Chapter 8
PIP-Spaces and Signal Processing

Contemporary signal processing makes an extensive use of function spaces, always with the aim of getting a precise control on smoothness and decay properties of functions. In this chapter, we will discuss several classes of such function spaces that have found interesting applications, namely, mixed-norm spaces, amalgam spaces, modulation spaces, or Besov spaces. It turns out that all those spaces come in families indexed by one or more parameters, that specify, for instance, the local behavior or the asymptotic properties. In general, a single space, taken alone, does not have an intrinsic meaning, it is the family as a whole that does, which brings us to the very topic of this volume. In addition, several rigged Hilbert spaces (also called Gel'fand triplets) have a particular interest, notably the one generated by the so-called Feichtinger algebra. This too deserves a detailed discussion in the sequel.

Note that, unlike the previous chapter, we will treat each class with the corresponding applications. Also, we will merely state the relevant results/propositions, referring the interested reader to the vast literature quoted in the Notes.

8.1 Mixed-Norm Lebesgue Spaces

The first type of function space is the family of mixed-norm Lebesgue spaces, already described briefly in Section 4.4, Example 4.4.5. For the commodity of the reader, we repeat the general definition.

Let (X, μ) and (Y, ν) be two σ-finite measure spaces and $1 \leqslant p, q \leqslant \infty$. Then, a function $f(x, y)$ measurable on the product space $X \times Y$ is said to belong to $L^{(p,q)}(X \times Y)$ if the number obtained by taking successively the p-norm in x and the q-norm in y, in that order, is finite. If $p, q < \infty$, the norm is given in (4.19). The analogous norm for p or $q = \infty$ is obvious.

The case $X = Y = \mathbb{R}^d$ with Lebesgue measure is the important one for signal processing. More generally, one can add a weight function m and obtain the spaces $L^{p,q}_m(\mathbb{R}^d)$ (we switch to a notation more suitable for the applications):

J.-P. Antoine and C. Trapani, *Partial Inner Product Spaces: Theory and Applications*, Lecture Notes in Mathematics 1986,
DOI 10.1007/978-3-642-05136-4_8,

$$L_m^{p,q}(\mathbb{R}^2) = \{f \text{ Lebesgue measurable on } \mathbb{R}^{2d} : \|f\|_{L_m^{p,q}} < \infty\},\ 1 \leqslant p,q < \infty,$$

where

$$\|f\|_{L_m^{p,q}} := \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |f(x,\omega)|^p \, m(x,\omega)^p dx \right)^{q/p} d\omega \right)^{1/q}.$$

Here m is a weight function, that is, a non-negative locally integrable function on $\mathbb{R}^{2d}$. In addition, m is assumed to be *v-moderate*, i.e., $m(z_1+z_2) \leqslant v(z_1)m(z_2)$, for all $z_1, z_2 \in \mathbb{R}^{2d}$, with v a submultiplicative weight function, that is, $v(z_1+z_2) \leqslant v(z_1)v(z_2)$, for all $z_1, z_2 \in \mathbb{R}^{2d}$. The typical weights are of polynomial growth: $v_s(z) = (1+|z|)^s$, $s \geqslant 0$.

Once again, things simplify when $p=q$: $L_m^{p,p}(\mathbb{R}^{2d}) = L_m^p(\mathbb{R}^{2d})$, a weighted L^p space.

The spaces $L_m^{p,q}(\mathbb{R}^{2d})$ have the properties inherited from the general case $L^{(p,q)}$, namely:

(i) *Completeness:* $L_m^{p,q}(\mathbb{R}^{2d})$ is a Banach space for the norm $\|\cdot\|_{L_m^{p,q}}$.
(ii) *Hölder's inequality:* If $f \in L_m^{p,q}(\mathbb{R}^{2d})$ and $h \in L_{1/m}^{\overline{p},\overline{q}}(\mathbb{R}^{2d})$, with $1/p + 1/\overline{p} = 1$, $1/q + 1/\overline{q} = 1$, then $f\,h \in L^1(\mathbb{R}^{2d})$ and

$$\left| \int_{\mathbb{R}^{2d}} \overline{f(z)}\, h(z)\, dz \right| \leqslant \|f\|_{L_m^{p,q}} \, \|h\|_{L_{1/m}^{\overline{p},\overline{q}}}.$$

(iii) *Duality:* If $p,q < \infty$, then $(L_m^{p,q})^\times = L_{1/m}^{\overline{p},\overline{q}}$.
(iv) *Translation invariance:* $L_m^{p,q}(\mathbb{R}^{2d})$ is invariant under translations $(T_z g)(w) = g(w-z)$, $z, w \in \mathbb{R}^{2d}$, if, and only if, m is v-moderate; then one has

$$\|T_z f\|_{L_m^{p,q}} \leqslant C v(z) \|f\|_{L_m^{p,q}}, \text{ for all } f \in L_m^{p,q}.$$

(v) *Convolution:* If m is v-moderate, $f \in L_v^1(\mathbb{R}^{2d})$, and $g \in L_m^{p,q}(\mathbb{R}^{2d})$, then

$$\|f * g\|_{L_m^{p,q}} \leqslant C \|f\|_{L_v^1} \, \|g\|_{L_m^{p,q}},$$

that is , $L_v^1 * L_m^{p,q} \subseteq L_m^{p,q}$. This property generalizes the usual one of L^p spaces, $L^1 * L^p \subseteq L^p$.

Concerning lattice properties of the family of $L_m^{p,q}$ spaces, we cannot expect more than for the L^p spaces. Two $L_m^{p,q}$ spaces are never comparable, even for the same weight m, so one has to take the lattice generated by intersection and duality. Nevertheless, properties (iv) and (v) mean that translation and convolution by L_v^1 are totally regular operators in whatever PIP-space is constructed out of the $L_m^{p,q}$ spaces.

A different type of mixed-norm spaces is obtained if one takes $X = Y = \mathbb{Z}^d$, with the counting measure. Thus one gets the space $\ell^{p,q}_m(\mathbb{Z}^{2d})$, which consists of all sequences $a = (a_{kn})$, $k, n \in \mathbb{Z}^d$, for which the following norm is finite:

$$\|a\|_{\ell^{p,q}_m} := \left(\sum_{n\in\mathbb{Z}^d} \Big(\sum_{k\in\mathbb{Z}^d} |a_{kn}|^p \, m(k,n)^p \Big)^{q/p} \right)^{1/q} .$$

Contrary to the continuous case, here we do have inclusion relations:

Lemma 8.1.1. *If* $p_1 \leqslant p_2, q_1 \leqslant q_2$ *and* $m_2 \leqslant C m_1$*, then* $\ell^{p_1,q_1}_{m_1} \subseteq \ell^{p_2,q_2}_{m_2}$.

Proof. First take $m_1 = m_2 \equiv 1$. Then the inclusion results from the obvious inequality $\|a\|_{\ell^{p_2,q_2}} \leqslant \|a\|_{\ell^{p_1,q_1}}$. Then, if the weights satisfy $m_2 \leqslant C m_1$, the result follows from the relations

$$\|a\|_{\ell^{p_2,q_2}_{m_2}} = \|a m_2\|_{\ell^{p_2,q_2}} \leqslant \|a m_2\|_{\ell^{p_1,q_1}} \leqslant C \|a m_1\|_{\ell^{p_1,q_1}} = C \|a\|_{\ell^{p_1,q_1}_{m_1}} . \quad ∎$$

As for the lattice properties, we have (for a fixed weight m)

$$\ell^{\min(p_1,p_2),\min(q_1,q_2)}_m \subset \ell^{p_1,q_1}_m \cap \ell^{p_2,q_2}_m ,$$
$$\ell^{p_1,q_1}_m + \ell^{p_2,q_2}_m \subset \ell^{\max(p_1,p_2),\max(q_1,q_2)}_m ,$$

but we conjecture that the equality is not obtained in general. Thus the set of spaces $\ell^{p,q}_m$, $1 \leqslant p, q \leqslant \infty$ is *not* a lattice, and one has to consider again the lattice it generates. For fixed m, however, one gets chains by varying either q or p, but not both.

Discrete mixed-norm spaces have been used extensively in functional analysis and signal processing. For instance, they are key to the proof that certain operators are bounded between two given function spaces, such as modulation spaces (see Section 8.3.1) or ℓ^p spaces. For instance, if $\{\psi_{j,k},\, (j,k) \in I\}$, is a wavelet basis or frame, mixed-norm spaces may be used to prove boundedness of the analysis operator $D : L^2(\mathbb{R}) \to \ell^2(I)$ given by $Df = \big(\langle \psi_{j,k} | f \rangle\big)_{(j,k)\in I}$.

In general, a mixed-norm space will prove useful whenever one has a signal consisting of sequences labeled by two indices that play different roles. An obvious example is time-frequency or time-scale analysis: a Gabor or wavelet basis (or frame) is written as $\{\psi_{j,k},\, j,k \in \mathbb{Z}\}$, where j indexes the scale or frequency and k the time. More generally, this applies whenever signals are expanded with respect to a dictionary with two indices. An example is provided by multichannel signals, where a first index labels dictionary elements and a second one labels channels. A variant is the case where indices are hierarchized. Coefficients are split into independent groups and coefficients within the same group are dependent. Thus one index labels groups and the other one elements within a group, and of course, the two are not interchangeable. Interesting applications of this procedure are found in the papers quoted in the Notes.

8.2 Amalgam Spaces

A situation intermediate between the mixed-norm spaces (for $m \equiv 1$) $L^{p,q}(\mathbb{R}^{2d})$ and the spaces $\ell^{p,q}(\mathbb{Z}^{2d})$ is that of the so-called *amalgam spaces*. They were introduced specifically to overcome the inability of the L^p norms to distinguish between the local and the global behavior of functions.

The simplest ones are the spaces $W(L^p, \ell^q)$ (sometimes denoted (L^p, ℓ^q) or $W(L^p, L^q)$) consisting of functions on $\mathbb{R}$ which are locally in L^p and have ℓ^q behavior at infinity, in the sense that the L^p norms over the intervals $(n, n+1)$ form an ℓ^q sequence. For $1 \leqslant p, q < \infty$, the norm

$$\|f\|_{p,q} = \left(\sum_{n=-\infty}^{\infty} \left[\int_n^{n+1} |f(x)|^p dx \right]^{q/p} \right)^{1/q}$$

makes $W(L^p, \ell^q)$ into a Banach space. The same is true for the obvious extensions to p and/or q equal to ∞. Notice that $W(L^p, \ell^p) = L^p$. Also it is easy to see that these spaces $W(L^p, \ell^q)$ are a particular case of the mixed-norm spaces $L^{(p,q)}(X \times Y)$. Taking indeed $X = [0,1]$ with Lebesgue measure, $Y = \mathbb{Z}$ with the counting measure, and $g(x,n) = f(x+n)$, one gets

$$\|g\|_{(p,q)} = \left(\sum_{n=-\infty}^{\infty} \left[\int_0^{1} |f(x+n)|^p dx \right]^{q/p} \right)^{1/q} = \|f\|_{p,q}.$$

Actually, an equivalent definition of the space $W(L^p, \ell^q)$ is obtained if one replaces the covering of $\mathbb{R}$ given by $\cup_{n=-\infty}^{\infty}[n, n+1] = \mathbb{R}$ by a so-called *bounded uniform partition of unity (BUPU)*. This means a family of functions $(\psi_i)_{i \in I}$, with I a countable index set, such that:

(1) $0 \leqslant \psi_i(x) \leqslant 1$, for all $i \in I$;
(2) There is a compact neighborhood W of 0 and a countable set of points $(x_i)_{i \in I}$ such that $\operatorname{supp} \psi_i \subseteq x_i + W$ for all $i \in I$;
(3) Every point $x \in \mathbb{R}$ belongs to a finite number of subsets $x_k + W$;
(4) $\sum_{i \in I} \psi_i(x) \equiv 1$, so that $\bigcup_{i \in I}(x_i + W) = \mathbb{R}$.

For instance, in $\mathbb{R}$, one may take $W = [0,1]$, $x_i = i \in \mathbb{Z}$, and $\psi_i = T_i \chi_W = \chi_{i+W}$ (which gives back the original partition)[1] or $W_\alpha := [0, \alpha]$ and $\psi_i = T_{\alpha i} \chi_{W_\alpha}$. Alternatively, one may replace χ_W by a nicer function with compact support, such as a triangular ('tent') function or a spline function, satisfying condition (4). Similar considerations apply, of course, to $\mathbb{R}^d$.

The corresponding norm then reads as

$$\|f\|'_{p,q} := \left(\sum_{i \in I} \left[\int_{\mathbb{R}} |f(x)\psi_i(x)|^p dx \right]^{q/p} \right)^{1/q},$$

[1] As usual, χ_W is the characteristic function of the set W.

and it is equivalent to the original one $\|\cdot\|_{p,q}$, which we denote henceforth by $\|\cdot\|'_{p,q} \asymp \|\cdot\|_{p,q}$. It should be noted that the choice of a particular BUPU is irrelevant: two different ones will give the same space $W(L^p, \ell^q)$, with equivalent norms.

