil estimators. AmerStat 39:193–198

Diebold FX, Pauly P (1987) [2.9; CS] Structural change and the combination of forecasts. J. Forecasting 6:21–40

Dielman TE (1983) [0.1] Pooled cross-sectional and time-series data: A survey of current statistical methodology. AmerStat 37:111–122

Draper NR, Guttman I, Lipow P (1977) [2.8] All-bias designs for spline functions, joined at the axes. JASA 72:424–429

Dufour J-M (1980) [2.1; EV] Dummy variables and predictive tests for structural change. Econ. Letters 6:241–247

Dufour J-M (1982) [2.1; C] Recursive stability analysis of linear regression relationships. An exploratory methodology. JEconmcs 19:31–76

Dufour J-M (1982) [2.1; E] Predictive tests for structural change and the St. Louis equation. ASAProBuEc, pp 323–327

Dufour J-M (1982) [2.1, 2.5; E] Generalized Chow tests for structural change: A coordinate-free approach. IntEconR 23:565–575

Duncan DB, Horn SD (1972) [3.3] Linear dynamic recursive estimation from the viewpoint of regression analysis. JASA 67:815–821

Dutkowsky D, Foote W (1985) [2.1, 2.3; E] Switching, aggregation, and the demand for borrowed reserves. REcon&St 67:331–335

Edler L, Berger J (1985) [2.1; V] Computational methods to determine a break point in linear regression. EDV Med. Biol. 16:128–134

El-Sayyad GM (1975) [2.4] A Bayesian analysis for the change-point problems. Egypt. Statist. J. 19:1–13

Enns PG, Wrobleski WJ (1974) [3.3, CE] A Bayesian model for explaining money supply growth rates. ASA, Proc. 1974 Annual Meeting, pp 361–366

Erlat H (1983) [2.1, 2.3, 2.5; CP] A note on testing for structural change in a single equation belonging to a simultaneous system. Econ. Letters 13:185–189

Ertel JE, Fowlkes EB (1976) [2.8; V] Some algorithms for linear spline and piecewise multiple linear regression. JASA 71:640–648

Esterby SR, El-Shaarawi AH (1981) [2.1, 2.3] Inference about the point of change in a regression model. ApplStat 30:277–285

Eubank RL (1984) [0.1, 2.8; CE] Approximate regression models and splines. CommStA 13:433–484

Farley JU, Hinich MJ (1970) [2.1; S] A test for a shifting slope coefficient in a linear model. JASA 65:1320–1329

Farley JU, Hinrich MJ (1970) [1.5; CS] Detecting "small" mean shifts in time series. Manag. Science 17:189–199

Farley JU, Hinich M, McGuire TW (1975) [2.1; CS] Some comparisons of tests for a shift in the slopes of a multivariate linear time series model. JEconmtcs 3:297–318

Fearn T (1975) [3.6; C] A Bayesian approach to growth curves. Biomtrka 62:89–100

Feder PI (1975) [2.3, 2.7, 2.8; A] On asymptotic distribution theory in segmented regression problems – identified case. AnlsStat 3:49–83

Feder PI (1975) [2.1; A] The log likelihood ratio in segmented regression. AnlsStat 3:84–97

Feder PI, Sylwester DL (1968) [2.3; A] On the asymptotic theory of least squares estimation in segmented regression: Identified case (abstract). AnnMathStat 39:1362

Ferreira PE (1975) [2.4; CS] A Bayesian analysis of a switching regression model: Known number of regimes. JASA 70:370–374

Fisher FM (1970) [2.1; C] Tests of equality between sets of coefficients in two linear regressions. Econmtca 38:361–366

Fisher GR (1983) [2.1; C] Tests for two separate regressions. JEconmtcs 21:117–132

Fisher L, Kamin JH (1985) [3.3; E] Forecasting systematic risk: Estimates of "raw" beta that take account of the tendency of beta to change and the heteroscedasticity. J. Financial and Quantitative Analysis 20:127–150

Fishman M (1986) [1.5; C] Bayesian mean square estimation of the instant of one-step shift of the mean level of white Gaussian noise. In: Telksnys L (ed) Detection of changes in random processes. Springer, New York, pp 52–55

Fomby TB, Carter-Hill R, Johnson SR (1984) [0.1, 2.1, 2.3, 2.5, 2.6] Varying coefficient models, cpt. 14. In: Advanced econometric methods. Springer, New York

Foster SA, Gorr WL (1986) [3.7; EMS] An adaptive filter for estimating spatially-varying parameters: Application to modeling police hours spent in response to calls for service. Manag. Science 32:878–889

Franzini L, Harvey AC (1983) [4.6; CT] Testing for deterministic trend and seasonal components in time series models. Biomtrka 70:673–682

Freebairn JW, Rausser GC (1974) [3.3; E] Updating parameter estimates: A least squares approach with an application to the inventory of beef cows. Rev. Marketing and Agricultural Economics 42:83–99

Freeman JM (1984) [2.1, 2.3; CE] Two-phase regression and goodness of fit. CommStA 13:1321–1334

Freeman JM (1986) [1.1; CE] An unknown change point and goodness of fit. The Statistician 35:335–344

Froehlich BR (1973) [3.2; CS] Some estimators for a random coefficient regression model. JASA 68:329–335

Gallant AR (1977) [2.3, 2.7; VA] Testing a nonlinear regression specification: a nonregular case. JASA 72:523–530

Gallant AR, Fuller WA (1973) [2.3, 2.7; VE] Fitting segmented polynomial regression models whose join points have to be estimated. JASA 68:144-147

Galpin JS, Hawkins DM (1984) [2.1, 2.3; T] The use of recursive residuals in checking model fit in linear regression. AmerStat 38:94–105

Garbade K (1977) [2.1, 2.3, 2.6; CSE] Two methods for examining the stability of regression coefficients. JASA 72:54–63

Gardner LA (1969) [1.1; V] On detecting changes in the mean of normal variates. AnnMathStat 40:116–126

Giles DEA (1981) Testing for parameter stability in structural econometric relationships. Econ. Letters 7:323–326

Ginsburgh V, Tishler A, Zang I (1980) [3.4; V] Alternative estimation methods for two-regime models. Europ. Econ. Rev. 13:207–228

Goldfeld SM, Quandt RE (1972) [2.1, 2.3] Nonlinear methods in econometrics. North-Holland, Amsterdam

Goldfeld SM, Quandt RE (1973) [2.1, 2.3, 2.5] A Markov model for switching regressions. JEconmtcs 1:3–16

Goldfeld SM, Quandt RE (1973) [2.1, 2.3, 2.5; C] The estimation of structural shifts by switching regressions. AnnEcSoMt 2:475–485

Goldfeld SM, Quandt RE (1975) [3.4; ES] Estimation in a disequilibrium model and the value of information. JEconmtcs 3:325–348

Goldfeld SM, Quandt RE (1976) Studies in nonlinear estimation. Ballinger, Cambridge/MA

Griffiths DA, Miller AJ (1973) [2.3, 2.5] Hyperbolic regression – a model based on two-phase piecewise linear regression with a smooth transition between regimes. CommStA 2:561–569

Griffiths WE (1972) [3.2; C] Estimation of actual response coefficients in the Hildreth-Houck random coefficient model. JASA 67:633–635

Griffiths WE, Drynan RG, Prakash S (1979) [2.3; E] Bayesian estimation of a random coefficient model. JEconmtcs 10:201–220

