Chapter 5
On the Determination of the Rouquier Blocks

The aim of this chapter is the determination of the Rouquier blocks of the cyclotomic Hecke algebras of all irreducible complex reflection groups. In the previous chapter, we saw that the Rouquier blocks have the property of "semi-continuity". This property allows us to obtain the Rouquier blocks for any cyclotomic Hecke algebra by actually calculating them in a small number of cases. Following the theory developed in the two previous chapters, we only need to determine the Rouquier blocks "associated with no and with each essential hyperplane" for all irreducible complex reflection groups.

For the exceptional irreducible complex reflection groups, the computations were made with the use of the GAP package CHEVIE (cf. [37]). In Section 5.2, we give the algorithm which has been used. This algorithm is heuristic and was applied only to the groups G_7, G_{11}, G_{19}, G_{26}, G_{28} and G_{32}. The results presented in the Appendix allow us to use Clifford theory in order to obtain the Rouquier blocks for the groups $G_4, \ldots, G_{22}$ and G_{25}. The remaining groups have already been studied by Malle and Rouquier in [53]. We have stored all the calculated data in a computer file and created GAP functions to display them. These functions are presented in this chapter and can be found on the author's webpage [24].

As far as the groups of the infinite series are concerned, Clifford theory again allows us to obtain the Rouquier blocks of the cyclotomic Hecke algebras of $G(de,e,r)$ (when $r > 2$ or $r = 2$ and e is odd) and $G(2fd,2f,2)$ from those of $G(de,1,r)$ and $G(2fd,2,2)$ respectively. Therefore, only the last two cases need to be studied thoroughly.

In Section 5.3, we determine the Rouquier blocks associated with the essential hyperplanes for the group $G(d,1,r)$. The algorithm of Lyle and Mathas (cf. [48]) for the determination of the blocks of an Ariki-Koike algebra over a field has played a key role in the achievement of this goal. The description of the Rouquier blocks for $G(d,1,r)$ is combinatorial and demonstrates an unexpected relation between them and the families of characters of the Weyl groups of type B_n, $n \leq r$.

In Section 5.4, we calculate the Rouquier blocks associated with no and with each essential hyperplane for the group $G(2d,2,2)$. The method used

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follows the same principles as the algorithm for the exceptional irreducible complex reflection groups.

Finally, in Section 5.5, we explain how exactly we apply the results of Clifford theory (Propositions 2.3.15 and 2.3.18) to obtain the Rouquier blocks of the cyclotomic Hecke algebras of the groups $G(de, e, r)$.

5.1 General Principles

Let W be an irreducible complex reflection group with field of definition K and let $\mathcal{H}$ be its generic Hecke algebra. We suppose that the assumptions 4.2.3 are satisfied. Following Theorem 4.2.4, we can find a set of indeterminates $\mathbf{v}$ such that the algebra $K(\mathbf{v})\mathcal{H}$ is split semisimple. Set $A := \mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$ and let us denote by $\mathcal{O}$ the Rouquier ring $\mathcal{R}_K(y)$ of K. Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_K$ lying over a prime number p which divides the order of the group W. We can determine the $\mathfrak{p}$-essential hyperplanes W from the factorization of the Schur elements of $\mathcal{H}$ over $K[\mathbf{v}, \mathbf{v}^{-1}]$.

Let $\phi_\emptyset : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ be a cyclotomic specialization such that the integers $n_{\mathcal{C},j}$ belong to no essential hyperplane for W. Such a cyclotomic specialization will be called *associated with no essential hyperplane*. The blocks of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}\mathcal{H}_{\phi_\emptyset}$ are called $\mathfrak{p}$-*blocks associated with no essential hyperplane* and coincide with the blocks of $A_{\mathfrak{p}A}\mathcal{H}$.

Let $\phi_H : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ be a cyclotomic specialization such that the integers $n_{\mathcal{C},j}$ belong to exactly one essential hyperplane H, corresponding to the essential monomial M. Such a cyclotomic specialization will be called *associated with the essential hyperplane* H. The blocks of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}\mathcal{H}_{\phi_H}$ are called $\mathfrak{p}$-*blocks associated with the essential hyperplane* H. If H is not $\mathfrak{p}$-essential for W, then the blocks of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}\mathcal{H}_{\phi_H}$ coincide with the $\mathfrak{p}$-blocks associated with no essential hyperplane. If H is $\mathfrak{p}$-essential for W, then the blocks of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}\mathcal{H}_{\phi_H}$ coincide with the blocks of $A_{\mathfrak{q}_M}\mathcal{H}$, where $\mathfrak{q}_M = (M-1)A + \mathfrak{p}A$. By Proposition 3.2.3, the $\mathfrak{p}$-blocks associated with the essential hyperplane H are unions of $\mathfrak{p}$-blocks associated with no essential hyperplane.

Following Proposition 4.4.4, the Rouquier blocks of $\mathcal{H}_{\phi_\emptyset}$ can be obtained as unions of $\mathfrak{p}$-blocks associated with no essential hyperplane, where $\mathfrak{p}$ runs over the set of prime ideals of $\mathbb{Z}_K$ lying over the prime divisors of $|W|$ (if $\mathfrak{p}$ is not $\phi_\emptyset$-bad, then the corresponding $\mathfrak{p}$-blocks are trivial). The Rouquier blocks of $\mathcal{H}_{\phi_\emptyset}$ are the *Rouquier blocks associated with no essential hyperplane*. Respectively, the Rouquier blocks of $\mathcal{H}_{\phi_H}$ are the *Rouquier blocks associated with the essential hyperplane* H. Like the $\mathfrak{p}$-blocks, the Rouquier blocks associated with the essential hyperplane H are unions of Rouquier blocks associated with no essential hyperplane.

The following result is a consequence of Theorem 3.3.2 and summarizes the results of Chapter 4.

Theorem 5.1.1. *Let $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ be a cyclotomic specialization which is not associated with no essential hyperplane. Let $\mathcal{E}$ be the set of all essential hyperplanes to which the integers $n_{\mathcal{C},j}$ belong. Let $\chi, \psi \in \mathrm{Irr}(W)$. The characters χ_ϕ and ψ_ϕ belong to the same block of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}\mathcal{H}_\phi$ if and only if there exist a finite sequence $\chi_0, \chi_1, \ldots, \chi_n \in \mathrm{Irr}(W)$ and a finite sequence $H_1, \ldots, H_n \in \mathcal{E}$ such that*

- $(\chi_0)_\phi = \chi_\phi$ *and* $(\chi_n)_\phi = \psi_\phi$,
- *for all j $(1 \leq j \leq n)$, $(\chi_{j-1})_\phi$ and $(\chi_j)_\phi$ are in the same $\mathfrak{p}$-block associated with the essential hyperplane H_j.*

Moreover, the characters χ_ϕ and ψ_ϕ belong to the same Rouquier block of $\mathcal{H}_\phi$ if and only if there exist a finite sequence $\chi_0, \chi_1, \ldots, \chi_n \in \mathrm{Irr}(W)$ and a finite sequence $H_1, \ldots, H_n \in \mathcal{E}$ such that

- $(\chi_0)_\phi = \chi_\phi$ *and* $(\chi_n)_\phi = \psi_\phi$,
- *for all j $(1 \leq j \leq n)$, $(\chi_{j-1})_\phi$ and $(\chi_j)_\phi$ are in the same Rouquier block associated with the essential hyperplane H_j.*

Thanks to the above theorem, in order to determine the Rouquier blocks of any cyclotomic Hecke algebra associated to W, we only need to consider a cyclotomic specialization associated with no and with each essential hyperplane and

- either calculate their $\mathfrak{p}$-blocks, for all prime ideals $\mathfrak{p}$ lying over the prime divisors of $|W|$, and use Proposition 4.4.4 in order to obtain their Rouquier blocks,
- or calculate directly their Rouquier blocks.

In the case of the exceptional groups, we will use the first method, whereas in the case of the groups of the infinite series, we will mostly use the second one. In both cases, we will need some criteria in order to determine the corresponding partitions of $\mathrm{Irr}(W)$ into blocks. These are results which have already been presented in previous chapters, but we are going to repeat here for the convenience of the reader. Once more, let $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ be a cyclotomic specialization and let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_K$ lying over the prime number p.

Proposition 2.4.18. An irreducible character $\chi \in \mathrm{Irr}(W)$ is a block of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}\mathcal{H}_\phi$ by itself if and only if $s_{\chi_\phi} \notin \mathfrak{p}\mathbb{Z}_K[y, y^{-1}]$.

Proposition 3.2.5. Let C be a block of $A_{\mathfrak{p}A}\mathcal{H}$. If M is an essential monomial for W which is not $\mathfrak{p}$-essential for any $\chi \in C$, then C is a block of $A_{\mathfrak{q}_M}\mathcal{H}$, where $\mathfrak{q}_M = (M-1)A + \mathfrak{p}A$.

Proposition 4.3.8. If $\chi, \psi \in \mathrm{Irr}(W)$ belong to the same block of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}\mathcal{H}_\phi$, then they are in the same p-block of W.

Proposition 4.4.6. If $\chi, \psi \in \mathrm{Irr}(W)$ are in the same block of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}\mathcal{H}_\phi$, then they are in the same Rouquier block of $\mathcal{H}_\phi$ and we have

$$a_{\chi_\phi} + A_{\chi_\phi} = a_{\psi_\phi} + A_{\psi_\phi}.$$

5.2 The Exceptional Irreducible Complex Reflection Groups

Let $W := G_n$ $(4 \leq n \leq 37)$ be an irreducible exceptional complex reflection group with field of definition K.

If $n \in \{23, 24, 27, 29, 30, 31, 33, 34, 35, 36, 37\}$, then W has only one hyperplane orbit $\mathcal{C}$ with $e_{\mathcal{C}} = 2$. The generic Hecke algebra of W is defined over a Laurent polynomial ring in two indeterminates $v_{\mathcal{C},0}$ and $v_{\mathcal{C},1}$ and the only essential monomial for W is $v_{\mathcal{C},0}v_{\mathcal{C},1}^{-1}$.

If ϕ is the "spetsial" cyclotomic specialization (see Example 4.3.2), then ϕ is associated with no essential hyperplane for W. The $\mathfrak{p}$-blocks, for all ϕ-bad prime ideals $\mathfrak{p}$, and the Rouquier blocks of the spetsial cyclotomic Hecke algebra of these groups have been calculated by Malle and Rouquier in [53].

If ϕ is a cyclotomic specialization associated with the unique essential hyperplane for W, then $\mathcal{H}_\phi$ is isomorphic to the group algebra $\mathbb{Z}_K W$. Its p-blocks are known from Brauer theory, whereas there exists a single Rouquier block (see also [58], §3, Remark 1).

Therefore, we will only study in detail the remaining cases.

5.2.1 Essential Hyperplanes

Let $\mathfrak{p}_1$, $\mathfrak{p}_2$ be two prime ideal of $\mathbb{Z}_K$ lying over the same prime number p. If Ψ is a K-cyclotomic polynomial, then $\Psi(1) \in \mathfrak{p}_1$ if and only if $\Psi_1 \in \mathfrak{p}_2$. We deduce that an essential hyperplane is $\mathfrak{p}_1$-essential for W if and only if it is $\mathfrak{p}_2$-essential for W. Therefore, we can talk about determining the p-essential hyperplanes for W, where p runs over the set of prime divisors of $|W|$.

Together with Jean Michel, we have programmed into the GAP package CHEVIE the Schur elements of the generic Hecke algebras of all exceptional irreducible complex reflection groups in factorized form (functions SchurModels and SchurData). Given a prime ideal $\mathfrak{p}$ of $\mathbb{Z}_K$, GAP provides us with a way to determine whether an element of $\mathbb{Z}_K$ belongs to $\mathfrak{p}$. Therefore, we can easily determine the p-essential monomials and thus, the p-essential hyperplanes for W.

In particular, we only need to follow this procedure for the groups G_7, G_{11}, G_{19}, G_{26}, G_{28} and G_{32}. In the Appendix, we give the specializations of the parameters which make

- $\mathcal{H}(G_7)$ the twisted symmetric algebra of some finite cyclic group over $\mathcal{H}(G_4)$, $\mathcal{H}(G_5)$ and $\mathcal{H}(G_6)$.
- $\mathcal{H}(G_{11})$ the twisted symmetric algebra of some finite cyclic group over $\mathcal{H}(G_8)$, $\mathcal{H}(G_9)$, $\mathcal{H}(G_{10})$, $\mathcal{H}(G_{12})$, $\mathcal{H}(G_{13})$, $\mathcal{H}(G_{14})$ and $\mathcal{H}(G_{15})$.
- $\mathcal{H}(G_{19})$ the twisted symmetric algebra of some finite cyclic group over $\mathcal{H}(G_{16})$, $\mathcal{H}(G_{17})$, $\mathcal{H}(G_{18})$, $\mathcal{H}(G_{20})$, $\mathcal{H}(G_{21})$ and $\mathcal{H}(G_{22})$.
- $\mathcal{H}(G_{26})$ the twisted symmetric algebra of the cyclic group C_2 over $\mathcal{H}(G_{25})$.

In all these cases, Proposition 2.3.15 implies that the Schur elements of the twisted symmetric algebra are scalar multiples of the Schur elements of the subalgebra. Due to the nature of the specializations, we can obtain the essential hyperplanes for the smaller group from the ones for the larger.

Example 5.2.1. The essential hyperplanes for G_7 are given in Example 5.2.3 (note that different letters represent different hyperplane orbits). The only 3-essential hyperplanes for G_7 are:

$$c_1 - c_2 = 0,\ c_0 - c_1 = 0,\ c_0 - c_2 = 0,$$
$$b_1 - b_2 = 0,\ b_0 - b_1 = 0,\ b_0 - b_2 = 0.$$

All its remaining essential hyperplanes are strictly 2-essential. From these, we can obtain the p-essential hyperplanes (where $p = 2, 3$)

- for G_6 by setting $b_0 = b_1 = b_2 = 0$,
- for G_5 by setting $a_0 = a_1 = 0$,
- for G_4 by setting $a_0 = a_1 = b_0 = b_1 = b_2 = 0$.

We have created the GAP function EssentialHyperplanes which is applied as follows:

```
gap> EssentialHyperplanes(W,p);
```

and returns

- the essential hyperplanes for W, if $p = 0$,
- the p-essential hyperplanes for W, if p divides the order of W,
- error, if p does not divide the order of W.

Example 5.2.2.

```
gap> W:=ComplexReflectionGroup(4);
gap> EssentialHyperplanes(W,0);
c_1-c_2=0
c_0-c_1=0
c_0-c_2=0
2c_0-c_1-c_2=0
c_0-2c_1+c_2=0
c_0+c_1-2c_2=0
gap> EssentialHyperplanes(W,2);
```

```
2c_0-c_1-c_2=0
c_0-2c_1+c_2=0
c_0+c_1-2c_2=0
c_0-c_1=0
c_1-c_2=0
c_0-c_2=0
gap> EssentialHyperplanes(W,3);
c_1-c_2=0
c_0-c_1=0
c_0-c_2=0
gap> EssentialHyperplanes(W,5);
Error, The number p should divide the order of the group.
```

5.2.2 Algorithm

Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_K$ lying over a prime number p which divides the order of the group W. In this section, we will present an algorithm for the determination of the $\mathfrak{p}$-blocks associated with no and with each essential hyperplane for W. We use again here the notation of Section 5.1.

If we are interested in calculating the blocks of $A_{\mathfrak{p}A}\mathcal{H}$, we follow the steps below:

1. We select the characters $\chi \in \mathrm{Irr}(W)$ whose generic Schur elements belong to $\mathfrak{p}A$. The remaining ones will be blocks of $A_{\mathfrak{p}A}\mathcal{H}$ by themselves, due to Proposition 2.4.18. Thus we form a first partition λ_1 of $\mathrm{Irr}(W)$; one part formed by the selected characters, each remaining character forming a part by itself.
2. We calculate the p-blocks of W. By Proposition 4.3.8, if two irreducible characters are not in the same p-block of W, then they cannot be in the same block of $A_{\mathfrak{p}A}\mathcal{H}$. We intersect the partition λ_1 with the partition obtained by the p-blocks of W and we obtain a finer partition, named λ_2.
3. We find a cyclotomic specialization $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ associated with no essential hyperplane by checking random values for the $n_{\mathcal{C},j}$. Following Proposition 4.4.6, we take the intersection of the partition we already have with the subsets of $\mathrm{Irr}(W)$, where the sum $a_{\chi_\phi} + A_{\chi_\phi}$ remains constant. This procedure is repeated several times, because sometimes the partition becomes finer after some repetitions. Finally, we obtain the partition λ_3, which is the finest of them all.

If we are interested in calculating the blocks of $A_{\mathfrak{q}_M}\mathcal{H}$ for some $\mathfrak{p}$-essential monomial M, the procedure is more or less the same:

1. We select the characters $\chi \in \mathrm{Irr}(W)$ for which M is a $\mathfrak{p}$-essential monomial. We form a first partition λ_1 of $\mathrm{Irr}(W)$; one part formed by the

selected characters, each remaining character forming a part by itself. The idea is that, by Proposition 3.2.5, if M is not $\mathfrak{p}$-essential for any character in a block C of $A_{\mathfrak{p}A}\mathcal{H}$, then C is a block of $A_{\mathfrak{q}_M}\mathcal{H}$. This explains step 4 below.

2. We calculate the p-blocks of W. By Proposition 4.3.8, if two irreducible characters are not in the same p-block of W, then they can not be in the same block of $A_{\mathfrak{q}_M}\mathcal{H}$. We intersect the partition λ_1 with the partition obtained by the p-blocks of W and we obtain a finer partition, named λ_2.
3. We find a cyclotomic specialization $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ associated with the $\mathfrak{p}$-essential hyperplane defined by M (again by checking random values for the $n_{\mathcal{C},j}$). We repeat the third step as described for $A_{\mathfrak{p}A}\mathcal{H}$ to obtain the partition λ_3.
4. We take the union of λ_3 and the partition defined by the blocks of $A_{\mathfrak{p}A}\mathcal{H}$.

The above algorithm is, due to step 3, heuristic. However, we will see in the next section that we only need to apply this algorithm to the groups G_7, G_{11}, G_{19}, G_{26}, G_{28} and G_{32}. In these cases, we have been able to determine (using again the criteria presented in Section 5.1) that the partition obtained at the end is minimal and corresponds to the blocks we are looking for.

Remark. Eventually, the above algorithm provides us with the correct Rouquier blocks for all exceptional irreducible complex reflection groups, except for G_{34}.

Remark. If $\mathfrak{p}_1$, $\mathfrak{p}_2$ *are two prime ideals of* $\mathbb{Z}_K$ *lying over the same prime number* p, *we have observed that, for all exceptional irreducible complex reflection groups, the* $\mathfrak{p}_1$*-blocks always coincide with the* $\mathfrak{p}_2$*-blocks. Therefore, we can talk about determining the* p*-blocks associated with no and with each essential hyperplane.*

5.2.3 Results

With the help of the GAP package CHEVIE, we developed an application which implements the algorithm of the previous section. Using Proposition 4.4.4, we have been able to determine the Rouquier blocks associated with no and with each essential hyperplane for the groups G_7, G_{11}, G_{19}, G_{26}, G_{28} and G_{32}.

Now, Clifford theory allows us to calculate the Rouquier blocks associated with no and with each essential hyperplane for the remaining exceptional irreducible complex reflection groups. In all the cases presented in the Appendix, the explicit calculation of the blocks of the twisted symmetric algebras with the use of the algorithm of the previous section has shown that the assumptions of Corollary 2.3.19 are satisfied. Moreover, in all these cases, if H is the twisted symmetric algebra of the finite cyclic group G over $\bar{H}$, then each irreducible character of H restricts to an irreducible character of $\bar{H}$. Using the notation of Proposition 2.3.15, this means that $|\bar{\Omega}| = 1$, whence the blocks of $\bar{H}$ are stable under the action of G. We deduce that the block-idempotents of H and $\bar{H}$ over the Rouquier ring coincide. In particular, if C is a block (of characters) of H, then $\{\mathrm{Res}^H_{\bar{H}}(\chi) \mid \chi \in C\}$ is a block of $\bar{H}$.

We will give here the example of G_7 and show how we obtain the blocks of G_6 from those of G_7. Nevertheless, let us first explain the notation of characters used by the CHEVIE package.

Let W be an exceptional irreducible complex reflection group. For $\chi \in \mathrm{Irr}(W)$, we set $d(\chi) := \chi(1)$ and we denote by $b(\chi)$ the valuation of the fake degree of χ (for the definition of the fake degree, see [17], 1.20). The irreducible characters χ of W are determined by the corresponding pairs $(d(\chi), b(\chi))$ and we write $\chi = \phi_{d,b}$, where $d := d(\chi)$ and $b := b(\chi)$. If two irreducible characters χ and χ' have $d(\chi) = d(\chi')$ and $b(\chi) = b(\chi')$, we use primes " $'$ " to distinguish them (following [52, 53]).