From condition (4) in the definition of a BUPU, we see that any function $f \in W(L^p, \ell^q)$ may be written as $f = \sum_{i\in I} f\psi_i$, where each component $f\psi_i$ has a compact support centered on x_i. This is the key step in the *localization* of functions or distributions, a fundamental tool in signal processing.

The spaces $W(L^p, \ell^q)$ obey the following (immediate) inclusion relations, with all embeddings continuous:

- If $q_1 \leqslant q_2$, then $W(L^p, \ell^{q_1}) \hookrightarrow W(L^p, \ell^{q_2})$.
- If $p_1 \leqslant p_2$, then $W(L^{p_2}, \ell^q) \hookrightarrow W(L^{p_1}, \ell^q)$.

From this it follows that the smallest space is $W(L^\infty, \ell^1)$ and the largest one is $W(L^1, \ell^\infty)$, and therefore

- If $p \geqslant q$, then $W(L^p, \ell^q) \subset L^p \cap L^q \subset L^s, \forall\, q < s < p$.
- If $p \leqslant q$, then $W(L^p, \ell^q) \supset L^p + L^q$.

Once again, Hölder's inequality is satisfied. Whenever $f \in W(L^p, \ell^q)$ and $g \in W(L^{\overline{p}}, \ell^{\overline{q}})$, with $1/p + 1/\overline{p} = 1$, $1/q + 1/\overline{q} = 1$, then $fg \in L^1$ and one has

$$\|fg\|_1 \leqslant \|f\|_{p,q}\, \|g\|_{\overline{p},\overline{q}}.$$

Therefore, one has the expected duality relation:

$$W(L^p, \ell^q)^\times = W(L^{\overline{p}}, \ell^{\overline{q}}), \quad \text{for } 1 \leqslant q, p < \infty.$$

The interesting fact is that, for $1 \leqslant p, q \leqslant \infty$, the set $\mathcal{J}$ of all amalgam spaces $\{W(L^p, \ell^q)\}$ may be represented by the points (p, q) of the *same* unit square J as in the example of the L^p spaces (Section 4.1.2), with the *same* order structure. However, $\mathcal{J}$ is not a lattice with respect to the order (4.3). One has indeed

$$W(L^p, \ell^q) \wedge W(L^{p'}, \ell^{q'}) \supset W(L^{p\vee p'}, \ell^{q\wedge q'}),$$
$$W(L^p, \ell^q) \vee W(L^{p'}, \ell^{q'}) \subset W(L^{p\wedge p'}, \ell^{q\vee q'}),$$

where again $\wedge$ means intersection with projective norm and $\vee$ means vector sum with inductive norm, but equality is not obtained. Thus, as in the previous case, one gets chains by varying either p or q, but not both.

Standard properties hold here too.

(i) Convolution: If $f \in W(L^{p_1}, \ell^{p_2})$ and $g \in W(L^{q_1}, \ell^{q_2})$, where $1/p_i + 1/q_i \geqslant 1$, $i = 1, 2$, then $f * g \in W(L^{r_1}, \ell^{r_2})$, where $1/r_i = 1/p_i + 1/q_i - 1$, and

$$\|f * g\|_{r_1,r_2} \leqslant C \|f\|_{p_1,p_2}\, \|g\|_{q_1,q_2}\,.$$

(ii) Fourier transform: If $f \in W(L^p, \ell^q)$, with $1 \leqslant p, q \leqslant 2$, then $\mathcal{F}f \in W(L^{\overline{q}}, \ell^{\overline{p}})$ and there is a constant $C_{p,q}$ such that

$$\|\mathcal{F}f\|_{\overline{q},\overline{p}} \leqslant C_{p,q}\, \|f\|_{p,q}.$$

(iii) Multiplication: Let $1 \leqslant p_i, q_i \leqslant \infty$ and $1/r_i = \max(0, 1/q_i - 1/p_i)$, $i = 1, 2,$. Then $fg \in W(L^{q_1}, \ell^{q_2})$ whenever $g \in W(L^{p_1}, \ell^{p_2})$ if, and only if, $f \in W(L^{r_1}, \ell^{r_2})$.

(iv) Translation invariance: $W(L^p, \ell^q)$ is invariant under translations $T_x, x \in \mathbb{R}^d$, and one has

$$\|T_x F\|_{p,q} \leqslant C_{pq} \|F\|_{p,q}, \quad \text{where } C_{pq} = \max\{2^{1/p-1/q}, 2^{1/q-1/p}\}.$$

Thus, here again, translation is a totally regular operator in any PIP-space constructed out of the $W(L^p, \ell^q)$ spaces.

The spaces $W(L^p, \ell^q)$ may be generalized considerably. The first obvious modification is to replace ℓ^q by a weighted space ℓ^q_m, with a suitable (v-moderate) weight m, thus getting the space $W(L^p, \ell^q_m)$. Actually, there is an alternative definition for that space. Let χ_Q denote the characteristic function of $Q := [0, 1]$ (or any compact interval). Then consider the following norm:

$$\begin{aligned} \|f\|''_{p,q} &= \| \|f \cdot T_x \chi_Q\|_{L^p} \|_{L^q_m} \\ &= \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(t)|^p\, \chi_Q(t-x)\, dt \right)^{q/p} m(x)^q\, dx \right)^{1/q} \\ &= \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} |f(t)|^p\, \chi_Q(t-x)\, m(x)^p\, dt\, dx \right)^{q/p} \right)^{1/q} . \end{aligned}$$

It turns out that this ('continuous') norm is equivalent to the ('discrete') norm $\| \cdot \|_{p,q}$, thus it defines the same space $W(L^p, \ell^q_m) \equiv W(L^p, L^q_m)$, but, in addition, this definition justifies the latter notation.

In such a setting, the properties listed above generalize in an obvious way. For instance, translation invariance of $W(L^p, \ell^q_m)$becomes

$$\|T_x f\|_{W(L^p, \ell^q_m)} \leqslant C\, v(x) \|F\|_{W(L^p, \ell^q_m)}, \quad \text{for all } x \in \mathbb{R}^d.$$

The next step consists in replacing the "local" space L^p by a suitable Banach space B. Consider first the weighted spaces L^p_m. First we note that L^1_m is a Banach algebra with respect to convolution (called the *Beurling algebra*). Moreover, as mentioned already, L^1_v acts on L^p_m through convolution, $L^1_v * L^p_m \subseteq L^p_m$.

Next, consider the space $\mathcal{F}L^1_m$ of Fourier transforms of functions $f \in L^1_m$, with norm $\|\mathcal{F}f\| = \|f\|^1_m$ and a symmetric v-moderate weight, $m(-x) =$

$m(x)$. This guarantees that $\mathcal{F}^{-1}L^1_m = \mathcal{F}L^1_m$. Applying a Fourier transform yields $\mathcal{F}L^1_v \cdot \mathcal{F}L^p_m = \mathcal{F}(L^1_v * L^p_m) \subseteq \mathcal{F}L^p_m$. The space $\mathcal{F}L^1_m$ possesses a BUPU $(\psi_i)_{i\in I}$, so that the construction above applies.

Let now $(B, \|\cdot\|_B)$ be a Banach space of functions (or distributions), invariant under translations $(T_x g)(y) = g(y-x)$ and modulations $(M_\omega g)(y) = e^{2\pi i y\omega} g(y)$, and such that $\mathcal{S}(\mathbb{R}) \hookrightarrow B \hookrightarrow \mathcal{S}^\times(\mathbb{R})$. In addition, assume that $\mathcal{F}L^1_m$ acts on B by pointwise multiplication. Then the (Wiener) generalized amalgam space $W(B, \ell^q_m)$ consists of all functions or distributions $f \in B_{\text{loc}}$ such that

$$\|f\|_{B,\ell^q_m} := \left(\sum_{i\in I} \|f\psi_i\|^q_B \, m(x_i)^q\right)^{1/q} = \big\|\big(\|f\psi_i\|_B\big)\big\|_{\ell^q_m} < \infty. \tag{8.1}$$

Here we have introduced the space $B_{\text{loc}} := \{f \in \mathcal{S}^\times(\mathbb{R}) : hf \in B$ for any $h \in \mathcal{D}(\mathbb{R})\}$. As usual, different BUPUs $(\psi_i)_{i\in I}$ and sets of points $(x_i)_{i\in I}$ yield equivalent norms, hence the same space. For $B \equiv L^p$, one recovers the usual space $W(L^p, \ell^q_m)$. An interesting case is $B = \mathcal{F}L^p_u$, which indeed satisfies all the conditions stated above, but many other spaces may be chosen, such as Besov spaces, Bessel potential spaces, etc.

The spaces $W(\mathcal{F}L^p_u, \ell^q_m)$ have many interesting properties. Given submultiplicative weights v and w, assume m is a v-moderate weight and u is a w-moderate weight. For this case, there is also an equivalent, 'continuous' norm, namely,

$$\|f\|'_{W(\mathcal{F}L^p_u, \ell^q_m)} = \left(\int_{\mathbb{R}} \left(\|f \cdot T_x h\|_{\mathcal{F}L^p_u}\right)^q m(x)^q \, dx\right)^{1/q},$$

where h can be any nonzero element of $W(\mathcal{F}L^1_\nu, \ell^1_w)$. Then the following holds:

(i) *Completeness:* $W(\mathcal{F}L^p_u, \ell^q_m)$ is a Banach space for the norm $\|\cdot\|_{W(\mathcal{F}L^p_u, \ell^q_m)}$.

(ii) *Invariance:* $W(\mathcal{F}L^p_u, \ell^q_m)$ is invariant under translation and modulation and the corresponding operators are bounded.

(iii) *Duality:* $W(\mathcal{F}L^p_u, \ell^q_m)^\times = W(\mathcal{F}L^{\overline{p}}_{1/u}, \ell^{\overline{q}}_{1/m})$, with the usual notation.

(iv) *Convolution, multiplication:* one has, with all embeddings continuous,

$$L^1_v * W(\mathcal{F}L^p_u, \ell^q_m) \hookrightarrow W(\mathcal{F}L^p_u, \ell^q_m),$$

$$\mathcal{F}L^1_w \cdot W(\mathcal{F}L^p_u, \ell^q_m) \hookrightarrow W(\mathcal{F}L^p_u, \ell^q_m).$$

(v) *The Fourier transform* $\mathcal{F}$ is an isomorphism between the spaces $W(\mathcal{F}L^1_w, \ell^1_v)$ and $W(\mathcal{F}L^1_v, \ell^1_w)$. More generally, for weight functions v, w at most of polynomial growth, and for given $\alpha, \beta \in \mathbb{R}$, $\mathcal{F}$ extends to an isomorphism between the spaces $W(\mathcal{F}L^p_u, \ell^p_m)$ and $W(\mathcal{F}L^p_m, \ell^p_u)$, where $u := w^\alpha$ and $m := v^\beta$. For proving this, one starts from the case

$p = q = 1$. Then, since v, w are assumed to be at most of polynomial growth, one transposes the relation to the dual spaces $W(\mathcal{F}L^\infty_{1/w}, \ell^\infty_{1/v})$ and $W(\mathcal{F}L^\infty_{1/v}, \ell^\infty_{1/w})$, and finally one gets the case (p,p) by interpolation.

(vi) *Inclusion relations :* the spaces $W(\mathcal{F}L^p_u, \ell^q_m)$ are ordered by inclusion as follows. If $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$ and $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$, then $W(\mathcal{F}L^{p_1}_u, \ell^{q_1}_m) \hookrightarrow W(\mathcal{F}L^{p_2}_u, \ell^{q_2}_m)$. Furthermore,

(a) if $p_1 < p_2$ and $u_2 \leqslant C\, u_1$, then $W(\mathcal{F}L^{p_1}_{u_1}, \ell^q_m) \hookrightarrow W(\mathcal{F}L^{p_2}_{u_2}, \ell^q_m)$;
(b) if $q_1 < q_2$ and $m_2 \leqslant C\, m_1$, then $W(\mathcal{F}L^p_u, \ell^{q_1}_{m_1}) \hookrightarrow W(\mathcal{F}L^p_u, \ell^{q_2}_{m_2})$; the same result holds if $q_1 < q_2$ and $m_2/m_1 \in L^r$, for $1/r = 1/q_1 - 1/q_1$.

The two statements (a) and (b) are proven essentially as in Lemma 8.1.1. The second part of (b) follows from the Hölder inequality.

We proceed now with further generalizations. The first step is to replace the global space ℓ^q_m by a solid, translation invariant Banach space Y. By this, we mean a Banach space of locally integrable functions, continuously embedded in $L^1_{\rm loc}$, and such that $f \in Y$, $g \in L^1_{\rm loc}$ and $|g(x)| \leqslant |f(x)|$ (a.e.) imply $g \in Y$ and $\|g\|_Y \leqslant \|f\|_Y$. Translation invariance here means that $\|T_x f\|_Y \leqslant c\, w_\gamma(x) \|f\|_Y$, where $w_\gamma(x) := (1 + |x|)^\gamma$. We also assume that $f * g \in Y$ for $f \in Y, g \in L^1_{w_\gamma}$ and $\|f * g\|_Y \leqslant \|g\|_{L^1_{w_\gamma}} \|f\|_Y$. Typical examples are the weighted spaces $L^p_{w_s}$ with $0 \leqslant |s| \leqslant \gamma$.

