Chapter 12
Milnor Number and Milnor Classes

Abstract Both Schwartz–MacPherson and Fulton–Johnson classes generalize Chern classes to the case of singular varieties. It is known that for local complete intersections with isolated singularities, the 0-degree SM and FJ classes differ by the local Milnor numbers [149] and all other classes coincide [155]. As we explain in the sequel, if V has nonisolated singularities, the difference $c_i^{SM}(V) - c_i^{FJ}(V)$ of the SM and FJ classes is, for each i, a homology class with support in the homology $H_{2i}(\mathrm{Sing}(V))$ of the singular set of V. That is the reason for which their difference was called in [30, 31] the *Milnor class* of degree i. These classes have been also considered, from different viewpoints, by other authors, most notably by P. Aluffi, T. Ohmoto, A. Parusiński, P. Pragacz, J. Schürmann, S. Yokura.

In this chapter we introduce the Milnor classes of a local complete intersection V of dimension $n \geq 1$ in a complex manifold M, defined by a regular section s of a holomorphic bundle N over M. The aim of this chapter is to show that, as mentioned above, the Milnor classes are localized at the connected components of the singular set of V: If S is such a component then one has Milnor classes $\mu_i(V, S)$ of V at S in degrees $i = 0, \cdots, \dim S$. The 0-degree class coincides with the *generalized Milnor number* of V at S, introduced by Parusiński in [127] (if V is a hypersurface in M). The sum of all the Milnor classes over the connected components of $\mathrm{Sing}(V)$ gives the global Milnor classes studied in [8, 126, 131, 169]. See [28] for another presentation.

The method we use for constructing the localized Milnor classes comes from [31] and uses Chern–Weil theory. The idea is to use stratified frames to localize at the singular set the Schwartz–MacPherson and the Fulton–Johnson classes, in such a way that the difference of these localizations is canonical.

J.-P. Brasselet et al., *Vector Fields on Singular Varieties*,
Lecture Notes in Mathematics 1987, DOI 10.1007/978-3-642-05205-7_12,

12.1 Milnor Classes

For most authors, Milnor classes are globally defined as elements in $H_*(V,\mathbb{Z})$, on the other hand in [31], these classes are localized at the singular set of V from the beginning. We explain this in a moment, first we introduce the global classes; there is one such class in each degree:

Definition 12.1.1. For each $r = 0, 1, \cdots, n-1$, the r-th *Milnor class* $\mu_r(V)$ of V is:

$$\mu_r(V) \;=\; (-1)^{n+1}\big(c_r^{\mathrm{SM}}(V) \;-\; c_r^{\mathrm{FJ}}(V)\big) \quad \text{in} \quad H_{2r}(V,\mathbb{Z}).$$

The difference class

$$\mu_*(V) = (-1)^{n-1}(c_*^{\mathrm{SM}}(V) - c_*^{\mathrm{FJ}}(V))$$

is called the (total) Milnor class of V.

In fact, FJ-classes and SM-classes coincide with the usual Chern classes in the regular part of V. Thus Milnor classes ought to be concentrated in the singular set $\mathrm{Sing}(V)$. The results of [31,129] prove that this is indeed the case. Since the results of [149, 155] prove that in the case of isolated singularities this contribution corresponds to the local Milnor number at each singular point, and this is a local invariant of the singularity (not a global one), we considered in [31] *Milnor classes localized* at the connected components of the singular set of V. For each connected component S of $\mathrm{Sing}(V)$, the r-th *Milnor class* $\mu_r(V,S)$ of V at S is a homology class in $H_{2r}(S,\mathbb{Z})$. There is one such class for each $r = 0, 1, \cdots, s$, where s is the dimension of the component S. The inclusion $S \hookrightarrow V$ maps the homology of S into that of V, and adding up the contributions in each dimension of all the connected components of $\mathrm{Sing}(V)$ we get the corresponding global Milnor classes.

For hypersurfaces, the 0-degree localized Milnor class $\mu_0(V,S) \in H_0(S) \simeq \mathbb{Z}$ coincides with the *generalized Milnor number* of Parusiński [127], that we will discuss in Sect. 12.4. Thus $\mu_0(V,S)$ can be also considered as a generalized Milnor number for complete intersections.