Gujarati D (1970) [2.1; E] Use of dummy variables in testing for equality between sets of coefficients in two linear regressions: A note. AmerStat 24:50–52

Gupta S (1982) [2.1; CS] Structural shift: A comparative study of alternative tests. ASAProBuEc, pp 328–331

Guthery SB (1974) [2.8; V] Partition regression. JASA 69:945–947

Guttman I, Menzefricke U (1982) On the use of loss functions in the change-point problem. Ann. Inst. Statist. Math. 34:319–326

Haccou P, Meelis E, van de Geer S (1988) [1.1; ACS] The likelihood ratio test for the change point problem for exponentially distributed random variables. Stochast. Proc. Appls. 27:121–139

Hackl P (1978) [2.1; E] Moving sums of residuals: A tool for testing constancy of regression relationship over time. In: Janssen JML et al (eds) Models and Decision Making in National Economies. North-Holland, Amsterdam, pp 219–225

Hackl P (1980) [2.1; CST] Testing the constancy of regression relationships over time. Vandenhoeck und Ruprecht, Göttingen

Halpern EF (1973) [2.8] Bayesian spline regression when the number of knots is unknown. JRSS-B 35:347–360

Halverson AL (1985) [2.5, 3.8; E] Switching regression estimates of a sequential production process: The case of underground coal mining. REcon&Statist 67:161–165

Harvey AC (1976) [2.1, 2.2] An alternative proof and generalization of a test for structural change. AmerStat 30:122–123

Harvey AC (1984) [3.3, 3.8; C] A unified view of statistical forecasting procedures (with discussion). J. Forecasting 3:245–283

Harvey AC, Phillips GDA (1979) [2.3, 2.6; V] Maximum likelihood estimation of regression models with autoregressive-moving average disturbances. Biomtrka 66:49–58

Harvey AC, Phillips GDA (1982) [3.3, 3.7; S] The estimation of regression models with time-varying parameters. In: Deistler M et al. (eds) Games, economic dynamics, and time series analysis. Physica, Wien, pp 306–321

Hatanaka M (1980) [3.3; A] A note on the application of the Kalman filter to regression models with some parameters varying over time and others unvarying. AstrlJSt 22:298–306

Havenner A, Swamy PAVB (1981) [3.2; CEA] A random coefficient approach to seasonal adjustment of economic time series. JEconmtcs 15:177–209

Hawkins DL (1984) [4.2; A] Sequential procedures for detecting deviations in the parameters of the autoregressive model from specified targets. SqtlAnly 3:121–154

Hawkins DL (1986) [1.1, 1.3; AT] A simple least squares method for estimating a change in mean. CommStB 15:655–679

Hawkins DM (1976) [2.3; VE] Point estimation of the parameters of piecewise regression models. ApplStat 25:51–57

Hawkins DM (1977) [1.1; A] Testing a sequence of observations for a shift in location. JASA 72:180–186

Hawkins DM (1981) A CUSUM for a scale parameter. J. Quality Technology 13:228–231

Hays PA, Upton DE (1986) [2.1, 2.2; E] A shifting regimes approach to the stationarity of the market model parameters of individual securities. J. Financial and Quantitative Analysis 21:307–322

Heath AB, Anderson JA (1983) [2.1, 2.3; ES] Estimation in a multivariable two-phase regression. CommStA 12:809–828

Heckman JJ (1978) [2.3; M] Dummy endogenous variables in a simultaneous equation system. Econmtca 46:931–959

Henderson R (1986) [1.1, 1.5; BE] Change-point problem with correlated observations, with an application in material accountancy. Technmcs 28:381–389

Herzberg AM, Hickie JS (1981) [1.1; ME] An investigation of Andrews' plots to show time variation of model parameters. JTimSrAn 2:233–262

Hildreth C, Houck JP (1968) [3.2; A] Some estimators for a linear model with random coefficients. JASA 63:584–595

Hines WGS (1976) [1.2, 1.1] A simple monitor of a system with sudden parameter changes. IEEEInfo IT-22:210–216

Hinich M, Farley JU (1966) [1.3; B] Theory and application of an estimation model for time series with nonstationary means. Manag. Science 12:648–658

Hinkley DV (1969) [2.1; A] Inference about the intersection in two-phase regression. Biomtrka 56:495–504

Hinkley DV (1970) [1.1; AT] Inference about the change-point in a sequence of random variables. Biomtrka 57:1–17

Hinkley DV (1971) [2.1, 2.3; ES] Inference in two-phase regression. JASA 66:736–743

Hinkley DV (1971) [1.1, 1.2, 1.4; AFS] Inference about the change-point from cumulative sum tests. Biomtrka 58:509–523

Hinkley DV (1972) [1.1; AT] Time-ordered classification. Biomtrka 59:509–523

Hinkley DV, Schechtman E (1987) [1.4, 1.6; CNE] Conditional bootstrap methods in the mean-shift model. Biomtrka 74:85–93

Hinkley DV, Chapman P, Ranger G (1980) [0.1] Change-point problem. Techn. Report, 382. School of Statistics, Univ. Minnesota

Hinkley DV, Hinkley EA (1970) [1.1, 1.3; AT] Inference about the change-point in a sequence of binomial variables. Biomtrka 57:477–488

Holbert D (1982) [2.4; E] A Bayesian analysis of a switching linear model. JEconmtcs 19:77–87

Holbert D, Broemeling LD (1977) [1.5, 2.4] Bayesian inferences related to shifting sequences and two-phase regression. CommStA 6:265–275

Honda Y (1982) [2.1; CS] On tests of equality between sets of coefficients in two linear regressions when disturbance variances are unequal. The Manchester School 50:116–125

Horn SD, Horn RA, Duncan DB (1975) [3.2; CV] Estimating heteroscedastic variances in linear models JASA 70:380–385

Hsiao C (1974) [3.2; AC] Statistical inference for a model with both random cross-sectional and time effects. IntEconR 15:12–30

Hsiao C (1975) [3.2; AC] Some estimation methods for a random coefficient model. Econmtca 43:305–325

Hsieh HK (1984) [1.3; NATC] Nonparametric tests for scale shift at an unknown time point. CommStA 13:1335–1355

Hsu DA (1977) [1.3, 1.2; TSC] Tests for variance shift at an unknown time point. ApplStat 26:279–284

Hsu DA (1979) [1.3; TE] Detecting shifts of parameter in gamma sequences with applications to stock prices and air traffic flow analysis. JASA 74:31–40

Hsu DA (1982) [2.4, 1.5; E] A Bayesian robust detection of shift in the risk structure of stock-market returns. JASA 77:29–39

Hsu DA (1982) [2.4; RE] Robust inferences for structural shift in regression models. JEconmtcs 19:89–107

Hudson DJ (1966) [2.3; V] Fitting segmented curves whose join points have to be estimated. JASA 61:1097–1129

Hwang H-S (1980) [2.1, 2.2, 2.5; EM] A test of a disequilibrium model. JEconmtcs 12:319–333

Inselmann EH (1968) [2.1] Test for several regression equations (abstract). AnnMathStat 39:1362

Irvine JM (1982) [2.1; CSR] Testing for changes in regime in regression models. ASAProBuEc, pp 317–322

James B, James KL, Siegmund D (1987) [1.1; ACT] Tests for a change-point. Biomtrka 74:71–83

James B, James KL, Siegmund D (1988) Conditional boundary crossing probabilities with applications to change-point problems. Ann. Prob 16:825–839