Example 5.2.3. The generic Hecke algebra of G_7 is

$$\mathcal{H}(G_7) = \left\langle S, T, U \;\middle|\; STU = TUS = UST, \begin{array}{l} (S - x_0)(S - x_1) = 0 \\ (T - y_0)(T - y_1)(T - y_2) = 0 \\ (U - z_0)(U - z_1)(U - z_2) = 0 \end{array} \right\rangle$$

Let

$$\phi : \begin{cases} x_i \mapsto (\zeta_2)^i q^{a_i} & (0 \le i < 2), \\ y_j \mapsto (\zeta_3)^j q^{b_j} & (0 \le j < 3), \\ z_k \mapsto (\zeta_3)^k q^{c_k} & (0 \le k < 3) \end{cases}$$

be a cyclotomic specialization of $\mathcal{H}(G_7)$. The only prime numbers which divide the order of G_7 are 2 and 3. Using the algorithm of the previous section, we have determined the Rouquier blocks associated with no and with each essential hyperplane for G_7. We present here only the non-trivial ones:

No essential hyperplane

$\{\phi_{2,9'}, \phi_{2,15}\}$, $\{\phi_{2,7'}, \phi_{2,13'}\}$, $\{\phi_{2,11'}, \phi_{2,5'}\}$, $\{\phi_{2,7''}, \phi_{2,13''}\}$, $\{\phi_{2,11''}, \phi_{2,5''}\}$, $\{\phi_{2,9''}, \phi_{2,3'}\}$, $\{\phi_{2,11'''}, \phi_{2,5'''}\}$, $\{\phi_{2,9'''}, \phi_{2,3''}\}$, $\{\phi_{2,7'''}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$c_1 - c_2 = 0$

$\{\phi_{1,4'}, \phi_{1,8'}\}$, $\{\phi_{1,8''}, \phi_{1,12'}\}$, $\{\phi_{1,12''}, \phi_{1,16}\}$, $\{\phi_{1,10'}, \phi_{1,14'}\}$, $\{\phi_{1,14''}, \phi_{1,18'}\}$, $\{\phi_{1,18''}, \phi_{1,22}\}$, $\{\phi_{2,9'}, \phi_{2,15}\}$, $\{\phi_{2,7'}, \phi_{2,11'}, \phi_{2,13'}, \phi_{2,5'}\}$, $\{\phi_{2,7''}, \phi_{2,13''}\}$, $\{\phi_{2,11''}, \phi_{2,9''}, \phi_{2,5''}, \phi_{2,3'}\}$, $\{\phi_{2,11'''}, \phi_{2,5'''}\}$, $\{\phi_{2,9'''}, \phi_{2,7'''}, \phi_{2,3''}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$c_0 - c_1 = 0$

$\{\phi_{1,0}, \phi_{1,4'}\}$, $\{\phi_{1,4''}, \phi_{1,8''}\}$, $\{\phi_{1,8'''}, \phi_{1,12''}\}$, $\{\phi_{1,6}, \phi_{1,10'}\}$, $\{\phi_{1,10''}, \phi_{1,14''}\}$, $\{\phi_{1,14'''}, \phi_{1,18''}\}$, $\{\phi_{2,9'}, \phi_{2,7'}, \phi_{2,15}, \phi_{2,13'}\}$, $\{\phi_{2,11'}, \phi_{2,5'}\}$, $\{\phi_{2,7''}, \phi_{2,11''}, \phi_{2,13''}, \phi_{2,5''}\}$, $\{\phi_{2,9''}, \phi_{2,3'}\}$, $\{\phi_{2,11'''}, \phi_{2,9'''}, \phi_{2,5'''}, \phi_{2,3''}\}$, $\{\phi_{2,7'''}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$c_0 - c_2 = 0$

$\{\phi_{1,0}, \phi_{1,8'}\}$, $\{\phi_{1,4''}, \phi_{1,12'}\}$, $\{\phi_{1,8'''}, \phi_{1,16}\}$, $\{\phi_{1,6}, \phi_{1,14'}\}$, $\{\phi_{1,10''}, \phi_{1,18'}\}$, $\{\phi_{1,14'''}, \phi_{1,22}\}$, $\{\phi_{2,9'}, \phi_{2,11'}, \phi_{2,15}, \phi_{2,5'}\}$, $\{\phi_{2,7'}, \phi_{2,13'}\}$, $\{\phi_{2,7''}, \phi_{2,9''}, \phi_{2,13''}, \phi_{2,3'}\}$, $\{\phi_{2,11''}, \phi_{2,5''}\}$, $\{\phi_{2,11'''}, \phi_{2,7'''}, \phi_{2,5'''}, \phi_{2,1}\}$, $\{\phi_{2,9'''}, \phi_{2,3''}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$b_1 - b_2 = 0$

$\{\phi_{1,4''}, \phi_{1,8'''}\}, \{\phi_{1,8''}, \phi_{1,12''}\}, \{\phi_{1,12'}, \phi_{1,16}\}, \{\phi_{1,10''}, \phi_{1,14'''}\}, \{\phi_{1,14''}, \phi_{1,18''}\},$
$\{\phi_{1,18'}, \phi_{1,22}\}, \{\phi_{2,9'}, \phi_{2,15}\}, \{\phi_{2,7'}, \phi_{2,13'}\}, \{\phi_{2,11'}, \phi_{2,5'}\}, \{\phi_{2,7''}, \phi_{2,11'''}, \phi_{2,13''}, \phi_{2,5'''}\},$
$\{\phi_{2,11''}, \phi_{2,9'''}, \phi_{2,5''}, \phi_{2,3''}\}, \{\phi_{2,9''}, \phi_{2,7'''}, \phi_{2,3'}, \phi_{2,1}\}, \{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\},$
$\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$b_0 - b_1 = 0$

$\{\phi_{1,0}, \phi_{1,4''}\}, \{\phi_{1,4'}, \phi_{1,8''}\}, \{\phi_{1,8'}, \phi_{1,12'}\}, \{\phi_{1,6}, \phi_{1,10''}\}, \{\phi_{1,10'}, \phi_{1,14''}\}, \{\phi_{1,14'}, \phi_{1,18'}\},$
$\{\phi_{2,9'}, \phi_{2,7''}, \phi_{2,15}, \phi_{2,13''}\}, \{\phi_{2,7'}, \phi_{2,11''}, \phi_{2,13'}, \phi_{2,5''}\}, \{\phi_{2,11'}, \phi_{2,9''}, \phi_{2,5'}, \phi_{2,3'}\},$
$\{\phi_{2,11'''}, \phi_{2,5'''}\}, \{\phi_{2,9'''}, \phi_{2,3''}\}, \{\phi_{2,7'''}, \phi_{2,1}\}, \{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}, \{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$b_0 - b_2 = 0$

$\{\phi_{1,0}, \phi_{1,8'''}\}, \{\phi_{1,4'}, \phi_{1,12''}\}, \{\phi_{1,8'}, \phi_{1,16}\}, \{\phi_{1,6}, \phi_{1,14'''}\}, \{\phi_{1,10'}, \phi_{1,18''}\}, \{\phi_{1,14'}, \phi_{1,22}\},$
$\{\phi_{2,9'}, \phi_{2,11'''}, \phi_{2,15}, \phi_{2,5'''}\}, \{\phi_{2,7'}, \phi_{2,9'''}, \phi_{2,13'}, \phi_{2,3''}\}, \{\phi_{2,11'}, \phi_{2,7'''}, \phi_{2,5'}, \phi_{2,1}\},$
$\{\phi_{2,7''}, \phi_{2,13''}\}, \{\phi_{2,11''}, \phi_{2,5''}\}, \{\phi_{2,9''}, \phi_{2,3'}\}, \{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}, \{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 - 2b_0 + b_1 + b_2 - 2c_0 + c_1 + c_2 = 0$

$\{\phi_{1,6}, \phi_{2,9'}, \phi_{2,15}, \phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}, \{\phi_{2,7'}, \phi_{2,13'}\}, \{\phi_{2,11'}, \phi_{2,5'}\}, \{\phi_{2,7''}, \phi_{2,13''}\},$
$\{\phi_{2,11''}, \phi_{2,5''}\}, \{\phi_{2,9''}, \phi_{2,3'}\}, \{\phi_{2,11'''}, \phi_{2,5'''}\}, \{\phi_{2,9'''}, \phi_{2,3''}\}, \{\phi_{2,7'''}, \phi_{2,1}\},$
$\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$

$a_0 - a_1 - 2b_0 + b_1 + b_2 + c_0 - 2c_1 + c_2 = 0$

$\{\phi_{1,10'}, \phi_{2,7'}, \phi_{2,13'}, \phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}, \{\phi_{2,9'}, \phi_{2,15}\}, \{\phi_{2,11'}, \phi_{2,5'}\}, \{\phi_{2,7''}, \phi_{2,13''}\},$
$\{\phi_{2,11''}, \phi_{2,5''}\}, \{\phi_{2,9''}, \phi_{2,3'}\}, \{\phi_{2,11'''}, \phi_{2,5'''}\}, \{\phi_{2,9'''}, \phi_{2,3''}\}, \{\phi_{2,7'''}, \phi_{2,1}\},$
$\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$

$a_0 - a_1 - 2b_0 + b_1 + b_2 + c_0 + c_1 - 2c_2 = 0$

$\{\phi_{1,14'}, \phi_{2,11'}, \phi_{2,5'}, \phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}, \{\phi_{2,9'}, \phi_{2,15}\}, \{\phi_{2,7'}, \phi_{2,13'}\}, \{\phi_{2,7''}, \phi_{2,13''}\},$
$\{\phi_{2,11''}, \phi_{2,5''}\}, \{\phi_{2,9''}, \phi_{2,3'}\}, \{\phi_{2,11'''}, \phi_{2,5'''}\}, \{\phi_{2,9'''}, \phi_{2,3''}\}, \{\phi_{2,7'''}, \phi_{2,1}\},$
$\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$

$a_0 - a_1 - b_0 - b_1 + 2b_2 - c_0 - c_1 + 2c_2 = 0$

$\{\phi_{1,16}, \phi_{2,7'''}, \phi_{2,1}, \phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}, \{\phi_{2,9'}, \phi_{2,15}\}, \{\phi_{2,7'}, \phi_{2,13'}\}, \{\phi_{2,11'}, \phi_{2,5'}\},$
$\{\phi_{2,7''}, \phi_{2,13''}\}, \{\phi_{2,11''}, \phi_{2,5''}\}, \{\phi_{2,9''}, \phi_{2,3'}\}, \{\phi_{2,11'''}, \phi_{2,5'''}\}, \{\phi_{2,9'''}, \phi_{2,3''}\},$
$\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 - b_0 - b_1 + 2b_2 - c_0 + 2c_1 - c_2 = 0$

$\{\phi_{1,12''}, \phi_{2,9'''}, \phi_{2,3''}, \phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}, \{\phi_{2,9'}, \phi_{2,15}\}, \{\phi_{2,7'}, \phi_{2,13'}\}, \{\phi_{2,11'}, \phi_{2,5'}\},$
$\{\phi_{2,7''}, \phi_{2,13''}\}, \{\phi_{2,11''}, \phi_{2,5''}\}, \{\phi_{2,9''}, \phi_{2,3'}\}, \{\phi_{2,11'''}, \phi_{2,5'''}\}, \{\phi_{2,7'''}, \phi_{2,1}\},$
$\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 - b_0 - b_1 + 2b_2 + 2c_0 - c_1 - c_2 = 0$

$\{\phi_{1,8'''}, \phi_{2,11'''}, \phi_{2,5'''}, \phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}, \{\phi_{2,9'}, \phi_{2,15}\}, \{\phi_{2,7'}, \phi_{2,13'}\}, \{\phi_{2,11'}, \phi_{2,5'}\},$
$\{\phi_{2,7''}, \phi_{2,13''}\}, \{\phi_{2,11''}, \phi_{2,5''}\}, \{\phi_{2,9''}, \phi_{2,3'}\}, \{\phi_{2,9'''}, \phi_{2,3''}\}, \{\phi_{2,7'''}, \phi_{2,1}\},$
$\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 - b_0 + b_2 - c_0 + c_1 = 0$

$\{\phi_{1,12''}, \phi_{1,6}, \phi_{2,9''}, \phi_{2,3'}\}, \{\phi_{2,9'}, \phi_{2,15}\}, \{\phi_{2,7'}, \phi_{2,13'}\}, \{\phi_{2,11'}, \phi_{2,5'}\}, \{\phi_{2,7''}, \phi_{2,13''}\},$
$\{\phi_{2,11''}, \phi_{2,5''}\},$
$\{\phi_{2,11'''}, \phi_{2,5'''}\}, \{\phi_{2,9'''}, \phi_{2,3''}\}, \{\phi_{2,7'''}, \phi_{2,1}\}, \{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}, \{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 - b_0 + b_2 - c_1 + c_2 = 0$

$\{\phi_{1,16}, \phi_{1,10'}, \phi_{2,7''}, \phi_{2,13''}\}, \{\phi_{2,9'}, \phi_{2,15}\}, \{\phi_{2,7'}, \phi_{2,13'}\}, \{\phi_{2,11'}, \phi_{2,5'}\}, \{\phi_{2,11''}, \phi_{2,5''}\},$
$\{\phi_{2,9''}, \phi_{2,3'}\},$
$\{\phi_{2,11'''}, \phi_{2,5'''}\}, \{\phi_{2,9'''}, \phi_{2,3''}\}, \{\phi_{2,7'''}, \phi_{2,1}\}, \{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}, \{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 - b_0 + b_2 + c_0 - c_2 = 0$

$\{\phi_{1,8'''}, \phi_{1,14'}, \phi_{2,11''}, \phi_{2,5''}\}, \{\phi_{2,9'}, \phi_{2,15}\}, \{\phi_{2,7'}, \phi_{2,13'}\}, \{\phi_{2,11'}, \phi_{2,5'}\}, \{\phi_{2,7''}, \phi_{2,13''}\},$
$\{\phi_{2,9''}, \phi_{2,3'}\},$
$\{\phi_{2,11'''}, \phi_{2,5'''}\}, \{\phi_{2,9'''}, \phi_{2,3''}\}, \{\phi_{2,7'''}, \phi_{2,1}\}, \{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}, \{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 - b_0 + b_1 - c_0 + c_2 = 0$

$\{\phi_{1,12'}, \phi_{1,6}, \phi_{2,9'''}, \phi_{2,3''}\}, \{\phi_{2,9'}, \phi_{2,15}\}, \{\phi_{2,7'}, \phi_{2,13'}\}, \{\phi_{2,11'}, \phi_{2,5'}\}, \{\phi_{2,7''}, \phi_{2,13''}\},$
$\{\phi_{2,11''}, \phi_{2,5''}\},$
$\{\phi_{2,9''}, \phi_{2,3'}\}, \{\phi_{2,11'''}, \phi_{2,5'''}\}, \{\phi_{2,7'''}, \phi_{2,1}\}, \{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}, \{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 - b_0 + b_1 + c_1 - c_2 = 0$

$\{\phi_{1,8''}, \phi_{1,14'}, \phi_{2,11'''}, \phi_{2,5'''}\}, \{\phi_{2,9'}, \phi_{2,15}\}, \{\phi_{2,7'}, \phi_{2,13'}\}, \{\phi_{2,11'}, \phi_{2,5'}\}, \{\phi_{2,7''}, \phi_{2,13''}\},$
$\{\phi_{2,11''}, \phi_{2,5''}\},$
$\{\phi_{2,9''}, \phi_{2,3'}\}, \{\phi_{2,9'''}, \phi_{2,3''}\}, \{\phi_{2,7'''}, \phi_{2,1}\}, \{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}, \{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 - b_0 + b_1 + c_0 - c_1 = 0$

$\{\phi_{1,4''}, \phi_{1,10'}, \phi_{2,7'''}, \phi_{2,1}\}, \{\phi_{2,9'}, \phi_{2,15}\}, \{\phi_{2,7'}, \phi_{2,13'}\}, \{\phi_{2,11'}, \phi_{2,5'}\}, \{\phi_{2,7''}, \phi_{2,13''}\},$
$\{\phi_{2,11''}, \phi_{2,5''}\},$
$\{\phi_{2,9''}, \phi_{2,3'}\}, \{\phi_{2,11'''}, \phi_{2,5'''}\}, \{\phi_{2,9'''}, \phi_{2,3''}\}, \{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}, \{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 - b_0 + 2b_1 - b_2 - c_0 - c_1 + 2c_2 = 0$

$\{\phi_{1,12'}, \phi_{2,9''}, \phi_{2,3'}, \phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}, \{\phi_{2,9'}, \phi_{2,15}\}, \{\phi_{2,7'}, \phi_{2,13'}\}, \{\phi_{2,11'}, \phi_{2,5'}\},$

$\{\phi_{2,7''},\phi_{2,13''}\}$, $\{\phi_{2,11''},\phi_{2,5''}\}$, $\{\phi_{2,11'''},\phi_{2,5'''}\}$, $\{\phi_{2,9'''},\phi_{2,3''}\}$, $\{\phi_{2,7'''},\phi_{2,1}\}$,
$\{\phi_{3,4},\phi_{3,8},\phi_{3,12}\}$

$a_0-a_1-b_0+2b_1-b_2-c_0+2c_1-c_2=0$
$\{\phi_{1,8''},\phi_{2,11''},\phi_{2,5''},\phi_{3,6},\phi_{3,10},\phi_{3,2}\}$, $\{\phi_{2,9'},\phi_{2,15}\}$, $\{\phi_{2,7'},\phi_{2,13'}\}$, $\{\phi_{2,11'},\phi_{2,5'}\}$,
$\{\phi_{2,7''},\phi_{2,13''}\}$, $\{\phi_{2,9''},\phi_{2,3'}\}$, $\{\phi_{2,11'''},\phi_{2,5'''}\}$, $\{\phi_{2,9'''},\phi_{2,3''}\}$, $\{\phi_{2,7'''},\phi_{2,1}\}$,
$\{\phi_{3,4},\phi_{3,8},\phi_{3,12}\}$

$a_0-a_1-b_0+2b_1-b_2+2c_0-c_1-c_2=0$
$\{\phi_{1,4''},\phi_{2,7''},\phi_{2,13''},\phi_{3,6},\phi_{3,10},\phi_{3,2}\}$, $\{\phi_{2,9'},\phi_{2,15}\}$, $\{\phi_{2,7'},\phi_{2,13'}\}$, $\{\phi_{2,11'},\phi_{2,5'}\}$,
$\{\phi_{2,11''},\phi_{2,5''}\}$, $\{\phi_{2,9''},\phi_{2,3'}\}$, $\{\phi_{2,11'''},\phi_{2,5'''}\}$, $\{\phi_{2,9'''},\phi_{2,3''}\}$, $\{\phi_{2,7'''},\phi_{2,1}\}$,
$\{\phi_{3,4},\phi_{3,8},\phi_{3,12}\}$

$a_0-a_1-b_1+b_2-c_0+c_2=0$
$\{\phi_{1,16},\phi_{1,10''},\phi_{2,7'},\phi_{2,13'}\}$, $\{\phi_{2,9'},\phi_{2,15}\}$, $\{\phi_{2,11'},\phi_{2,5'}\}$, $\{\phi_{2,7''},\phi_{2,13''}\}$, $\{\phi_{2,11''},\phi_{2,5''}\}$,
$\{\phi_{2,9''},\phi_{2,3'}\}$,
$\{\phi_{2,11'''},\phi_{2,5'''}\}$, $\{\phi_{2,9'''},\phi_{2,3''}\}$, $\{\phi_{2,7'''},\phi_{2,1}\}$, $\{\phi_{3,6},\phi_{3,10},\phi_{3,2}\}$, $\{\phi_{3,4},\phi_{3,8},\phi_{3,12}\}$

$a_0-a_1-b_1+b_2+c_1-c_2=0$
$\{\phi_{1,12''},\phi_{1,18'},\phi_{2,9'},\phi_{2,15}\}$, $\{\phi_{2,7'},\phi_{2,13'}\}$, $\{\phi_{2,11'},\phi_{2,5'}\}$, $\{\phi_{2,7''},\phi_{2,13''}\}$, $\{\phi_{2,11''},\phi_{2,5''}\}$,
$\{\phi_{2,9''},\phi_{2,3'}\}$,
$\{\phi_{2,11'''},\phi_{2,5'''}\}$, $\{\phi_{2,9'''},\phi_{2,3''}\}$, $\{\phi_{2,7'''},\phi_{2,1}\}$, $\{\phi_{3,6},\phi_{3,10},\phi_{3,2}\}$, $\{\phi_{3,4},\phi_{3,8},\phi_{3,12}\}$

$a_0-a_1-b_1+b_2+c_0-c_1=0$
$\{\phi_{1,8'''},\phi_{1,14''},\phi_{2,11'},\phi_{2,5'}\}$, $\{\phi_{2,9'},\phi_{2,15}\}$, $\{\phi_{2,7'},\phi_{2,13'}\}$, $\{\phi_{2,7''},\phi_{2,13''}\}$, $\{\phi_{2,11''},\phi_{2,5''}\}$,
$\{\phi_{2,9''},\phi_{2,3'}\}$,
$\{\phi_{2,11'''},\phi_{2,5'''}\}$, $\{\phi_{2,9'''},\phi_{2,3''}\}$, $\{\phi_{2,7'''},\phi_{2,1}\}$, $\{\phi_{3,6},\phi_{3,10},\phi_{3,2}\}$, $\{\phi_{3,4},\phi_{3,8},\phi_{3,12}\}$

$a_0-a_1=0$
$\{\phi_{1,0},\phi_{1,6}\}$, $\{\phi_{1,4'},\phi_{1,10'}\}$, $\{\phi_{1,8'},\phi_{1,14'}\}$, $\{\phi_{1,4''},\phi_{1,10''}\}$, $\{\phi_{1,8''},\phi_{1,14''}\}$, $\{\phi_{1,12'},\phi_{1,18'}\}$,
$\{\phi_{1,8'''},\phi_{1,14'''}\}$, $\{\phi_{1,12''},\phi_{1,18''}\}$, $\{\phi_{1,16},\phi_{1,22}\}$, $\{\phi_{2,9'},\phi_{2,15}\}$, $\{\phi_{2,7'},\phi_{2,13'}\}$,
$\{\phi_{2,11'},\phi_{2,5'}\}$, $\{\phi_{2,7''},\phi_{2,13''}\}$, $\{\phi_{2,11''},\phi_{2,5''}\}$, $\{\phi_{2,9''},\phi_{2,3'}\}$, $\{\phi_{2,11'''},\phi_{2,5'''}\}$,
$\{\phi_{2,9'''},\phi_{2,3''}\}$, $\{\phi_{2,7'''},\phi_{2,1}\}$, $\{\phi_{3,6},\phi_{3,4},\phi_{3,10},\phi_{3,8},\phi_{3,2},\phi_{3,12}\}$