Each connected component S has a contribution $\mu_r(V,S)$ to the global Milnor class $\mu_r(V)$ up to the dimension of S. Therefore, if $\mathrm{Sing}(V)$ has dimension 0, then all Milnor classes vanish in dimensions $r > 0$, *i.e.*, the SM and FJ classes coincide for all $r > 0$. If some component has dimension 1, then we have corresponding Milnor classes in dimensions 0, 1, and so on.

Since for isolated singularities the “Milnor classes” are just the Milnor numbers, which can be regarded as the number of vanishing cycles in the local Milnor fibers, it was natural to ask in [31] whether Milnor classes are related to the *vanishing homology.* Answers were given in [31] in particular cases, one of them is the Lefschetz type Theorem 12.3.1.

12.2 Localization of Milnor Classes

Let V be a local complete intersection of dimension n defined by a section of a vector bundle N over the ambient complex manifold M of dimension $m = n + k$, as in the previous section. We introduce the Milnor classes of V at a connected component S of $\mathrm{Sing}(V)$. For $r \geq 1$, let $v^{(r)}$ be an r-frame on $(U \setminus S) \cap D^{(2q)}$, where U is a neighborhood of S in V such that $U \setminus S \subset V_0$ and $q = m - r + 1$.

Definition 12.2.1. The $(r-1)$-st *Milnor class* $\mu_{r-1}(V, S)$ of V at S is defined by

$$\mu_{r-1}(V, S) = (-1)^{n+1} \left(\mathrm{Sch}(v^{(r)}, S) - \mathrm{Vir}(v^{(r)}, S)\right) \qquad \text{in} \quad H_{2r-2}(S),$$

which is independent of the choice of $v^{(r)}$ by (10.5.1) and (11.4.3).

We call $\mu_*(V, S) = \sum_{r \geq 0} \mu_r(V, S) \in H_*(S)$ the total Milnor class of V at S. Note that $\mu_r(V, S) = 0$ for $r > \dim_{\mathbb{C}} S$. Since there exist always frames as in Theorems 10.5.2 and 11.4.4, we have:

Theorem 12.2.1. *For a subvariety V of a complex manifold M as above,*

$$c_*(V) = c_*^{\mathrm{FJ}}(V) + (-1)^{n+1} \sum_S i_* \mu_*(V, S) \qquad \textit{in} \quad H_*(V),$$

where the sum is taken over the connected components S of $\mathrm{Sing}(V)$.

In particular, if the singularities of V are isolated points, then the Milnor classes are zero, except in degree 0 where they coincide with the usual Milnor numbers of [79,116,121]. Hence, in this case the SM classes and the FJ classes of V coincide in all dimensions, except in degree 0, where their difference is given by the sum of the usual Milnor numbers, recovering the formula in [149, 155].

Remark 12.2.1. 1. The classes $\mathrm{PH}(v^{(r)}, S)$, $\mathrm{Sch}(v^{(r)}, S)$ and $\mathrm{Vir}(v^{(r)}, S)$ may be defined for an r-frame $v^{(r)}$ on the intersection of a neighborhood of $\partial\mathcal{T}$ (in V) and $D^{(2q)}$, where $\mathcal{T} = \widehat{\mathcal{T}} \cap V$ with $\widehat{\mathcal{T}}$ a cellular tube around S.

2. If $r = 1$, *i.e.*, $v^{(1)} = (v)$, $\mathrm{PH}(v, S)$, $\mathrm{Sch}(v, S)$ and $\mathrm{Vir}(v, S)$ are called and denoted, respectively, the Poincaré–Hopf index $\mathrm{Ind}_{\mathrm{PH}}(v, S)$, the Schwartz index $\mathrm{Ind}_{\mathrm{Sch}}(v, S)$ and the virtual index $\mathrm{Ind}_{\mathrm{Vir}}(v, S)$ of the vector field v [71,111,148,149]. The corresponding Milnor class $\mu_0(V, S)$ is a number which will be discussed in Sect. 12.4.

12.3 Differential Geometric Point of View

In this section, we give a Lefschetz type formula for the Milnor classes at a nonsingular connected component S of the singular set of V under the assumption that V satisfies the Whitney condition along S. For the detailed proof, we refer to [31].