Jayatissa WA (1977) [2.1; A] Test of equality between sets of coefficients in two linear regressions when disturbance variances are unequal. Econmtca 45:1291–1292

Johnson KH (1974) [3.1, 3.2; CS] On estimating models with random coefficients. ASA Proc 1974 Annual Meeting pp 420–425

Johnson LW (1977) [0.1] Stochastic parameter regression: An annotated bibliography. IntStRvw 45:257–272

Johnson LW (1978) [3.4; S] Some simulation results for the linear expenditure system with random marginal budget shares. JStCmpSm 7:225–236

Johnson LW (1980) [0.1] Stochastic parameter regression: An additional annotated bibliography. IntStRvw 48:95–102

Johnson RA, Bagshaw M (1974) [1.1, 1.2, 4.1; AS] The effect of serial correlation on the performance of CUSUM tests. Technmcs 16:103–112

Judge GG, Griffiths WE, Carter Hill R, Lütkepohl H, Lee TC (1985) Varying and random coefficient models, chp 19 in: Introduction to the theory and practice of econometrics, 2nd ed. John Wiley, New York

Jurgutis M (1986) [1.4, 4.4; CS] Comparison of statistical properties of estimates of the instant of change in autoregressive sequence parameters. In: Telksnys L (ed) Detection of changes in random processes. Springer, New York, pp 63–74

Kahl DR, Ledolter J (1983) [3.9] A recursive Kalman filter forecasting approach. Manag. Science 29:1325–1333

Kalman RE (1960) [3.3, 4.4; M] A new approach to linear filtering and prediction problems. J. Basic Engineering 82:35–45

Kalman RE, Bucy RS (1961) [3.3, 4.4; M] New results in linear filtering and prediction theory. J Basic Engineering 83:95–108

Kaminskas VA, Sipenite DA (1975) Detection of a parameter change of an autoregression progress. TANLitov 4:143–148

Kaminskas V, Sidlanskas K (1986) [4.1, 4.4] Sequential detection of a change in the properties of an autoregressive time series. In: Telksnys L (ed) Detection of changes in random processes. Springer, New York, pp 74–79

Kander Z, Zacks S (1966) [1.1; BT] Test procedures for possible changes in parameters of statistical distributions occurring at unknown time points. AnnMathStat 37:1196–1210

Kastenbaum MA (1959) [2.3] A confidence interval on the abscisse of the point of intersection of two fitted linear regressions. Biomtrcs 15:323–324

Katayama S, Ohtani K, Toyoda T (1987) Estimation of structural change in the import and export equations: An international comparison. Economic Studies Quarterly 38:148–158

Kelejian HH (1974) Random parameters in a simultaneous equation framework: Identification and estimation. Econmtca 42:517–527

Kelejian HH, Stephen SW (1983) [3.2; A] Inference in random coefficient panel data models: A correction and clarification of the literature. IntEconR 24:249–254

Kenett R, Pollak M (1983) [1.2; CT] On sequential detection of a shift in the probability of a rare event. JASA 78:389–395

Kiefer NM (1978) [2.3, 2.5, 2.7, 3.4; AVC] Discrete parameter variation: Efficient estimation of a switching regression model. Econmtca 46:427–434

Kiefer NM (1980) [3.2; A] Estimation of fixed effect models for time series of cross-sections with arbitrary intertemporal covariance. JEconmtcs 14:195–202

Kiefer NM (1980) A note of switching regression and logistic discrimination. Econmtca 48:1065–1069

King ML (1987) [2.1, 2.6; E] An alternative test for regression coefficient stability. REcon&St 69:379–381

Kligene N (1977) On the estimation of the change-point in the autoregressive sequence. In: Transactions of the 7th Prague conference on information theory, statistical decision function, random processes and of the 1974 European meeting of statisticians. Academica, Prague, pp 325–334

Kligene N, Telksnys L (1983) [0.1, 1.1; UMCT] Methods of detecting instants of change of random process properties. Autom. Rem. Contr. 44:1241:1316

Kohn R, Ansley CF (1984) [4.6; V] A note on Kalman filtering for the seasonal moving average model. Biomtrka 71:648–650

Kool CJM (1988) [2.9; E] Time-variation in economic relationships: A forecasting problem. Proc. of the 8th International Symposium on Forecasting. Amsterdam (ISF)

Koskela E, Viren M (1984) [2.1; CE] Testing the stability of the Hall consumption function specification. Econ. Letters 14:289–293

Krämer W, Ploberger W, Alt R (1989) Testing for structural change in dynamic models. Econmtca 57

Kuh E (1974) An essay on aggregation theory and practice. In: Sellekaerts W (ed), Econometrics and economic theory: Essays in honour of Jan Tinbergen. Macmillan, London, pp 57–99

Kumar KD, Nicklin EH, Paulson AS (1979) [1.4; CS] Comment on "Estimating mixtures of normal distributions and switching regressions". JASA 74:52–55

Land M, Broemeling LD (1983) [2.4, 2.5, 2.9, 3.9; B] Bayesian forecasting with changing linear models. CommStA 12:1421–1430

Laumas GS (1977) [3.5; E] Liquidity functions for the United States manufactoring corporations. Southern Economic Journal 44:271–276

Laumas GS (1978) [3.5; E] A test of the stability of the demand for money. Scott. J. Polit. Economy 25:239–251

Laumas GS, Mehra YP (1977) [3.5; E] The stability of the demand for money functions 1900–1974. J. Finance 32:911–916

LaMotte LR, McWhorter A (1978) An exact test for the presence of random walk coefficients in a linear regression model. JASA 73:816–820

Ledolter J (1981) [4.3; S] Recursive estimation and adaptive forecasting in ARIMA models with time varying coefficients. In: Findley DF (ed) Applied Time Series Analysis II. Academic Press, New York, pp 449–471

Ledolter J, Box GEP, Tiao GC (1976) Topics in time series analysis for various aspects of parameter changes in ARIMA model. Techn. Report No. 449. University of Wisconsin, Madison

Lee AFS, Heghinian SM (1977) [1.5; BE] A shift of the mean level in a sequence of independent normal random variables. A Bayesian approach. Technmcs 19:503–506

Lee LF, Porter RH (1984) [2.1, 2.3, 2.5; E] Switching regression models with imperfect sample separation information. With an application on cartel stability. Econmtca 52:391–418

Leone RP (1980) Analyzing shifts in time series. In: Bogocci et al. (eds) Proceedings of the American Marketing Assoc. of Educators' Conference, pp 375–379

Leone RP (1987) [4.7, 4.8; E] Forecasting the effect of an environmental change on market performance: An intervention time-series approach. Intern. J. Forecasting 3:463–478

Lerman PM (1980) [2.3, 2.7; E] Fitting segmented regression models by grid search. ApplStat 29:77–84

Lipeikiene J (1986) [4.4; RE] The M-estimate of the instant of change in the properties of autoregressive sequences. In: Telksnys L (ed) Detection of changes in random processes. Springer, New York, pp 108–118

Little JDC (1966) [3.3; BERS] A model of adaptive control of promotional spending. Operations Research 14:1075–1097

Liu L-M, Tiao GC (1980) [4.4] Random coefficient first-order autoregressive models. JEconmtcs 13:305–325

Liu L-M (1981) [3.6; E] Estimation of random coefficient regression models. JStCmpSm 13:27–39

Liu L-M, Hanssen DM (1981) [3.6; CE] A Bayesian approach to time-varying cross-sectional regression models. JEconmtcs 15:341–356