Fixing a window function $h \in \mathcal{D}(\mathbb{R})$ (that is, a positive C^∞ 'bump' function with compact support), we consider the *control function* $F_h(x) := \|T_x h \cdot f\|_B$. Then, given B, Y as above, we define the *generalized Wiener amalgam space*

$$W(B, Y) := \{f \in B_{\rm loc} : F_h \in Y\}$$

with norm $\|f\|_{B,Y} := \|F_h\|_Y$. As usual, there is an equivalent (discrete) formulation in terms of a BUPU $\{\psi_i, x_i\}$, as in (8.1). Namely, considering the function $F_W \in Y$ defined by

$$F_W(x) := \sum_{i\,:\, x \in x_i + W} \|f\psi_i\|_B \cdot \chi_{x_i + W}(x),$$

the norm $\|F_W\|_Y$ of the function F_W in Y defines an equivalent norm on $W(B, Y)$. If Y is a sequence space, this norm reads simply

$$\|f\|'_{B,Y} = \big\| \big(\|f\psi_i\|_B\big) \big\|_Y \,.$$

Taking $Y = \ell^q_m$, we recover the previous class of amalgam spaces.

Among the (very general) results about these spaces (see the literature), an interesting one concerns convolution between amalgam spaces. Assume (B_1, B_2, B_3) and (Y_1, Y_2, Y_3) are Banach convolution triples, i.e., that

$$\|f * g\|_{B_3} \leqslant C_1 \|g\|_{B_1} \|f\|_{B_2}, \text{ for all } g \in B_1, f \in B_2,$$

and

$$\|F \cdot G\|_{Y_3} \leqslant C_2 \|G\|_{Y_1} \, \|F\|_{Y_2}, \text{ for all } G \in Y_1, F \in Y_2.$$

Then $(W(B_1, Y_1), W(B_2, Y_2), W(B_3, Y_3))$ is a Banach convolution triple, that is, one has, for some constant $C_3 > 0$,

$$\|f * g\|_{B_3,Y_3} \leqslant C_3 \|g\|_{B_1,Y_1} \, \|f\|_{B_2,Y_2}.$$

This is a far reaching generalization of Young's identity.

The last step in the generalization process is to replace $\mathbb{R}^d$ by a locally compact abelian group G with Haar measure dx and dual $\widehat{G}$ (which consists of all continuous unitary characters of G). In that case, the Fourier transform becomes

$$\mathcal{F}f(\chi) := \int_G \overline{\chi(x)} \, f(x) \, dx,$$

where $\chi \in \widehat{G}$ is a unitary character of G. Hence, if L^p is a space of functions on G, its Fourier transform $\mathcal{F}L^p$ is a space of functions on $\widehat{G}$. Translations are as usual, while modulation becomes multiplication by characters. With these modifications, the whole theory goes through.

Summarizing, we see that mixed-norm spaces and amalgam spaces consist of large families of (mostly) reflexive Banach spaces, indexed by one or several real indices. Among these, one finds plenty of Banach chains and lattices of Banach spaces. On the corresponding PIP-space structures, operations like translation and modulation may be seen as regular operators, Fourier transform and convolution may also be reinterpreted, etc. Thus the whole theory may be rewritten in PIP-space language. It remains to be seen to what extent this approach improves and/or simplifies it. We will see in the next sections that the natural framework for Gabor or time-frequency analysis is the family of so-called modulation spaces $M^{p,q}_m$, but it turns out that these may often be replaced by amalgam spaces $W(L^p, \ell^q_m)$. Thus, these too have a important role in signal processing.

8.3 Modulation Spaces

Among the function spaces that play a central role in signal processing, several classes are closely related to well-known integral transforms like the Gabor and the wavelet transforms, namely, the modulation spaces and the Besov spaces, respectively. We treat them successively.

8.3.1 General Modulation Spaces

Modulation spaces are closely linked to, and in fact defined in terms of, the Short-Time Fourier (or Gabor) Transform.

Given a C^∞ window function $g \neq 0$, the *Short-Time Fourier Transform (STFT)* of $f \in L^2(\mathbb{R}^d)$ is defined by

$$(V_g f)(x,\omega) = \langle M_\omega T_x g | f \rangle := \int_{\mathbb{R}^d} \overline{g(y-x)}\, f(y)\, e^{-2\pi i y\omega}\, dy, \quad x,\omega \in \mathbb{R}^d, \tag{8.2}$$

where, as usual, $(T_x g)(y) = g(y-x)$ (translation) and $(M_\omega h)(y) = e^{2\pi i y\omega} h(y)$ (modulation).

Notice that an equivalent definition is often used, namely $(\widetilde{V}_g f)(x,\omega) = \langle T_x M_\omega g | f \rangle$, the connection between the two resulting from the identity

$$T_x M_\omega = e^{-2\pi i x\omega} M_\omega T_x.$$

Then, given a v-moderate weight function $m(x,\omega)$, the modulation space $M_m^{p,q}$ is defined in terms of a mixed-norm of a STFT:

$$M_m^{p,q}(\mathbb{R}^d) = \{ f \in \mathcal{S}^\times(\mathbb{R}^d) : V_g f \in L_m^{p,q}(\mathbb{R}^{2d}) \}, \quad 1 \leqslant p,q \leqslant \infty.$$

For $p = q$, one writes $M_m^p \equiv M_m^{p,p}$. The space $M_m^{p,q}$ is a Banach space for the norm

$$\|f\|_{M_m^{p,q}} := \|V_g f\|_{L_m^{p,q}}$$

Actually, the original definition was slightly more restrictive, in that it used the weight function $m_s(x,\omega) = w_s(\omega) = (1+|\omega|)^s,\ s \geqslant 0$, (or, equivalently, $\widetilde{m}_s(x,\omega) = (1+|\omega|^2)^{s/2}$), so that the norm reads

$$\|f\|_{M_{w_s}^{p,q}} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |\langle M_\omega T_x g | f \rangle|^p \, dx \right)^{q/p} (1+|\omega|)^{sq}\, d\omega \right)^{1/q}.$$

Equivalently, one may define the modulation spaces as the inverse Fourier transform of a Wiener amalgam space:

$$M_{w_s}^{p,q} = \mathcal{F}^{-1}(W(L^p, \ell_{w_s}^q)).$$

This space is independent of the choice of window g, in the sense that different window functions define equivalent norms.

The class of modulation spaces $M_{w_s}^{p,q}$ contains several well-known spaces, such as:

(i) The Bessel potential spaces or fractional Sobolev spaces $H^s = M_{\widetilde{m}_s}^2$:

$$H^s(\mathbb{R}^d) = M_{\widetilde{m}_s}^2(\mathbb{R}^d) = \{ f \in \mathcal{S}^\times : \int_{\mathbb{R}^d} |\widehat{f}(t)|^2\, (1+|t|^2)^s\, dt < \infty \},\ s \in \mathbb{R}.$$

(ii) $L^2(\mathbb{R}^d) = M^2(\mathbb{R}^d)$.

(iii) The Feichtinger algebra $\mathcal{S}_0 = M^1$, that we shall describe in detail in Section 8.3.2.

The main properties of the modulation spaces $M_m^{p,q}(\mathbb{R}^{2d})$ follow from the similar ones of the spaces $L_m^{p,q}(\mathbb{R}^{2d})$.

(i) Duality: if $1 \leqslant p, q < \infty$, then $(M_m^{p,q})^\times = M_{1/m}^{\overline{p},\overline{q}}$, with the usual notation.
(ii) Translation invariance: $M_m^{p,q}$ is invariant under time-frequency shifts and

$$\|T_x M_\omega f\|_{M_m^{p,q}} \leqslant C v(x,\omega) \|f\|_{L_m^{p,q}}$$

(m is assumed to be v-moderate).
(iii) Fourier transform: If $p = q$ and $m(\omega, -x) \leqslant Cm(x, \omega)$, then M_m^p is invariant under the Fourier transform.
(iv) Density: If $|m(z)| \leqslant (1 + |z|)^N$ and $1 \leqslant p, q < \infty$, then $\mathcal{S}(\mathbb{R}^d)$ is dense in $M_m^{p,q}(\mathbb{R}^d)$.

The lattice properties of the family $\{M_m^{p,q}, 1 \leqslant p, q \leqslant \infty\}$ are, of course, the same as those of the mixed-norm spaces $L_m^{p,q}$. Here also, statements (ii) and (iii) may be translated in PIP-space language, in terms of totally regular operators.

Similar inclusion relations hold:

Lemma 8.3.1. *If $p_1 \leqslant p_2$, $q_1 \leqslant q_2$, and $m_2 \leqslant Cm_1$, for some constant $C > 0$, then $M_{m_1}^{p_1,q_1} \subseteq M_{m_2}^{p_2,q_2}$.*

In particular, one has

$$M_v^1 \subseteq M_m^{p,q} \subseteq M_{1/v}^\infty.$$

The proof follows immediately from Lemma 8.1.1.

By construction, modulation spaces are function spaces well-adapted to *Gabor analysis*. A wealth of information about the spaces and their application in Gabor analysis may be found in the monograph of Gröchenig [Grö01]. Here we just indicate a few relevant points.

Given a nonzero window function $g \in L^2(\mathbb{R}^d)$ and lattice parameters $\alpha, \beta > 0$, the set of vectors

$$\mathcal{G}(g, \alpha, \beta) = \{M_{n\beta} T_{k\alpha} g,\ k, n \in \mathbb{Z}^d\}$$

is called a *Gabor system.* The system $\mathcal{G}(g, \alpha, \beta)$ is a *Gabor frame* if there exist two constants $\mathsf{m} > 0$ and $\mathsf{M} < \infty$ such that

$$\mathsf{m}\|f\|^2 \leqslant \sum_{k,n \in \mathbb{Z}^d} |\langle M_{n\beta} T_{k\alpha} g | f \rangle|^2 \leqslant \mathsf{M}\|f\|^2, \text{ for all } f \in L^2(\mathbb{R}^d).$$

The associated Gabor frame operator $S_{g,g}$ is given by

$$S_{g,g}f := \sum_{k,n\in\mathbb{Z}^d} \langle M_{n\beta}T_{k\alpha}g|f\rangle \, M_{n\beta}T_{k\alpha}g. \tag{8.3}$$

The main results of the Gabor time-frequency analysis stem from the following proposition.

Proposition 8.3.2. *If $\mathcal{G}(g,\alpha,\beta)$ is a Gabor frame, there exists a dual window $\breve{g} = S^{-1}g$ such that $\mathcal{G}(\breve{g},\alpha,\beta)$ is a frame, called the* dual frame. *Then one has, for every $f \in L^2(\mathbb{R}^d)$,*

$$f = \sum_{k,n\in\mathbb{Z}^d} \langle M_{n\beta}T_{k\alpha}g|f\rangle \, M_{n\beta}T_{k\alpha}\breve{g} \tag{8.4}$$

$$= \sum_{k,n\in\mathbb{Z}^d} \langle M_{n\beta}T_{k\alpha}\breve{g}|f\rangle \, M_{n\beta}T_{k\alpha}g, \tag{8.5}$$

with unconditional convergence in $L^2(\mathbb{R}^d)$.

The two relations (8.4) and (8.5) mean that the function f may be reconstructed from suitable samples of its STFT. But this raises a number of questions:

(1) For which values of α,β is $\mathcal{G}(g,\alpha,\beta)$ a frame? For which class of windows g?

The answer is that $\mathcal{G}(g,\alpha,\beta)$ is a frame if $g \in W(L^\infty,\ell^1)$ and α,β are sufficiently small (for the technical meaning of 'sufficiently small ', see [Grö01, Sec.6.5]). In particular, $\alpha\beta \leqslant 1$ is a necessary condition.

(2) Can one replace the regular lattice $\mathbb{Z}^d$ by an irregular set of points in $\mathbb{R}^d$?

The answer is positive, but this is a difficult problem, related to irregular sampling and number theory.

(3) Under which conditions are the operators associated to Gabor frames (analysis, synthesis, frame operator) well-defined and bounded?