$a_0-a_1+b_1-b_2-c_0+c_1=0$
$\{\phi_{1,8''},\phi_{1,14'''},\phi_{2,11'},\phi_{2,5'}\}$, $\{\phi_{2,9'},\phi_{2,15}\}$, $\{\phi_{2,7'},\phi_{2,13'}\}$, $\{\phi_{2,7''},\phi_{2,13''}\}$, $\{\phi_{2,11''},\phi_{2,5''}\}$,
$\{\phi_{2,9''},\phi_{2,3'}\}$,
$\{\phi_{2,11'''},\phi_{2,5'''}\}$, $\{\phi_{2,9'''},\phi_{2,3''}\}$, $\{\phi_{2,7'''},\phi_{2,1}\}$, $\{\phi_{3,6},\phi_{3,10},\phi_{3,2}\}$, $\{\phi_{3,4},\phi_{3,8},\phi_{3,12}\}$

$a_0-a_1+b_1-b_2-c_1+c_2=0$
$\{\phi_{1,12'},\phi_{1,18''},\phi_{2,9'},\phi_{2,15}\}$, $\{\phi_{2,7'},\phi_{2,13'}\}$, $\{\phi_{2,11'},\phi_{2,5'}\}$, $\{\phi_{2,7''},\phi_{2,13''}\}$, $\{\phi_{2,11''},\phi_{2,5''}\}$,
$\{\phi_{2,9''},\phi_{2,3'}\}$,
$\{\phi_{2,11'''},\phi_{2,5'''}\}$, $\{\phi_{2,9'''},\phi_{2,3''}\}$, $\{\phi_{2,7'''},\phi_{2,1}\}$, $\{\phi_{3,6},\phi_{3,10},\phi_{3,2}\}$, $\{\phi_{3,4},\phi_{3,8},\phi_{3,12}\}$

$a_0-a_1+b_1-b_2+c_0-c_2=0$
$\{\phi_{1,4''},\phi_{1,22},\phi_{2,7'},\phi_{2,13'}\}$, $\{\phi_{2,9'},\phi_{2,15}\}$, $\{\phi_{2,11'},\phi_{2,5'}\}$, $\{\phi_{2,7''},\phi_{2,13''}\}$, $\{\phi_{2,11''},\phi_{2,5''}\}$,
$\{\phi_{2,9''},\phi_{2,3'}\}$,
$\{\phi_{2,11'''},\phi_{2,5'''}\}$, $\{\phi_{2,9'''},\phi_{2,3''}\}$, $\{\phi_{2,7'''},\phi_{2,1}\}$, $\{\phi_{3,6},\phi_{3,10},\phi_{3,2}\}$, $\{\phi_{3,4},\phi_{3,8},\phi_{3,12}\}$

$a_0-a_1+b_0-2b_1+b_2-2c_0+c_1+c_2=0$
$\{\phi_{1,10''},\phi_{2,7''},\phi_{2,13''},\phi_{3,4},\phi_{3,8},\phi_{3,12}\}$, $\{\phi_{2,9'},\phi_{2,15}\}$, $\{\phi_{2,7'},\phi_{2,13'}\}$, $\{\phi_{2,11'},\phi_{2,5'}\}$,
$\{\phi_{2,11''},\phi_{2,5''}\}$,
$\{\phi_{2,9''},\phi_{2,3'}\}$, $\{\phi_{2,11'''},\phi_{2,5'''}\}$, $\{\phi_{2,9'''},\phi_{2,3''}\}$, $\{\phi_{2,7'''},\phi_{2,1}\}$, $\{\phi_{3,6},\phi_{3,10},\phi_{3,2}\}$

$a_0-a_1+b_0-2b_1+b_2+c_0-2c_1+c_2=0$
$\{\phi_{1,14''},\phi_{2,11''},\phi_{2,5''},\phi_{3,4},\phi_{3,8},\phi_{3,12}\}$, $\{\phi_{2,9'},\phi_{2,15}\}$, $\{\phi_{2,7'},\phi_{2,13'}\}$, $\{\phi_{2,11'},\phi_{2,5'}\}$,
$\{\phi_{2,7''},\phi_{2,13''}\}$, $\{\phi_{2,9''},\phi_{2,3'}\}$, $\{\phi_{2,11'''},\phi_{2,5'''}\}$, $\{\phi_{2,9'''},\phi_{2,3''}\}$, $\{\phi_{2,7'''},\phi_{2,1}\}$,
$\{\phi_{3,6},\phi_{3,10},\phi_{3,2}\}$

$a_0-a_1+b_0-2b_1+b_2+c_0+c_1-2c_2=0$
$\{\phi_{1,18'},\phi_{2,9''},\phi_{2,3'},\phi_{3,4},\phi_{3,8},\phi_{3,12}\}$, $\{\phi_{2,9'},\phi_{2,15}\}$, $\{\phi_{2,7'},\phi_{2,13'}\}$, $\{\phi_{2,11'},\phi_{2,5'}\}$,
$\{\phi_{2,7''},\phi_{2,13''}\}$, $\{\phi_{2,11''},\phi_{2,5''}\}$, $\{\phi_{2,11'''},\phi_{2,5'''}\}$, $\{\phi_{2,9'''},\phi_{2,3''}\}$, $\{\phi_{2,7'''},\phi_{2,1}\}$,
$\{\phi_{3,6},\phi_{3,10},\phi_{3,2}\}$

$a_0-a_1+b_0-b_1-c_0+c_1=0$
$\{\phi_{1,4'},\phi_{1,10''},\phi_{2,7'''},\phi_{2,1}\}$, $\{\phi_{2,9'},\phi_{2,15}\}$, $\{\phi_{2,7'},\phi_{2,13'}\}$, $\{\phi_{2,11'},\phi_{2,5'}\}$, $\{\phi_{2,7''},\phi_{2,13''}\}$,
$\{\phi_{2,11''},\phi_{2,5''}\}$,
$\{\phi_{2,9''},\phi_{2,3'}\}$, $\{\phi_{2,11'''},\phi_{2,5'''}\}$, $\{\phi_{2,9'''},\phi_{2,3''}\}$, $\{\phi_{3,6},\phi_{3,10},\phi_{3,2}\}$, $\{\phi_{3,4},\phi_{3,8},\phi_{3,12}\}$

$a_0-a_1+b_0-b_1-c_1+c_2=0$
$\{\phi_{1,8'},\phi_{1,14''},\phi_{2,11'''},\phi_{2,5'''}\}$, $\{\phi_{2,9'},\phi_{2,15}\}$, $\{\phi_{2,7'},\phi_{2,13'}\}$, $\{\phi_{2,11'},\phi_{2,5'}\}$, $\{\phi_{2,7''},\phi_{2,13''}\}$,
$\{\phi_{2,11''},\phi_{2,5''}\}$,
$\{\phi_{2,9''},\phi_{2,3'}\}$, $\{\phi_{2,9'''},\phi_{2,3''}\}$, $\{\phi_{2,7'''},\phi_{2,1}\}$, $\{\phi_{3,6},\phi_{3,10},\phi_{3,2}\}$, $\{\phi_{3,4},\phi_{3,8},\phi_{3,12}\}$

$a_0-a_1+b_0-b_1+c_0-c_2=0$
$\{\phi_{1,0},\phi_{1,18'},\phi_{2,9'''},\phi_{2,3''}\}$, $\{\phi_{2,9'},\phi_{2,15}\}$, $\{\phi_{2,7'},\phi_{2,13'}\}$, $\{\phi_{2,11'},\phi_{2,5'}\}$, $\{\phi_{2,7''},\phi_{2,13''}\}$,
$\{\phi_{2,11''},\phi_{2,5''}\}$,
$\{\phi_{2,9''},\phi_{2,3'}\}$, $\{\phi_{2,11'''},\phi_{2,5'''}\}$, $\{\phi_{2,7'''},\phi_{2,1}\}$, $\{\phi_{3,6},\phi_{3,10},\phi_{3,2}\}$, $\{\phi_{3,4},\phi_{3,8},\phi_{3,12}\}$

$a_0 - a_1 + b_0 - b_2 - c_0 + c_2 = 0$
$\{\phi_{1,8'}, \phi_{1,14'''}, \phi_{2,11''}, \phi_{2,5''}\}$, $\{\phi_{2,9'}, \phi_{2,15}\}$, $\{\phi_{2,7'}, \phi_{2,13'}\}$, $\{\phi_{2,11'}, \phi_{2,5'}\}$, $\{\phi_{2,7''}, \phi_{2,13''}\}$, $\{\phi_{2,9''}, \phi_{2,3'}\}$,
$\{\phi_{2,11'''}, \phi_{2,5'''}\}$, $\{\phi_{2,9'''}, \phi_{2,3''}\}$, $\{\phi_{2,7'''}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 + b_0 - b_2 + c_1 - c_2 = 0$
$\{\phi_{1,4'}, \phi_{1,22}, \phi_{2,7''}, \phi_{2,13''}\}$, $\{\phi_{2,9'}, \phi_{2,15}\}$, $\{\phi_{2,7'}, \phi_{2,13'}\}$, $\{\phi_{2,11'}, \phi_{2,5'}\}$, $\{\phi_{2,11''}, \phi_{2,5''}\}$, $\{\phi_{2,9''}, \phi_{2,3'}\}$,
$\{\phi_{2,11'''}, \phi_{2,5'''}\}$, $\{\phi_{2,9'''}, \phi_{2,3''}\}$, $\{\phi_{2,7'''}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 + b_0 - b_2 + c_0 - c_1 = 0$
$\{\phi_{1,0}, \phi_{1,18''}, \phi_{2,9''}, \phi_{2,3'}\}$, $\{\phi_{2,9'}, \phi_{2,15}\}$, $\{\phi_{2,7'}, \phi_{2,13'}\}$, $\{\phi_{2,11'}, \phi_{2,5'}\}$, $\{\phi_{2,7''}, \phi_{2,13''}\}$, $\{\phi_{2,11''}, \phi_{2,5''}\}$,
$\{\phi_{2,11'''}, \phi_{2,5'''}\}$, $\{\phi_{2,9'''}, \phi_{2,3''}\}$, $\{\phi_{2,7'''}, \phi_{2,1}\}$, $\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 + b_0 + b_1 - 2b_2 - 2c_0 + c_1 + c_2 = 0$
$\{\phi_{1,14'''}, \phi_{2,11'''}, \phi_{2,5'''}, \phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$, $\{\phi_{2,9'}, \phi_{2,15}\}$, $\{\phi_{2,7'}, \phi_{2,13'}\}$, $\{\phi_{2,11'}, \phi_{2,5'}\}$,
$\{\phi_{2,7''}, \phi_{2,13''}\}$, $\{\phi_{2,11''}, \phi_{2,5''}\}$, $\{\phi_{2,9''}, \phi_{2,3'}\}$, $\{\phi_{2,9'''}, \phi_{2,3''}\}$, $\{\phi_{2,7'''}, \phi_{2,1}\}$,
$\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$

$a_0 - a_1 + b_0 + b_1 - 2b_2 + c_0 - 2c_1 + c_2 = 0$
$\{\phi_{1,18''}, \phi_{2,9'''}, \phi_{2,3''}, \phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$, $\{\phi_{2,9'}, \phi_{2,15}\}$, $\{\phi_{2,7'}, \phi_{2,13'}\}$, $\{\phi_{2,11'}, \phi_{2,5'}\}$,
$\{\phi_{2,7''}, \phi_{2,13''}\}$, $\{\phi_{2,11''}, \phi_{2,5''}\}$, $\{\phi_{2,9''}, \phi_{2,3'}\}$, $\{\phi_{2,11'''}, \phi_{2,5'''}\}$, $\{\phi_{2,7'''}, \phi_{2,1}\}$,
$\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$

$a_0 - a_1 + b_0 + b_1 - 2b_2 + c_0 + c_1 - 2c_2 = 0$
$\{\phi_{1,22}, \phi_{2,7'''}, \phi_{2,1}, \phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$, $\{\phi_{2,9'}, \phi_{2,15}\}$, $\{\phi_{2,7'}, \phi_{2,13'}\}$, $\{\phi_{2,11'}, \phi_{2,5'}\}$,
$\{\phi_{2,7''}, \phi_{2,13''}\}$, $\{\phi_{2,11''}, \phi_{2,5''}\}$, $\{\phi_{2,9''}, \phi_{2,3'}\}$, $\{\phi_{2,11'''}, \phi_{2,5'''}\}$, $\{\phi_{2,9'''}, \phi_{2,3''}\}$,
$\{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$

$a_0 - a_1 + 2b_0 - b_1 - b_2 - c_0 - c_1 + 2c_2 = 0$
$\{\phi_{1,8'}, \phi_{2,11'}, \phi_{2,5'}, \phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{2,9'}, \phi_{2,15}\}$, $\{\phi_{2,7'}, \phi_{2,13'}\}$, $\{\phi_{2,7''}, \phi_{2,13''}\}$,
$\{\phi_{2,11''}, \phi_{2,5''}\}$, $\{\phi_{2,9''}, \phi_{2,3'}\}$, $\{\phi_{2,11'''}, \phi_{2,5'''}\}$, $\{\phi_{2,9'''}, \phi_{2,3''}\}$, $\{\phi_{2,7'''}, \phi_{2,1}\}$,
$\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 + 2b_0 - b_1 - b_2 - c_0 + 2c_1 - c_2 = 0$
$\{\phi_{1,4'}, \phi_{2,7'}, \phi_{2,13'}, \phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{2,9'}, \phi_{2,15}\}$, $\{\phi_{2,11'}, \phi_{2,5'}\}$, $\{\phi_{2,7''}, \phi_{2,13''}\}$,
$\{\phi_{2,11''}, \phi_{2,5''}\}$, $\{\phi_{2,9''}, \phi_{2,3'}\}$, $\{\phi_{2,11'''}, \phi_{2,5'''}\}$, $\{\phi_{2,9'''}, \phi_{2,3''}\}$, $\{\phi_{2,7'''}, \phi_{2,1}\}$,
$\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

$a_0 - a_1 + 2b_0 - b_1 - b_2 + 2c_0 - c_1 - c_2 = 0$
$\{\phi_{1,0}, \phi_{2,9'}, \phi_{2,15}, \phi_{3,6}, \phi_{3,10}, \phi_{3,2}\}$, $\{\phi_{2,7'}, \phi_{2,13'}\}$, $\{\phi_{2,11'}, \phi_{2,5'}\}$, $\{\phi_{2,7''}, \phi_{2,13''}\}$,
$\{\phi_{2,11''}, \phi_{2,5''}\}$, $\{\phi_{2,9''}, \phi_{2,3'}\}$, $\{\phi_{2,11'''}, \phi_{2,5'''}\}$, $\{\phi_{2,9'''}, \phi_{2,3''}\}$, $\{\phi_{2,7'''}, \phi_{2,1}\}$,
$\{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}$

Now, by Lemma A.1.1, the generic Hecke algebra $\mathcal{H}(G_6)$ of G_6 is isomorphic to the subalgebra $\bar{H} := \langle S, U \rangle$ of the following specialization H of $\mathcal{H}(G_7)$

$$H := \left\langle S, T, U \;\middle|\; STU = TUS = UST,\ T^3 = 1,\ \begin{array}{l}(S - x_0)(S - x_1) = 0\\(U - z_0)(U - z_1)(U - z_2) = 0\end{array} \right\rangle$$

The algebra H is the twisted symmetric algebra of the cyclic group C_3 over the symmetric subalgebra $\bar{H}$ and this holds for all further specializations of the parameters. If we denote by ϕ the characters of H and by ψ the characters of $\bar{H}$, we have

$$\begin{array}{ll}
\mathrm{Ind}_{\bar{H}}^{H}(\psi_{1,0}) = \phi_{1,0} + \phi_{1,4''} + \phi_{1,8'''} & \mathrm{Ind}_{\bar{H}}^{H}(\psi_{1,4}) = \phi_{1,4'} + \phi_{1,8''} + \phi_{1,12''} \\
\mathrm{Ind}_{\bar{H}}^{H}(\psi_{1,8}) = \phi_{1,8'} + \phi_{1,12'} + \phi_{1,16} & \mathrm{Ind}_{\bar{H}}^{H}(\psi_{1,6}) = \phi_{1,6} + \phi_{1,10''} + \phi_{1,14'''} \\
\mathrm{Ind}_{\bar{H}}^{H}(\psi_{1,10}) = \phi_{1,10'} + \phi_{1,14''} + \phi_{1,18''} & \mathrm{Ind}_{\bar{H}}^{H}(\psi_{1,14}) = \phi_{1,14'} + \phi_{1,18'} + \phi_{1,22} \\
\mathrm{Ind}_{\bar{H}}^{H}(\psi_{2,5''}) = \phi_{2,9'} + \phi_{2,13''} + \phi_{2,5'''} & \mathrm{Ind}_{\bar{H}}^{H}(\psi_{2,3''}) = \phi_{2,7'} + \phi_{2,11''} + \phi_{2,3''} \\
\mathrm{Ind}_{\bar{H}}^{H}(\psi_{2,3'}) = \phi_{2,11'} + \phi_{2,7'''} + \phi_{2,3'} & \mathrm{Ind}_{\bar{H}}^{H}(\psi_{2,7}) = \phi_{2,7''} + \phi_{2,11'''} + \phi_{2,15} \\
\mathrm{Ind}_{\bar{H}}^{H}(\psi_{2,1}) = \phi_{2,9''} + \phi_{2,5'} + \phi_{2,1} & \mathrm{Ind}_{\bar{H}}^{H}(\psi_{2,5'}) = \phi_{2,9'''} + \phi_{2,13'} + \phi_{2,5''} \\
\mathrm{Ind}_{\bar{H}}^{H}(\psi_{3,2}) = \phi_{3,6} + \phi_{3,10} + \phi_{3,2} & \mathrm{Ind}_{\bar{H}}^{H}(\psi_{3,4}) = \phi_{3,4} + \phi_{3,8} + \phi_{3,12}.
\end{array}$$

Let

$$\theta : \begin{cases} x_i \mapsto (\zeta_2)^i q^{a_i} & (0 \le i < 2), \\ z_k \mapsto (\zeta_3)^k q^{c_k} & (0 \le k < 3) \end{cases}$$

be a cyclotomic specialization of $\mathcal{H}(G_6)$. Let us consider the corresponding cyclotomic specialization of $\mathcal{H}(G_7)$

$$\vartheta : \begin{cases} x_i \mapsto (\zeta_2)^i q^{a_i} & (0 \le i < 2), \\ y_j \mapsto (\zeta_3)^j & (0 \le j < 3), \\ z_k \mapsto (\zeta_3)^k q^{c_k} & (0 \le k < 3). \end{cases}$$

Then $(\mathcal{H}(G_7))_\vartheta$ is the twisted symmetric algebra of the cyclic group C_3 over the symmetric subalgebra $(\mathcal{H}(G_6))_\theta$. Therefore, the essential hyperplanes for G_6 are obtained from the essential hyperplanes for G_7 by setting $b_0 = b_1 = b_2 = 0$. If now, for example, θ is associated with no essential hyperplane for G_6, then the Rouquier blocks of $(\mathcal{H}(G_7))_\vartheta$ are:

$$\{\phi_{1,0}, \phi_{1,4''}, \phi_{1,8'''}\},\ \{\phi_{1,4'}, \phi_{1,8''}, \phi_{1,12''}\},\ \{\phi_{1,8'}, \phi_{1,12'}, \phi_{1,16}\},$$
$$\{\phi_{1,6}, \phi_{1,10''}, \phi_{1,14'''}\},\ \{\phi_{1,10'}, \phi_{1,14''}, \phi_{1,18''}\},\ \{\phi_{1,14'}, \phi_{1,18'}, \phi_{1,22}\},$$
$$\{\phi_{2,9'}, \phi_{2,13''}, \phi_{2,5'''}, \phi_{2,7''}, \phi_{2,11'''}, \phi_{2,15}\},$$
$$\{\phi_{2,7'}, \phi_{2,11''}, \phi_{2,3''}, \phi_{2,9'''}, \phi_{2,13'}, \phi_{2,5''}\},$$
$$\{\phi_{2,11'}, \phi_{2,7'''}, \phi_{2,3'}, \phi_{2,9''}, \phi_{2,5'}, \phi_{2,1}\},\ \{\phi_{3,6}, \phi_{3,10}, \phi_{3,2}\},\ \{\phi_{3,4}, \phi_{3,8}, \phi_{3,12}\}.$$

By Clifford theory, the Rouquier blocks of $(\mathcal{H}(G_6))_\theta$, *i.e.*, the Rouquier blocks associated with no essential hyperplane for G_6 are:

$$\{\psi_{1,0}\},\ \{\psi_{1,4}\},\ \{\psi_{1,8}\},\ \{\psi_{1,6}\},\ \{\psi_{1,10}\},\ \{\psi_{1,14}\},$$
$$\{\psi_{2,5''}, \psi_{2,7}\},\ \{\psi_{2,3''}, \psi_{2,5'}\},\ \{\psi_{2,3'}, \psi_{2,1}\},\ \{\psi_{3,2}\},\ \{\psi_{3,4}\}.$$