Lo AW, Newey WK (1985) [2.1; M] A large-sample Chow test for the linear simultaneous equation. Econ. Letters 18:351–353

Lockwood LJ (1986) [3.2; C] Estimation of covariance components for random-walk regression parameters. Econ. Letters 21:251–255

Lodon G (1971) [1.1, 1.2; A] Procedures for reacting to a change in distribution. AnnMathStat 42:1897–1908

Lombard F (1981) [2.1, 2.2; N] An invariance principle for sequential nonparametric test statistics under contiguous alternatives. S. African Statist. J. 15:129–152

Lombard F (1983) [2.1, 2.2; N] Asymptotic distributions of rank statistics in the change-point problem. S. African Statist. J. 17:83–105

Lombard F (1986) [1.1, 1.4; NES] The change-point problem for angular data: A nonparametric approach. Technmcs 28:391–397

Lombard F (1987) [0.1, 1.1, 1.3; ACNT] Rank tests for changepoint problems. Biomtrka 74:615–624

Lombard F (1988) [1.1, 1.4; E] Detecting change points by Fourier Analysis. Technmcs 30:305–310

Lubrano M (1985) [2.4; CSE] Bayesian analysis of switching regression models. JEconmtcs 29:69–95

Lütkepohl H (1985) [4.1, 4.3; ACES] Prediction tests for structural stability. Techn. Report, Univ. Hamburg

Lütkepohl H (1987) [4.1, 4.3, 4.7; CES] Prediction tests for structural stability of multiple time series. Paper presented at the 1987 European Meeting of the Econometric Society in Copenhagen, Denmark

Machak JA, Spivey WA, Wrobleski WJ (1985) [3.4; CE] A framework for time varying parameter regression modeling. JBES 3:104–111

MacNeill IB (1974) [1.3; AT] Test for change of parameter at unknown times and distributions of some related functionals on Brownian motion. AnlsStat 2:950–962

MacNeill IB (1978)[2.1; A] Properties of sequences of partial sums of polynomial regression residuals with applications to tests for change of regression at unknown times. AnlsStat 6:422–433

Maddala GS, Nelson FD (1975) Switching regression models with exogenous and endogenous switching. ASAProBuEc, pp 423–426

Maddala GS, Trost RP (1981) [2.3; MC] Alternative formulations of the Nerlove-Press models. JEconmtcs 16:35–49

Mahajan YL (1977) Estimation of the monetarist model using varying parameter framework and its implication. ASA Proc. 1977 Annual Metting, pp 595–599

Mahajan YL, Mahajan LS (1977) [3.8; C] Efficiency of varying parameter estimator in a reducible simultaneous equation system. ASA Proc. 1977 Annual Meeting, 349–353

Malinauskas V, Lipeika A (1986) [4.4; MVE] Determination of the instants of changes in the properties of long multivariate autoregressive random sequences. In: Telksnys L (ed) Detection of changes in random processes. Springer, New York, pp 118–124

Maronna R, Yohai VJ (1978) [2.1; ATS] A bivariate test for the detection of a systematic change in mean. JASA 73:640–645

Matthews DE, Farewell VT, Pyke R (1985) Asymptotic score-statistic processes and tests for constant hazard against a change-point alternative. AnlsStat 13:583–591

McAleer M, Fisher G (1982) [2.1; AE] Testing separate regression models subject to specification error. JEconmtcs 19:125–145

McCabe BPM, Harrison MJ (1980) [2.1, 2.2; CS] Testing the constancy of regression relationships over time using least squares residuals. ApplStat 29:142–148

McGee VE, Carleton WT (1970) [2.3; VES] Piecewise regression. JASA 65:1109–1124

McGilchrist CA, Woodyer KD (1975) [1.1, 1.2; N] Note on a distribution-free CUSUM technique. Technmcs 17:321–325

McWhorter A, Narasimham GVL, Simonds RR (1977) [3.3; CEU] An empirical examination of the predictive performance of an econometric model with random coefficients. IntStRvw 45:243–255

McWhorter A, Spivey WA, Wrobleski WJ (1976) [3.3; RS] A sensitivity analysis of varying parameter econometric models. IntStRvw 44:265–282

Medvedev G, Kazachenok V (1986) [2.3; E] Estimation of the discontinuous regression function. In: Telksnys L (ed) Detection of changes in random processes. Springer, New York, pp 124–129

Mehra RK (1974) [3.4; C] Identification in control and econometrics: Similarities and differences. AnnEcSoMt 3:21–47

Mehta JS, Narasimham GVL, Swamy PAVB (1978) [2.3; E] Estimation of a dynamic demand function for gasoline with different schemes of parameter variation. JEconmtcs 7:263–279

Meinhold RJ, Singpurwalla ND (1983) [3.3; BE] Understanding the Kalman filter. AmerStat 37:123–127

Menzefricke U (1981) [1.5; E] A Bayesian analysis of a change in the precision of a sequence of independent normal random variables at an unknown time point. ApplStat 30:141–146

Mikhail WM, Ghazal GA (1979) [3.8] Testing the equality of coefficients estimated by two-stage least squares. Egypt. Statist. J. 23:52–70

Mizrach B, Santomero AM (1986) [2.1, 2.9; CE] The stability of money demand and forecasting through changes in regimes. REcon&St 68:324–328

Moen DH, Broemeling LD (1984) [2.4; ME] Testing for a change in the regression matrix of a multivariate linear model. CommStA 13:1521–1531

Moen DH, Broemeling LD (1985) [2.4, 2.9; E] The uncertainty of forecasting: Models with structural change versus those without changing parameters. CommStA 14:2029–2040

Moen DH, Salazar D, Broemeling LD (1985) [2.4; EM] Structural changes in multivariate regression models. CommStA 14:1757–1768

Moustakides GV, Benveniste A (1986) [4.1, 4.3; A] Detecting changes in the AR parameters of nonstationary ARMA processes. Stochastics 16:137–155

Muliere P, Scarsini M (1985) [1.1; BN] Change-point problems: A Bayesian non-parametric approach. Aplik. Matematiky 30:397–402

Murthy GVSN (1976) [3.2] On the estimation of general functional form with random coefficients. SankhyaC 38:37–43

Mustafi CK (1968) [1.1, 1.4; A] Inference problems about parameters which are subjected to changes over time. AnnMathStat 39:840–854

Nadler J, Robbins NB (1971) [1.1, 1.2] Some characteristics of Page's two-sided procedure for detecting a change in a location parameter. AnnMathStat 42:538–551

Nelder JA (1968) [3.4; CE] Regression, model-building and invariance. JRSS-A 131:303–315

Nicholls DF, Pagan AR (1985) [0.1, 3.1, 3.4, 3.8] Varying coefficient models. In: Hannan EJ et al (eds) Time series in the time domain. Handbook of Statistics 5. North Holland, Amsterdam

Nicholls DF, Quinn BG (1981) [4.4, 4.6; M] Multiple autoregressive models with random coefficients. JMultiAn 11:185–198

Nikiforov IV (1979) [4.1; S] Cumulative sums for detection of changes in random process characteristics. Autom. Rem. Contr. 40:192–200

Nikiforov IV (1980) [1.1; C] Modification and analysis of the cumulative sum procedure. Autom. Rem. Contr. 41:1247–1252

Nikiforov IV (1986) [1.2; C] Sequential detection of changes in time series properties based on a modified cumulative sum algorithm. In: Telksnys L (ed) Detection of changes in random processes. Springer, New York, pp 142–150