Here, the analysis operator $C_g : L^2(\mathbb{R}^{2d}) \to \ell^2(\mathbb{Z}^{2d})$ is defined by $(C_g f)_{kn} = \langle M_{n\beta}T_{k\alpha}g|f\rangle$ and the synthesis operator $D_g : \ell^2(\mathbb{Z}^{2d}) \to L^2(\mathbb{R}^{2d})$ by $D_g c = \sum_{k,n\in\mathbb{Z}^d} c_{kn} M_{n\beta}T_{k\alpha}g$. Then, one has $C_g^* = D_g$. As a slight generalization of (8.3), we still call Gabor frame operator the operator

$$\begin{aligned} S_{g,\breve{g}}f &:= D_{\breve{g}}C_g \\ &= \sum_{k,n\in\mathbb{Z}^d} \langle M_{n\beta}T_{k\alpha}g|f\rangle \, M_{n\beta}T_{k\alpha}\breve{g}. \end{aligned}$$

This is where the modulation spaces $M^{p,q}_m$ turn out to be the natural class of function spaces. First, the optimal space for window functions is M^1_v. Next one has (in a somewhat abbreviated form):

(i) If $g \in M^1_v$, then C_g is bounded from $M^{p,q}_m$ into $\ell^{p,q}_{\widetilde{m}}(\mathbb{Z}^{2d})$, for all v-moderate weights m, $1 \leqslant p,q \leqslant \infty$ and all lattice constants α, β. Here $\widetilde{m}(k,n) = m(k\alpha, n\beta)$.
(ii) If $g \in M^1_v$, then D_g is bounded from $\ell^{p,q}_{\widetilde{m}}(\mathbb{Z}^{2d})$ into $M^{p,q}_m$, for all p,q. If $p,q < \infty$, the series expressing D_g converges unconditionally in $M^{p,q}_m$.
(iii) If $g, \breve{g} \in W(L^\infty, \ell^1)$, then the Gabor frame operator $S_{g,\breve{g}}$ is bounded on every $L^p(\mathbb{R}^{2d})$, $1 \leqslant p \leqslant \infty$.
(iv) If $g, \breve{g} \in M^1_v$, then $S_{g,\breve{g}}$ is bounded on $M^{p,q}_m$ for all $1 \leqslant p,q \leqslant \infty$, all v-moderate weights m, and all α, β.
(v) If $\breve{g}$ is a dual window of g, that is, $S_{g,\breve{g}} = 1$ on L^2, then the two expansions (8.4) and (8.5) converge unconditionally in $M^{p,q}_m$ if $p,q < \infty$.
(vi) If $g \in \mathcal{S}$, then a tempered distribution $f \in \mathcal{S}^\times(\mathbb{R}^d)$ belongs to $M^{p,q}_m$ if, and only if, $C_g f \in \ell^{p,q}_{\widetilde{m}}$.

Notice once again that statements (iii) and (iv) can be translated into PIP-space language, by saying that $S_{g,\breve{g}}$ is a totally regular operator in the chain $\{L^p, 1 \leqslant p \leqslant \infty\}$, resp. any PIP-space built from modulation spaces.

Most of these results are highly nontrivial and their proof requires deep analysis. As for the result (v), it is a first example where membership of f in the modulation space $M^{p,q}_m$ is *characterized* by membership of the sequence of its Gabor coefficients $C_g f$ in $\ell^{p,q}_{\widetilde{m}}$. This type of result is quite strong and in general valid only for the pair $(L^2 \leftrightarrow \ell^2)$. Here, in fact, lies the power of Gabor analysis, and of wavelet analysis as well, as we shall see below.

These results should suffice to convince the reader that the modulation spaces $M^{p,q}_m$ are the 'natural' spaces for Gabor analysis. Actually, most of this remains true if one replaces modulation spaces by amalgam spaces $W(L^p, \ell^q_m)$. Second, it is obvious that most of the statements have a distinctly PIP-space flavor: it is not some individual space $M^{p,q}_m$ or $W(L^p, \ell^q_m)$ that counts, but the whole family, with many operators being regular in the sense of PIP-spaces.

8.3.2 The Feichtinger Algebra

A particularly interesting case of modulation space is the space M^1, the smallest of them, which consists of all functions with integrable Gabor transform. This space is also known as the *Feichtinger algebra*, denoted by $\mathcal{S}_0(\mathbb{R}^d)$, and it plays an important role in abstract harmonic analysis. As for general modulation spaces, $\mathcal{S}_0$ may also be defined as an amalgam space, namely $\mathcal{S}_0 = W(\mathcal{F}L^1, \ell^1)$.

By definition, $f \in \mathcal{S}_0$ if $V_{g_0} f$ is integrable, where g_0 is the Gaussian (which could be replaced by any function in $\mathcal{S}$). The space $\mathcal{S}_0$ has many interesting properties, for instance:

(i) $\mathcal{S}_0$ is a Banach space for the norm $\|f\|_{\mathcal{S}_0} = \|V_{g_0} f\|_1$, and $\mathcal{S} \hookrightarrow \mathcal{S}_0 \hookrightarrow L^2$, with all embeddings continuous with dense range.
(ii) $\mathcal{S}_0$ is a Banach algebra with respect to pointwise multiplication and convolution.
(iii) Time-frequency shifts $T_x M_\omega$ are isometric on $\mathcal{S}_0 : \|T_x M_\omega f\|_{\mathcal{S}_0} = \|f\|_{\mathcal{S}_0}$. $\mathcal{S}_0$ is continuously embedded in any Banach space with the same property and containing g_0, thus it is the smallest Banach space with this property.
(iv) The Fourier transform is an isometry on $\mathcal{S}_0 : \|\mathcal{F} f\|_{\mathcal{S}_0} = \|f\|_{\mathcal{S}_0}$.

Next we turn to the (conjugate) dual $\mathcal{S}_0^\times$ of $\mathcal{S}_0$. Since $\mathcal{S}_0 = M^1$, we have $\mathcal{S}_0^\times = M^\infty$, a Banach space with norm $\|f\|_{\mathcal{S}_0^\times} = \|V_g f\|_\infty$. The space $\mathcal{S}_0^\times$ contains both the δ function and the pure frequency $\chi_\omega(x) = e^{-2\pi i x \omega}$.

In virtue of (i) above, we have

$$\mathcal{S} \hookrightarrow \mathcal{S}_0 \hookrightarrow L^2 \hookrightarrow \mathcal{S}_0^\times \hookrightarrow \mathcal{S}^\times, \tag{8.6}$$

where all embeddings are continuous and have dense range. In the terminology of Section 5.2.1, $\mathcal{S}_0$ and $\mathcal{S}_0^\times$, are interspaces for the RHS $\mathcal{S} \hookrightarrow L^2 \hookrightarrow \mathcal{S}^\times$. In the quintuplet of spaces (8.6), the central triplet

$$\mathcal{S}_0(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d) \hookrightarrow \mathcal{S}_0^\times(\mathbb{R}^d) \tag{8.7}$$

is the prototype of a *Banach Gel'fand triple*, that is a RHS (or LBS) in which the extreme spaces are (nonreflexive) Banach spaces. By (iii) and (iv) above, both time-frequency shifts and Fourier transform are isomorphisms of $\mathcal{S}_0^\times(\mathbb{R}^d)$, and indeed of the three spaces of the triple (8.7), and the Parseval formula holds:

$$\langle f|g\rangle = \langle \mathcal{F} f|\mathcal{F} g\rangle, \quad \text{for all } (f, g) \in \mathcal{S}_0(\mathbb{R}^d) \times \mathcal{S}_0^\times(\mathbb{R}^d).$$

Things becomes even more interesting in the discrete case. First there is the following striking result.

Theorem 8.3.3. *Let $\mathcal{G}(g, \alpha, \beta)$ be a Gabor frame with $g \in \mathcal{S}_0(\mathbb{R}^d)$. Then the dual window $\breve{g} = S_{g,\breve{g}}^{-1} g$, where $S_{g,\breve{g}}$ is the Gabor frame operator, belongs to $\mathcal{S}_0(\mathbb{R}^d)$ as well.*

Next, one can characterize membership of a function f in $\mathcal{S}_0$ or $\mathcal{S}_0^\times(\mathbb{R}^d)$ in terms of its Gabor coefficients. For simplicity, we put $\alpha = \beta = 1$ (so that we are in the so-called critical case $\alpha\beta = 1$).

Proposition 8.3.4. *Let $\mathcal{G}(g,1,1)$ be a Gabor frame with $g \in \mathcal{S}_0(\mathbb{R}^d)$. Then f belongs to $\mathcal{S}_0(\mathbb{R}^d)$ if and only if the sequence of Gabor coefficients $(\langle M_n T_k g|f\rangle)_{k,n\in\mathbb{Z}^d}$ belongs to $\ell^1(\mathbb{Z}^d)$. In addition, one has the equivalence of norms:*

$$C_1\|f\|_{\mathcal{S}_0} \leqslant \sum_{k,n\in\mathbb{Z}^d} |\langle M_n T_k g|f\rangle| \leqslant C_2\|f\|_{\mathcal{S}_0}, \text{ for all } f \in \mathcal{S}_0(\mathbb{R}^d).$$

Similarly, $f \in \mathcal{S}_0^\times(\mathbb{R}^d)$ if and only if the sequence of Gabor coefficients $(\langle M_n T_k g|f\rangle)_{k,n\in\mathbb{Z}^d}$ belongs to $\ell^\infty(\mathbb{Z}^d)$.

This result can be formalized in terms of the so-called *localization operators.* Let g be a window function and σ a bounded non-negative function on $\mathbb{R}^{2d}$. Then the localization operator associated to the symbol σ is the operator H_σ defined by

$$H_\sigma f := \int_{\mathbb{R}^{2d}} \sigma(x,\omega)\, V_g f(x,\omega)\, M_\omega T_x g \, dx\, d\omega.$$

If $\sigma \equiv 1$ and $\|g\|_2 = 1$, then $H_\sigma = 1$ and the relation above is nothing but the inversion formula of the STFT. If σ has compact support $\Omega \subset \mathbb{R}^{2d}$, then, intuitively, $H_\sigma f$ represents the part of f that lives in Ω, hence the name. This statement can be made precise as follows. Given a time-frequency shift $T_j, j \in \mathbb{Z}^{2d}$, consider the collection of localization operators $\{H_j := H_{T_j\sigma},\ j \in \mathbb{Z}^{2d}\}$. Then the map $f \mapsto \{H_j f\}$ can be interpreted as the decomposition of f into (localized) components $H_j f$ living essentially on $\operatorname{supp} T_j\sigma = j + \operatorname{supp}\sigma$ in the time-frequency plane and the norm $\|H_j f\|_2^2$ is the energy of that component.

Using this concept, one has the following fundamental result.

Theorem 8.3.5. *Let $\sigma \in L^1(\mathbb{R}^{2d})$ be a non-negative symbol satisfying the condition*

$$A \leqslant \sum_{j\in\mathbb{Z}^{2d}} T_j\sigma \leqslant B, \quad a.e.,$$

for two constants $A, B > 0$ and assume that $g \in \mathcal{S}_0(\mathbb{R}^d)$. Then $f \in \mathcal{S}_0(\mathbb{R}^d)$ if and only if $\sum_{j\in\mathbb{Z}^{2d}} \|H_j f\|_2 < \infty$ and this quantity defines an equivalent norm on $\mathcal{S}_0(\mathbb{R}^d)$.

Similarly, the following norm equivalences characterize $\mathcal{S}_0^\times$ and L^2:

$$\|f\|_{\mathcal{S}_0^\times} \asymp \sup_{j\in\mathbb{Z}^{2d}} \|H_j f\|_2, \quad \|f\|_2^2 \asymp \sum_{j\in\mathbb{Z}^{2d}} \|H_j f\|_2^2.$$

Using the notion of Gel'fand triples, this result takes a simpler form.

Corollary 8.3.6. *Under the conditions of Theorem 8.3.5, the map $\iota : f \mapsto (\|H_j f\|_2)_{j\in\mathbb{Z}^{2d}}$ is an isomorphism between the Gel'fand triple $(\mathcal{S}_0, L^2, \mathcal{S}_0^\times)$ and a closed subspace of the triple $(\ell^1, \ell^2, \ell^\infty)$.*

Actually, one can go further. Since $\mathcal{S}_0 = M^1$ and $\mathcal{S}_0^\times = M^\infty$, all the modulation spaces $M^p, 1 \leqslant p \leqslant \infty$ may be obtained by interpolation between $\mathcal{S}_0$

and $\mathcal{S}_0^\times$, so that the statement of Corollary 8.3.6 extends to the whole chain. Thus the map ι is a monomorphism from the LHS $\{M^p\}$ into the LHS $\{\ell^p\}$.

The Feichtinger algebra $\mathcal{S}_0$ is often used in time-frequency analysis and it is considered by many authors as the natural space of test functions. Indeed, the Banach Gel'fand triple $(\mathcal{S}_0, L^2, \mathcal{S}_0^\times)$ often replaces advantageously Schwartz' space RHS $\mathcal{S} \hookrightarrow L^2 \hookrightarrow \mathcal{S}^\times$.

8.4 Besov Spaces

Besov spaces were introduced around 1960 for providing a precise control on the smoothness of solutions of certain partial differential equations. Later on, it was discovered that they are closely linked to wavelet analysis, exactly as the (much more recent) modulation spaces are structurally adapted to Gabor analysis. In fact, there are many equivalent definitions of Besov spaces. We begin by a 'discrete' formulation, based on a dyadic partition of unity.

Let us consider a weight function $\varphi \in \mathcal{S}(\mathbb{R})$ with the following properties:

- supp $\varphi = \{\xi : 2^{-1} \leqslant |\xi| \leqslant 2\}$,
- $\varphi(\xi) > 0$ for $2^{-1} < |\xi| < 2$,
- $\sum_{j=-\infty}^{\infty} \varphi(2^{-j}\xi) = 1$ $(\xi \neq 0)$.

Then one defines the following functions by their Fourier transform:

- $\widehat{\varphi_j}(\xi) = \varphi(2^{-j}\xi), \; j \in \mathbb{Z}$: high "frequency" for $j > 0$, low "frequency" for $j < 0$,
- $\widehat{\psi}(\xi) = 1 - \sum_{j=1}^{\infty} \varphi(2^{-j}\xi)$: low "frequency" part.