In the same way, we obtain the Rouquier blocks associated with each essential hyperplane for G_6. Here we present only the non-trivial ones:

No essential hyperplane

$\{\psi_{2,5''}, \psi_{2,7}\}, \{\psi_{2,3''}, \psi_{2,5'}\}, \{\psi_{2,3'}, \psi_{2,1}\}$

$c_1 - c_2 = 0$

$\{\psi_{1,4}, \psi_{1,8}\}, \{\psi_{1,10}, \psi_{1,14}\}, \{\psi_{2,5''}, \psi_{2,7}\}, \{\psi_{2,3''}, \psi_{2,3'}, \psi_{2,1}, \psi_{2,5'}\}$

$c_0 - c_1 = 0$

$\{\psi_{1,0}, \psi_{1,4}\}, \{\psi_{1,6}, \psi_{1,10}\}, \{\psi_{2,5''}, \psi_{2,3''}, \psi_{2,7}, \psi_{2,5'}\}, \{\psi_{2,3'}, \psi_{2,1}\}$

$c_0 - c_2 = 0$

$\{\psi_{1,0}, \psi_{1,8}\}, \{\psi_{1,6}, \psi_{1,14}\}, \{\psi_{2,5''}, \psi_{2,3'}, \psi_{2,7}, \psi_{2,1}\}, \{\psi_{2,3''}, \psi_{2,5'}\}$

$a_0 - a_1 - 2c_0 + c_1 + c_2 = 0$

$\{\psi_{1,6}, \psi_{2,5''}, \psi_{2,7}, \psi_{3,4}\}, \{\psi_{2,3''}, \psi_{2,5'}\}, \{\psi_{2,3'}, \psi_{2,1}\}$

$a_0 - a_1 + c_0 - 2c_1 + c_2 = 0$

$\{\psi_{1,10}, \psi_{2,3''}, \psi_{2,5'}, \psi_{3,4}\}, \{\psi_{2,5''}, \psi_{2,7}\}, \{\psi_{2,3'}, \psi_{2,1}\}$

$a_0 - a_1 + c_0 + c_1 - 2c_2 = 0$

$\{\psi_{1,14}, \psi_{2,3'}, \psi_{2,1}, \psi_{3,4}\}, \{\psi_{2,5''}, \psi_{2,7}\}, \{\psi_{2,3''}, \psi_{2,5'}\}$

$a_0 - a_1 - c_0 - c_1 + 2c_2 = 0$

$\{\psi_{1,8}, \psi_{2,3'}, \psi_{2,1}, \psi_{3,2}\}, \{\psi_{2,5''}, \psi_{2,7}\}, \{\psi_{2,3''}, \psi_{2,5'}\}$

$a_0 - a_1 - c_0 + 2c_1 - c_2 = 0$

$\{\psi_{1,4}, \psi_{2,3''}, \psi_{2,5'}, \psi_{3,2}\}, \{\psi_{2,5''}, \psi_{2,7}\}, \{\psi_{2,3'}, \psi_{2,1}\}$

$a_0 - a_1 + 2c_0 - c_1 - c_2 = 0$

$\{\psi_{1,0}, \psi_{2,5''}, \psi_{2,7}, \psi_{3,2}\}, \{\psi_{2,3''}, \psi_{2,5'}\}, \{\psi_{2,3'}, \psi_{2,1}\}$

$a_0 - a_1 - c_0 + c_1 = 0$

$\{\psi_{1,4}, \psi_{1,6}, \psi_{2,3'}, \psi_{2,1}\}, \{\psi_{2,5''}, \psi_{2,7}\}, \{\psi_{2,3''}, \psi_{2,5'}\}$

$a_0 - a_1 - c_1 + c_2 = 0$

$\{\psi_{1,8}, \psi_{1,10}, \psi_{2,5''}, \psi_{2,7}\}, \{\psi_{2,3''}, \psi_{2,5'}\}, \{\psi_{2,3'}, \psi_{2,1}\}$

$a_0 - a_1 + c_0 - c_2 = 0$
$\{\psi_{1,0}, \psi_{1,14}, \psi_{2,3''}, \psi_{2,5'}\}$, $\{\psi_{2,5''}, \psi_{2,7}\}$, $\{\psi_{2,3'}, \psi_{2,1}\}$
$a_0 - a_1 - c_0 + c_2 = 0$
$\{\psi_{1,8}, \psi_{1,6}, \psi_{2,3''}, \psi_{2,5'}\}$, $\{\psi_{2,5''}, \psi_{2,7}\}$, $\{\psi_{2,3'}, \psi_{2,1}\}$
$a_0 - a_1 + c_1 - c_2 = 0$
$\{\psi_{1,4}, \psi_{1,14}, \psi_{2,5''}, \psi_{2,7}\}$, $\{\psi_{2,3''}, \psi_{2,5'}\}$, $\{\psi_{2,3'}, \psi_{2,1}\}$
$a_0 - a_1 + c_0 - c_1 = 0$
$\{\psi_{1,0}, \psi_{1,10}, \psi_{2,3'}, \psi_{2,1}\}$, $\{\psi_{2,5''}, \psi_{2,7}\}$, $\{\psi_{2,3''}, \psi_{2,5'}\}$
$a_0 - a_1 = 0$
$\{\psi_{1,0}, \psi_{1,6}\}$, $\{\psi_{1,4}, \psi_{1,10}\}$, $\{\psi_{1,8}, \psi_{1,14}\}$, $\{\psi_{2,5''}, \psi_{2,7}\}$, $\{\psi_{2,3''}, \psi_{2,5'}\}$, $\{\psi_{2,3'}, \psi_{2,1}\}$, $\{\psi_{3,2}, \psi_{3,4}\}$

Since it will take too many pages to give here the Rouquier blocks associated with all essential hyperplanes for all exceptional irreducible complex reflection groups, and in order to make it easier to work with them, we have stored these data in a computer file and created two GAP functions which display them. These functions are called AllBlocks and DisplayAllBlocks and they can be found on the author's webpage, along with explanations for their use. Here is an example of the use of the second one on the group G_4.

Example 5.2.4.

```
gap> W:=ComplexReflectionGroup(4);
gap> DisplayAllBlocks(W);
No essential hyperplane
[["phi{1,0}"],["phi{1,4}"],["phi{1,8}"], ["phi{2,5}"],
["phi{2,3}"],["phi{2,1}"],["phi{3,2}"]]
c_1-c_2=0
[["phi{1,0}"],["phi{1,4}","phi{1,8}","phi{2,5}"],
["phi{2,3}","phi{2,1}"],["phi{3,2}"]]
c_0-c_1=0
[["phi{1,0}","phi{1,4}","phi{2,1}"],["phi{1,8}"],
["phi{2,5}","phi{2,3}"],["phi{3,2}"]]
c_0-c_2=0
[["phi{1,0}","phi{1,8}","phi{2,3}"],["phi{1,4}"],
["phi{2,5}","phi{2,1}"],["phi{3,2}"]]
2c_0-c_1-c_2=0
[["phi{1,0}","phi{2,5}","phi{3,2}"],
["phi{1,4}"],["phi{1,8}"], ["phi{2,3}"],["phi{2,1}"]]
c_0-2c_1+c_2=0
[["phi{1,0}"],["phi{1,4}","phi{2,3}","phi{3,2}"],
["phi{1,8}"], ["phi{2,5}"],["phi{2,1}"]]
c_0+c_1-2c_2=0
[["phi{1,0}"],["phi{1,4}"],["phi{1,8}","phi{2,1}",
"phi{3,2}"], ["phi{2,5}"],["phi{2,3}"]]
```

Let W be any exceptional irreducible complex reflection group. Now that we have the Rouquier blocks associated with no and with each essential hyperplane for W, we can determine the Rouquier blocks of any cyclotomic Hecke algebra associated to W with the use of Theorem 5.1.1. We have also created the GAP functions RouquierBlocks and DisplayRouquierBlocks (corresponding to AllBlocks and DisplayAllBlocks) which, given a cyclotomic specialization $\phi : u_{\mathcal{C},j} \mapsto \zeta_{e_\mathcal{C}}^{j} q^{n_{\mathcal{C},j}}$, check to which essential hyperplanes the integers $n_{\mathcal{C},j}$ belong and, using the stored data, apply Theorem 5.1.1 to return the Rouquier blocks of $\mathcal{H}_\phi$. We will give here an example of their use on G_4.

Example 5.2.5. The generic Hecke algebra of G_4 has a presentation of the form

$$\mathcal{H}(G_4) = \left\langle S, T \;\middle|\; TST = TST, \begin{array}{c} (S-u_0)(S-u_1)(S-u_2) = 0 \\ (T-u_0)(T-u_1)(T-u_2) = 0 \end{array} \right\rangle.$$

If we want to calculate the Rouquier blocks of the cyclotomic Hecke algebra

$$\mathcal{H}_\phi = \left\langle S, T \;\middle|\; TST = TST, \begin{array}{c} (S-1)(S-\zeta_3 q)(S-\zeta_3^2 q^2) = 0 \\ (T-1)(T-\zeta_3 q)(T-\zeta_3^2 q^2) = 0 \end{array} \right\rangle,$$

we use the following commands (the way to define a cyclotomic Hecke algebra in CHEVIE is explained in the GAP manual, cf., for example, [55]):

```
gap> W:=ComplexReflectionGroup(4);
gap> H:=Hecke(W,[[1,E(3)*q,E(3)^2*q^2]]);
gap> DisplayRouquierBlocks(H);
[["phi{1,0}"],["phi{1,4}","phi{2,3}","phi{3,2}"],
["phi{1,8}"],["phi{2,5}"],[ "phi{2,1}"]]
```

5.3 The Groups $G(d, 1, r)$

The group $G(d,1,r)$ is the group of all $r \times r$ monomial matrices with non-zero entries in μ_d. It is isomorphic to the wreath product $\mu_d \wr \mathfrak{S}_r$ and its field of definition is $\mathbb{Q}(\zeta_d)$.

We will start by introducing some combinatorial notation and results (cf. [18], §3A) which will be useful for the description of the Rouquier blocks of the cyclotomic Ariki-Koike algebras, *i.e.,* the cyclotomic Hecke algebras associated to the group $G(d,1,r)$.

5.3.1 Combinatorics

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_h)$ be a *partition*, *i.e.,* a finite decreasing sequence of positive integers

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_h \geq 1.$$

The integer

$$|\lambda| := \lambda_1 + \lambda_2 + \cdots + \lambda_h$$

is called *the size of* λ. We also say that λ *is a partition of* $|\lambda|$. The integer h is called *the height of* λ and we set $h_\lambda := h$. To each partition λ we associate its *β-number*, $\beta_\lambda = (\beta_1, \beta_2, \ldots, \beta_h)$, defined by

$$\beta_1 := h + \lambda_1 - 1,\ \beta_2 := h + \lambda_2 - 2,\ \ldots,\ \beta_h := h + \lambda_h - h.$$

Example 5.3.1. If $\lambda = (4, 2, 2, 1)$, then $\beta_\lambda = (7, 4, 3, 1)$.

Let n be a non-negative integer. The *n-shifted β-number* of λ is the sequence of numbers defined by

$$\beta_\lambda[n] := (\beta_1 + n, \beta_2 + n, \ldots, \beta_h + n, n - 1, n - 2, \ldots, 1, 0).$$

We have $\beta_\lambda[0] = \beta_\lambda$.

Example 5.3.2. If $\lambda = (4, 2, 2, 1)$, then $\beta_\lambda[3] = (10, 7, 6, 4, 2, 1, 0)$.

Multipartitions

Let d be a positive integer and let $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(d-1)})$ be a d-partition, *i.e.,* a family of d partitions indexed by the set $\{0, 1, \ldots, d-1\}$. We set

$$h^{(a)} := h_{\lambda^{(a)}},\ \ \beta^{(a)} := \beta_{\lambda^{(a)}}$$

and we have

$$\lambda^{(a)} = (\lambda_1^{(a)}, \lambda_2^{(a)}, \ldots, \lambda_{h^{(a)}}^{(a)}).$$

The integer

$$|\lambda| := \sum_{a=0}^{d-1} |\lambda^{(a)}|$$

is called *the size of* λ. We also say that λ *is a d-partition of* $|\lambda|$.

Ordinary Symbols

Let $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(d-1)})$ be a d-partition. We call *d-height of* λ the family $(h^{(0)}, h^{(1)}, \ldots, h^{(d-1)})$ and we define the *height of* λ to be the integer

$$h_\lambda := \max\{h^{(a)} \mid 0 \le a \le d-1\}.$$

Definition 5.3.3. The *ordinary standard symbol* of λ is the family of numbers defined by

$$B_\lambda = \left(B_\lambda^{(0)}, B_\lambda^{(1)}, \ldots, B_\lambda^{(d-1)}\right),$$

where, for all a $(0 \leq a \leq d-1)$, we have

$$B_\lambda^{(a)} := \beta^{(a)}[h_\lambda - h^{(a)}].$$

An *ordinary symbol* of λ is a symbol obtained from the ordinary standard symbol by shifting all the rows by the same integer.

The ordinary standard symbol of a d-partition λ is of the form

$$\begin{array}{lcllll} B_\lambda^{(0)} & = & b_1^{(0)} & b_2^{(0)} & \ldots & b_{h_\lambda}^{(0)} \\ B_\lambda^{(1)} & = & b_1^{(1)} & b_2^{(1)} & \ldots & b_{h_\lambda}^{(1)} \\ & \vdots & & & & \\ B_\lambda^{(d-1)} & = & b_1^{(d-1)} & b_2^{(d-1)} & \ldots & b_{h_\lambda}^{(d-1)}. \end{array}$$

The *ordinary content* of a d-partition of ordinary standard symbol B_λ is the multiset

$$\mathrm{Cont}_\lambda := B_\lambda^{(0)} \cup B_\lambda^{(1)} \cup \cdots \cup B_\lambda^{(d-1)}$$

or (with the above notation) the polynomial defined by

$$\mathrm{Cont}_\lambda(x) := \sum_{0 \leq a < d} \sum_{1 \leq i \leq h_\lambda} x^{b_i^{(a)}}.$$

Example 5.3.4. If $d = 2$ and $\lambda = ((2,1),(3))$, then

$$B_\lambda = \begin{pmatrix} 3 & 1 \\ 4 & 0 \end{pmatrix}.$$

We have $\mathrm{Cont}_\lambda = \{0,1,3,4\}$ or $\mathrm{Cont}_\lambda(x) = 1 + x + x^3 + x^4$.

Charged Symbols

Let us suppose that we have a given "weight system", *i.e.,* a family of integers

$$m := (m^{(0)}, m^{(1)}, \ldots, m^{(d-1)}).$$

Let $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(d-1)})$ be a d-partition. We call (d,m)-*charged height of* λ the family $(hc^{(0)}, hc^{(1)}, \ldots, hc^{(d-1)})$, where

$$hc^{(0)} := h^{(0)} - m^{(0)},\ hc^{(1)} := h^{(1)} - m^{(1)},\ \ldots,\ hc^{(d-1)} := h^{(d-1)} - m^{(d-1)}.$$

We define the m-*charged height of* λ to be the integer

$$hc_\lambda := \max\{hc^{(a)} \,|\, 0 \leq a \leq d-1\}.$$

Definition 5.3.5. The *m-charged standard symbol* of λ is the family of numbers defined by

$$Bc_\lambda = \left(Bc_\lambda^{(0)}, Bc_\lambda^{(1)}, \ldots, Bc_\lambda^{(d-1)}\right),$$

where, for all a $(0 \leq a \leq d-1)$, we have

$$Bc_\lambda^{(a)} := \beta^{(a)}[hc_\lambda - hc^{(a)}].$$

An *m-charged symbol* of λ is a symbol obtained from the m-charged standard symbol by shifting all the rows by the same integer.

Remark. The ordinary symbols correspond to the weight system

$$m^{(0)} = m^{(1)} = \cdots = m^{(d-1)} = 0.$$

The m-charged standard symbol of λ is a tableau of numbers arranged into d rows indexed by the set $\{0, 1, \ldots, d-1\}$ such that the a^{th} row has length equal to $hc_\lambda + m^{(a)}$. For all a $(0 \leq a \leq d-1)$, we set $l^{(a)} := hc_\lambda + m^{(a)}$ and we denote by

$$Bc_\lambda^{(a)} = bc_1^{(a)}\ bc_2^{(a)}\ \ldots\ bc_{l^{(a)}}^{(a)}$$

the a^{th} row of the m-charged standard symbol.

The *m-charged content* of a d-partition of m-charged standard symbol Bc_λ is the multiset

$$\mathrm{Contc}_\lambda := Bc_\lambda^{(0)} \cup Bc_\lambda^{(1)} \cup \cdots \cup Bc_\lambda^{(d-1)}$$

or (with the above notation) the polynomial defined by

$$\mathrm{Contc}_\lambda(x) := \sum_{0 \leq a < d} \sum_{1 \leq i \leq l^{(a)}} x^{bc_i^{(a)}}.$$

Example 5.3.6. If $d = 2$, $\lambda = ((2,1),(3))$ and $m = (-1,2)$, then

$$Bc_\lambda = \begin{pmatrix} 3\ 1 \\ 7\ 3\ 2\ 1\ 0 \end{pmatrix}.$$

We have $\mathrm{Contc}_\lambda = \{\{0,1,1,2,3,3,7\}\}$ or $\mathrm{Contc}_\lambda(x) = 1 + 2x + x^2 + 2x^3 + x^7$.

5.3.2 Ariki-Koike Algebras

The *generic Ariki-Koike algebra* associated to $G(d,1,r)$ (cf. [4, 19]) is the algebra $\mathcal{H}_{d,r}$ generated over the Laurent polynomial ring in $d+1$ indeterminates

$$\mathcal{L}_d := \mathbb{Z}[u_0, u_0^{-1}, u_1, u_1^{-1}, \ldots, u_{d-1}, u_{d-1}^{-1}, x, x^{-1}]$$

by the elements $\mathbf{s}, \mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_{r-1}$ satisfying the relations:

- $\mathbf{s}\mathbf{t}_1\mathbf{s}\mathbf{t}_1 = \mathbf{t}_1\mathbf{s}\mathbf{t}_1\mathbf{s},\ \mathbf{s}\mathbf{t}_j = \mathbf{t}_j\mathbf{s},$ for $j \neq 1$,
- $\mathbf{t}_j\mathbf{t}_{j+1}\mathbf{t}_j = \mathbf{t}_{j+1}\mathbf{t}_j\mathbf{t}_{j+1},\ \mathbf{t}_i\mathbf{t}_j = \mathbf{t}_j\mathbf{t}_i,$ for $|i-j| > 1$,
- $(\mathbf{s} - u_0)(\mathbf{s} - u_1)\cdots(\mathbf{s} - u_{d-1}) = 0$,
- $(\mathbf{t}_j - x)(\mathbf{t}_j + 1) = 0$, for all $j = 1, 2, \ldots, r-1$.

Remark. If the last relation in the above definition is replaced by

$$(\boldsymbol{t}_j - x)(\boldsymbol{t}_j - 1) = 0,$$

then we obtain a presentation of the generic Hecke algebra of $G(d,1,r)$. However, the second "$-$" becomes a "$+$" when we specialize via a cyclotomic specialization, so we might as well consider the generic Ariki-Koike algebra instead.

For every d-partition λ of r, we consider the free $\mathcal{L}_d$-module which has as basis the family of standard tableaux of λ. We can give to this module the structure of a $\mathcal{H}_{d,r}$-module (cf. [3,4,39]) and hence obtain the *Specht module* $\mathbf{Sp}^\lambda$ associated to λ.

Let $\mathcal{K}_d$ be the field of fractions of $\mathcal{L}_d$. The $\mathcal{K}_d\mathcal{H}_{d,r}$-module $\mathcal{K}_d\mathbf{Sp}^\lambda$, obtained by extension of scalars, is absolutely irreducible and every irreducible $\mathcal{K}_d\mathcal{H}_{d,r}$-module is isomorphic to a module of this type. Thus, $\mathcal{K}_d$ is a splitting field for $\mathcal{H}_{d,r}$. We denote by χ_λ the (absolutely) irreducible character of the $\mathcal{K}_d\mathcal{H}_{d,r}$-module $\mathcal{K}_d\mathbf{Sp}^\lambda$.

5.3.3 Rouquier Blocks, Charged Content and Residues

Let q be an indeterminate and let

$$\phi : \begin{cases} u_a \mapsto \zeta_d^a q^{m_a} \ (0 \leq a < d), \\ x \ \ \mapsto q^n \end{cases}$$

be a cyclotomic specialization of $\mathcal{H}_{d,r}$. Since the algebra $\mathcal{K}_d\mathcal{H}_{d,r}$ is split, we can deduce easily from Theorem 4.2.4 and Proposition 4.3.4 that the algebra $\mathbb{Q}(\zeta_d, q)(\mathcal{H}_{d,r})_\phi$ is split semisimple. Therefore, the Rouquier blocks of $(\mathcal{H}_{d,r})_\phi$ are the blocks of the algebra $\mathcal{R}_{\mathbb{Q}(\zeta_d)}(q)(\mathcal{H}_{d,r})_\phi$, where

$$\mathcal{R}_{\mathbb{Q}(\zeta_d)}(q) = \mathbb{Z}[\zeta_d][q, q^{-1}, (q^n - 1)^{-1}_{n\geq 1}].$$

Theorem 3.13 in [18] gives a description of the Rouquier blocks of $(\mathcal{H}_{d,r})_\phi$ when $n \neq 0$. However, in the proof it is supposed that $1 - \zeta_d$ always belongs to a prime ideal of $\mathbb{Z}[\zeta_d]$. This is not correct, unless d is a power of a prime number. Therefore, we will state here the part of the theorem that is correct and only for the case $n = 1$.