Norberg R (1977) [3.4; A] Inference in random coefficient regression models with one-way and nested classifications. ScandJSt 4:71–80

Oberhofer W (1980) [2.3; A] Die Nichtkonsistenz der M-L Schätzer im "Switching Regression" Problem. Metrika 27:1–13

Ohtani K (1982) [2.1, 2.3; S] Bayesian estimation of the switching regression model with autocorrelated errors. JEconmtcs, 18:251–261

Ohtani K (1987) [2.1, 2.3; S] Some small sample properties of tests for structural stability in a simultaneous equation. Econ. Letters 22:229–232

Ohtani K, Katayama S (1986) [2.5, 3.4; E] A gradual switching regression model with autocorrelated errors. Econ Letters 21:169–172

Ohtani K, Katayama S (1986) An alternative gradual switching regression model and its application. Economic Studies Quarterly 36:148–153

Ohtani K, Toyoda T (1985) [2.1; CS] Small sample properties of tests of equality between sets of coefficients in two linear regressions under heteroscedasticity. IntEconR 26:37–44

Otter PW (1978) [3.3; CE] The discrete Kalman filter applied to linear regression models: Statistical considerations and an application. Statistica Neerlandica 32:41–56

Pagan AR (1980) [3.4; AC] Some identification and estimation results for regression models with stochastically varying coefficients. JEconmtcs 13:341–363

Pagan AR, Tanaka K (1979) A further test for assessing the stability of regression coefficient. Canberra: Australian Nat. University

Page ES (1954) [1.2; C] Continuous inspection schemes. Biomtrka 41:100–115

Page ES (1955) [1.2; TS] A test for a change in a parameter occurring at an unknown time point. Biomtrka 42:523–527

Page ES (1957) [1.2; E] On problems in which a change of a parameter occurs at an unknown time point. Biomtrka 44:248–252

Parhizgari AM, Davis PS (1978) [3.2; CE] The residential demand for electricity: A varying parameters approach. Appl. Economics 10:331–340

Park SH (1978) [2.3, 2.8; V] Experimental designs for fitting segmented polynomial regression models. Technmcs 20:151–154

Parke DW, Zagardo J (1985) [3.2; E] Stochastic coefficient regression estimates of the sources of shifts into MMDA deposits using cross-section data. JEconmtcs 29:327–340

Parsons LJ, Schultz RL (1976) [3.5; E] Marketing models and econometric research. North-Holland, Amsterdam

Patterson KD (1987) [2.1; E] The specification and stability of the demand for money in the United Kingdom. Economica 54:41–55

Pesaran MH, Smith RP, Yeo JS (1985) [2.1; C] Testing for structural stability and predictive failure: A review. The Manchester School 53:280–295

Pettitt AN (1979) [1.1; NC] A non-parametric approach to the change-point problem. ApplStat 28:126–135

Pettitt AN (1980) [1.3, 1.4; ACS] A simple cumulative sum type statistic for the change-point problem with zero-one observations. Biomtrka 67:79–84

Phillips GDA, McCabe BP (1983) The independence of tests for structural change in regressions models. Econ. Letters 12:283–287

Picard D (1985) [1.1, 1.4; ACET] Testing and estimating change-points in time series. AdvAppPr 17:841–867

Ploberger W (1983) [2.1; AC] Testing the constancy of parameters in linear models. Techn. Report, Technische Universität, Vienna

Ploberger W, Kontrus K, Krämer W (1987) [2.1, 2.3; ACE] A perturbation test for structural stability in the linear regression model. Paper presented at the 1987 European Meeting of the Econometric Society in Denmark, Copenhagen

Ploberger W, Krämer W (1986) [2.1, 2.3; A] On studentizing a test for structural change. Econ. Letters 20:341–344

Ploberger W, Krämer W (1987) [2.1] Mean adjustment and the CUSUM test for structural change. Econ. Letters 25:255–258

Poirier DJ (1973) [2.8; E] Piecewise regression using cubic splines. JASA 68:515–524

Poirier DJ (1976) [0.1, 2.3, 2.8; B] The econometrics of structural change. North-Holland, Amsterdam

Poirier DJ, Ruud PA (1981) [2.5; C] On the apropriateness of endogenous switching. JEconmtcs 16:249–256

Pole AM, Smith AFM (1985) [2.1, 2.4, 2.5; BECS] A Bayesian analysis of some threshold switching models. JEconmtcs 29:97–119

Pollak M (1985) [1.1, 1.2; ACE] Optimal detection of a change in distribution. AnlsStat 13:206–227

Pollak M, Siegmund D (1985) [1.1; CE] A diffusion process and its application to detecting a change in the shift of Brownian motion. Biomtrka 72:267–280

Praagman J (1986) Efficiency of change-point tests. PhD thesis, University of Technology, Eindhoven

Quandt RE (1958) [2.1, 2.3; EV] The estimation of the parameters of a linear regression system obeying two separate regimes. JASA 53:873–880

Quandt RE (1960) [2.1, 2.3; AST] Test of the hypothesis that a linear system obeys two separate regimes. JASA 55:324–330

Quandt RE (1972) [2.1, 2.3, 2.5; CE] A new approach to estimating switching regressions. JASA 67:306–310

Quandt RE, Ramsay JB (1978) [1.4, 2.3; CE] Estimating mixtures of normal distributions and switching regressions (invited paper, with comments) JASA 73:730–752

Quinn BG, Nicholls DF (1982) [4.3; A] Testing for the randomness of autoregressive coefficients. JTimSrAn 3:123–135

Raferty AE, Akman VE (1986) [1.5; E] Bayesian analysis of a Poisson process with a change-point. Biomtrka 73:85–90

Raj B (1975) [3.8; CS] Linear regression with random coefficients: The finite sample and convergence properties. JASA 70:127–137

Raj B, Srivastava VK, Upadhyaya S (1980) [3.2; AE] The efficiency of estimating a random coefficient model. JEconmtcs 12:285–299

Raj B, Ullah A (1981) [3.2; ACE] Econometrics, a varying coefficients approach. Croom Helm, London

Ramirez MM (1984) [2.1; E] A modification to some proposed tests in relation to the problem of switching regression models. CommStA 13:901–914

Rao CR (1965) [3.1; C] The theory of least squares when the parameters are stochastic and its application to the analysis of growth curves. Biomtrka 52:447–458

Rao PSRS (1972) [2.3] On two phase regression estimators. SankhyaA 34:473–476

Rao ULG (1982) [3.2] A note on the unbiasedness of Swamy's estimator for the random coefficient regression model. JEconmtcs 18:395–401

Rausser GC, Mundlak Y (1978) Structural change, parameter variation, and agricultural forecasting. Harvard University, Cambridge

Rea JD (1978) [2.1] Indeterminancy of the Chow test when the number of observations is insufficient. Econmtca 46:229

Reinsel G (1979) [3.1, 3.2; C] A note on the estimation of the adaptive regression model. IntEconR 20:193–202

Richard J-F (1980) [2.1, 2.3; M] Models with several regimes and changes in exogeneity. Rev. Econ. Studies 47:1–20

Riddell WC (1980) [2.3; V] Estimating switching regression: A computational note. JStCmpSm 10:95–101

Robison DE (1964) [2.3] Estimates for the points of intersection of two polynomial regressions. JASA 59:214–224

Ronchetti E, Rousseeuw PJ (1985) [2.3; R] Change-of-variance sensitivities in regression analysis. ZeitWahr 68:503–519