Given the weight function φ, the inhomogeneous Besov space B_{pq}^s is defined as

$$B_{pq}^s = \{f \in \mathcal{S}^\times : \|f\|_{pq}^s < \infty\}, \tag{8.8}$$

where $\|\cdot\|_{pq}^s$ denotes the norm

$$\|f\|_{pq}^s := \|\psi * f\|_p + \left(\sum_{j=1}^{\infty} (2^{sj} \|\varphi_j * f\|_p)^q \right)^{1/q}, \; s \in \mathbb{R}, 1 \leqslant p, q \leqslant \infty. \tag{8.9}$$

The space B_{pq}^s is a Banach space and it does not depend on the choice of the weight function φ, since a different choice defines an equivalent norm. Note that $B_{22}^s = H^s$, the (fractional) Sobolev space or Bessel potential space.

For $f \in B_{pq}^s$, one may write the following (weakly converging) expansion, known as a *dyadic Littlewood–Paley decomposition*:

$$f = \psi * f + \sum_{j=1}^{\infty} \varphi_j * f. \tag{8.10}$$

Clearly the first term represents the (relatively uninteresting) low "frequency" part of the function, whereas the second term analyzes in detail the high "frequency" component.

An equivalent, 'continuous', definition is based on the notion of *modulus of smoothness*. For $f \in L^p$ and $h > 0$, this is the quantity

$$\omega_p(f,h) := \|f(\cdot + h) - f(\cdot)\|_p \ .$$

Then, for $0 < s < 1$ and $q < \infty$, the space B^s_{pq} consists of all functions $f \in L^p$ for which the following norm is finite:

$$\|f\|_{B^s_{pq}} := \|f\|_p + \Big(\int_0^\infty [h^{-s}\omega_p(f,h)]^q \, \frac{dh}{h} \Big)^{1/q}.$$

This norm is equivalent to the norm (8.9). A similar norm may be defined for $s > 1$ and for $q = \infty$.

Another equivalent norm (again for $0 < s < 1$) yet is the following:

$$\|f\|_{B^s_{pq}} \asymp \|f\|_p + \Big(\sum_{j=0}^{\infty} [2^{sj}\omega_p(f, 2^{-j})]^q \Big)^{1/q}.$$

Besov spaces enjoy many familiar properties (for more details, we refer to the literature):

(i) *Inclusion relations:* The following relations hold, where all embeddings are continuous:

- $\mathcal{S} \hookrightarrow B^s_{pq} \hookrightarrow \mathcal{S}^\times$;
- $B^s_{pq} \hookrightarrow L^p$, if $1 \leqslant p, q \leqslant \infty$ and $s > 0$;
- for $s_1 < s_2$, $B^{s_2}_{pq} \hookrightarrow B^{s_1}_{pq}$ $(1 \leqslant q, p \leqslant \infty)$;
- for $1 \leqslant q_1 < q_2 \leqslant \infty$, $B^s_{pq_1} \hookrightarrow B^s_{pq_2}$ $(s \in \mathbb{R}, 1 \leqslant p \leqslant \infty)$;
- for $s - 1/p = s_1 - 1/p_1$, $B^s_{pq} \hookrightarrow B^{s_1}_{p_1 q_1}$ $(s, s_1 \in \mathbb{R}, 1 \leqslant p \leqslant p_1 \leqslant \infty, 1 \leqslant q \leqslant q_1 \leqslant \infty)$.

In the terminology of Section 5.2.1, the first statement means that the spaces B^s_{pq} are interspaces for the RHS $\mathcal{S} \hookrightarrow L^2 \hookrightarrow \mathcal{S}^\times$. The inclusion relations above mean that the family of spaces B^s_{pq} contains again many chains of Banach spaces, but no more.

(ii) *Interpolation:* Besov spaces enjoy nice interpolation properties, in all three parameters s, p, q.

(iii) *Duality:* one has $(B^s_{pq})^\times = B^{-s}_{\overline{p}\,\overline{q}}$ $(s \in \mathbb{R})$.

(iv) *Translation and dilation invariance:* every space B^s_{pq} is invariant under translation and dilation.

(v) *Regularity shift:* let $J^\sigma : \mathcal{S}^\times \to \mathcal{S}^\times$ denote the operator $J^s f = \mathcal{F}^{-1}\left\{(1 + |\cdot|^2)^{s/2}\mathcal{F}f\right\}$, $s \in \mathbb{R}$. Then J^σ is an isomorphism from B^s_{pq} onto $B^{s-\sigma}_{pq}$. Thus J^σ is totally regular for $\sigma \leqslant 0$, but not for $\sigma > 0$.

It is also useful to consider the homogeneous Besov space $\dot{B}^s_{pq}$, defined as the set of all $f \in \mathcal{S}^\times$ for which $\|f\dot{\|}^s_{pq} < \infty$, where the quasi-norm $\| \cdot \dot{\|}^s_{pq}$ is defined by

$$\|f\dot{\|}^s_{pq} := \left(\sum_{j=-\infty}^{\infty} (2^{sj} \|\varphi_j * f\|_p)^q \right)^{1/q}$$

(this is only a quasi-norm since $\|f\dot{\|}^s_{pq} = 0$ if and only if $\operatorname{supp} \widehat{f} = \{0\}$, i.e., f is a polynomial). Note that, if $0 \notin \operatorname{supp} \widehat{f}$, then $f \in \dot{B}^s_{pq}$ if and only if $f \in B^s_{pq}$.

The spaces $\dot{B}^s_{pq}$ have properties similar to the previous ones and, in addition, one has $B^s_{pq} = L^p \cap \dot{B}^s_{pq}$ for $s > 0$, $1 \leqslant p, q \leqslant \infty$. In particular, every space $\dot{B}^s_{pq}$ is invariant under translation and dilation, which is not surprising, since these spaces are in fact based on the $ax+b$ group, consisting precisely of dilations and translations of the real line, via the coorbit space construction (see Section 8.5(ii) below).

Besov spaces are well-adapted to *wavelet analysis*, because the definition (8.8) essentially relies on a dyadic partition (powers of 2). Historically, the connection was made with the *discrete* wavelet analysis, for that reason. Indeed, there exists an equivalent definition given in terms of decay of wavelet coefficients. More precisely, if a function f is expanded in a wavelet basis, the decay properties of the wavelet coefficients allow to characterize precisely to which Besov space the function f belongs. In addition, the Besov spaces may also be characterized in terms of the *continuous* wavelet transform. These properties will be discussed below.

In order to go into details, we have to some recall basic facts about the wavelet transform (for simplicity, we restrict ourselves to one dimension). Whereas the STFT is defined in terms of translation and modulation, the continuous wavelet transform is based on translations and dilations:[2]

$$(W_\psi s)(b, a) = a^{-1} \int_{-\infty}^{\infty} \overline{\psi\,(a^{-1}(x-b))}\, s(x)\, dx, \quad a > 0,\, b \in \mathbb{R},\, s \in L^2(\mathbb{R}). \tag{8.11}$$

In this relation, the wavelet ψ is assumed to satisfy the admissibility condition

$$c_\psi := \int_{-\infty}^{\infty} d\omega\; |\omega|^{-1} |\widehat{\psi}(\omega)|^2 < \infty,$$

which implies $\int_{-\infty}^{\infty} \psi(x)\, dx = 0$. In addition, the wavelet ψ is said to have N *vanishing moments* ($N \in \mathbb{N}$) if it verifies the conditions

[2] This is the so-called L^1 normalization. It is more frequent to use the L^2 normalization, in which the prefactor is $a^{-1/2}$ instead of a^{-1}.

$$\int_{\mathbb{R}} x^n \psi(x)\, dx = 0, \quad \text{for} \quad n = 0, 1, \ldots, N-1.$$

This property improves the efficiency of ψ at detecting singularities in a signal, since the wavelet ψ is then blind to polynomials up to order $N-1$, which constitute the smoothest part of the signal, i.e., the part which contains the smallest amount of information.

However, discretizing the two parameters a and b in (8.11) leads in general only to frames. In order to get orthogonal wavelet bases, one relies on the so-called *multiresolution analysis* of $L^2(\mathbb{R})$. This is defined as an increasing sequence of closed subspaces of $L^2(\mathbb{R})$:

$$\ldots \subset \mathcal{V}_{-2} \subset \mathcal{V}_{-1} \subset \mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \subset \ldots \tag{8.12}$$

with $\bigcap_{j \in \mathbb{Z}} \mathcal{V}_j = \{0\}$ and $\bigcup_{j \in \mathbb{Z}} \mathcal{V}_j$ dense in $L^2(\mathbb{R})$, and such that

(1) $f(x) \in \mathcal{V}_j \Leftrightarrow f(2x) \in \mathcal{V}_{j+1}$;
(2) There exists a function $\phi \in \mathcal{V}_0$, called a *scaling function*, such that the family $\{\phi(\cdot - k),\, k \in \mathbb{Z}\}$ is an orthonormal basis of $\mathcal{V}_0$.

Combining conditions (1) and (2), one sees that $\{\phi_{jk} \equiv 2^{j/2}\phi(2^j \cdot - k),\, k \in \mathbb{Z}\}$ is an orthonormal basis of $\mathcal{V}_j$. The space $\mathcal{V}_j$ can be interpreted as an *approximation* space at resolution 2^j. Defining $\mathcal{W}_j$ as the orthogonal complement of $\mathcal{V}_j$ in $\mathcal{V}_{j+1}$, i.e., $\mathcal{V}_j \oplus \mathcal{W}_j = \mathcal{V}_{j+1}$, we see that $\mathcal{W}_j$ contains the additional *details* needed to improve the resolution from 2^j to 2^{j+1}. Thus one gets the decomposition $L^2(\mathbb{R}) = \bigoplus_{j \in \mathbb{Z}} \mathcal{W}_j$. The crucial theorem then asserts the existence of a function ψ, called the *mother wavelet*, explicitly computable from ϕ, such that $\{\psi_{jk} \equiv 2^{j/2}\psi(2^j \cdot - k),\ k \in \mathbb{Z}\}$ constitutes an orthonormal basis of $\mathcal{W}_j$ and thus $\{\psi_{jk} \equiv 2^{j/2}\psi(2^j \cdot - k),\ j, k \in \mathbb{Z}\}$ is an orthonormal basis of $L^2(\mathbb{R})$: these are the *orthonormal wavelets*. Thus the expansion of an arbitrary function $f \in L^2$ into an orthogonal wavelet basis $\{\psi_{jk},\ j, k \in \mathbb{Z}\}$ reads

$$f = \sum_{j,k \in \mathbb{Z}} c_{jk}\, \psi_{jk}, \quad \text{with } c_{jk} = \langle \psi_{jk} | f \rangle. \tag{8.13}$$

Additional regularity conditions can be imposed to the scaling function ϕ. Given $r \in \mathbb{N}$, the multiresolution analysis corresponding to ϕ is called r-*regular* if

$$\left| \frac{d^n \phi}{dx^n} \right| \leqslant c_m (1 + |x|^m), \text{ for all } n \leqslant r \text{ and all integers } m \in \mathbb{N}.$$

Well-known examples include the Haar wavelets, the B-splines, and the various Daubechies wavelets.

As a result of the 'dyadic' definition (8.8)-(8.9), it is natural that Besov spaces can be characterized in terms of an r-regular multiresolution analysis $\{\mathcal{V}_j\}$. Let $E_j : L^2 \to \mathcal{V}_j$ be the orthogonal projection on $\mathcal{V}_j$ and $D_j = E_{j+1} - E_j$ that on $\mathcal{W}_j$. Let $0 < s < r$ and $f \in L^p(\mathbb{R})$. Then, $f \in B^s_{pq}(\mathbb{R})$ if, and only if, $\|D_j f\|_p = 2^{-js}\, \delta_j$, where $(\delta_j) \in \ell^q(\mathbb{N})$, and one has

$$\|f\|^s_{pq} \asymp \|E_0 f\|_p + \Big(\sum_{j\in\mathbb{Z}} 2^{jsq} \|D_j f\|^q_p\Big)^{1/q}.$$

Specializing to $p = q = 2$, one gets a similar result for Sobolev spaces: given $f \in H^{-r}(\mathbb{R})$ and $|s| < r$, $f \in H^s(\mathbb{R})$ if, and only if, $E_0 f \in L^2(\mathbb{R})$ and $\|D_j f\|_2 = 2^{-js}\, \epsilon_j$, $j \in \mathbb{N}$, where $(\epsilon_j) \in \ell^2(\mathbb{N})$.

But there is more. Indeed, modulation spaces and Besov spaces admit decomposition of elements into wavelet bases and each space can be uniquely characterized by the decay properties of the wavelet coefficients. To be precise, let $\{\psi_{jk},\, j, k \in \mathbb{Z}\}$ be an orthogonal wavelet basis coming from an r-regular multiresolution analysis based on the scaling function ϕ. Then the following results are typical:

(i) *Inhomogeneous Besov spaces:* $f \in B^s_{pq}(\mathbb{R})$ if it can be written as

$$f(x) = \sum_{k\in\mathbb{Z}} \beta_k \phi(x-k) + \sum_{j\geqslant 0, k\in\mathbb{Z}} c_{jk}\, \psi_{jk},$$

where $(\beta_k) \in \ell^p$ and $\big(\sum_{k\in\mathbb{Z}} |c_{jk}|^p\big)^{1/p} = 2^{-j(s+1/2-1/p)}\, \gamma_j$, with $(\gamma_j) \in \ell^q(\mathbb{Z})$.