Theorem 5.3.7. *Let ϕ be a cyclotomic specialization of $\mathcal{H}_{d,r}$ such that $\phi(x) = q$. Let λ and μ be two d-partitions of r. If the irreducible characters $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$, then* $\mathrm{Contc}_\lambda = \mathrm{Contc}_\mu$ *with respect to the weight system $m = (m_0, m_1, \ldots, m_{d-1})$. The converse holds when d is a power of a prime number.*

Remark. In [25]*, we have proved that the converse of Theorem* 5.3.7*, and thus the description by Broué and Kim, holds when ϕ is the spetsial cyclotomic specialization and d is any positive integer.*

Set $\mathcal{O} := \mathcal{R}_{\mathbb{Q}(\zeta_d)}(q)$. Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}[\zeta_d]$ lying over a prime number p. By Proposition 4.4.2, the ring $\mathcal{O}$ is a Dedekind ring, whence $\mathcal{O}_{\mathfrak{p}\mathcal{O}}$ is a discrete valuation ring. Let us denote by $k_\mathfrak{p}$ the residue field of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}$ and by $\pi_\mathfrak{p}$ the canonical surjection $\mathcal{O}_{\mathfrak{p}\mathcal{O}} \twoheadrightarrow k_\mathfrak{p}$. Following Corollary 2.1.14, the morphism $\pi_\mathfrak{p}$ induces a block bijection between $\mathcal{O}_{\mathfrak{p}\mathcal{O}}(\mathcal{H}_{d,r})_\phi$ and $k_\mathfrak{p}(\mathcal{H}_{d,r})_\phi$.

Definition 5.3.8. The *diagram* of a d-partition λ is the set

$$[\lambda] := \{(i, j, a) \,|\, (0 \leq a \leq d-1)(1 \leq i \leq h^{(a)})(1 \leq j \leq \lambda_i^{(a)})\}.$$

A *node* of λ is any ordered triple $(i, j, a) \in [\lambda]$. The $\mathfrak{p}$-*residue* of the node $x = (i, j, a)$ with respect to ϕ is

$$\mathrm{res}_{\mathfrak{p},\phi}(x) := \begin{cases} (\pi_\mathfrak{p}(j-i), \pi_\mathfrak{p}(\phi(u_a))), & \text{if } n = 0 \text{ and } \pi_\mathfrak{p}(\phi(u_a)) \neq \pi_\mathfrak{p}(\phi(u_b)) \\ & \text{for } b \neq a, \\ \pi_\mathfrak{p}(\phi(u_a x^{j-i})), & \text{otherwise.} \end{cases}$$

Let $\mathrm{Res}_{\mathfrak{p},\phi} := \{\mathrm{res}_{\mathfrak{p},\phi}(x) \,|\, x \in [\lambda] \text{ for some } d\text{-partition } \lambda \text{ of } r\}$ be the set of all possible residues. For any d-partition λ of r and $f \in \mathrm{Res}_{\mathfrak{p},\phi}$, we set

$$C_f(\lambda) := |\{x \in [\lambda] \,|\, \mathrm{res}_{\mathfrak{p},\phi}(x) = f\}|.$$

Definition 5.3.9. Let λ and μ be two d-partitions of r. We say that λ and μ are $\mathfrak{p}$-*residue equivalent* with respect to ϕ if $C_f(\lambda) = C_f(\mu)$ for all $f \in \mathrm{Res}_{\mathfrak{p},\phi}$.

Then [48], Theorem 2.13 implies the following:

Theorem 5.3.10. *Let λ and μ be two d-partitions of r. The irreducible characters $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same block of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}(\mathcal{H}_{d,r})_\phi$ if and only if λ and μ are $\mathfrak{p}$-residue equivalent with respect to ϕ.*

Corollary 5.3.11. *Let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be two prime ideals of $\mathbb{Z}[\zeta_d]$ lying over the same prime number p. Then the blocks of $\mathcal{O}_{\mathfrak{p}_1\mathcal{O}}(\mathcal{H}_{d,r})_\phi$ coincide with the blocks of $\mathcal{O}_{\mathfrak{p}_2\mathcal{O}}(\mathcal{H}_{d,r})_\phi$.*

Proof. Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}[\zeta_d]$ lying over p and let $a,b,c,d \in \mathbb{Z}$ such that $0 \leq a \leq b \leq d-1$. We have $\pi_{\mathfrak{p}}(\zeta_d^a q^c) = \pi_{\mathfrak{p}}(\zeta_d^b q^d)$ if and only if $c = d$ and $\pi_{\mathfrak{p}}(\zeta_d^a) = \pi_{\mathfrak{p}}(\zeta_d^b)$. If $\pi_{\mathfrak{p}}(\zeta_d^a) = \pi_{\mathfrak{p}}(\zeta_d^b)$, then the element $\zeta_d^a - \zeta_d^b$ belongs to all the prime ideals lying over p. Following the definition of $\mathfrak{p}$-residue, we deduce that two d-partitions λ and μ of r are $\mathfrak{p}_1$-residue equivalent with respect to ϕ if and only if λ and μ are $\mathfrak{p}_2$-residue equivalent with respect to ϕ. ■

Theorem 5.3.10, combined with Proposition 4.4.4, gives the following:

Proposition 5.3.12. *Let λ and μ be two d-partitions of r. The irreducible characters $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$ if and only if there exist a finite sequence $\lambda_{(0)}, \lambda_{(1)}, \ldots, \lambda_{(m)}$ of d-partitions of r and a finite sequence $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ of ϕ-bad prime ideals for $G(d,1,r)$ such that*

- $\lambda_{(0)} = \lambda$ *and* $\lambda_{(m)} = \mu$,
- *for all i $(1 \leq i \leq m)$, the d-partitions $\lambda_{(i-1)}$ and $\lambda_{(i)}$ are $\mathfrak{p}_i$-residue equivalent with respect to ϕ.*

5.3.4 Essential Hyperplanes

The Schur elements of the algebra $\mathcal{K}_d\mathcal{H}_{d,r}$ have been independently calculated by Geck, Iancu and Malle [36] and by Mathas [54]. Following their description by Theorem A.7.2, we deduce that the essential hyperplanes for $G(d,1,r)$ are of the form

- $N = 0$,
- $kN + M_s - M_t = 0$, where $0 \leq s < t < d$ and $-r < k < r$.

The hyperplane $N = 0$ is always essential for $G(d,1,r)$. Let H be a hyperplane of the form $kN + M_s - M_t = 0$, where $0 \leq s < t < d$ and $-r < k < r$. The hyperplane H is essential for $G(d,1,r)$ if and only if there exists a prime ideal $\mathfrak{p}$ of $\mathbb{Z}[\zeta_d]$ such that $\zeta_d^s - \zeta_d^t \in \mathfrak{p}$. In this case, H is $\mathfrak{p}$-essential for $G(d,1,r)$. In particular, if $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are two prime ideals of $\mathbb{Z}[\zeta_d]$ lying over the same prime number p, then H is $\mathfrak{p}_1$-essential if and only if it is $\mathfrak{p}_2$-essential.

Example 5.3.13. The hyperplane $M_0 = M_1$ is (2)-essential for $G(2,1,r)$, whereas it is not essential for $G(6,1,r)$, for all $r > 0$.

5.3.5 Results

Now we are going to determine the Rouquier blocks associated with no and with each essential hyperplane for $G(d,1,r)$. All the results presented in this section have been first published in [25].

Proposition 5.3.14. *The Rouquier blocks associated with no essential hyperplane for $G(d, 1, r)$ are trivial.*

Proof. Let ϕ be a cyclotomic specialization associated with no essential hyperplane for $G(d, 1, r)$. By Theorem A.7.2, the coefficients of the Schur elements of $\mathcal{K}_d\mathcal{H}_{d,r}$ are units in $\mathbb{Z}[\zeta_d]$. We deduce that there are no ϕ-bad prime ideals, whence every irreducible character is a Rouquier block by itself. ■

Proposition 5.3.15. *Let λ, μ be two d-partitions of r. The following two assertions are equivalent:*

(i) The irreducible characters χ_λ and χ_μ are in the same Rouquier block associated with the essential hyperplane $N = 0$.
(ii) We have $|\lambda^{(a)}| = |\mu^{(a)}|$ for all $a = 0, 1, \ldots, d-1$.

Proof. Let

$$\phi : \begin{cases} u_a \mapsto \zeta_d^a q^{m_a} \ (0 \leq a < d), \\ x \ \mapsto 1 \end{cases}$$

be a cyclotomic specialization associated with the essential hyperplane $N = 0$.

$(i) \Rightarrow (ii)$ Due to Proposition 5.3.12, it is enough to prove that if two d-partitions λ, μ of r are $\mathfrak{p}$-residue equivalent with respect to ϕ for some prime ideal $\mathfrak{p}$ of $\mathbb{Z}[\zeta_d]$, then $|\lambda^{(a)}| = |\mu^{(a)}|$ for all $a = 0, 1, \ldots, d-1$. Since the integers m_a $(0 \leq a < d)$ do not belong to another essential hyperplane for $G(d, 1, r)$, we have $\pi_\mathfrak{p}(\zeta_d^a q^{m_a}) \neq \pi_\mathfrak{p}(\zeta_d^b q^{m_b})$ for all $0 \leq a < b < d$. If $x = (i, j, a)$ is a node of λ or μ, then $\mathrm{res}_{\mathfrak{p},\phi}(x) = (\pi_\mathfrak{p}(j - i), \pi_\mathfrak{p}(\zeta_d^a q^{m_a}))$. Since λ and μ are $\mathfrak{p}$-residue equivalent, the number of nodes of λ whose $\mathfrak{p}$-residue's second entry is $\pi_\mathfrak{p}(\zeta_d^a q^{m_a})$ must be equal to the number of nodes of μ whose $\mathfrak{p}$-residue's second entry is $\pi_\mathfrak{p}(\zeta_d^a q^{m_a})$, for all $a = 0, 1, \ldots, d-1$. We deduce that

$$\begin{aligned} |\lambda^{(a)}| &= |\{(i, j, a) \mid (1 \leq i \leq h_\lambda^{(a)})(1 \leq j \leq \lambda_i^{(a)})\}| \\ &= |\{(i, j, a) \mid (1 \leq i \leq h_\mu^{(a)})(1 \leq j \leq \mu_i^{(a)})\}| = |\mu^{(a)}| \end{aligned}$$

for all $a = 0, 1, \ldots, d-1$.

$(ii) \Rightarrow (i)$ Let $a \in \{0, 1, \ldots, d-1\}$. It is enough to show that if λ and μ are two distinct d-partitions of r such that

$$|\lambda^{(a)}| = |\mu^{(a)}| \ \text{ and } \ \lambda^{(b)} = \mu^{(b)} \ \text{ for all } b \neq a,$$

then $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$. Set $l := |\lambda^{(a)}| = |\mu^{(a)}|$. The partitions $\lambda^{(a)}$ and $\mu^{(a)}$ correspond to two distinct irreducible characters of the group $\mathfrak{S}_l$. The cyclotomic Ariki-Koike algebra obtained from $\mathcal{H}_{1,l}$ via a cyclotomic specialization associated with the hyperplane $N = 0$ is isomorphic to the group algebra $\mathbb{Z}[\mathfrak{S}_l]$. For any finite group, it is known that 1 is the only block-idempotent of its group algebra over $\mathbb{Z}$ (see also [58], §3, Remark 1). Thus, all irreducible characters of $\mathfrak{S}_l$ belong to the same Rouquier block of $\mathbb{Z}[\mathfrak{S}_l]$. Proposition 5.3.12 implies that there exist a finite sequence $\nu_{(0)}, \nu_{(1)}, \ldots, \nu_{(m)}$ of partitions of l and a finite sequence $p_1, p_2, \ldots, p_m$ of prime numbers dividing the order of $\mathfrak{S}_l$ such that

- $\nu_{(0)} = \lambda^{(a)}$ and $\nu_{(m)} = \mu^{(a)}$,
- for all i $(1 \leq i \leq m)$, $\nu_{(i-1)}$ and $\nu_{(i)}$ are (p_i)-residue equivalent with respect to the cyclotomic specialization of $\mathcal{H}_{1,l}$ associated with the essential hyperplane $N = 0$.

For all i $(1 \leq i \leq m)$, we define $\nu_{d,i}$ to be the d-partition of r such that

$$\nu_{d,i}^{(a)} := \lambda_{(i)} \text{ and } \nu_{d,i}^{(b)} := \lambda^{(b)} \text{ for all } b \neq a.$$

Let $\mathfrak{p}_i$ be a prime ideal of $\mathbb{Z}[\zeta_d]$ lying over the prime number p_i. Then we have

- $\nu_{d,0} = \lambda$ and $\nu_{d,m} = \mu$,
- for all i $(1 \leq i \leq m)$, $\nu_{d,i-1}$ and $\nu_{d,i}$ are $\mathfrak{p}_i$-residue equivalent with respect to ϕ.

By Proposition 5.3.12, the characters $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$. ■

Proposition 5.3.16. *Let λ, μ be two d-partitions of r and let H be an essential hyperplane for $G(d,1,r)$ of the form $kN + M_s - M_t = 0$, where $0 \leq s < t < d$ and $-r < k < r$. The irreducible characters χ_λ and χ_μ are in the same Rouquier block associated with the hyperplane H if and only if the following conditions are satisfied:*

(1) We have $\lambda^{(a)} = \mu^{(a)}$ for all $a \notin \{s,t\}$.
(2) If $\lambda^{st} := (\lambda^{(s)}, \lambda^{(t)})$ and $\mu^{st} := (\mu^{(s)}, \mu^{(t)})$, then $\mathrm{Contc}_{\lambda^{st}} = \mathrm{Contc}_{\mu^{st}}$ with respect to the weight system $(0,k)$.

Proof. Let

$$\phi : \begin{cases} u_a \mapsto \zeta_d^a q^{m_a} \ (0 \leq j < d), \\ x \ \mapsto q^n \end{cases}$$

be a cyclotomic specialization associated with the essential hyperplane H. We can assume, without loss of generality, that $n = 1$. We can also assume that $m_s = 0$ and $m_t = k$.

Suppose that $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ belong to the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$. By Theorem 5.3.7, we have $\mathrm{Contc}_\lambda = \mathrm{Contc}_\mu$ with respect to the weight system $m = (m_0, m_1, \ldots, m_{d-1})$. Since the m_a, $a \notin \{s,t\}$, can take any value (as long as they do not belong to another essential hyperplane), the equality $\mathrm{Contc}_\lambda = \mathrm{Contc}_\mu$ yields the first condition. Moreover, the m-charged standard symbols Bc_λ and Bc_μ must have the same cardinality, whence $hc_\lambda = hc_\mu$. Therefore, we obtain

$$Bc_\lambda^{(a)} = \beta_\lambda^{(a)}[hc_\lambda - hc_\lambda^{(a)}] = \beta_\mu^{(a)}[hc_\mu - hc_\mu^{(a)}] = Bc_\mu^{(a)} \text{ for all } a \notin \{s, t\},$$

whence we deduce the following equality between multisets:

$$Bc_\lambda^{(s)} \cup Bc_\lambda^{(t)} = Bc_\mu^{(s)} \cup Bc_\mu^{(t)}.$$

We can assume that the m_a, $a \notin \{s, t\}$, are sufficiently large so that

$$hc_\lambda \in \{hc_\lambda^{(s)}, hc_\lambda^{(t)}\} \text{ and } hc_\mu \in \{hc_\mu^{(s)}, hc_\mu^{(t)}\}.$$

If $\lambda^{st} := (\lambda^{(s)}, \lambda^{(t)})$ and $\mu^{st} := (\mu^{(s)}, \mu^{(t)})$, then we have

$$Bc_{\lambda^{st}}^{(0)} = Bc_\lambda^{(s)},\ Bc_{\lambda^{st}}^{(1)} = Bc_\lambda^{(t)},\ Bc_{\mu^{st}}^{(0)} = Bc_\mu^{(s)},\ Bc_{\mu^{st}}^{(1)} = Bc_\mu^{(t)}$$

with respect to the weight system $(0, k)$. We obtain $\mathrm{Contc}_{\lambda^{st}} = \mathrm{Contc}_{\mu^{st}}$ with respect to the weight system $(0, k)$.

Now let us suppose that the conditions (1) and (2) are satisfied. Since H is an essential hyperplane for $G(d, 1, r)$, there exists a prime ideal $\mathfrak{p}$ of $\mathbb{Z}[\zeta_d]$ such that $\zeta_d^s - \zeta_d^t \in \mathfrak{p}$. We are going to show that the partitions λ and μ are $\mathfrak{p}$-residue equivalent with respect to ϕ. Thanks to the first condition, we only need to compare the $\mathfrak{p}$-residues of the nodes with third entry s or t.

Set $l := |\lambda^{st}|$. The first condition yields that $|\mu^{st}| = l$. Let $\mathcal{H}_{2,l}$ be the generic Ariki-Koike algebra associated to the group $G(2, 1, l)$. The algebra $\mathcal{H}_{2,l}$ is defined over the Laurent polynomial ring

$$\mathbb{Z}[U_0, U_0^{-1}, U_1, U_1^{-1}, X, X^{-1}].$$

Let us consider the cyclotomic specialization

$$\vartheta : U_0 \mapsto 1, U_1 \mapsto -q^k, X \mapsto q.$$

Due to Theorem 5.3.7, condition (2) implies that the characters $(\chi_{\lambda^{st}})_\vartheta$ and $(\chi_{\mu^{st}})_\vartheta$ belong to the same Rouquier block of $(\mathcal{H}_{2,l})_\vartheta$. We deduce that $kN + M_0 - M_1 = 0$ is a (2)-essential hyperplane for $G(2, 1, l)$ and that ϑ is associated with this hyperplane. Following Proposition 5.3.12, λ^{st} and μ^{st} must be (2)-residue equivalent with respect to ϑ. We have

- $(i, j, 0) \in [\lambda^{st}]$ (respectively $[\mu^{st}]$) if and only if $(i, j, s) \in [\lambda]$ (respectively $[\mu]$). Moreover, $\mathrm{res}_{(2),\vartheta}(i, j, 0) = \pi_{(2)}(q^{j-i})$, whereas $\mathrm{res}_{\mathfrak{p},\phi}(i, j, s) = \pi_\mathfrak{p}(\zeta_d^s q^{j-i})$.
- $(i, j, 1) \in [\lambda^{st}]$ (respectively $[\mu^{st}]$) if and only if $(i, j, t) \in [\lambda]$ (respectively $[\mu]$). Moreover, $\mathrm{res}_{(2),\vartheta}(i, j, 1) = \pi_{(2)}(-q^{k+j-i})$, whereas $\mathrm{res}_{\mathfrak{p},\phi}(i, j, s) = \pi_\mathfrak{p}(\zeta_d^t q^{k+j-i})$.

Note that we have $\pi_{(2)}(1) = \pi_{(2)}(-1)$ and $\pi_{\mathfrak{p}}(\zeta_d^s) = \pi_{\mathfrak{p}}(\zeta_d^t)$. We deduce that λ^{st} and μ^{st} are (2)-residue equivalent with respect to ϑ if and only if λ and μ are $\mathfrak{p}$-residue equivalent with respect to ϕ. ■

The following result is a corollary of the above proposition. However, we will show that it can also be obtained independently, with the use of the Morita equivalences established in [30].

Corollary 5.3.17. *Let λ, μ be two d-partitions of r and let H be an essential hyperplane for $G(d, 1, r)$ of the form $kN + M_s - M_t = 0$, where $0 \leq s < t < d$ and $-r < k < r$. Let*

$$\phi : \begin{cases} u_a \mapsto \zeta_d^a q^{m_a} \ (0 \leq a < d), \\ x \ \ \mapsto q^n \end{cases}$$

be a cyclotomic specialization associated with the essential hyperplane H. The irreducible characters $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same Rouquier block of $(\mathcal{H}_{d,r})_\phi$ if and only if the following conditions are satisfied:

(1) We have $\lambda^{(a)} = \mu^{(a)}$ for all $a \notin \{s, t\}$.
(2) If $\lambda^{st} := (\lambda^{(s)}, \lambda^{(t)})$, $\mu^{st} := (\mu^{(s)}, \mu^{(t)})$ and $l := |\lambda^{st}| = |\mu^{st}|$, then the characters $(\chi_{\lambda^{st}})_\vartheta$ and $(\chi_{\mu^{st}})_\vartheta$ belong to the same Rouquier block of the cyclotomic Ariki-Koike algebra of $G(2, 1, l)$ obtained via the specialization

$$\vartheta : U_0 \mapsto q^{m_s}, U_1 \mapsto -q^{m_t}, X \mapsto q^n.$$

Proof. Set $\mathcal{O} := \mathcal{R}_{\mathbb{Q}(\zeta_d)}(q)$. Since H is an essential hyperplane for $G(d, 1, r)$, there exists a prime ideal $\mathfrak{p}$ of $\mathbb{Z}[\zeta_d]$ such that $\zeta_d^s - \zeta_d^t \in \mathfrak{p}$. Due to Corollary 5.3.11, the Rouquier blocks of $(\mathcal{H}_{d,r})_\phi$ coincide with the blocks of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}(\mathcal{H}_{d,r})_\phi$.