Let $\widehat{U}$ be a tubular neighborhood of S in M with C^∞ projection $\widehat{\rho} : \widehat{U} \to S$. We set $U = \widehat{U} \cap V$ and $U_0 = U \setminus S$ and denote by ρ and ρ_0, respectively, the restrictions of $\widehat{\rho}$ to U and U_0. From the Whitney condition, we see that the fibers of ρ are transverse to V and that S is a deformation retract of U with retraction ρ. We identify $\rho_0^*(N|_S)$ with N_{U_0}, and $\widehat{\rho}^*(N|_S)$ with $N|_{\widehat{U}}$. The bundle $T\widehat{\rho}$ of vectors in $T\widehat{U}$ tangent to the fibers of $\widehat{\rho}$ admits a complex structure, since it is C^∞ isomorphic with the normal bundle of the complex submanifold S in V. Let $\widehat{\mathcal{T}}$ be a (D)-cellular tube around S in $\widehat{U}$ and $\widehat{\mathcal{R}}$ a (D')-cellular tube in $\widehat{\mathcal{T}}$ as in Sect. 10.5.2. We set $\mathcal{T} = \widehat{\mathcal{T}} \cap V$ and $\mathcal{R} = \widehat{\mathcal{R}} \cap V$ as before.

Let s denote the complex dimension of S and let $v^{(r-1)}$ be an $(r-1)$-frame on the $2(s-r+1)$-skeleton $S \cap D^{(2q)}$ of S. In what follows, we set $\ell = s-r+1$. By the Schwartz construction, there exists a radial r-field $v_0^{(r)} = (v_0^{(r-1)}, v_0)$ on $\widehat{\mathcal{T}} \cap D^{(2q)}$ such that $v_0^{(r-1)}$ extends $v^{(r-1)}$. The radial vector field v_0 is tangent to U_0 and possibly has singularities in the barycenters of 2ℓ-cells in $S \cap D^{(2q)}$. We may assume that v_0 is tangent to the fibers of $\widehat{\rho}$ near $\partial\widehat{\mathcal{R}}$.

Let v be a vector field on $U_0 \cap D^{(2q)}$ which is nonsingular and tangent to the fibers of ρ in a neighborhood U_0' of $\partial\mathcal{R}$ so that $v^{(r)} = (v_0^{(r-1)}, v)$ is an r-frame on $U_0' \cap D^{(2q)}$. For example, the above v_0 has these properties.

For a point x in $S \cap D^{(2q)}$, let $\widehat{U}_x$ denote the fiber of $\widehat{\rho}$ at x and set $U_x = \widehat{U}_x \cap V$, which is the fiber of ρ at x. We also set $\mathcal{R}_x = \mathcal{R} \cap U_x$. The restriction of v to U_x determines the Schwartz index $\mathrm{Ind}_{\mathrm{Sch}}(v, S)$ and the virtual index $\mathrm{Ind}_{\mathrm{Vir}}(v, S)$ on U_x. By the Whitney condition, these indices do not depend on x.

Recall that we have the difference $d_S(v_0^{(r)}, v^{(r)})$ in $H_{2r-2}(S)$. We also have the difference $d(v_0, v)$, which is an integer, of v_0 and v as vector fields on U_x.

Lemma 12.3.1. *We have*

$$d_S(v_0^{(r)}, v^{(r)}) = d(v_0, v) \cdot c_{r-1}(S).$$

Proof. We consider the exact sequence of vector bundles on U_0:

$$0 \longrightarrow T\rho_0 \longrightarrow TU_0 \longrightarrow \rho_0^* TS \longrightarrow 0,$$

where $T\rho_0$ denotes the bundle of vectors in TU_0 tangent to the fibers of ρ_0. We may assume that $v_0^{(r)}$ and $v^{(r)}$ are r-frames on a neighborhood W of $U_0' \cap D^{(2q)}$. Let ∇_1^ρ and ∇_2^ρ be, respectively, v_0-trivial and v-trivial connections for $T\rho_0$

on W. Also let ∇^S be a $v_0^{(r-1)}$-trivial connection for TS on a neighborhood of $S \cap D^{(2q)}$. We take connections ∇_1 and ∇_2 for TU_0 so that $(\nabla_1^\rho, \nabla_1, \rho_0^* \nabla^S)$ and $(\nabla_2^\rho, \nabla_2, \rho_0^* \nabla^S)$ are both compatible with the above sequence. Thus ∇_1 is $v_0^{(r)}$-trivial and ∇_2 is $v^{(r)}$-trivial on W. By Lemma 10.5.1, the homology class $d_S(v_0^{(r)}, v^{(r)})$ is determined by

$$c^p(\nabla_1, \nabla_2) = \sum_{i+j=p} c^i(\nabla_1^\rho, \nabla_2^\rho) \cdot \rho_0^* c^j(\nabla^S). \tag{12.3.1}$$