(ii) *Homogeneous Besov spaces:* let $|s| < r$. Then, if $f \in \dot{B}^s_{pq}(\mathbb{R})$, its wavelet coefficients c_{jk} verify $\big(\sum_{k\in\mathbb{Z}} |c_{jk}|^p\big)^{1/p} = 2^{-j(s+1/2-1/p)}\, \gamma_j$, where $(\gamma_j) \in \ell^q(\mathbb{Z})$. Conversely, if this condition is satisfied, then $f = g + P$, where $g \in \dot{B}^s_{pq}$ and P is a polynomial.

We conclude this section with some examples of unconditional wavelet bases, as announced in Section 3.4.4. For precise definitions, we refer to the literature.

- The Haar wavelet basis is defined by the scaling function $\phi_H = \chi_{[0,1]}$ and the mother wavelet $\psi_H = \chi_{[0,1/2]} - \chi_{[1/2,1]}$. It is a standard result that the Haar system is an unconditional basis for every $L^p(\mathbb{R})$, $1 < p < \infty$.
- The Lemarié-Meyer wavelet basis is an unconditional basis for all L^p spaces, Sobolev spaces, homogeneous Besov spaces $\dot{B}^s_{pq}$ $(1 \leqslant p, q < \infty)$.
- There is a class of wavelet bases (Wilson bases of exponential decay) that are unconditional bases for every modulation space $M^{p,q}_m$, $1 \leqslant p, q < \infty$, but *not* for L^p, $1 < p < \infty$, $p \neq 2$.

The characterization of Besov spaces in terms of discrete wavelet coefficients is standard, but there exists also an interesting one in terms of the continuous wavelet transform (CWT). A preliminary step is to reformulate the CWT (8.11) in L^p. The result is that, for any admissible wavelet ψ, the CWT map $W_\psi : f(x) \mapsto (W_\psi f)(b,a)$ is a bounded linear operator from $L^p(\mathbb{R})$ into $L^p(\mathbb{R}) \times L^2(\mathbb{R}_+^*, \frac{da}{a})$ and one has

$$\|f\|_p \asymp \left(\int_{-\infty}^{+\infty} \left(\int_0^{+\infty} |(W_\psi f)(b,a)|^2 \frac{da}{a} \right)^{p/2} db \right)^{1/p}.$$

The familiar Parseval formula extends to this context too (for pairs of vectors belonging to L^p, resp. $L^{\overline{p}}$, with $1/p + 1/\overline{p} = 1$, as usual), and so does the reconstruction formula.

Now we come back to Besov spaces. For simplicity, we quote the results only in the simplest case, which is s non-integer, denoting by $[s]$ the integer part of s. Then one has:

Proposition 8.4.1. *(1) Let $f \in B^s_{pq}(\mathbb{R})$, $1 \leqslant p,q < \infty$, s non-integer. Let ψ be a wavelet such that $(x^{s-[s]}\psi) \in L^1(\mathbb{R})$, with $[s]+1$ vanishing moments. Then the wavelet transform of f satisfies the condition*

$$\int_0^\infty \left(a^{-s} \|(W_\psi f)(\cdot, a)\| \right)^q \frac{da}{a} < \infty.$$

(2) Conversely, let $s > 0$, non-integer, and ψ a real-valued $C^{[s]+1}$ wavelet, with all derivatives rapidly decreasing. If $f, f', \ldots, f^{[s]} \in L^p(\mathbb{R}), 1 < p < \infty$, and if $a^{-s}\|W_\psi f(\cdot, a)\|_p \in L^q(\mathbb{R}_+^*, \frac{da}{a}), 1 \leqslant q \leqslant \infty$, then f belongs to $B^s_{pq}(\mathbb{R})$.

Thus, as expected, the behavior at small scales of the wavelet transform indeed characterizes Besov spaces.

8.4.1 α-Modulation Spaces

The α-modulation spaces ($\alpha \in [0,1]$) are spaces intermediate between modulation and Besov spaces, to which they reduce for $\alpha = 0$ and $\alpha \to 1$, respectively. A possible definition of these spaces runs as follows. Whereas the modulation spaces are defined in terms of the Gabor transform, the α-modulation spaces rely on the so-called *flexible Gabor-wavelet transform*, that is,

$$(V_\psi^\alpha f)(x,\omega) = \langle T_x M_\omega D_{w_{-\alpha}(\omega)} \psi | f \rangle, \tag{8.14}$$

where D_a is the unitary dilation operator:

$$D_a f(x) = a^{-d/2} f(a^{-1}x),\ a > 0,\ f \in L^2(\mathbb{R}^d), \tag{8.15}$$

and $w_{-\alpha}$ is, as usual, the weight function $w_{-\alpha}(\omega) = (1 + |\omega|)^{-\alpha}$, $\alpha \in [0, 1)$. Clearly, for $\alpha = 0$, this reduces to the Gabor transform, whereas, for $\alpha = 1$, one gets a simple variant of the wavelet transform. The intermediate case $\alpha = 1/2$ appears in the literature under the name of *FBI transform* (for Fourier-Bros-Iagolnitzer).

Then, for $s \in \mathbb{R}$, for all $1 \leqslant p, q \leqslant \infty$ and for $\alpha \in [0, 1]$, one can define the α-modulation space via the relation

$$M^{p,q}_{s+\alpha(1/q-1/2),\alpha} := \{f \in \mathcal{S}^{\times}(\mathbb{R}^d) : V^{\alpha}_{\psi} f \in L^{p,q}_{w_s}(\mathbb{R}^{2d})\} \tag{8.16}$$

with the norm

$$\|f\|_{M^{p,q}_{s+\alpha(1/q-1/2),\alpha}} = \|V^{\alpha}_{\psi} f\|^{p,q}_{w_s}.$$

Here $L^{p,q}_{w_s}(\mathbb{R}^{2d})$ denotes, as usual, the weighted mixed-norm L^2 space with weight $w_s = (1 + |\omega|)^s$:

$$\|F\|_{L^{p,q}_{w_s}} := \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |F(x, \omega)|^p dx \right)^{q/p} (1 + |\omega|)^{sq} d\omega \right)^{1/q}, 1 \leqslant p, q < \infty.$$

The usual modifications apply when $p = \infty$ or $q = \infty$.

For $\alpha = 0$, the space $M^{p,q}_{s,0}$ coincides with the modulation space $M^{p,q}_{w_s}$. For $\alpha \to 1$, the space $M^{p,q}_{s,\alpha}$ tends to the inhomogeneous Besov space $B^s_{p,q}$. Thus we may write $M^{p,q}_{s,1} = \lim_{\alpha \to 1} M^{p,q}_{s,\alpha}$, where the limit is to be taken in a geometrical (but somewhat imprecise) sense. In order to appreciate the true signification of these facts in signal processing, one needs some group-theoretical technology that we will introduce in the next section.

8.5 Coorbit Spaces

Coorbit spaces constitute a far reaching generalization of the function spaces described above. They provide a unified description of a number of function spaces useful in signal processing, some examples of which will be detailed at the end of this section. The construction is based on integrable group representations and thus requires a substantial amount of new concepts. Therefore our treatment will be very sketchy here, since otherwise this would lead us too far from our main subject. In particular, propositions will be stated without proof.

The starting point is the notion of (square) integrable group representation. Let G be a locally compact group with left Haar measure dg and U a (strongly continuous) irreducible representation in a Hilbert space $\mathcal{H}$. For a fixed nonzero vector $\eta \in \mathcal{H}$, denote by $V_\eta \phi$ the representation coefficient (matrix element) $V_\eta \phi(g) := \langle U(g)\eta | \phi \rangle$, a continuous bounded function on G.

Then the representation U is said to be *square integrable*, resp. *integrable*, if there exists at least one nonzero vector $\eta \in \mathcal{H}$ (called admissible) such that $V_\eta \eta \in L^2(G, dg)$, resp. $V_\eta \eta \in L^1(G, dg)$. Every integrable representation is square integrable.

Let U be a square integrable representation in $\mathcal{H}$ and let $\mathcal{A}$ denote the set of all admissible vectors. The crucial fact is the existence of *orthogonality relations*. Namely, there exists a unique positive, self-adjoint, invertible operator[3] C in $\mathcal{H}$, with dense domain $\mathcal{D}(C)$ equal to $\mathcal{A}$, such that, for any two admissible vectors η and η' and arbitrary vectors $\phi, \phi' \in \mathcal{H}$, one has

$$\int_G \overline{V_{\eta'}\phi'(g)}\, V_\eta \phi(g)\, dg = \langle C\eta | C\eta' \rangle \, \langle \phi' | \phi \rangle. \tag{8.17}$$

Furthermore $C = \lambda I,\ \lambda > 0$, if, and only if, G is unimodular. As an important consequence of the relations (8.17), one has the convolution identity

$$V_\eta \phi * V_{\eta'} \phi' = \langle C\eta | C\phi' \rangle V_{\eta'} \phi,\ \forall\, \eta, \phi' \in \mathcal{D}(C),\ \eta', \phi \in \mathcal{H}. \tag{8.18}$$

Here the convolution on G is defined as

$$(\chi * \xi)(g) = \int_G \chi(g_1) \xi(g_1^{-1} g)\, dg_1.$$

In particular, normalizing the vector $\eta \in \mathcal{D}(C)$ by $\|C\eta\| = 1$, one gets the *reproduction formula*

$$V_\eta \phi = V_\eta \phi * V_\eta \eta. \tag{8.19}$$

In other words, the function $K(g, g_1) = \langle U(g_1^{-1} g)\eta | \eta \rangle$ is a reproducing kernel on G.

Given a fixed admissible vector $\eta \in \mathcal{H}$, the map $V_\eta : \phi \mapsto V_\eta \phi(g)$ is called the *coherent state transform* or *CS transform* on G (sometimes called abusively the wavelet transform). The map V_η is an isometry from $\mathcal{H}$ into $L^2(G, dg)$ and satisfies $V_\eta(U(g)\phi) = L_g V_\eta \phi$, where L_g is the left regular representation. In other words, V_η intertwines U and L, and U is equivalent to a subrepresentation of L, hence U belongs to the discrete series of G.

The orthogonal projection from $L^2(G)$ onto the range of V_η is given by the convolution operator $\xi \mapsto \xi * V_\eta \eta$. Thus a function $\xi \in L^2(G)$ belongs to the range of V_η, i.e., $\xi = V_\eta \phi$ for some $\phi \in \mathcal{H}$ if and only if $\xi * V_\eta \eta = \xi$.

Now we are ready for defining coorbit spaces. We start with a unitary, irreducible, integrable representation U of G in $\mathcal{H}$. Given a weight function w (i.e., a positive continuous submultiplicative function) on G, define the following set of analyzing vectors:

$$\mathcal{A}_w := \{\eta \in \mathcal{H} : V_\eta \eta \in L^1_w(G)\}.$$

[3] The operator C is called the Duflo-Moore operator.

Since U is irreducible, $\mathcal{A}_w$ is a dense subspace of $\mathcal{H}$, assumed to be nontrivial. Then, fixing a nonzero vector $\eta \in \mathcal{A}_w$, one defines the space

$$\mathcal{H}_w^1 := \{\phi \in \mathcal{H} : V_\eta \phi \in L_w^1(G)\}.$$

$\mathcal{H}_w^1$ is a U-invariant Banach space for the norm

$$\|\phi\|_{\mathcal{H}_w^1} := \|V_\eta \phi\|_w^1.$$

It is dense in $\mathcal{H}$ and independent of the choice of the vector $\eta \in \mathcal{A}_w$.

Next one considers the conjugate dual $(\mathcal{H}_w^1)^\times$ of $\mathcal{H}_w^1$ (called the *reservoir*) and thus one gets the triplet

$$\mathcal{H}_w^1 \hookrightarrow \mathcal{H} \hookrightarrow (\mathcal{H}_w^1)^\times. \tag{8.20}$$

In other words, we obtain a RHS whose extreme spaces are Banach spaces, thus a Banach Gel'fand triple and a PIP-space. The action of U on $\mathcal{H}_w^1$ can be extended to $(\mathcal{H}_w^1)^\times$ by duality:

$$\langle \phi, U(g)\psi \rangle := \langle U(g^{-1})\phi, \psi \rangle, \quad \text{for } \phi \in \mathcal{H}_w^1,\ \psi \in (\mathcal{H}_w^1)^\times.$$

Therefore the CS transform can also be extended as $V_\eta \psi(g) := \langle U(g)\eta, \psi \rangle$ for $\psi \in (\mathcal{H}_w^1)^\times$. This extension has the following properties, which clearly mean that we are in a PIP-space-setting.