From now on, all algebras are considered over the ring $\mathcal{O}_{\mathfrak{p}\mathcal{O}}$. Following [30], Theorem 1.1, we obtain that the algebra $(\mathcal{H}_{d,r})_\phi$ is Morita equivalent to the algebra

$$A := \bigoplus_{\substack{n_1,\ldots,n_{d-1} \geq 0 \\ n_1+\ldots+n_{d-1}=r}} (\mathcal{H}_{2,n_1})_{\phi'} \otimes \mathcal{H}(\mathfrak{S}_{n_2})_{\phi''} \otimes \ldots \otimes \mathcal{H}(\mathfrak{S}_{n_{d-1}})_{\phi''},$$

where ϕ' is the restriction of ϕ to $\mathbb{Z}[u_s, u_s^{-1}, u_t, u_t^{-1}, x, x^{-1}]$ and ϕ'' is the restriction of ϕ to $\mathbb{Z}[x, x^{-1}]$. Therefore, $(\mathcal{H}_{d,r})_\phi$ and A have the same blocks.

Since $n \neq 0$, the blocks of $\mathcal{H}(\mathfrak{S}_{n_2})_{\phi''}, \ldots, \mathcal{H}(\mathfrak{S}_{n_2})_{\phi''}$ are trivial. Thus, we obtain that the irreducible characters $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same (Rouquier) block of $(\mathcal{H}_{d,r})_\phi$ if and only if the following conditions are satisfied:

(1) We have $\lambda^{(a)} = \mu^{(a)}$ for all $a \notin \{s, t\}$.
(2) If $\lambda^{st} := (\lambda^{(s)}, \lambda^{(t)})$, $\mu^{st} := (\mu^{(s)}, \mu^{(t)})$ and $l := |\lambda^{st}| = |\mu^{st}|$, then the characters $(\chi_{\lambda^{st}})_{\phi'}$ and $(\chi_{\mu^{st}})_{\phi'}$ belong to the same block of $(\mathcal{H}_{2,l})_{\phi'}$.

Theorem 5.3.10 implies that the second condition holds if and only if the 2-partitions λ^{st} and μ^{st} are $\mathfrak{p}$-residue equivalent with respect to ϕ'. Using the same reasoning as in the proof of Proposition 5.3.16, we obtain that λ^{st} and μ^{st} are $\mathfrak{p}$-residue equivalent with respect to ϕ' if and only if they are (2)-residue equivalent with respect to ϑ, *i.e.,* if and only if the characters $(\chi_{\lambda^{st}})_\vartheta$ and $(\chi_{\mu^{st}})_\vartheta$ belong to the same Rouquier block of $(\mathcal{H}_{2,l})_\vartheta$. ■

Example 5.3.18. Let $d := 3$ and $r := 3$. The irreducible characters of $G(3,1,3)$ are parametrized by the 3-partitions of 3. The generic Ariki-Koike algebra associated to $G(3,1,3)$ is the algebra $\mathcal{H}_{3,3}$ generated over the Laurent polynomial ring in 4 indeterminates

$$\mathbb{Z}[u_0, u_0^{-1}, u_1, u_1^{-1}, u_2, u_2^{-1}, x, x^{-1}]$$

by the elements $\mathbf{s}, \mathbf{t}_1, \mathbf{t}_2$ satisfying the relations:

- $\mathbf{st}_1\mathbf{st}_1 = \mathbf{t}_1\mathbf{st}_1\mathbf{s}$, $\mathbf{st}_2 = \mathbf{t}_2\mathbf{s}$, $\mathbf{t}_1\mathbf{t}_2\mathbf{t}_1 = \mathbf{t}_2\mathbf{t}_1\mathbf{t}_2$,
- $(\mathbf{s} - u_0)(\mathbf{s} - u_1)(\mathbf{s} - u_2) = 0$,
- $(\mathbf{t}_1 - x)(\mathbf{t}_1 + 1) = (\mathbf{t}_2 - x)(\mathbf{t}_2 + 1) = 0$.

Let

$$\phi : \begin{cases} u_a \mapsto \zeta_3^a q^{m_a} \ (0 \le a \le 2), \\ x \ \ \mapsto q^n \end{cases}$$

be a cyclotomic specialization of $\mathcal{H}_{3,3}$. The essential hyperplanes for $G(3,1,3)$ are:

- $N = 0$.
- $kN + M_0 - M_1 = 0$ for $k \in \{-2,-1,0,1,2\}$.
- $kN + M_0 - M_2 = 0$ for $k \in \{-2,-1,0,1,2\}$.
- $kN + M_1 - M_2 = 0$ for $k \in \{-2,-1,0,1,2\}$.

Let us suppose that $m_0 = 0, m_1 = 0, m_2 = 5$ and $n = 1$. These integers belong only to the essential hyperplane $M_0 - M_1 = 0$. Following Proposition 5.3.16, two irreducible characters $(\chi_\lambda)_\phi$, $(\chi_\mu)_\phi$ are in the same Rouquier block of $(\mathcal{H}_{3,3})_\phi$ if and only if

(1) We have $\lambda^{(2)} = \mu^{(2)}$.
(2) If $\lambda^{01} := (\lambda^{(0)}, \lambda^{(1)})$ and $\mu^{01} := (\mu^{(0)}, \mu^{(1)})$, then $\mathrm{Contc}_{\lambda^{01}} = \mathrm{Contc}_{\mu^{01}}$ with respect to the weight system $(0,0)$, *i.e.,* $\mathrm{Cont}_{\lambda^{01}} = \mathrm{Cont}_{\mu^{01}}$.

The first condition yields immediately that the irreducible characters corresponding to the 3-partitions $(\emptyset, \emptyset, (1,1,1))$, $(\emptyset, \emptyset, (2,1))$ and $(\emptyset, \emptyset, (3))$ are singletons. Moreover, we have:

$$B_{((1,1,1),\emptyset)} = \begin{pmatrix} 3\ 2\ 1 \\ 2\ 1\ 0 \end{pmatrix}, \ B_{(\emptyset,(1,1,1))} = \begin{pmatrix} 2\ 1\ 0 \\ 3\ 2\ 1 \end{pmatrix},$$

$$B_{((2,1),\emptyset)} = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}, \; B_{(\emptyset,(2,1))} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix},$$

$$B_{((3),\emptyset)} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \; B_{(\emptyset,(3))} = \begin{pmatrix} 0 \\ 3 \end{pmatrix},$$

$$B_{((1,1),(1))} = \begin{pmatrix} 2 & 1 \\ 2 & 0 \end{pmatrix}, \; B_{((1),(1,1))} = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix},$$

$$B_{((2),(1))} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \; B_{((1),(2))} = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

$$B_{((1,1),\emptyset)} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \; B_{(\emptyset,(1,1))} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},$$

$$B_{((2),\emptyset)} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \; B_{(\emptyset,(2))} = \begin{pmatrix} 0 \\ 2 \end{pmatrix},$$

$$B_{((1),\emptyset)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; B_{(\emptyset,(1))} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$B_{((1),(1))} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Hence, the Rouquier blocks of $(\mathcal{H}_{3,3})_\phi$ are:

1. $\{\chi_{((1),(1),(1))}\}$,
2. $\{\chi_{(\emptyset,\emptyset,(1,1,1))}\}$,
3. $\{\chi_{(\emptyset,\emptyset,(2,1))}\}$,
4. $\{\chi_{(\emptyset,\emptyset,(3))}\}$,
5. $\{\chi_{((1,1,1),\emptyset,\emptyset)}, \chi_{(\emptyset,(1,1,1),\emptyset)}\}$,
6. $\{\chi_{((2,1),\emptyset,\emptyset)}, \chi_{(\emptyset,(2,1),\emptyset)}\}$,
7. $\{\chi_{((3),\emptyset,\emptyset)}, \chi_{(\emptyset,(3),\emptyset)}\}$,
8. $\{\chi_{((1,1),(1),\emptyset)}, \chi_{((1),(1,1),\emptyset)}\}$,
9. $\{\chi_{((2),(1),\emptyset)}, \chi_{((1),(2),\emptyset)}\}$,
10. $\{\chi_{((1,1),\emptyset,(1))}, \chi_{(\emptyset,(1,1),(1))}\}$,
11. $\{\chi_{((2),\emptyset,(1))}, \chi_{(\emptyset,(2),(1))}\}$,
12. $\{\chi_{((1),\emptyset,(1,1))}, \chi_{(\emptyset,(1),(1,1))}\}$,
13. $\{\chi_{((1),\emptyset,(2))}, \chi_{(\emptyset,(1),(2))}\}$.

By definition, these are the Rouquier blocks associated with the $(1-\zeta_3)$-essential hyperplane $M_0 - M_1 = 0$.

If we now take $m_0 = m_1 = m_2 = 0$ and $n = 1$, then the Rouquier blocks of $(\mathcal{H}_{3,3})_\phi$ are unions of the Rouquier blocks associated with the essential hyperplanes $M_0 - M_1 = 0$, $M_0 - M_2 = 0$ and $M_1 - M_2 = 0$. Following Theorem 5.1.1, the Rouquier blocks of $(\mathcal{H}_{3,3})_\phi$ are:

1. $\{\chi_{((1),(1),(1))}\}$,
2. $\{\chi_{((1,1,1),\emptyset,\emptyset)}, \chi_{(\emptyset,(1,1,1),\emptyset)}, \chi_{(\emptyset,\emptyset,(1,1,1))}\}$,
3. $\{\chi_{((2,1),\emptyset,\emptyset)}, \chi_{(\emptyset,(2,1),\emptyset)}, \chi_{(\emptyset,\emptyset,(2,1))}\}$,
4. $\{\chi_{((3),\emptyset,\emptyset)}, \chi_{(\emptyset,(3),\emptyset)}, \chi_{(\emptyset,\emptyset,(3))}\}$,
5. $\{\chi_{((1,1),(1),\emptyset)}, \chi_{((1),(1,1),\emptyset)}, \chi_{((1,1),\emptyset,(1))}, \chi_{((1),\emptyset,(1,1))}, \chi_{(\emptyset,(1,1),(1))}, \chi_{(\emptyset,(1),(1,1))}\}$,
6. $\{\chi_{((2),(1),\emptyset)}, \chi_{((1),(2),\emptyset)}, \chi_{((2),\emptyset,(1))}, \chi_{((1),\emptyset,(2))}, \chi_{(\emptyset,(2),(1))}, \chi_{(\emptyset,(1),(2))}\}$.

5.4 The Groups $G(2d, 2, 2)$

Let $d \geq 1$. The group $G(2d, 2, 2)$ has $4d$ irreducible characters of degree 1,

$$\chi_{ijk} \ (0 \leq i, j \leq 1,\ 0 \leq k < d),$$

and $d^2 - d$ irreducible characters of degree 2,

$$\chi^1_{kl},\ \chi^2_{kl} \ (0 \leq k \neq l < d),$$

where $\chi^{1,2}_{kl} = \chi^{1,2}_{lk}$. The field of definition of $G(2d, 2, 2)$ is $\mathbb{Q}(\zeta_{2d})$.

The generic Hecke algebra of the group $G(2d, 2, 2)$ is the algebra $\mathcal{H}_{2d}$ generated over the Laurent polynomial ring in $d + 4$ indeterminates

$$\mathbb{Z}[x_0, x_0^{-1}, x_1, x_1^{-1}, y_0, y_0^{-1}, y_1, y_1^{-1}, z_0, z_0^{-1}, z_1, z_1^{-1}, \ldots, z_{d-1}, z_{d-1}^{-1}]$$

by the elements $\mathbf{s}, \mathbf{t}, \mathbf{u}$ satisfying the relations:

- $\mathbf{stu} = \mathbf{tus} = \mathbf{ust}$,
- $(\mathbf{s} - x_0)(\mathbf{s} - x_1) = (\mathbf{t} - y_0)(\mathbf{t} - y_1) = (\mathbf{u} - z_0)(\mathbf{u} - z_1) \cdots (\mathbf{u} - z_{d-1}) = 0$.

5.4.1 Essential Hyperplanes

Let

$$\phi : \begin{cases} x_i \mapsto (-1)^i q^{a_i} & (0 \leq i < 2), \\ y_j \mapsto (-1)^j q^{b_j} & (0 \leq j < 2), \\ z_k \mapsto \zeta_d^k q^{c_k} & (0 \leq k < d) \end{cases}$$

be a cyclotomic specialization of $\mathcal{H}_{2d}$.

The essential hyperplanes for $G(2d, 2, 2)$ are determined by the Schur elements of $\mathcal{H}_{2d}$. The Schur elements of $\mathcal{H}_{2d}$ have been calculated by Malle

([49], Theorem 3.11). Following their description (see Subsection 6.7.3), the essential hyperplanes for $G(2d, 2, 2)$ are:

- $A_0 - A_1 = 0$ (2-essential),
- $B_0 - B_1 = 0$ (2-essential),
- $C_k - C_l = 0$, where $0 \leq k < l < d$ and $\zeta_d^k - \zeta_d^l$ belongs to a prime ideal $\mathfrak{p}$ of $\mathbb{Z}[\zeta_{2d}]$ ($\mathfrak{p}$-essential),
- $A_i - A_{1-i} + B_j - B_{1-j} + C_k - C_l = 0$, where $0 \leq i, j \leq 1$, $0 \leq k < l < d$ and $\zeta_d^k - \zeta_d^l$ belongs to a prime ideal $\mathfrak{p}$ of $\mathbb{Z}[\zeta_{2d}]$ ($\mathfrak{p}$-essential).

Remark. When we say that a hyperplane is 2*-essential, we mean that it is* $\mathfrak{I}$*-essential for all prime ideals* $\mathfrak{I}$ *of* $\mathbb{Z}[\zeta_{2d}]$ *lying over* 2*.*

5.4.2 Results

In order to determine the Rouquier blocks associated with no and with each essential hyperplane for $G(2d, 2, 2)$, we are going to use Proposition 4.4.6. Following that result, if two irreducible characters χ_ϕ and ψ_ϕ belong to the same Rouquier block of $(\mathcal{H}_{2d})_\phi$, then

$$a_{\chi_\phi} + A_{\chi_\phi} = a_{\psi_\phi} + A_{\psi_\phi}.$$

Using the formulas for the Schur elements of $\mathcal{H}_{2d}$ given in the Appendix, we can obtain the value of the sum $a_{\chi_\phi} + A_{\chi_\phi}$ for all $\chi \in \mathrm{Irr}(G(2d, 2, 2))$:

Proposition 5.4.1. *Let* $\chi \in \mathrm{Irr}(G(2d, 2, 2))$*. If* χ *is a linear character* χ_{ijk}*, then*

$$a_{\chi_\phi} + A_{\chi_\phi} = d(a_i - a_{1-i} + b_j - b_{1-j} + 2c_k) - 2\sum_{l=0}^{d-1} c_l.$$

If χ *is a character* $\chi_{kl}^{1,2}$ *of degree* 2*, then*

$$a_{\chi_\phi} + A_{\chi_\phi} = d(c_k + c_l) - 2\sum_{m=0}^{d-1} c_m.$$

Now we are ready to prove our main result ([26], Theorem 4.3):

Theorem 5.4.2. *For the group* $G(2d, 2, 2)$*, we have that:*

(1) The non-trivial Rouquier blocks associated with no essential hyperplane are

$$\{\chi_{kl}^1, \chi_{kl}^2\} \text{ for all } 0 \leq k < l < d.$$

(2) The non-trivial Rouquier blocks associated with the 2*-essential hyperplane* $A_0 = A_1$ *are*

$$\{\chi_{0jk}, \chi_{1jk}\} \text{ for all } 0 \leq j \leq 1 \text{ and } 0 \leq k < d,$$

$$\{\chi^1_{kl}, \chi^2_{kl}\} \text{ for all } 0 \leq k < l < d.$$

(3) The non-trivial Rouquier blocks associated with the 2*-essential hyperplane* $B_0 = B_1$ *are*

$$\{\chi_{i0k}, \chi_{i1k}\} \text{ for all } 0 \leq i \leq 1 \text{ and } 0 \leq k < d,$$

$$\{\chi^1_{kl}, \chi^2_{kl}\} \text{ for all } 0 \leq k < l < d.$$

(4) The non-trivial Rouquier blocks associated with the $\mathfrak{p}$*-essential hyperplane* $C_k = C_l$ $(0 \leq k < l < d)$ *are*

$$\{\chi_{ijk}, \chi_{ijl}\} \text{ for all } 0 \leq i, j \leq 1,$$

$$\{\chi^1_{km}, \chi^2_{km}, \chi^1_{lm}, \chi^2_{lm}\} \text{ for all } 0 \leq m < d \text{ with } m \notin \{k, l\},$$
$$\{\chi^1_{kl}, \chi^2_{kl}\},$$
$$\{\chi^1_{rs}, \chi^2_{rs}\} \text{ for all } 0 \leq r < s < d \text{ with } r, s \notin \{k, l\}.$$

(5) The non-trivial Rouquier blocks associated with the $\mathfrak{p}$*-essential hyperplane* $A_i - A_{1-i} + B_j - B_{1-j} + C_k - C_l = 0$ $(0 \leq i, j \leq 1,\ 0 \leq k < l < d)$ *are*

$$\{\chi_{ijk}, \chi_{1-i,1-j,l}, \chi^1_{kl}, \chi^2_{kl}\},$$

$$\{\chi^1_{rs}, \chi^2_{rs}\} \text{ for all } 0 \leq r < s < d \text{ with } (r, s) \neq (k, l).$$

Proof. Let

$$\phi : \begin{cases} x_i \mapsto (-1)^i q^{a_i} & (0 \leq i < 2), \\ y_j \mapsto (-1)^j q^{b_j} & (0 \leq j < 2), \\ z_k \mapsto \zeta_d^k q^{c_k} & (0 \leq k < d) \end{cases}$$

be a cyclotomic specialization of $\mathcal{H}_{2d}$.

(1) If ϕ is a cyclotomic specialization associated with no essential hyperplane, then, by Proposition 2.4.18, each linear character is a Rouquier block by itself, whereas any character of degree 2 is not. Due to the formulas of Proposition 5.4.1, Proposition 4.4.6 yields that the character χ^1_{kl} $(0 \leq k < l < d)$ can be in the same Rouquier block only with the character χ^2_{kl}.

(2) Suppose that ϕ is a cyclotomic specialization associated with the essential hyperplane $A_0 = A_1$. Since the hyperplane $A_0 = A_1$ is not essential for the characters of degree 2, Proposition 3.2.5 implies that $\{\chi^1_{kl}, \chi^2_{kl}\}$ is a Rouquier block of $(\mathcal{H}_{2d})_\phi$ for all $0 \leq k < l < d$. Moreover, the hyperplane $A_0 = A_1$ is 2-essential for all characters of degree 1 and thus, due to Proposition 2.4.18, there exist no linear character which is a block by itself. Due to the formulas of Proposition 5.4.1, Proposition 4.4.6 yields that the character

χ_{0jk} $(0 \le j \le 1, 0 \le k < d)$ can be in the same Rouquier block only with the character χ_{1jk}.

(3) If ϕ is a cyclotomic specialization associated with the essential hyperplane $B_0 = B_1$, we proceed as in the previous case.

(4) If ϕ is a cyclotomic specialization associated with the $\mathfrak{p}$-essential hyperplane $C_k = C_l$, where $0 \le k < l < d$, then the Rouquier blocks of $(\mathcal{H}_{2d})_\phi$ are unions of the Rouquier blocks associated with no essential hyperplane, due to Proposition 3.2.3. Hence, the characters χ^1_{rs} and χ^2_{rs} are in the same Rouquier block of $(\mathcal{H}_{2d})_\phi$ for all $0 \le r < s < d$. Now, the hyperplane $C_k = C_l$ is $\mathfrak{p}$-essential for the following characters:

- χ_{ijk}, χ_{ijl}, for all $0 \le i, j \le 1$,
- $\chi^{1,2}_{km}$, $\chi^{1,2}_{lm}$, for all $0 \le m < d$ with $m \notin \{k, l\}$.

Due to the formulas of Proposition 5.4.1, Proposition 4.4.6 yields that

- the character χ_{ijk} $(0 \le i, j \le 1)$ can be in the same Rouquier block only with the character χ_{ijl},
- the character χ^1_{km} $(0 \le m < d$ with $m \notin \{k, l\})$ can be in the same Rouquier block only with the characters $\chi^2_{km}, \chi^1_{lm}, \chi^2_{lm}$.

It remains to show that $\{\chi^1_{km}, \chi^2_{km}\}$ $(0 \le m < d$ with $m \notin \{k, l\})$ is not a Rouquier block of $(\mathcal{H}_{2d})_\phi$. Following [49], Table 3.10, there exists an element T_1 of $\mathcal{H}_{2d}$ such that

$$\chi^1_{km}(T_1) = \chi^2_{km}(T_1) = x_0 + x_1.$$

Suppose that $\{\chi^1_{km}, \chi^2_{km}\}$ is a Rouquier block of $(\mathcal{H}_{2d})_\phi$ and set $y^{|\mu(\mathbb{Q}(\zeta_{2d}))|} := q$. Then, by Corollary 2.2.13, we must have

$$\frac{\phi(\chi^1_{km}(T_1))}{\phi(s_{\chi^1_{km}})} + \frac{\phi(\chi^2_{km}(T_1))}{\phi(s_{\chi^2_{km}})} \in \mathcal{O},$$

where $\mathcal{O}$ denotes the Rouquier ring of $\mathbb{Q}(\zeta_{2d})$. We have

$$\frac{\phi(\chi^1_{km}(T_1))}{\phi(s_{\chi^1_{km}})} + \frac{\phi(\chi^2_{km}(T_1))}{\phi(s_{\chi^2_{km}})} = \phi(x_0 + x_1) \cdot \left(\frac{1}{\phi(s_{\chi^1_{km}})} + \frac{1}{\phi(s_{\chi^2_{km}})} \right),$$

where

$$\phi(x_0 + x_1) = q^{a_0} - q^{a_1} = y^{a_0|\mu(\mathbb{Q}(\zeta_{2d}))|} - y^{a_1|\mu(\mathbb{Q}(\zeta_{2d}))|}.$$

Since ϕ is associated with the hyperplane $C_k = C_l$, we must have $a_0 \ne a_1$, whence $\phi(x_0 + x_1)^{-1} \in \mathcal{O}$. We deduce that

$$\frac{1}{\phi(s_{\chi^1_{km}})} + \frac{1}{\phi(s_{\chi^2_{km}})} \in \mathcal{O}.$$

Using the formulas for the description of the Schur elements of $\chi_{km}^{1,2}$ given in the Appendix, we can easily calculate that the above element does not belong to the Rouquier ring.