Rosenberg B (1972) [3.8; C] The estimation of stationary stochastic regression parameters reexamined. JASA 67:650–654

Rosenberg B (1973) [3.4] Linear regression with randomly dispersed parameters. Biomtrka 60:65–72

Rosenberg B (1973) [3.4; BC] A survey of stochastic parameter regression. AnnEcSoMt 2:381–397

Rosenberg B (1973) [3.6; CV] The analysis of a cross-section of time series by stochastically convergent parameter regression. AnnEcSoMt 2:399–428

Rosenberg B (1977) [3.3] Estimation error covariance in regression with sequentially varying parameters. AnnEcSoMt 6:457–462

Salazar D (1982) [4.5, 2.4, 2.5; E] Structural changes in time series models. JEconmtcs 19:147–163

Salazar D, Broemeling LD, Chi A (1981) [2.4; 3.6; ER] Parameter changes in a regression model with autocorrelated errors. CommStA 10:1751–1758

Salmon M, Wallis KF (1982) Model validation and forecast comparison: Theoretical and practical considerations. In: Chow GC, Corsi P (eds) Evaluating the reliability of macro-economic models. Wiley, New York

Sant DT (1977) [3.2; A] Generalized least squares applied to time varying parameter models. AnnEcSoMt 6:301–311

Sarris AH (1973) [3.3] A Bayesian approach to estimation of time-varying regression coefficients. AnnEcSoMt 2:501–523

Sastri T (1986) A recursive algorithm for adaptive estimation and parameter change detection of time series models. J. Oper. Rsr. 37:987–999

Schechtman E (1982) [1.1; NCS] A nonparametric test for detecting changes in location. CommStA 11:1475–1482

Schechtman E (1983) [1.4; NS] A conservative nonparametric distribution-free confidence bound for the shift in the changepoint problem. CommStA 12:2455–2464

Schechtman E, Wolfe DA (1985) [1.1, 1.4; NS] Multiple changepoints problem: Nonparametric procedures for estimation of the points of change. CommStB 14:615–631

Schmidt P (1982) [1.4, 2.3; AE] An improved version of the Quandt-Ramsey MGF estimator for mixtures of normal distributions and switching regressions. Econmtca 50, 501–516

Schmidt P, Sickles R (1977) [2.1; E] Some further evidence on the use of the Chow test under heteroscedasticity. Econmtca 45:1293–1298

Schneeberger H (1973) [2.1] Punkt-, Intervallprognose und Test auf Strukturbruch mit Hilfe der Regressionsanalyse. In: Mertens P (ed) Prognoserechnung. Physica, Würzburg, pp 143–158

Schneider W (1986) [3.3, 4.4; AV] Der Kalmanfilter als Instrument zur Diagnose und Schätzung variabler Parameter in ökonometrischen Modellen. Physica, Heidelberg

Schulze U (1982) [2.3; BXCV] Estimation in segmented regression: Known number of regimes. MaOpfStS 13:295–316

Schulze U (1984) [2.1, 2.3] A method of estimation of change points in multiphasic growth models. BiomtrcJ 26:495–504

Schulze U (1987) [2.1, 2.2, 2.3; CEPT] Mehrphasenregression: Stabilitätsprüfung, Schätzung, Hypothesenprüfung. Akademie-Verlag, Berlin

Schweder T (1976) [2.1; E] Some "optimal" methods to detect structural shift or outliers in regression. JASA 71:491–501

Segen J, Sanderson AC (1980) [1.1, 1.2, 1.3, 4.1, 4.2; AM] Detecting change in a time-series. IEEEInfo, IT-26:249–255

Sen A, Srivastava MS (1973) [1.1; N] On multivariate tests for detecting change in mean. SankhyaA 35:173–186

Sen A, Srivastava MS (1975) [1.1; ABCN] On tests for detecting change in mean. AnlsStat 3:98–108

Sen A, Srivastava MS (1975) [1.1; BXC] Some one-sided tests for change in level. Technmcs 17:61–64

Sen PK (1977) [1.1, 1.3; NA] Tied-down Wiener process approximations for aligned rank order processes and some applications. AnlsStat 5:1107–1123

Sen PK (1980) [2.1; NA] Asymptotic theory of some tests for a possible change in the regression slope occurring at an unknown time point. ZeitWahr 52:203–218

Sen PK (1982) [2.1; NA] Asymptotic theory of some tests for constancy of regression relationships over time. MaOpfStS 13:21–31

Sen PK (1983) On some recursive residual rank tests for change-points. In: Rizvi MH et al (eds) Recent advances in statistics: Papers in honor of Herman Chernoff on his sixtieth birthday. Academic Press, New York, pp 371–391

Sen PK (1982–83) [1.1, 1.2; A] Tests for change-points based on recursive U-statistics. SqtlAnly 1:263–284

Sen PK (1984) [2.1, 2.2; NA] Nonparametric procedures for some miscellaneous problems. In: Krishnaiah PR, Sen PK (eds) Nonparametric methods. Handbook of Statistics, vol 4. North-Holland, Amsterdam, pp 699–739

Sen PK (1984) [2.1, 2.2; AMN] Recursive M-tests for the constancy of multivariate regression relationships over time. SqtlAnly 3:191–211

Senkus A (1978) The determination of the change-point in an autoregressive sequence. In: Trans. of the 8th Prague conf. on information theory, statistical decision functions, random processes. Academica, Prague, pp 193–200

Shaban SA (1980) [0.1] Change point problem and two-phase regression: An annotated bibliography. IntStRvw 48:83–93

Shiryaev AN (1963) On optimum methods in quickest detection problems. ThProbAp 8:22–46

Shiryaev AN (1965) Some exact formulas in a "disorder" problem. ThProbAp 10:348–354

Siegmund D (1988) [1.4; ACS] Confidence sets in change-point problems. IntStRvw 56:31–48

Simos EO (1977) [3.5; E] The demand for money specification based on the stability criterion: Empirical evidence for the Italian economy. Riv Int Sc Econ Comm 24:943–951

Singh B, Nagar AL, Choudhry NK, Raj B (1976) [3.2, 3.4; CEV] On the estimation of structural change: A generalization of the random coefficients regression models. IntEconR 17:340-361

Singh B, Ullah A (1974) [3.2; AC] Estimation of seemingly unrelated regressions with random coefficients. JASA 69:191–195

Singpurwalla ND (1974) [2.3] Estimation of the join point in a heteroscedastic regression model arising in accelerated life tests. CommStA 3:853–863

Smith AFM (1975) [1.5; E] A Bayesian approach to inference about a change-point in a sequence of random variables. Biomtrka 62:407–416

Smith AFM (1977) [2.4, 4.3; E] A Bayesian analysis of some time-varying models. In: Barra JR et al (eds) Recent developments in statistics. North-Holland, Amsterdam, pp 257–267

Smith AFM (1980) Change-point problems: Approaches and applications. In: Bernardo JM et al. (eds) Bayesian statistics. Valencia University Press, Valencia, pp 83–98

Smith AFM, Cook DG (1980) [2.4, 3.6; E] Straight lines with a change-point: A Bayesian analysis of some renal transplant data. ApplStat 29:180–189

Smith PL (1982) [2.4, 3.6; E] Hypothesis testing in B-spline regression. CommStA 11:143–157

Spjøtvoll E (1977) Random coefficients regression models. A review. Statistics 8:69–93