Proposition 8.5.1. *(i) The inner product of $\mathcal{H}$ extends to a sesquilinear U-invariant pairing between $\mathcal{H}_w^1$ and $(\mathcal{H}_w^1)^\times$. For any $\psi \in (\mathcal{H}_w^1)^\times$, the CS transform $V_\eta \psi(g) := \langle U(g)\eta, \psi \rangle$ is a continuous function in $L_{1/w}^\infty(G)$.*

(ii) The map $V_\eta : (\mathcal{H}_w^1)^\times \to L_{1/w}^\infty(G)$ is one-to-one and intertwines U and L, i.e., one has $V_\eta(U(g)\psi) = L_g V_\eta \psi$, $\forall\, \psi \in (\mathcal{H}_w^1)^\times$.

(iii) If η is normalized by $\|C\eta\| = 1$, the reproducing formula holds true:

$$V_\eta \psi = V_\eta \psi * V_\eta \eta, \quad \textit{for all } \psi \in (\mathcal{H}_w^1)^\times.$$

Let now Y be a solid Banach function space (actually, a Köthe function space) on G, that is, a Banach space of functions on G, continuously embedded in $L_{\text{loc}}^1(G)$, and satisfying the solidity condition (Section 4.4). Then the *coorbit space of Y under the representation U* is the space

$$\mathcal{C}oY := \{\psi \in (\mathcal{H}_w^1)^\times \text{ with } V_\eta \psi \in Y\}.$$

As natural norm, one takes $\|\psi\|_{\mathcal{C}oY} := \|V_\eta \psi\|_Y$. The basic properties of these spaces are as follows.

Theorem 8.5.2. *(i) $\mathcal{C}oY$ is a U-invariant Banach space, continuously embedded into $(\mathcal{H}_w^1)^\times$.*

(ii) *$\mathcal{C}oY$ is independent of the choice of the analyzing vector $\eta \in \mathcal{A}_w$, i.e., different vectors define the same space with an equivalent norm.*
(iii) *$\mathcal{C}oY$ is independent of the reservoir $(\mathcal{H}^1_w)^\times$, i.e., if w_1 is another weight with $w(g) \leqslant Cw_1(g)$ for all $g \in G$ and $\mathcal{A}_{w_1} \neq \{0\}$, then both weights generate the same space $\mathcal{C}oY$.*

The proof of this theorem relies on the following proposition, which is crucial for the applications.

Proposition 8.5.3. *(i) Given $\eta \in \mathcal{A}_w$, a function $\Psi \in Y$ is of the form $V_\eta\psi$ for some $\psi \in \mathcal{C}oY$ if and only if Ψ satisfies the reproducing formula, i.e., $\Psi = \Psi * V_\eta\eta$. It follows that*
(ii) *$V_\eta : \mathcal{C}oY \to Y$ establishes an isometric isomorphism between $\mathcal{C}oY$ and the closed subspace $Y * V_\eta\eta$ of Y, whereas $\Psi \mapsto \Psi * V_\eta\eta$ defines a bounded projection from Y onto that subspace.*
(iii) *Every function $\Psi = \Psi * V_\eta\eta$ is continuous and belongs to $L^\infty_{1/w}(G)$.*

Further interesting properties of coorbit spaces are summarized in the following

Proposition 8.5.4. *(i) $\mathcal{C}oL^\infty_{1/w} = (\mathcal{H}^1_w)^\times$.*
(ii) *$\mathcal{C}oL^2 = \mathcal{H}$.*
(iii) *Assume that Y has an absolutely continuous norm (i.e., $Y^\times = Y^\alpha$, the Köthe dual, see Section 4.4). Then*

$$(\mathcal{C}oY)^\times = \mathcal{C}oY^\alpha = \mathcal{C}oY^\times.$$

As a consequence, $\mathcal{C}oY$ is reflexive if Y is reflexive.

Besides the fact that coorbit spaces provide a unified description of a number of useful function spaces, their advantage is that, for all these spaces, the coorbit language yields interesting *atomic decompositions*. This means that every element in a given space of functions or distributions can be represented as a sum of simpler functions, called *atoms*. Then many properties of the space, such as duality, interpolation, operator theory, growth and smoothness properties, can be characterized in terms of such atoms. Furthermore, the atoms are obtained in a unified way by the action of a group on the space, this being, of course, the coherent state formalism. In turn, such atomic decompositions may be used as a discretization technique that allows to obtain in a simpler way various types of *frames* in the spaces in question. This is of crucial importance for the applications, in particular approximation theory.

The key to the atomic decompositions is that one can associate to each Banach function space Y a sequence space Y_d that characterizes the properties of Y. One starts with a discrete set of points $X = (x_i)_{i\in I}$ in G such as the one used in the definition of a BUPU in Section 8.2, that is,

(i) For a given neighborhood U_o of the identity in G, the family X is U_o-dense, i.e., $(x_iU_o)_{i\in I}$ covers G.

(ii) The family X is relatively separated, that is, for any relatively compact set W with nonempty interior, $\sup_{i\in I} \mathsf{card}\{k : x_k W \cap x_i W \neq \emptyset\} < \infty$ (here card stands for the cardinality of the set).

Then, given a Banach function space Y as before and a discrete family $X = (x_i)_{i\in I}$, one defines the *associated discrete Banach space* as

$$Y_d := \{\Lambda = (\lambda_i)_{i\in I} \text{ with } \sum_{i\in I} \lambda_i \, \chi_{x_i W} \in Y\}$$

with the natural norm $\|\Lambda\|_{Y_d} := \| \sum_{i\in I} |\lambda_i| \, \chi_{x_i W} \|_Y$. The space Y_d does not depend of the choice of W, different sets yield the same space Y_d with equivalent norms. If the functions of compact support are dense in Y, then the finite sequences form a dense subspace of Y_d. To give an example, if $Y = L^p_m$, then $Y_d = \ell^p_m$, with the weights $m(i) = m(x_i)$.

Using this tool, the central result is the atomic decomposition in $\mathcal{C}oY$. Let $X = (x_i)_{i\in I}$ be a discrete family as above and Y_d the discrete Banach space associated to Y, with $\Lambda = (\lambda_i)_{i\in I}$. Then, roughly speaking, one has:

(i) *Analysis:* There exists a bounded operator $A : \mathcal{C}oY \to Y_d$, thus

$$\|Af\|_{Y_d} \leqslant C_0 \|f\|_{\mathcal{C}oY},$$

such that every $f \in \mathcal{C}oY$ can be represented as $f = \sum_{i\in I} \lambda_i U(x_i)\eta$, where $Af = \Lambda = (\lambda_i)_{i\in I}$.

(ii) *Synthesis:* Conversely, every element $\Lambda \in Y_d$ defines an element $f = \sum_{i\in I} \lambda_i U(x_i)\eta$ in $\mathcal{C}oY$ with

$$\|f\|_{\mathcal{C}oY} \leqslant C_1 \|\Lambda\|_{Y_d}.$$

In both cases, convergence is in the sense of the norm of $\mathcal{C}oY$, if the finite sequences are dense in Y_d, in the weak*-sense of $(\mathcal{H}^1_w)^\times$ otherwise.

Associated discrete Banach spaces are the key to a number of interesting results about coorbit spaces. In fact, the Banach space structure of $\mathcal{C}oY$ is closely related to that of Y_d, although it is not known whether the two are always isomorphic. For instance:

(1) $\mathcal{C}oY \subseteq \mathcal{C}oZ$ if, and only if, $Y_d \subseteq Z_d$. In particular, $\mathcal{C}oY = \mathcal{C}oZ$ if and only if $Y_d = Z_d$.
(2) $\mathcal{C}oY$ shares with Y_d all properties which are inherited by closed subspaces and finite direct sums of Banach spaces.
(3) $\mathcal{C}oY$ is reflexive if and only if Y_d is reflexive.
(4) Whenever $\mathcal{C}oY \subseteq \mathcal{C}oZ$, the inclusion $J : \mathcal{C}oY \to \mathcal{C}oZ$ is automatically continuous. The same is true for $J_d : Y_d \to Z_d$.
(5) Moreover, J is compact (resp. Hilbert-Schmidt, nuclear) if and only if J_d is compact (resp. Hilbert-Schmidt, nuclear)

In order to go further into the properties of the class of coorbit spaces, in the PIP-space spirit, we need one more qualification. A coorbit space $\mathcal{C}oY$ is called *minimal* if $\mathcal{H}^1_w$ is norm dense in $\mathcal{C}oY$. It is called *maximal* if it is not properly contained in another coorbit space defining the same norm on $\mathcal{H}^1_w$. Then one has:

Proposition 8.5.5. *(i) $\mathcal{C}oY$ is a minimal coorbit space if and only if the finite sequences are dense in Y_d, if and only if $(Y_d)^\times = (Y_d)^\alpha$.*
(ii) $\mathcal{C}oY$ is a maximal coorbit space if and only if $\mathcal{C}oY = \mathcal{C}oZ^\alpha$ for some Banach function space Z, if and only if $Y_d = Z^\alpha_d$ for some appropriate sequence space Z.

Finally, there is a result that pertains to lattice properties of the coorbit spaces (there are further results concerning the hereditary properties of interpolation methods).

Proposition 8.5.6. *(i) The family of all minimal coorbit spaces is closed with respect to finite intersections and sums.*
(ii) The family of all maximal coorbit spaces is closed with respect to intersections and sums.
(iii) The family of all reflexive coorbit spaces is closed with respect to duality, intersections and sums.

For instance, for two minimal coorbit spaces, one has:

$$(\mathcal{C}oY^1 \cap \mathcal{C}oY^2)^\times = (\mathcal{C}oY^1)^\times + (\mathcal{C}oY^2)^\times$$
$$(\mathcal{C}oY^1 + \mathcal{C}oY^2)^\times = (\mathcal{C}oY^1)^\times \cap (\mathcal{C}oY^2)^\times,$$

and all four spaces are minimal coorbit spaces. Similarly for reflexive spaces.

It is clear that further PIP-space-type results could be obtained by combining the coorbit space methodology with the theory of Köthe sequence spaces developed in Section 4.3.

We are going now to indicate very briefly a number of examples of coorbit spaces of interest for signal processing.

(i) The Weyl-Heisenberg group and modulation spaces

The (reduced) Weyl-Heisenberg group is $\mathbb{H}_d = \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{T}$, with elements $h = (x, y, \tau)$ and group law $h_1 h_2 = (x_1, y_1, \tau_1)(x_2, y_2, \tau_2) := (x_1 + x_2, y_1 + y_2, \tau_1 \tau_2 e^{iy_1 x_2})$. The group $\mathbb{H}_d$ is unimodular, with Haar measure $dh = dx\, dy\, d\tau$. The relevant representation is the so-called *Schrödinger representation*, which forms the basis of nonrelativistic quantum mechanics, namely,

$$\big(U(x, y, \tau)\phi\big)(z) := \tau\big(M_y T_x \phi\big)(z) = \tau e^{2\pi i x y} \phi(z - x),\ z \in \mathbb{R}^d,\ \phi \in L^2(\mathbb{R}^d, dz). \tag{8.21}$$

Notice there are many different normalizations in the literature, both for the group and the representation. It follows that $V_\eta \phi$ is simply the Gabor transform, multiplied by the innocuous factor $\tau \in \mathbb{T}$, and one shows that the Schrödinger representation is integrable. To that effect, one proves that

$$\int_{\mathbb{H}_d} |V_\eta \phi(h)| \, dh < \infty,$$

whenever both functions η, ϕ have compactly supported Fourier transforms. Choose now the weight function $w_s(x, y, \tau) := (1 + |y|)^s$, $s \geqslant 0$, and the weighted L^p spaces $L^p_{w_s}$. Then it turns out that the corresponding coorbit spaces are $\mathcal{C}oL^p_{w_s} = M^{p,p}_{w_s}(\mathbb{R}^{2d})$, belonging to the family of modulation spaces. For $p = 1$, in particular, one gets the Feichtinger algebra, $\mathcal{C}oL^1 = \mathcal{S}_0$. For $p = 2$, one recovers the fractional Sobolev or Bessel potential spaces H^s. Finally, applying the discretization procedure mentioned above, one gets for the atomic decomposition simply the familiar Gabor frames.

In addition to the representation (8.21), the Weyl-Heisenberg group admits other, nonequivalent, unitary irreducible representations (UIRs), namely, $U^k(x, y, \tau) := \tau^k M_y T_x$, $k \in \mathbb{Z} \setminus \{0\}$. However, the Stone-von Neumann uniqueness theorem says that any unitary irreducible representation of $\mathbb{H}_d$ is equivalent to some U^k and, moreover, all these representations yield the same coorbit spaces.

(ii) The affine group and Besov spaces

The full affine group of the line is $G_{\text{aff}} = \mathbb{R} \rtimes \mathbb{R}_* := \{(b, a) : b \in \mathbb{R}, \, a \neq 0, \}$, with the natural action $x \mapsto ax + b$ and group law $(b, a)(b', a') = (b + ab', aa')$. The group G_{aff} is non-unimodular, the left Haar measure is $d\mu(b, a) = |a|^{-2} da \, db$ and the right Haar measure is $d\mu_r(b, a) = |a|^{-1} da \, db$.