We recall the commutative diagram

$$\begin{array}{ccc} H^{2q}(\widehat{U}, \widehat{U} \setminus S) & \xrightarrow[\hat{\rho}_*]{\sim} & H^{2\ell}(S) \\ \wr \big\downarrow A_M & & \wr \big\downarrow \\ H_{2r-2}(S) & \xrightarrow{=} & H_{2r-2}(S), \end{array} \tag{12.3.2}$$

where the first row is the inverse of the Thom isomorphism, given by integration along the fibers of $\widehat{\rho}$, and the second column is Poincaré duality. The dual of the first row in (12.3.2) gives an isomorphism

$$H_{2q}(\widehat{U}, \widehat{U} \setminus S) \xleftarrow{\sim} H_{2\ell}(S),$$

which shows that every relative $2q$-cycle γ (is homologous to a cycle which) fibers over a 2ℓ-cycle ζ of S. By the projection formula, we get from (12.3.1) (note that the rank of the bundle $T\rho_0$ is $n-s$):

$$\int_{\gamma \cap \partial \mathcal{R}} c^p(\nabla_1, \nabla_2) = \int_{\partial \mathcal{R}_x} c^{n-s}(\nabla_1^\rho, \nabla_2^\rho) \cdot \int_\zeta c^\ell(\nabla^S),$$

where x is a point in ζ. Noting that the first factor in the right hand side is $d(v_0, v)$, we proved the lemma, in view of (12.3.2).

Since $\mathrm{Ind}_{\mathrm{Sch}}(v_0^{(r)}, S) = c_{r-1}(S)$ and $\mathrm{Ind}_{\mathrm{Sch}}(v_0, x) = 1$, from Lemma 12.3.1, we have the following:

Theorem 12.3.3. *Let S be a nonsingular component of* $\mathrm{Sing}(V)$ *such that V satisfies the Whitney condition along S, then,*

$$\mathrm{Sch}(v^{(r)}, S) = \mathrm{Ind}_{\mathrm{Sch}}(v, x) \cdot c_{r-1}(S).$$

Now we wish to obtain a formula for the virtual class analogous to the one in Theorem 12.3.3. First, we consider the exact sequence of vector bundles on U_0:

$$0 \longrightarrow T\rho_0 \longrightarrow T\widehat{\rho}|_{U_0} \longrightarrow N_{U_0} \longrightarrow 0. \tag{12.3.4}$$

We compute the Chern classes $c^j(\tau_\rho)$ of the virtual bundle $\tau_\rho = (T\widehat{\rho} - N)|_U$ on U and will see that there is a canonical lifting $c_S^j(\tau_\rho)$ in $H^{2j}(U, U \setminus S)$, for $j > n - s = \operatorname{rank} T\rho_0$, of $c^j(\tau_\rho) \in H^{2j}(U)$. For this, we consider the covering $\mathcal{U}$ of $\widehat{U}$ consisting of U itself and a tubular neighborhood $\widehat{U}_0$ of U_0 and represent $c^j(\tau_{\rho_M})$, $\tau_{\rho_M} = T\widehat{\rho} - N$, as a Čech-de Rham cocycle on $\mathcal{U}$ (cf. [102, 156], here we use the notation in [156, Ch.II]).

Let ∇_0^ρ be a connection for $T\rho_0$. Let ∇^N be a connection for $N|_S$ and take a connection $\nabla_0^{\hat{\rho}}$ for $T\widehat{\rho}|_{U_0}$ so that $(\nabla_0^\rho, \nabla_0^{\hat{\rho}}, \rho_0^*\nabla^N)$ is compatible with (12.3.4). Let $\hat{\nabla}^{\hat{\rho}}$ be a connection for $T\widehat{\rho}$ on $\widehat{U}$. We set $\nabla^{\widehat{\rho}\bullet} = (\hat{\nabla}^{\hat{\rho}}, \widehat{\rho}^*\nabla^N)$ and $\nabla_0^{\widehat{\rho}\bullet} = (\nabla_0^{\widehat{\rho}}, \rho_0^*\nabla^N)$. Then $c^j(\tau_{\widehat{\rho}})$ is represented by a cocycle in $A^{2j}(\mathcal{U}) = A^{2j}(\widehat{U}_0) \oplus A^{2j}(\widehat{U}) \oplus A^{2j-1}(\widehat{U}_0)$, where $A^*(\ )$ denotes the space of differential forms on the relevant open set, given by

$$c^j(\nabla_\star^\bullet) = (c^j(\nabla_0^{\widehat{\rho}\bullet}), c^j(\nabla^{\widehat{\rho}\bullet}), c^j(\nabla_0^{\widehat{\rho}\bullet}, \nabla^{\widehat{\rho}\bullet})).$$