Sprent P (1961) [2.1; E] Some hypotheses concerning two phase regression lines. Biomtrcs 17:643–645

Srivastava MS, Worsley KJ (1986) [1.1; MET] Likelihood ratio tests for a change in the multivariate normal mean. JASA 81:199–204

Srivastava TN (1967) A problem in life testing with changing failure rate. Defence Sci. J. 17:163–168

Srivastava TN (1975) [1.6; A] Life tests with periodic change in failure rate: Grouped observations. JASA 70:394–397

Srivastava VK, Mishra GD, Chaturvedi A (1981) [3.2; V] Estimation of linear regression model with random coefficients ensuring almost non-negativity of variance estimators. BiomtrcJ 23:3–8

Steyn I (1987) [3.3; CE] Recursive estimation of parameters in the Kalman filter. Report AE8/87, Univ. of Amsterdam

Stoker TM (1985) [2.3] Aggregation, structural change and cross-section estimation. JASA 80:720–729

Sunder S (1975) [3.5; E] Stock price and risk related to accounting changes in inventory valuation. Account Rev 50:305–315

Sunder S (1980) [2.1; E] Stationarity of market risk: Random coefficients tests for individual stocks. J Finance 35:883–896

Swamy PAVB (1970) [3.2; ACE] Efficient inference in a random coefficient regression model. Econmtca 38:311–323

Swamy PAVB (1971) [3.2; ACE] Statistical inference in random coefficient regression models. Springer, Berlin

Swamy PAVB (1973) [3.4, 3.8; C] Criteria, constraints and multicollinearity in random coefficient regression models. AnnEcSoMt 2:429–450

Swamy PAVB (1974) Linear models with random coefficients. In: Zarembka P (ed) Frontiers in Econometrics. Academic Press, New York, pp 143–168

Swamy PAVB, Mehta JS (1975) [2.3, 2.4, 3.6; CM] Bayesian and non-Bayesian analysis of switching regressions and of random coefficient regression models. JASA 70:593–602

Swamy PAVB, Mehta JS (1977) [3.2; A] Estimation of linear models with time and cross-sectionally varying coefficients. JASA 72:890–898

Swamy PAVB, Mehta JS (1977) Estimation of common coefficients in two regression equations. JEconmtcs 10:1–14

Swamy PAVB, Tinsley PA (1980) [3.9; E] Linear prediction and estimation methods for regression models with stationary stochastic coefficients. JEconmtcs 12:103–142

Talwar PP (1983) [1.1; TCR] Detecting a shift in location. Some robust tests. JEconmtcs 23:353–367

Talwar PP, Gentle JE (1981) [1.3; CR] Detecting a scale shift in random sequence at an unknown time point. ApplStat 30:301–304

Tanaka K (1983) [2.1, 2.6; A] Non-normality of the Lagrange multiplier statistics for testing the constancy of regression coefficients. Econmtca 51:1577–1582

Telksnys L (ed) (1986) Detection of changes in random processes. Springer, New York

Telksnys L (1986) [1.1, 4.1; CE] The reliability and resolution of detecting the instants of change in properties of random sequences. In: Telksnys L (ed) Detection of changes in random processes. Springer, New York, pp 175–183

Terza JV, Okoruwa AA (1985) [2.7; EV] An algorithm for the estimation of Poisson regressions involving structural change. CommStB 14:853–866

Tishler A, Zang I (1979) [3.4; V] A switching regression method using inequality conditions. JEconmtcs 11:259–274

Tishler A, Zang I (1981) [2.1, 2.3, 2.5; EV] A maximum likelihood method for piecewise regression models with a continuous dependent variable. ApplStat 30:116–124

Tishler A, Zang I (1981) A new maximum likelihood algorithm for piecewise regression. JASA 76:980–987

Toyoda T (1974) [2.1; T] Use of the Chow test under heteroscedasticity. Econmtca 42:601–608

Toyoda T, Ohtani K (1986) Testing equality between sets of coefficients after a preliminary test for equality of disturbance variances in two linear regressions. JEconmtcs 31:67–80

Trifonov A, Butejko V (1986) [1.1, 1.4; S] The effectiveness of detection algorithms and estimates of change in a Wiener process. In: Telksnys L (ed) Detection of changes in random processes. Springer, New York, pp 184–194

Tsurumi H (1977) [2.4; E] A Bayesian test of a parameter shift and an application. JEconmtcs 6:371–80

Tsurumi H (1978) [2.4; ME] A Bayesian test of a parameter shift in a simultaneous equation with an application to a macro savings function. Econ. Stud. Quarterly 24:216–230

Tsurumi H (1980) [2.4, 2.5; M] A Bayesian estimation of structural shifts by gradual switching regressions with an application to the US gasoline market. In: Zellner A (ed) Bayesian analysis in econometrics and statistics: Essays in honor of Harold Jeffreys. North-Holland, Amsterdam, pp 213–240

Tsurumi H (1982) [2.4; MC] A Bayesian and maximum likelihood analysis of a gradual switching regression in a simultaneous equation framework. JEconmtcs 19:165–182

Tsurumi H (1983) [2.3, 2.4, 2.5; CSR] A Bayesian and maximum likelihood analysis of a gradual switching regression model with sampling experiments. Econ. Stud. Quarterly 34:237–248

Tsurumi H, Sheflin N (1984) [2.4; E] Bayesian tests of a parameter shift under heteroscedasticity: Weighted-t vs, double-t approaches. CommStA 13:1003–1013

Tsurumi H, Sheflin N (1985) [2.4; ES] Some tests for the constancy of regressions under heteroscedasticity. JEconmtcs 27:221–234

Tsurumi H, Shiba T (1982) [3.6; CS] A Bayesian analysis of a random coefficient model in a simple Keynesian model. JEconmtcs 18:239–249

Van Dobben de Bruyn CS (1968) [1.1, 1.2; VT] Cumulative sum tests: Theory and practice. Griffin, London

Vostrikova LJ (1981) [1.1; M] Detecting "disorder" in multidimensional random processes. Soviet Mathematics Doklady 24:55–59

Waragai T, Akiba H (1987) [4.3; E] Structural changes in the foreign exchange market: An application of AIC to seven major exchange rates. Paper presented at the 1987 European Meeting of the Econometric Society in Denmark, Copenhagen

Watson MW, Engle RF (1983) [3.4; CE] Alternative algorithms for the estimation of dynamic factor, mimic and varying coefficient regression models. JEconmtcs 23:385–400

Watson MW, Engle RF (1985) [2.1, 2.6; ACES] Testing for regression coefficient stability with a stationary AR(1) alternative. REcon&St 67:341–346

Watson PK (1983) [3.3; CE] Kalman filtering as an alternative to ordinary least squares: Some theoretical considerations and empirical results. Emp. Economics 8:71–85

Watt PA (1979) [2.1; CS] Tests of equality between sets of coefficients in two linear regressions when disturbance variances are unequal: Some small sample properties. The Manchester School 47:391–396

Watts DG, Bacon DW (1974) [2.3, 2.5] Using an hyperbola as a transition model to fit two-regime straight-line data. Technmcs 16:369–373

Weerahandi S (1987) [2.1; C] Testing regression equality with unequal variances. Econmtca 55:1211–1215

Westlund A (1983) [3.9; R] A note on partial structural variability and prediction bias effects in ID-systems. In: Proc 44th ISI Meeting (Madrid), pp 329–332

Westlund A (1984) [2.1, 2.2; E] Sequential moving sums of squares of OLS residuals in parameter stability testing. Quality and Quantity 18:261–273