Up to unitary equivalence, G_{aff} has a unique UIR, acting in $L^2(\mathbb{R}, dx)$, namely,

$$\big(U(b, a)\psi\big)(x) := \big(T_b D_a \psi\big)(x) = |a|^{-1/2} \, \psi\left(\frac{x - b}{a}\right), \; \psi \in L^2(\mathbb{R}, dx) \quad (8.22)$$

where D_a is the unitary dilation operator defined in (8.15). We may also write

$$(V_\eta \phi)(b, a) = \langle T_b D_a \eta | \phi \rangle = (D_a \eta^\nabla * \phi)(b),$$

where $\eta^\nabla(x) := \overline{\eta(-x)}$. This representation is square integrable, even integrable. This is shown as in the Weyl-Heisenberg case, starting with a function $\eta \in L^2(\mathbb{R})$ such that supp $\widehat{\eta}$ is compact and bounded away from 0. Then $V_\eta \eta \in L^1_w$ for many weights w, in particular $r_s(b, a) := |a|^{-s}$, $s \in \mathbb{R}$. Then it follows that $\phi \in \mathcal{C}oL^p_{r_s}$ if and only if

$$\int_{\mathbb{R}_*} \|D_a \eta^\nabla * \phi\|_p \, |a|^{-sp} \frac{da}{|a|^2} < \infty.$$

This means that $\mathcal{C}oL^p_{r_s} = \dot{B}^{s-1/2-1/p}_{pp}$, a homogeneous Besov space. Of course, the resulting atomic decompositions are simply wavelet expansions.

The extension to the multidimensional case is easy. One starts with the similitude group of $\mathbb{R}^d$, consisting of translations, rotations and dilations, namely, $\mathrm{SIM}(d) = \mathbb{R}^d \rtimes (\mathbb{R}^+_* \times \mathrm{SO}(d))$. Again this group has, up to unitary equivalence, a unique UIR, acting in $L^2(\mathbb{R}^d)$:

$$\big(U(b,a,R)\psi\big)(x) = a^{-d/2}\psi(a^{-1}R^{-1}(x-b)),\ a > 0,\, b \in \mathbb{R}^d,\, R \in \mathrm{SO}(d).$$

Then the analysis is the same as for $d = 1$ and leads to multidimensional wavelet expansions.

(iii) SL(2,$\mathbb{R}$) and Bergman spaces

The group $\mathrm{SL}(2,\mathbb{R})$ is the group of all real 2×2 matrices of determinant equal to 1 and it is unimodular. It has a family of square integrable representations (the discrete series), acting in Hilbert spaces of functions analytic in the upper half-plane $\mathbb{C}^+ := \{z = x + iy \in \mathbb{C}, y > 0\}$. The representation spaces are special cases of the so-called *Bergman spaces* $A^{p,\beta}$, $1 \leqslant p < \infty, \beta > 1$, defined as follows:

$$A^{p,\beta} := \{f \text{ analytic in } \mathbb{C}^+ : \|f\|^p_{p,\beta} = \iint_{\mathbb{C}^+} |f(z)|^p \, y^\beta \frac{dxdy}{y^2} < \infty\}. \quad (8.23)$$

For any integer $m \geqslant 2$, the discrete series representation U_m is defined on $A^{2,m}$ by

$$\left(U_m \begin{pmatrix} a & b \\ c & d \end{pmatrix} f\right)(z) := f\Big(\frac{dz-b}{-cz+a}\Big)(-cz+a)^{-m}. \quad (8.24)$$

Consider now the simpler functions $f_m(z) := (z+i)^{-m}$. The following properties are known:

(i) $f_m \in A^{2,m}$ for all $m \geqslant 2$.
(ii) $V_{f_m} f_m = \langle U_m(\cdot) f_m | f_m\rangle \in L^1(\mathrm{SL}(2,\mathbb{R}))$, $\forall\, m \geqslant 3$, that is, $f_m \in \mathcal{H}^1(U_m)$, but $\mathcal{H}^1(U_2) = \{0\}$.
(iii) For $m \geqslant 3$, one has $f_m \in A^{p,pm/2}$, $1 \leqslant p < \infty$, and U_m acts isometrically on $A^{p,pm/2}$.

As a consequence, the Bergman spaces are coorbit spaces of $L^p(\mathrm{SL}(2,\mathbb{R}))$ under the representation U_m, namely, $A^{p,pm/2} = \mathcal{C}o(L^p, U_m)$.

(iv) The Weyl-Heisenberg group and Fock-Bargmann spaces

The Fock-Bargmann space $\mathfrak{F} \equiv \mathfrak{F}^0$, introduced in Section 1.1.3, Example (v), may be generalized as follows:

$$\mathfrak{F}^{(p)} = \{f(z) \text{ entire on } \mathbb{C}^d : \|f\|^p_{\mathfrak{F}^{(p)}} := \int_{\mathbb{C}^d} |f(z)|^p \, d\mu(z) < \infty\}, \tag{8.25}$$

where $d\mu(z) = \pi^{-d} e^{-|z|^2} d\nu(z)$ is the Gaussian measure on $\mathbb{C}^d$. Thus $\mathfrak{F}^{(2)} = \mathfrak{F}$ is a Hilbert space, on which the Weyl-Heisenberg group acts via the following representation:

$$\big(U(x,y,\tau)f\big)(z) = \tau e^{-i\pi xy} e^{-|w|^2/2} e^{wz} f(z - \overline{w}), \quad w = x + iy \in \mathbb{C}^d, \ f \in \mathfrak{F}^{(2)}. \tag{8.26}$$

Choosing the function $g(z) \equiv 1$, one sees that

$$|(V_g f)(x,y,\tau)| = |f(\overline{w})| e^{-|w|^2/2},$$

so that $V_g f \in L^p(\mathbb{H}_d)$ if and only if $f \in \mathfrak{F}^{(p)}$. In other words, $\mathcal{C}o(L^p(\mathbb{H}_d)) = \mathfrak{F}^{(p)}$. As for atomic decompositions, one gets all sorts of sampling theorems for entire functions.

As a last remark, it should be mentioned that the whole theory of coorbit spaces may be generalized to the case of a representation of a locally compact group G which is only integrable modulo a subgroup H. In that case, the analysis takes place not on the group G itself, but on the quotient manifold $X = G/H$. A good example is the two-sphere $S^2 = \mathrm{SO}(3)/\mathrm{SO}(2)$.

Notes for Chapter 8

Section 8.1. Mixed norm spaces are described in detail in Benedek–Panzone [44], Bertrandias–Datry–Dupuis [45, 46]. For their applications in functional analysis and, in particular, the Schur tests of boundedness, we refer to Samarah *et al.* [174]. A nice application in signal processing, in the context of sparse representations of signals, may be found in Kowalski–Torrésani [130]. Here the authors consider hierarchized indices, as described in the text, and use (1,2)- and (2,1)-norms. The idea is that ℓ^1 norms favor sparsity, whereas ℓ^2 norms do not.

- Amalgam spaces (L^p, ℓ^q) are discussed in detail in the review papers by Fournier–Stewart [97] and Holland [123]. Some information may also be found in the monograph of Gröchenig [Grö01, Sec.11.1]. In the notation $W(L^p, \ell^q)$ or $W(L^p, L^q))$, W stands for Wiener, since this author was the first to consider a space of this type. Indeed he introduced the spaces $W(L^1, \ell^2)$ and $W(L^2, \ell^1)$ in [190], then $W(L^1, \ell^\infty)$ and $W(L^\infty, \ell^1)$ in [191] and the textbook [Wie33]. Weighted Wiener amalgams are reviewed by Heil [122]. The general theory was developed by Feichtinger [79–81], using the notion of bounded uniform partition of unity (BUPU). It is often applied to Gabor analysis, see for instance in Gröchenig–Heil–Okoudjou [113] and in Feichtinger–Weisz [92].

Section 8.3. Time-frequency analysis, more precisely Gabor analysis, and modulation spaces are studied in the monograph of Gröchenig [Grö01, Sec.11.1], that we follow closely. A good review paper on modulation spaces is Feichtinger [82].

- The Feichtinger algebra $\mathcal{S}_0$ was introduced by Feichtinger [78]. A comprehensive study may be found in Feichtinger–Zimmermann [93]. Banach Gel'fand triples and their application in Gabor analysis, in particular to localization operators, are studied by Dörfler–Feichtinger–Gröchenig [74] and Feichtinger–Luef–Cordero [91]. The concept extends naturally to locally compact abelian groups, see for instance, Feichtinger–Kozek [90] or the monograph of Reiter–Stegeman [RS00]. As a replacement of the standard Schwartz' space RHS $(\mathcal{S}, L^2, \mathcal{S}^\times)$, the Banach Gel'fand triple $(\mathcal{S}_0, L^2, \mathcal{S}_0^\times)$ has found a natural role in the connection between Gabor analysis and noncommutative geometry [145, 146].

Section 8.4. Besov spaces are described in the monographs of Bergh–Löfström [BL76, Sec.6.2] and Triebel [Tri78b, vol.II, Chap.2]. For Littlewood-Paley dyadic decompositions, see Stein [Ste70, Sec. IV.5].

- A standard reference for wavelet analysis is the textbook of Daubechies [Dau92]. The characterization of Besov spaces in terms of the decay of discrete wavelet coefficients is analyzed in the monograph of Meyer [Mey90, Chap.II.9 and Chap.VI.10]. The analogous result in terms of the continuous wavelet transform is due to Perrier–Basdevant [164].
- Several examples of unconditional wavelet bases are given by Gröchenig [112]. The case of Wilson bases is due to Feichtinger–Gröchenig–Walnut [87].
- The flexible Gabor-wavelet transform was introduced by Feichtinger–Fornasier [89] as a transform intermediate between the Gabor ($\alpha = 0$) and the wavelet transforms ($\alpha = 1$). The case ($\alpha = 1/2$) is called the Fourier–Bros–Iagolnitzer or FBI transform. For the latter, we refer to the monograph of Delort [Del92]. The α-modulation spaces based on this transform have been introduced independently by P. Gröbner [83, Grö92] and Päivärinta–Somersalo [161], and further analyzed by Dahlke *et al.* [64] and Fornasier *et al.* [96]. Actually they are a particular case of the family of function spaces intermediate between modulation and Besov spaces introduced by Nazaret–Holschneider [151]. There is a considerable literature about the α-modulation spaces, mostly in the context of pseudodifferential operators. Typical examples are the papers by Borup [53] and Borup–Nielsen [54, 55]. As a general reference for pseudodifferential operators, we may mention the classical text of Shubin [Shu01] or Folland's monograph [Fol89]. On the other hand, α-modulation spaces provide an intrinsic adaptivity which is useful for the analysis of very complex signals, containing both stationary components and transients. A nice example is their use for disentangling car crash signals [159].

Section 8.5. Coorbit spaces were introduced by Feichtinger–Gröchenig [84–86, 88]. Here we follow closely [85] and [86], in particular Proposition 8.5.1, Theorem 8.5.2 and Proposition 8.5.3 are taken from the former paper.

- For information about square integrable representations, see the review paper of Ali–Antoine–Gazeau–Mueller [6], the textbook of Ali–Antoine–Gazeau [AAG00], especially Chapter 8, and the papers by Grossmann–Morlet–Paul [120, 121]. A deeper analysis, including integrable representations, may be found in Warner's treatise [War72, Sec. 4.5.9].
- Coorbit spaces on quotient manifolds have been constructed by Dahlke [63, 64], using the theory of square integrable representations modulo a subgroup developed by Ali–Antoine–Gazeau [3]-[6]. See also the textbook of Ali–Antoine–Gazeau [AAG00].
- The Weyl-Heisenberg group $\mathbb{H}_d$ is often denoted G_{WH} in the physics literature, in particular in Ali *et al.* [6] and in the textbook [AAG00]. For instance, there one writes $\tau = e^{i\theta} \in \mathbb{T}$ and uses a different normalization, namely,
  $$g = (\theta, q, p),\ \theta \in \mathbb{R},\ (q, p) \in \mathbb{R}^{2d},$$
  with multiplication law
  $$g_1 g_2 = (\theta_1 + \theta_2 + \xi\big((q_1, p_1); (q_2, p_2)\big),\ q_1 + q_2,\ p_1 + p_2),$$
  where ξ is the multiplier function
  $$\xi\big((q_1, p_1); (q_2, p_2)\big) = \frac{1}{2}(p_1 q_2 - p_2 q_1).$$
  There the Schrödinger representation takes the form
  $$\big(U(\theta, q, p)\phi\big)(x) = e^{i\theta} e^{ip(x-\frac{q}{2})}\phi(x - q),\ \phi \in L^2(\mathbb{R}, dx).$$
  A standard reference for the Weyl-Heisenberg group is Folland's monograph [Fol89].
- For representations of SL(2,$\mathbb{R}$), see, for instance, the monograph of Lang [Lan75, chap. IX]. Actually, there are two versions of the discrete series representations mentioned here. In the SL(2,$\mathbb{R}$) presentation, the standard representation space is the Bergman space $A^{2,m}$ of functions analytic in the upper half-plane. But, since SL(2,$\mathbb{R}$) is isomorphic to SU(1,1), there is an equivalent version for which the representation space is the Bergman space $\mathcal{K}_{m-2}$ of functions analytic in the unit disk. The spaces $\mathcal{K}_\alpha$ are defined in (4.55).