Note that, since $\widehat{U}_0$ retracts to U_0, it suffices to give forms on U_0. Since the family $(\nabla_0^\rho, \nabla_0^{\widehat{\rho}}, \rho_0^*\nabla^N)$ is compatible with (12.3.4), we have

$$c^j(\nabla_0^{\widehat{\rho}\bullet}) = c^j(\nabla_0^\rho),$$

which vanishes for $j > n-s$ by the rank reason. Thus, for $j > n-s$, the cocycle $c^j(\nabla_\star^\bullet)$ is in $A^{2j}(\mathcal{U}, \widehat{U}_0) = \{0\} \oplus A^{2j}(\widehat{U}) \oplus A^{2j-1}(\widehat{U}_0)$. Since the cohomology of $A^*(\mathcal{U}, \widehat{U}_0)$ is canonically isomorphic with $H^*(U, U \setminus S)$ [156, Ch.VI, 4], this cocycle defines a class, denoted $c_S^j(\tau_\rho)$, in $H^{2j}(U, U \setminus S)$, which is mapped to $c^j(\tau_\rho)$ by the canonical homomorphism $H^{2j}(U, U \setminus S) \to H^{2j}(U)$. The class $c_S^j(\tau_\rho)$ does not depend on the choices of various connections. It should be also noted that it does not depend on the frames we discussed earlier. Denoting by $A^{2i}(S)$ the space of $2i$-forms on S, we have the integration along the fibers of ρ [156, Ch.II, 5] $\rho_* : A^{2(n-s+i)}(\mathcal{U}, \widehat{U}_0) \to A^{2i}(S)$, which commutes with the differentials and induces a map on the cohomology level :

$$\rho_* : H^{2(n-s+i)}(U, U \setminus S) \longrightarrow H^{2i}(S).$$

On the cocycle level, ρ_* assigns to $c^{n-s+i}(\nabla_\star^\bullet)$, $i > 0$, the $2i$-form α^i on S given by

$$\alpha^i = \rho_* c^{n-s+i}(\nabla_M^{\widehat{\rho}\bullet}) + (\partial\rho)_* c^{n-s+i}(\nabla_M^{\widehat{\rho}\bullet}, \nabla_0^{\widehat{\rho}\bullet}), \tag{12.3.5}$$

where ρ_* and $(\partial\rho)_*$ denote the integration along the fibers of $\rho|_{\mathcal{R}}$ and $\rho|_{\partial\mathcal{R}}$.

We note that, in the following formulas, the classes $\rho_* c_S^{n-s+i}(\tau_\rho)$ for $i = 1, \ldots, k-1$ are involved and they do not appear if $k = 1$ (*i.e.*, V is a hypersurface). We denote by $[\ \ ]^i$ the component of degree $2i$ of the relevant cohomology class.

Theorem 12.3.6. *With the hypotheses of Theorem 12.3.3, we have*

$$\mathrm{Vir}(v^{(r)}, S) = \Bigg[\Bigg(\mathrm{Ind}_{\mathrm{Vir}}(v,x)\cdot(c^*(N) - c^k(N)) + \mathrm{Ind}_{\mathrm{Sch}}(v,x)\cdot c^k(N)$$

$$+ \sum_{j=1}^{k-1}\sum_{i=1}^{j} c^{j-i}(N)\cdot\rho_* c_S^{n-s+i}(\tau_\rho)\Bigg)\cdot c^*(N)^{-1}\cdot c^*(S)\Bigg]^{\ell} \frown [S].$$

From Theorems 12.3.3 and 12.3.6, we get the following Lefschetz type formula for the Milnor class.