Westlund A (1984) [2.1, 2.2; E] On structural stability testing by a ratio of moving sums of squares of OLS-residuals. Adv. Mod. Sim. 2:27–45

Westlund A (1985) [2.1; CS] On the power of some tests for examining the stability of regression coefficients. JStCmpSm 21:275–289

Westlund AH (1987) [3.9; R] Partial parameter instability and small sample prediction bias. In: Puri ML et al (eds) New perspectives in theoretical and applied statistics. John Wiley, New York, pp 383–395

Westlund A, Brännäs K (1979) [3.3] On the recursive estimation of stochastic and time-varying parameters in econometric systems. In: Iracki K et al. (eds) Optimization techniques. Springer, Berlin, pp 414–422

Westlund A, Brännäs K (1982) [3.3; RS] Robustness properties of a Kalman filter estimator in interdependent systems with time-varying parameters. In: Charatsis EG (ed) Selected papers on contemporary econometric problems. North-Holland, Amsterdam, pp 49–73

Westlund A, Zackrisson U (1986) [2.9; E] On the prediction of structurally varying systems. Techn. Forecasting and Social Change 30:63–72

Westlund A, Brännäs K, Eklof JA, Stenlund H (1981) [3.3; RS] Econometrics and stochastic control in macro-economic planning. Almqvist and Wiksell, Stockholm

Westlund AH, Törnkvist B (1985) [2.1; CT] On the identification of time for parameter variabilities. Metron 43:21–40

Wheeler FP (1984) [4.1] Detecting change in the time scale of short term fluctuations of a time series. In: Anderson OD (ed) Time series analysis: Theory and practice 5. North-Holland, Amsterdam, pp 269–279

Wichern DW, Miller RB, Hsu DA (1976) [4.1; SE] Changes of variance in first-order autoregressive time series models. With an application. ApplStat 25:248–256

Wilson AL (1978) [2.1; C] When is the Chow test UMP? AmerStat 32:66–68

Wilton DA (1975) [2.3, 2.5] Structural shift with an inter-structural transition function. Canad. J. Economics 8:423–432

Wimmer G (1980) [3.8; CE] Estimation of random regression parameters. BiomtrcJ 22:131–139

Wittink DR (1977) [3.1; E] Exploring territorial differences in the relationship between marketing variables. J. Market. Research 14:145–155

Wold S (1974) [2.8] Spline functions in data analysis. Technmcs 16:1–11

Wolfe DA, Schechtman E (1984) [1.1; NCS] Nonparametric statistical procedures for the changepoint problem. UStPlInf 9:389–396

Wolff CCP (1987) [2.9, 3.3; E] Time-varying parameters and the out-of-sample forecasting performance of structural exchange rate models. JBES 5:87–98

Woodward RH, Goldsmith PL (1964) [1.1, 1.2] Cumulative sum techniques. Oliver and Boyd, Edinburgh

Worsley KJ (1979) [1.1; T] On the likelihood ratio test for a shift in location of normal populations. JASA 74:365–367

Worsley KJ (1982) [1.1] An improved Bonferroni inequality and applications. Biomtrka 69:297–302

Worsley KJ (1983) [2.3] Testing for a two-phase multiple regression. Technmcs 25:35–42

Worsley KJ (1983) [1.1; ETA] The power of likelihood ratio and cumulative sum tests for a change in a binomial probability. Biomtrka 70:455–464

Worsley KJ (1986) Confidence regions and tests for a change-point in a sequence of exponential family random variables. Biomtrka 73:91–104

Wright RL, Dielman T, Nantell TJ (1977) [3.2; E] Analysis of stock repurchases with a random coefficient regression model. ASA Proc 1977 Annual Meeting, pp 345–348

Yao Y-C (1984) [1.5; CSV] Estimation of a noisy discrete-time step function: Bayes and empirical Bayes approaches. AnlsStat 12:1434–1447

Yao Y-C (1988) [1.1, 1.4, 1.6; A] Estimating the numer of change-points via Schwarz' criterion. Statist. Prob. Letters 6:181–189

Yokum TJ Jr., Wildt AR (1987) [2.9; CE] Forecasting sales response for multiple time horizons and temporally aggregated data: A comparison of constant and stochastic coefficient. Intern. J. Forecasting 3:479–488

Yoshida M (1984) [1.1, 1.2] Probability maximizing approach to a secretary problem with random change-point of the distribution law of the observed process. JAppProb 21:98–107

Zacks S (1983) Survey of classical and Bayesian approaches to the change-point problem: Fixed sample and sequential procedures of testing and estimation. In: Rizvi MH et al. (eds) Recent advances in statistics. Academic Press, New York, pp 245–269

4 List of Journal Abbreviations

AmerStat	American Statistician
AnlsStat	Annals of Statistics
AnnEcSoMt	Annals of Economic and Social Measurement
AnnMathStat	The Annals of Mathematical Statistics
ApplStat	Applied Statistics
ASAProBuEc	ASA Proceedings of Business and Economic Statistics Section
AstrlJSt	Australian Journal of Statistics
BiomtrcJ	Biometrical Journal
Biomtrcs	Biometrics
Biomtrka	Biometrika
CommStA	Communications in Statistics, Part A – Theory and Methods
DecisnSc	Decision Siences
Econmtca	Econometrica
IEEEAuCn	IEEE Transactions on Automatic Control
IEEEInfo	IEEE Transactions on Information Theory
IntEconR	International Economic Review
IntStRvw	International Statistical Review
JAppProb	Journal of Applied Probability
JASA	Journal of the American Statistical Association

JBES	Journal of Business and Economic Statistics
JEconmtcs	Journal of Econometrics
JIMaAppl	Journal of the Institute of Mathematics and its Applications
JMultiAn	Journal of Multivariate Analysis
JRRS-B	Journal of the Royal Statistical Society, Series B
JStCmpSm	Journal of Statistical Computation and Simulation
JStPlInf	Journal of Statistical Planning and Inference
JTimSrAn	Journal of Time Series Analysis
MaOpfStS	Mathematische Operationsforschung und Statistik, Series Statistics
REcon&St	Review of Economics and Statistics
ScandJSt	Scandinavian Journal of Statistics
SqtlAnly	Sequential Analysis
Technmcs	Technometrics
ThProbAp	Theory of Probability and its Applications
ZeitWahr	Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete

References

Broemeling LD, Tsurumi H (1987) See Chapter 3

Chow GC (1984) See Chapter 3

Hackl P (ed) (1988) Analysis and forecasting of economic structural change. Springer, New York

Hackl P, Westlund AH (1985) Statistical analysis of "Structural Change". An annotated bibliography. Collaborative Paper CP-85-31, Laxenburg/Austria: International Institute for Systems Analysis

Hinkley DV, Chapman P, Ranger G (1980) See Chapter 3

Johnson LW (1977) See Chapter 3

Johnson LW (1980) See Chapter 3

Judge GG, Griffiths WE, Carter Hill R, Lütkepohl H, Lee TC (1985) See Chapter 3

Nicholls DF, Pagan AR (1985) See Chapter 3

Oberhofer W, Kmenta J (1973) Estimation of standard errors of characteristic roots of a dynamic econometric model. Econmtca 41:171–177

Schulze U (1987) See Chapter 3

Shaban SA (1980) See Chapter 3

Theil H, Boot JCG (1962) The final form of econometric equation systems. Rev Intern Statist Inst 30:136–152