(5) Suppose that ϕ is a cyclotomic specialization associated with the $\mathfrak{p}$-essential hyperplane $A_i - A_{1-i} + B_j - B_{1-j} + C_k - C_l = 0$, where $0 \leq i,j \leq 1$ and $0 \leq k < l < d$. We have to distinguish two cases:

(a) If $\mathfrak{p}$ is lying over an odd prime number, then this hyperplane is $\mathfrak{p}$-essential for only three characters: χ_{ijk}, $\chi_{1-i,1-j,l}$ and either χ_{kl}^1 or χ_{kl}^2. If $\mathcal{O}$ is the Rouquier ring of $\mathbb{Q}(\zeta_{2d})$, then, by Proposition 2.4.18, these three characters belong to the same block of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}(\mathcal{H}_{2d})_\phi$. All the remaining characters are blocks of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}(\mathcal{H}_{2d})_\phi$ by themselves. Since the Rouquier blocks of $(\mathcal{H}_{2d})_\phi$ are unions of the Rouquier blocks associated with no essential hyperplane, we obtain the desired result.

(b) If $\mathfrak{p}$ lies over 2, then the hyperplane $A_i - A_{1-i} + B_j - B_{1-j} + C_k - C_l = 0$ is $\mathfrak{p}$-essential for the characters χ_{ijk}, $\chi_{1-i,1-j,l}$, χ_{kl}^1 and χ_{kl}^2. Using the same reasoning as in case (4), we can show that the set $\{\chi_{ijk}, \chi_{1-i,1-j,l}, \chi_{kl}^1, \chi_{kl}^2\}$ is a Rouquier block of $(\mathcal{H}_{2d})_\phi$ (and not a union of two Rouquier blocks). Due to Proposition 3.2.5, the remaining Rouquier blocks associated with no essential hyperplane remain as they are. ■

Example 5.4.3. Let $d := 2$. The group $G(4,2,2)$ has 8 irreducible characters of degree 1, χ_{ijk} $(0 \leq i,j,k \leq 1)$, and 2 irreducible characters of degree 2, $\chi_{01}^{1,2}$. The generic Hecke algebra of the group $G(4,2,2)$ is the algebra $\mathcal{H}_4$ generated over the Laurent polynomial ring in 6 indeterminates

$$\mathbb{Z}[x_0, x_0^{-1}, x_1, x_1^{-1}, y_0, y_0^{-1}, y_1, y_1^{-1}, z_0, z_0^{-1}, z_1, z_1^{-1}]$$

by the elements $\mathbf{s}, \mathbf{t}, \mathbf{u}$ satisfying the relations:

- $\mathbf{stu} = \mathbf{tus} = \mathbf{ust}$,
- $(\mathbf{s} - x_0)(\mathbf{s} - x_1) = (\mathbf{t} - y_0)(\mathbf{t} - y_1) = (\mathbf{u} - z_0)(\mathbf{u} - z_1) = 0$.

Let

$$\phi : \begin{cases} x_i \mapsto (-1)^i q^{a_i} & (0 \leq i < 2), \\ y_j \mapsto (-1)^j q^{b_j} & (0 \leq j < 2), \\ z_k \mapsto (-1)^k q^{c_k} & (0 \leq k < 2) \end{cases}$$

be a cyclotomic specialization of $\mathcal{H}_4$. The essential hyperplanes for $G(4,2,2)$ are:

- $H_1 : A_0 = A_1$,
- $H_2 : B_0 = B_1$,
- $H_3 : C_0 = C_1$,
- $H_4 : A_0 - A_1 + B_0 - B_1 + C_0 - C_1 = 0$,
- $H_5 : A_0 - A_1 + B_1 - B_0 + C_0 - C_1 = 0$,

- $H_6 : A_1 - A_0 + B_0 - B_1 + C_0 - C_1 = 0$,
- $H_7 : A_1 - A_0 + B_1 - B_0 + C_0 - C_1 = 0$.

The only non-trivial Rouquier block associated with no essential hyperplane is $\{\chi^1_{01}, \chi^2_{01}\}$. The Rouquier blocks associated with

- H_1 are: $\{\chi_{000}, \chi_{100}\}$, $\{\chi_{001}, \chi_{101}\}$, $\{\chi_{010}, \chi_{110}\}$, $\{\chi_{011}, \chi_{111}\}$, $\{\chi^1_{01}, \chi^2_{01}\}$.
- H_2 are: $\{\chi_{000}, \chi_{010}\}$, $\{\chi_{001}, \chi_{011}\}$, $\{\chi_{100}, \chi_{110}\}$, $\{\chi_{101}, \chi_{111}\}$, $\{\chi^1_{01}, \chi^2_{01}\}$.
- H_3 are: $\{\chi_{000}, \chi_{001}\}$, $\{\chi_{010}, \chi_{011}\}$, $\{\chi_{100}, \chi_{101}\}$, $\{\chi_{110}, \chi_{111}\}$, $\{\chi^1_{01}, \chi^2_{01}\}$.
- H_4 are: $\{\chi_{001}\}$, $\{\chi_{010}\}$, $\{\chi_{011}\}$, $\{\chi_{100}\}$, $\{\chi_{101}\}$, $\{\chi_{110}\}$, $\{\chi_{000}, \chi_{111}, \chi^1_{01}, \chi^2_{01}\}$.
- H_5 are: $\{\chi_{000}\}$, $\{\chi_{001}\}$, $\{\chi_{011}\}$, $\{\chi_{100}\}$, $\{\chi_{110}\}$, $\{\chi_{111}\}$, $\{\chi_{010}, \chi_{101}, \chi^1_{01}, \chi^2_{01}\}$.
- H_6 are: $\{\chi_{000}\}$, $\{\chi_{001}\}$, $\{\chi_{010}\}$, $\{\chi_{101}\}$, $\{\chi_{110}\}$, $\{\chi_{111}\}$, $\{\chi_{100}, \chi_{011}, \chi^1_{01}, \chi^2_{01}\}$.
- H_7 are: $\{\chi_{000}\}$, $\{\chi_{010}\}$, $\{\chi_{011}\}$, $\{\chi_{100}\}$, $\{\chi_{101}\}$, $\{\chi_{111}\}$, $\{\chi_{110}, \chi_{001}, \chi^1_{01}, \chi^2_{01}\}$.

Let us take $a_0 = 2$, $a_1 = 4$, $b_0 = 3$, $b_1 = 1$ and $c_0 = c_1 = 0$. These integers belong to the essential hyperplanes H_3, H_4 and H_7. By Theorem 5.1.1, the Rouquier blocks of $(\mathcal{H}_4)_\phi$ are

$$\{\chi_{000}, \chi_{001}, \chi_{110}, \chi_{111}, \chi^1_{01}, \chi^2_{01}\},\ \{\chi_{010}, \chi_{011}\},\ \{\chi_{100}, \chi_{101}\}.$$

5.5 The Groups $G(de, e, r)$

All the results in this section have first appeared in [26].

5.5.1 The groups $G(de, e, r)$, $r > 2$

We define the Hecke algebra of $G(de, e, r)$, $r > 2$, to be the algebra $\mathcal{H}_{de,e,r}$ generated over the Laurent polynomial ring in $d + 1$ indeterminates

$$\mathbb{Z}[v_0, v_0^{-1}, v_1, v_1^{-1}, \ldots, v_{d-1}, v_{d-1}^{-1}, x, x^{-1}]$$

by the elements $a_0, a_1, \ldots, a_r$ satisfying the relations:

- $(a_0 - v_0)(a_0 - v_1) \cdots (a_0 - v_{d-1}) = (a_j - x)(a_j + 1) = 0$, for $j = 1, \ldots, r$,
- $a_1 a_3 a_1 = a_3 a_1 a_3$, $a_j a_{j+1} a_j = a_{j+1} a_j a_{j+1}$, for $j = 2, \ldots, r-1$,
- $a_1 a_2 a_3 a_1 a_2 a_3 = a_3 a_1 a_2 a_3 a_1 a_2$,
- $a_1 a_j = a_j a_1$, for $j = 4, \ldots, r$,
- $a_i a_j = a_j a_i$, for $2 \leq i < j \leq r$ with $j - i > 1$,
- $a_0 a_j = a_j a_0$, for $j = 3, \ldots, r$,
- $a_0 a_1 a_2 = a_1 a_2 a_0$,
- $\underbrace{a_2 a_0 a_1 a_2 a_1 a_2 a_1 \cdots}_{e+1 \text{ factors}} = \underbrace{a_0 a_1 a_2 a_1 a_2 a_1 a_2 \cdots}_{e+1 \text{ factors}}$.

Let

$$\phi : \begin{cases} v_j \mapsto \zeta_d^j q^{n_j} \ (0 \le j < d), \\ x \ \mapsto q^n \end{cases}$$

be a cyclotomic specialization of $\mathcal{H}_{de,e,r}$. Following Theorem 5.1.1, the Rouquier blocks of $(\mathcal{H}_{de,e,r})_\phi$ coincide with the Rouquier blocks of $(\mathcal{H}_{de,e,r})_{\phi^e}$, where

$$\phi^e : \begin{cases} v_j \mapsto \zeta_d^j q^{en_j} \ (0 \le j < d), \\ x \ \mapsto q^{en}, \end{cases}$$

since the integers $\{(n_j)_{0 \le j < d}, n\}$ and $\{(en_j)_{0 \le j < d}, en\}$ belong to the same essential hyperplanes for $G(de, e, r)$.

We now consider the generic Ariki-Koike algebra $\mathcal{H}_{de,r}$ generated over the ring

$$\mathbb{Z}[u_0, u_0^{-1}, u_1, u_1^{-1}, \ldots, u_{de-1}, u_{de-1}^{-1}, x, x^{-1}]$$

by the elements $\mathbf{s}, \mathbf{t}_1, \mathbf{t}_2, \ldots, \mathbf{t}_{r-1}$ satisfying the relations described in Subsection 5.3.2. Let us consider the following cyclotomic specialization of $\mathcal{H}_{de,r}$:

$$\vartheta : \begin{cases} u_a \mapsto \zeta_{de}^a q^{m_a} \ (0 \le a < de, m_a := n_{a \bmod d}), \\ x \ \mapsto q^{en}. \end{cases}$$

Following Lemma A.7.1, the algebra $(\mathcal{H}_{de,r})_\vartheta$ is the twisted symmetric algebra of the cyclic group C_e over the symmetric subalgebra $(\mathcal{H}_{de,e,r})_{\phi^e}$.

From now on, set $\mathcal{H} := (\mathcal{H}_{de,r})_\vartheta$, $\bar{\mathcal{H}} := (\mathcal{H}_{de,e,r})_{\phi^e}$, $G := C_e$, $K := \mathbb{Q}(\zeta_{de})$ and let $\mathcal{R}_K(q)$ be the Rouquier ring of K. Applying Proposition 2.3.18 yields:

Proposition 5.5.1. *The block-idempotents of* $(Z\mathcal{R}_K(q)\bar{\mathcal{H}})^G$ *coincide with the block-idempotents of* $(Z\mathcal{R}_K(q)\mathcal{H})^{G^\vee}$.

The action of the cyclic group $G^\vee$ of order e on $\mathrm{Irr}(K(q)\mathcal{H})$ corresponds to the action generated by the cyclic permutation by d-packages on the de-partitions of r (cf., for example, [51], §4.A):

$$\begin{aligned} \tau_d : (&\lambda^{(0)}, \ldots, \lambda^{(d-1)}, \lambda^{(d)}, \ldots, \lambda^{(2d-1)}, \ldots, \lambda^{(de-d)}, \ldots, \lambda^{(de-1)}) \\ \mapsto (&\lambda^{(de-d)}, \ldots, \lambda^{(de-1)}, \lambda^{(0)}, \ldots, \lambda^{(d-1)}, \ldots, \lambda^{(de-2d)}, \ldots, \lambda^{(de-d-1)}). \end{aligned}$$

The de-partitions which are fixed by the action of $G^\vee$, *i.e.,* the de-partitions which are of the form

$$(\lambda^{(0)}, \ldots, \lambda^{(d-1)}, \lambda^{(0)}, \ldots, \lambda^{(d-1)}, \ldots, \lambda^{(0)}, \ldots, \lambda^{(d-1)}),$$

where the first d partitions are repeated e times, are called *d-stuttering*.

Proposition 5.5.2. *If λ is a de-partition of r, then the characters χ_λ and $\chi_{\tau_d(\lambda)}$ belong to the same Rouquier block of $\mathcal{H}$. In particular, the blocks of $\mathcal{R}_K(q)\mathcal{H}$ are stable under the action of $G^\vee$.*

Proof. The symmetric group $\mathfrak{S}_{de}$ acts naturally on the set of de-partitions of r, and thus on $\mathrm{Irr}(K(q)\mathcal{H})$: If $\tau \in \mathfrak{S}_{de}$ and $\lambda = (\lambda^{(0)}, \lambda^{(1)}, \ldots, \lambda^{(de-1)})$ is a de-partition of r, then $\tau(\lambda) := (\lambda^{(\tau(0))}, \lambda^{(\tau(1))}, \ldots, \lambda^{(\tau(de-1))})$. The action of $G^\vee$ on $\mathrm{Irr}(K(q)\mathcal{H})$ corresponds to the action of the cyclic subgroup of order e of $\mathfrak{S}_{de}$ generated by the element

$$\tau_d = \prod_{j=0}^{d-1} \prod_{i=1}^{e-1} \sigma_{j,i}$$

where $\sigma_{j,i}$ denotes the transposition $(j, j+id)$. In order to prove that the characters χ_λ and $\chi_{\tau_d(\lambda)}$ belong to the same Rouquier block of $\mathcal{H}$, it suffices to show that the characters χ_λ and $\chi_{\sigma_{j,i}(\lambda)}$ are in the same Rouquier block of $\mathcal{H}$ for all $j\,(0 \leq j < d)$ and $i\,(0 \leq i < e)$.

Following Theorem 5.1.1, the Rouquier blocks of $\mathcal{H}$ are unions of the Rouquier blocks associated with all the essential hyperplanes of the form

$$M_s = M_t \;\; (0 \leq s < t < de,\; s \equiv t \bmod d).$$

Recall that the hyperplane $M_s = M_t$ is actually essential for $G(de,1,r)$ if and only if the element $\zeta_{de}^s - \zeta_{de}^t$ belongs to a prime ideal of $\mathbb{Z}[\zeta_{de}]$.

Suppose that $e = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$, where the p_k are distinct prime numbers. For $k \in \{1, 2, \ldots, m\}$, we set $c_k := e/p_k^{a_k}$. Then $\gcd(c_k) = 1$ and, by Bezout's theorem, there exist integers $(b_k)_{1 \leq k \leq m}$ such that $\sum_{k=1}^m b_k c_k = 1$. We have $i = \sum_{k=1}^m i_k$, where $i_k := i b_k c_k$. The element $1 - \zeta_e^{i_k}$ belongs to all the prime ideals of $\mathbb{Z}[\zeta_{de}]$ lying over the prime number p_k. Now set

$$l_0 := 0 \;\text{ and }\; l_k := (l_{k-1} + i_k) \bmod e, \;\text{ for all } k\,(1 \leq k \leq m).$$

We have that the element $\zeta_{de}^{j+l_{k-1}d} - \zeta_{de}^{j+l_k d} = \zeta_{de}^{j+l_{k-1}d}(1 - \zeta_e^{i_k})$ belongs to all the prime ideals of $\mathbb{Z}[\zeta_{de}]$ lying over the prime number p_k. Therefore, the hyperplane $M_{j+l_{k-1}d} = M_{j+l_k d}$ is essential for $G(de,1,r)$ for all $k\,(1 \leq k \leq m)$. Moreover, if we denote by $\tau_{j,i,k}$ the transposition $(j + l_{k-1}d, j + l_k d)$, then

- we have $\lambda^{(a)} = \tau_{j,i,k}(\lambda)^{(a)}$ for all $a \notin \{j + l_{k-1}d, j + l_k d\}$,
- the 2-partitions $(\lambda^{(j+l_{k-1}d)}, \lambda^{(j+l_k d)})$ and $(\tau_{j,i,k}(\lambda)^{(j+l_{k-1}d)}, \tau_{j,i,k}(\lambda)^{(j+l_k d)})$ $= (\lambda^{(j+l_k d)}, \lambda^{(j+l_{k-1}d)})$ have the same ordinary content.

By Proposition 5.3.16, the characters χ_λ and $\chi_{\tau_{j,i,k}(\lambda)}$ belong to the same Rouquier block associated with the essential hyperplane $M_{j+l_{k-1}d} = M_{j+l_k d}$ and thus, to the same Rouquier block of $\mathcal{H}$. We have

$$\sigma_{j,i} = \tau_{j,i,1} \circ \tau_{j,i,2} \circ \cdots \circ \tau_{j,i,m-1} \circ \tau_{j,i,m} \circ \tau_{j,i,m-1} \circ \cdots \circ \tau_{j,i,2} \circ \tau_{j,i,1}.$$

Consequently, the characters χ_λ and $\chi_{\sigma_{j,i}(\lambda)}$ belong to the same Rouquier block of $\mathcal{H}$ for all $j\,(0 \leq j < d)$ and $i\,(0 \leq i < e)$. ■

Thanks to the above result, Proposition 5.5.1 now reads:

Corollary 5.5.3. *The block-idempotents of* $(Z\mathcal{R}_K(q)\bar{\mathcal{H}})^G$ *coincide with the block-idempotents of* $\mathcal{R}_K(q)\mathcal{H}$.

The following theorem demonstrates how we obtain the Rouquier blocks of $\bar{\mathcal{H}}$ from the Rouquier blocks of $\mathcal{H}$ (already determined in Section 5.3).

Theorem 5.5.4. *Let* λ *be a de-partition of* r *and* χ_λ *the corresponding irreducible character of* $G(de, 1, r)$. *We define* $\mathrm{Irr}(K(q)\bar{\mathcal{H}})_\lambda$ *to be the subset of* $\mathrm{Irr}(K(q)\bar{\mathcal{H}})$ *with the property:*

$$\mathrm{Res}^{K(q)\mathcal{H}}_{K(q)\bar{\mathcal{H}}}(\chi_\lambda) = \sum_{\bar{\chi}\in\mathrm{Irr}(K(q)\bar{\mathcal{H}})_\lambda} \bar{\chi}.$$

Then

(1) If λ *is d-stuttering and* χ_λ *is a block of* $\mathcal{R}_K(q)\mathcal{H}$ *by itself, then there are* e *irreducible characters* $(\bar{\chi})_{\bar{\chi}\in\mathrm{Irr}(K(q)\bar{\mathcal{H}})_\lambda}$. *Each of these characters is a block of* $\mathcal{R}_K(q)\bar{\mathcal{H}}$ *by itself.*
(2) The other blocks of $\mathcal{R}_K(q)\mathcal{H}$ *are in bijective correspondence with the remaining blocks of* $\mathcal{R}_K(q)\bar{\mathcal{H}}$ *via the map of Proposition 2.3.18, i.e., the corresponding block-idempotents of* $\mathcal{R}_K(q)\mathcal{H}$ *coincide with the remaining block-idempotents of* $\mathcal{R}_K(q)\bar{\mathcal{H}}$.

Proof. If λ is a d-stuttering partition, then it is the only element in its orbit Ω under the action of $G^\vee$. Set $\bar{\Omega} := \mathrm{Irr}(K(q)\bar{\mathcal{H}})_\lambda$. By Proposition 2.3.15, we have $|\Omega||\bar{\Omega}| = |G| = e$, whence $|\bar{\Omega}| = e$. Moreover, if $\bar{\chi} \in \bar{\Omega}$, then its Schur element $s_{\bar{\chi}}$ is equal to the Schur element s_λ of χ_λ. If χ_λ is a block of $\mathcal{R}_K(q)\mathcal{H}$ by itself, then, Propositions 4.4.4 and 2.4.18 imply that $s_\lambda = s_{\bar{\chi}}$ is invertible in $\mathcal{R}_K(q)$. Thus, $\bar{\chi}$ is a block of $\mathcal{R}_K(q)\bar{\mathcal{H}}$ by itself.