Corollary 12.3.1. *Let S be a nonsingular connected component of* $\mathrm{Sing}(V)$ *such that V satisfies the Whitney condition along S. Then*

$$\mu_*(V, S) = \Bigg((-1)^s \mu(V\cap H, x)\cdot(c^*(N) - c^k(N))$$

$$+ (-1)^n \sum_{j=1}^{k-1}\sum_{i=1}^{j} c^{j-i}(N)\cdot\rho_* c_S^{n-s+i}(\tau_\rho)\Bigg)\cdot c^*(N)^{-1}\cdot c^*(S)\frown [S],$$

where H denotes an $(m-s)$-dimensional plane transverse to S in M. In particular, if $k = 1$,

$$\mu_*(V, S) = (-1)^s \mu(V\cap H, x)\cdot c^*(N)^{-1}\cdot c^*(S)\frown [S].$$

Also, for arbitrary k,

$$\mu_s(V, S) = (-1)^s \mu(V\cap H, x)\cdot [S].$$

Remark 12.3.1. 1. In [131], the Milnor class of a hypersurface V is defined by $\mu_*(V) = (-1)^n\big(c_*(V) - c^*(\tau_V)\frown [V]\big)$ and a formula for this is given as a sum of the contributions from the strata of a stratification of V. This result was obtained earlier for the Milnor number $\mu_0(V)$ in [130] and, for the Milnor class, it was conjectured in [169]. If the stratum is a nonsingular component of $\mathrm{Sing}(V)$, its contribution coincides with the one given in the second formula above.

2. In fact, the formulas in Corollary 12.3.1 hold under an assumption weaker than the Whitney condition. Namely, we only need that there is a Whitney stratification of M compatible with V and S such that the $2(\ell-r)$-skeleton $S\cap D^{2(q-1)}$ of S is in the top dimensional stratum of S. Accordingly, under this assumption, we have a formula for $\mu_r(V, S)$ taking the terms of corresponding dimension in the above formulas (see [31]).

12.4 Generalized Milnor Number

As in the previous sections, let $V \subset M$ be defined by a holomorphic section of a vector bundle of rank k and let S be a connected component of $\mathrm{Sing}(V)$.

Definition 12.4.1. The *generalized Milnor number* $\mu(V, S)$ of V at S is defined as

$$\mu(V, S) = (-1)^{n+1}\big(\mathrm{Ind}_{\mathrm{Sch}}(v, S) - \mathrm{Ind}_{\mathrm{Vir}}(v, S)\big),$$

where v is a vector field on a neighborhood U of S in V, nonsingular on $U \setminus S$.

This definition does not depend on the choice of the vector field v and is equal to $\mu_0(V, S)$ in Definition 12.2.1. If (V, a) is an isolated complete intersection singularity germ, for a radial vector field v_0, $\mathrm{Ind}_{\mathrm{Sch}}(v_0, a) = 1$ and $\mathrm{Ind}_{\mathrm{Vir}}(v_0, a) = \chi(\mathbf{F})$, where $\mathbf{F}$ denotes the Milnor fiber. Thus the above Milnor number coincides with the usual one in [79, 116, 121].

We recall that the classical Milnor number of an isolated singular point [121] has been generalized to the case of nonisolated hypersurface singularities by A. Parusiński [127] in the following way. Recall that a hypersurface V in M is always defined by a holomorphic section s of a line bundle N over M. There is a canonical vector bundle homomorphism $\pi : TM|_V \to N|_V$ which extends the one in (11.4.1). Note that $\mathrm{Sing}(V)$ coincides with the set of points in V where π fails to be surjective. Now let ∇' be a connection for N of type $(1, 0)$. This means that in the decomposition $\nabla' = \nabla^{(1,0)} + \nabla^{(0,1)}$ of ∇' into the $(1, 0)$ and $(0, 1)$ components, we have $\nabla^{(0,1)} = \bar{\partial}$. Since s is holomorphic, we have $\nabla' s = \nabla^{(1,0)} s$, which is a C^∞ section t of $T^*M \otimes N$. Write $\tilde{\pi} : TM \to N$ the corresponding bundle homomorphism. Let S be a compact component of $\mathrm{Sing}(V)$ and $\widehat{U}$ a neighborhood of S in M disjoint from the other components. It is shown in [127] that S coincides with a connected component of the zero set of t. Then Parusiński defines the Milnor number $\mu_S(V)$ to be the intersection number in $\widehat{U}$ of the section t of $T^*M \otimes N$ with the zero section. We refer to [31] for the proof of the following

Theorem 12.4.1. *For a hypersurface V, we have*

$$\mu_S(V) = \mu(V, S).$$