If λ is d-stuttering and χ_λ is not a block of $\mathcal{R}_K(q)\mathcal{H}$ by itself, then, due to Theorem 5.1.1, there exists a de-partition $\mu \neq \lambda$ such that χ_λ and χ_μ belong to the same Rouquier block associated with an essential hyperplane H for $G(de, 1, r)$ such that the integers $\{(m_a)_{0\leq a<de}, en\}$ belong to H. If H is $N = 0$, then, by Proposition 5.3.15, we have $|\lambda^{(a)}| = |\mu^{(a)}|$ for all $a = 0, 1, \ldots, de-1$. Since $\lambda \neq \mu$, there exists $b \in \{0, 1, \ldots, de-1\}$ such that $\lambda^{(b)} \neq \mu^{(b)}$. If ν is the de-partition of r obtained from λ by replacing $\lambda^{(b)}$ with $\mu^{(b)}$, then χ_λ and χ_ν belong to the same block of $\mathcal{R}_K(q)\mathcal{H}$ and ν is not d-stuttering. If H is of the form $kN + M_s - M_t = 0,$ where $-r<k<r$ and $0 \leq s < t < de$, then $\lambda^{(a)} = \mu^{(a)}$ for all $a \notin \{s, t\}$. If $s \not\equiv t \bmod d$ or $e > 2$, then μ cannot be d-stuttering. Suppose now that $s \equiv t \bmod d$ and $e = 2$. The description of s_λ by Theorem A.7.2 implies that the hyperplane $M_s = M_t$ is not essential for χ_λ. Due to Proposition 3.2.5, we deduce that $k \neq 0$. Since the integers $\{(m_a)_{0\leq a<de}, en\}$ belong to H and $m_s = m_t$, we must have $n = 0$. If μ is d-stuttering, then $\mu^{(s)} = \mu^{(t)}$ and $|\mu^{(s)}| = |\mu^{(t)}| = |\lambda^{(t)}| = |\lambda^{(s)}|$. Let ν be the de-partition obtained from λ by replacing $\lambda^{(t)}$ with $\mu^{(t)}$. Then ν is

not d-stuttering and the characters χ_λ and χ_ν belong to the same Rouquier block associated with the essential hyperplane $N = 0$. Since $n = 0$, Theorem 5.1.1 implies that χ_λ and χ_ν belong to the same block of $\mathcal{R}_K(q)\mathcal{H}$. We will now show that the blocks of $\mathcal{R}_K(q)\mathcal{H}$ which contain at least one character corresponding to a not d-stuttering partition are in bijective correspondence with the remaining blocks of $\mathcal{R}_K(q)\bar{\mathcal{H}}$ via the map of Proposition 2.3.18.

Suppose that λ is not a d-stuttering partition and b is the block containing χ_λ. Let $\bar{\chi} \in \mathrm{Irr}(K(q)\bar{\mathcal{H}})_\lambda$ and let $\bar{b}$ be the block of $\mathcal{R}_K(q)\bar{\mathcal{H}}$ which contains $\bar{\chi}$. In order to establish the desired bijection, we have to show that $\bar{b}$ is stable under the action of G, *i.e.*, that $\bar{b} = \mathrm{Tr}(G, \bar{b}) := \sum_{g \in G/G_{\bar{b}}} g(\bar{b})$. By Proposition 2.3.15, we have that $b = \mathrm{Tr}(G, \bar{b})$.

If $\lambda = (\lambda^{(0)}, \ldots, \lambda^{(d-1)}, \lambda^{(d)}, \ldots, \lambda^{(2d-1)}, \ldots, \lambda^{(ed-d)}, \ldots, \lambda^{(ed-1)})$, then, for $i = 0, 1, \ldots, e-1$, we define the d-partition $\lambda_{(i)}$ by

$$\lambda_{(i)} := (\lambda^{(id)}, \lambda^{(id+1)}, \ldots, \lambda^{(id+d-1)})$$

and we have

$$\lambda = (\lambda_{(0)}, \lambda_{(1)}, \ldots, \lambda_{(e-1)}).$$

Since λ is not d-stuttering, there exists $m \in \{1, \ldots, e-1\}$ such that $\lambda_{(0)} \neq \lambda_{(m)}$. If p is any prime divisor of e, we denote by $\lambda(p)$ the de-partition obtained from λ by exchanging $\lambda_{(m)}$ and $\lambda_{(e/p)}$. Set

$$\sigma_p := \prod_{j=0}^{d-1} \sigma_{j,m} \cdot \sigma_{j,e/p} \cdot \sigma_{j,m},$$

where $\sigma_{j,i}$ denotes the transposition $(j, j+id)$ for all i $(0 \leq i < e)$. Then $\lambda(p) = \sigma_p(\lambda)$. In the proof of Proposition 5.5.2, we showed that the characters χ_λ and $\chi_{\sigma_{j,i}(\lambda)}$ are in the same Rouquier block of $\mathcal{H}$ for all j $(0 \leq j < d)$ and i $(0 \leq i < e)$. Therefore, the characters χ_λ and $\chi_{\lambda(p)}$ belong to the same block of $\mathcal{R}_K(q)\mathcal{H}$. Moreover, by construction, the de-partition $\lambda(p)$ is not fixed by the generator $\tau_d^{e/p}$ of the unique subgroup of order p of $G^\vee$. Thus, the order of the stabilizer $G^\vee_{\chi_{\lambda(p)}}$ of $\chi_{\lambda(p)}$ is prime to p.

By Proposition 2.3.15, we know that for each $\bar{\chi}_p \in \mathrm{Irr}(K(q)\bar{\mathcal{H}})_{\lambda(p)}$, we have $|G^\vee_{\chi_{\lambda(p)}}||G_{\bar{\chi}_p}| = e$. Hence, $|G_{\bar{\chi}_p}|$ is divisible by the largest power of p dividing e. Since $b = \mathrm{Tr}(G, \bar{b})$, the elements of $\mathrm{Irr}(K(q)\bar{\mathcal{H}})_{\lambda(p)}$ belong to blocks of $\mathcal{R}_K(q)\bar{\mathcal{H}}$ which are conjugate of $\bar{b}$ by G, whose stabilizer is $G_{\bar{b}}$. Following Lemma 2.3.16, we deduce that, for any prime number p, $|G_{\bar{b}}|$ is divisible by the largest power of p dividing e. Thus, $G_{\bar{b}} = G$ and $\mathrm{Tr}(G, \bar{b}) = \bar{b}$. ∎

Example 5.5.5. Let $d := 1$, $e := 3$ and $r := 3$. The Hecke algebra of $G(3,3,3)$ is the algebra $\mathcal{H}_{3,3,3}$ generated over the Laurent polynomial ring $\mathbb{Z}[x, x^{-1}]$ by the elements a_1, a_2, a_3 satisfying the relations:

- $a_1a_2a_1 = a_2a_1a_2$, $a_1a_3a_1 = a_3a_1a_3$, $a_2a_3a_2 = a_3a_2a_3$,
- $a_1a_2a_3a_1a_2a_3 = a_3a_1a_2a_3a_1a_2$,
- $(a_1 - x)(a_1 + 1) = (a_2 - x)(a_2 + 1) = (a_3 - x)(a_3 + 1) = 0$.

Let $\phi : x \mapsto q^n$ with $n \neq 0$ be a cyclotomic specialization of $\mathcal{H}_{3,3,3}$. We can apply Theorem 5.5.4 and obtain the Rouquier blocks of $(\mathcal{H}_{3,3,3})_\phi$ from the Rouquier blocks of $(\mathcal{H}_{3,3})_\vartheta$, where $\mathcal{H}_{3,3}$ is the generic Ariki-Koike algebra associated to $G(3,1,3)$ and

$$\vartheta : \begin{cases} u_a \mapsto \zeta_3^a \ (0 \leq a \leq 2), \\ x \ \ \mapsto q^n. \end{cases}$$

Since $n \neq 0$, the Rouquier blocks of $(\mathcal{H}_{3,3})_\vartheta$ coincide with the Rouquier blocks of $(\mathcal{H}_{3,3})_\theta$, where

$$\theta : \begin{cases} u_a \mapsto \zeta_3^a \ (0 \leq a \leq 2), \\ x \ \ \mapsto q. \end{cases}$$

The latter have been calculated in Example 5.3.18 and are:

1. $\{\chi_{((1),(1),(1))}\}$,
2. $\{\chi_{((1,1,1),\emptyset,\emptyset)}, \chi_{(\emptyset,(1,1,1),\emptyset)}, \chi_{(\emptyset,\emptyset,(1,1,1))}\}$,
3. $\{\chi_{((2,1),\emptyset,\emptyset)}, \chi_{(\emptyset,(2,1),\emptyset)}, \chi_{(\emptyset,\emptyset,(2,1))}\}$,
4. $\{\chi_{((3),\emptyset,\emptyset)}, \chi_{(\emptyset,(3),\emptyset)}, \chi_{(\emptyset,\emptyset,(3))}\}$,
5. $\{\chi_{((1,1),(1),\emptyset)}, \chi_{((1),(1,1),\emptyset)}, \chi_{((1,1),\emptyset,(1))}, \chi_{((1),\emptyset,(1,1))}, \chi_{(\emptyset,(1,1),(1))}, \chi_{(\emptyset,(1),(1,1))}\}$,
6. $\{\chi_{((2),(1),\emptyset)}, \chi_{((1),(2),\emptyset)}, \chi_{((2),\emptyset,(1))}, \chi_{((1),\emptyset,(2))}, \chi_{(\emptyset,(2),(1))}, \chi_{(\emptyset,(1),(2))}\}$.

Set $\mathcal{H} := (\mathcal{H}_{3,3})_\vartheta$, $\bar{\mathcal{H}} := (\mathcal{H}_{3,3,3})_\phi$ and $K := \mathbb{Q}(\zeta_3)$. We have that

$$\mathrm{Irr}(K(q)\bar{\mathcal{H}}) = \{\psi_1, \psi_2, \ldots, \psi_{10}\},$$

where

- $\psi_1 = \mathrm{Res}^{K(q)\mathcal{H}}_{K(q)\bar{\mathcal{H}}}(\chi_{((1,1,1),\emptyset,\emptyset)}) = \mathrm{Res}^{K(q)\mathcal{H}}_{K(q)\bar{\mathcal{H}}}(\chi_{(\emptyset,(1,1,1),\emptyset)}) = \mathrm{Res}^{K(q)\mathcal{H}}_{K(q)\bar{\mathcal{H}}}(\chi_{(\emptyset,\emptyset,(1,1,1))})$,
- $\psi_2 = \mathrm{Res}^{K(q)\mathcal{H}}_{K(q)\bar{\mathcal{H}}}(\chi_{((2,1),\emptyset,\emptyset)}) = \mathrm{Res}^{K(q)\mathcal{H}}_{K(q)\bar{\mathcal{H}}}(\chi_{(\emptyset,(2,1),\emptyset)}) = \mathrm{Res}^{K(q)\mathcal{H}}_{K(q)\bar{\mathcal{H}}}(\chi_{(\emptyset,\emptyset,(2,1))})$,
- $\psi_3 = \mathrm{Res}^{K(q)\mathcal{H}}_{K(q)\bar{\mathcal{H}}}(\chi_{((3),\emptyset,\emptyset)}) = \mathrm{Res}^{K(q)\mathcal{H}}_{K(q)\bar{\mathcal{H}}}(\chi_{(\emptyset,(3),\emptyset)}) = \mathrm{Res}^{K(q)\mathcal{H}}_{K(q)\bar{\mathcal{H}}}(\chi_{(\emptyset,\emptyset,(3))})$,
- $\psi_4 = \mathrm{Res}^{K(q)\mathcal{H}}_{K(q)\bar{\mathcal{H}}}(\chi_{((1,1),(1),\emptyset)}) = \mathrm{Res}^{K(q)\mathcal{H}}_{K(q)\bar{\mathcal{H}}}(\chi_{(\emptyset,(1,1),(1))}) = \mathrm{Res}^{K(q)\mathcal{H}}_{K(q)\bar{\mathcal{H}}}(\chi_{((1),\emptyset,(1,1))})$,
- $\psi_5 = \mathrm{Res}^{K(q)\mathcal{H}}_{K(q)\bar{\mathcal{H}}}(\chi_{((1),(1,1),\emptyset)}) = \mathrm{Res}^{K(q)\mathcal{H}}_{K(q)\bar{\mathcal{H}}}(\chi_{(\emptyset,(1),(1,1))}) = \mathrm{Res}^{K(q)\mathcal{H}}_{K(q)\bar{\mathcal{H}}}(\chi_{((1,1),\emptyset,(1))})$,
- $\psi_6 = \mathrm{Res}^{K(q)\mathcal{H}}_{K(q)\bar{\mathcal{H}}}(\chi_{((2),(1),\emptyset)}) = \mathrm{Res}^{K(q)\mathcal{H}}_{K(q)\bar{\mathcal{H}}}(\chi_{(\emptyset,(2),(1))}) = \mathrm{Res}^{K(q)\mathcal{H}}_{K(q)\bar{\mathcal{H}}}(\chi_{((1),\emptyset,(2))})$,
- $\psi_7 = \mathrm{Res}^{K(q)\mathcal{H}}_{K(q)\bar{\mathcal{H}}}(\chi_{((1),(2),\emptyset)}) = \mathrm{Res}^{K(q)\mathcal{H}}_{K(q)\bar{\mathcal{H}}}(\chi_{(\emptyset,(1),(2))}) = \mathrm{Res}^{K(q)\mathcal{H}}_{K(q)\bar{\mathcal{H}}}(\chi_{((2),\emptyset,(1))})$,
- $\psi_8 + \psi_9 + \psi_{10} = \mathrm{Res}^{K(q)\mathcal{H}}_{K(q)\bar{\mathcal{H}}}(\chi_{((1),(1),(1))})$.

Following Theorem 5.5.4, the Rouquier blocks of $\bar{\mathcal{H}}$ are

$$\{\psi_1\},\ \{\psi_2\},\ \{\psi_3\},\ \{\psi_4, \psi_5\},\ \{\psi_6, \psi_7\},\ \{\psi_8\},\ \{\psi_9\},\ \{\psi_{10}\}.$$

5.5.2 The Groups $G(de, e, 2)$

If the integer e is odd, then everything that we said in the previous section applies to the case of $G(de, e, 2)$. Hence, we can obtain the Rouquier blocks of the cyclotomic Hecke algebras of $G(de, e, 2)$ from those of $G(de, 1, 2)$.

If e is even, then Clifford theory allows us to obtain the Rouquier blocks of the cyclotomic Hecke algebras of $G(de, e, 2)$ from those of $G(de, 2, 2)$.

Let $f, d \geq 1$. We denote by $\mathcal{H}_{2fd,2f,2}$ the generic Hecke algebra of $G(2fd, 2f, 2)$ generated over the Laurent polynomial ring in $d + 4$ indeterminates

$$\mathbb{Z}[x_0, x_0^{-1}, x_1, x_1^{-1}, y_0, y_0^{-1}, y_1, y_1^{-1}, u_0, u_0^{-1}, u_1, u_1^{-1} \ldots, u_{d-1}, u_{d-1}^{-1}]$$

by the elements S, T, U satisfying the relations:

- $STU = UST,\ TUS(TS)^{f-1} = U(ST)^f$,
- $(S - x_0)(S - x_1) = (T - y_0)(T - y_1) = (U - u_0)(U - u_1) \cdots (U - u_{d-1}) = 0.$

Let

$$\phi : \begin{cases} x_i \mapsto (-1)^i q^{a_i} & (0 \leq i \leq 1), \\ y_j \mapsto (-1)^j q^{b_j} & (0 \leq j \leq 1), \\ u_h \mapsto \zeta_d^h q^{e_h} & (0 \leq h < d) \end{cases}$$

be a cyclotomic specialization of $\mathcal{H}_{2fd,2f,2}$. Following Theorem 5.1.1, the Rouquier blocks of $(\mathcal{H}_{2fd,2f,2})_\phi$ coincide with the Rouquier blocks of $(\mathcal{H}_{2fd,2f,2})_{\phi^f}$, where

$$\phi^f : \begin{cases} x_i \mapsto (-1)^i q^{fa_i} & (0 \leq i \leq 1), \\ y_j \mapsto (-1)^j q^{fb_j} & (0 \leq j \leq 1), \\ u_h \mapsto \zeta_d^h q^{fe_h} & (0 \leq h < d), \end{cases}$$

since the integers in $\{a_i, b_j, e_h\}$ and $\{fa_i, fb_j, fe_h\}$ belong to the same essential hyperplanes for $G(2fd, 2f, 2)$.

We now consider the generic Hecke algebra $\mathcal{H}_{2fd}$ of $G(2fd, 2, 2)$ generated over the ring

$$\mathbb{Z}[x_0, x_0^{-1}, x_1, x_1^{-1}, y_0, y_0^{-1}, y_1, y_1^{-1}, z_0, z_0^{-1}, z_1, z_1^{-1} \ldots, z_{fd-1}, z_{fd-1}^{-1}]$$

by the elements $\mathbf{s}, \mathbf{t}, \mathbf{u}$ satisfying the relations:

- $\mathbf{stu} = \mathbf{tus} = \mathbf{ust}$,
- $(\mathbf{s} - x_0)(\mathbf{s} - x_1) = (\mathbf{t} - y_0)(\mathbf{t} - y_1) = (\mathbf{u} - z_0)(\mathbf{u} - z_1) \cdots (\mathbf{u} - z_{fd-1}) = 0.$

Let us consider the following cyclotomic specialization of $\mathcal{H}_{2fd}$:

$$\vartheta : \begin{cases} x_i \mapsto (-1)^i q^{fa_i} & (0 \leq i \leq 1), \\ y_j \mapsto (-1)^j q^{fb_j} & (0 \leq j \leq 1), \\ z_k \mapsto \zeta_{fd}^k q^{c_k} & (0 \leq k < fd, c_k := e_{k \bmod d}). \end{cases}$$

Following Lemma A.7.3, the algebra $(\mathcal{H}_{2fd})_{\vartheta}$ is the twisted symmetric algebra of the cyclic group C_f over the symmetric subalgebra $(\mathcal{H}_{2fd,2f,2})_{\phi^f}$.

From now on, set $\mathcal{H} := (\mathcal{H}_{2fd})_{\vartheta}$, $\bar{\mathcal{H}} := (\mathcal{H}_{2fd,2f,2})_{\phi^f}$, $G := C_f$, $K := \mathbb{Q}(\zeta_{2fd})$, $y^{|\mu(K)|} := q$ and let $\mathcal{R}_K(y)$ be the Rouquier ring of K. Applying Proposition 2.3.18 yields:

Proposition 5.5.6. *The block-idempotents of $(Z\mathcal{R}_K(y)\bar{\mathcal{H}})^G$ coincide with the block-idempotents of $(Z\mathcal{R}_K(y)\mathcal{H})^{G^\vee}$.*

The action of the cyclic group $G^\vee$ of order f on $\mathrm{Irr}(K(y)\mathcal{H})$ corresponds to the action

$$\chi_{i,j,k} \mapsto \chi_{i,j,k+d} \quad (0 \leq i, j \leq 1,\, 0 \leq k < fd),$$

$$\chi^{1,2}_{k,l} \mapsto \chi^{1,2}_{k+d,l+d} \quad (0 \leq k < l < fd),$$

where all the indexes are considered mod fd.

Let $\chi \in \mathrm{Irr}(K(y)\mathcal{H})$. If we denote by Ω the orbit of χ under the action of $G^\vee$, then $|\Omega| = f$. We define $\bar{\Omega}$ to be the subset of $\mathrm{Irr}(K(y)\bar{\mathcal{H}})$ with the property:

$$\mathrm{Res}^{K(y)\mathcal{H}}_{K(y)\bar{\mathcal{H}}}(\chi) = \sum_{\bar{\chi} \in \bar{\Omega}} \bar{\chi}.$$

By Proposition 2.3.15, we know that $|\Omega||\bar{\Omega}| = f$, whence $|\bar{\Omega}| = 1$. Since $\bar{\Omega}$ is also the orbit of $\bar{\chi}$ under the action of G, we deduce that the block-idempotents of $\mathcal{R}_K(y)\bar{\mathcal{H}}$ are fixed by the action of G.

With the help of the following lemma, we will show that the Rouquier blocks of $\mathcal{H}$ are also stable under the action of $G^\vee$. Here the results of Theorem 5.4.2 are going to be used as definitions.

Lemma 5.5.7. *Let k_1, k_2, k_3 be three distinct elements of $\{0, 1, \ldots, fd-1\}$. If the blocks of $\mathcal{R}_K(y)\mathcal{H}$ are unions of the Rouquier blocks associated with the (not necessarily essential) hyperplanes $C_{k_1} = C_{k_2}$ and $C_{k_2} = C_{k_3}$, then they are also unions of the Rouquier blocks associated with the (not necessarily essential) hyperplane $C_{k_1} = C_{k_3}$.*

Proof. We only need to show that

(a) the characters χ_{i,j,k_1} and χ_{i,j,k_3} are in the same block of $\mathcal{R}_K(y)\mathcal{H}$ for all $0 \leq i, j \leq 1$,
(b) the characters $\chi^{1,2}_{k_1,m}$ and $\chi^{1,2}_{k_3,m}$ are in the same block of $\mathcal{R}_K(y)\mathcal{H}$ for all $0 \leq m < fd$ with $m \notin \{k_1, k_3\}$.

Since the blocks of $\mathcal{R}_K(y)\mathcal{H}$ are unions of the Rouquier blocks associated with the hyperplanes $C_{k_1} = C_{k_2}$ and $C_{k_2} = C_{k_3}$, Theorem 5.4.2 implies that

(1) χ_{i,j,k_1} and χ_{i,j,k_2} are in the same block of $\mathcal{R}_K(y)\mathcal{H}$ for all $0 \leq i, j \leq 1$,
(2) χ_{i,j,k_2} and χ_{i,j,k_3} are in the same block of $\mathcal{R}_K(y)\mathcal{H}$ for all $0 \leq i, j \leq 1$,

(3) $\chi_{k_1,m}^{1,2}$ and $\chi_{k_2,m}^{1,2}$ are in the same block of $\mathcal{R}_K(y)\mathcal{H}$ for all $0 \leq m < fd$ with $m \notin \{k_1, k_2\}$,
(4) $\chi_{k_2,m}^{1,2}$ and $\chi_{k_3,m}^{1,2}$ are in the same block of $\mathcal{R}_K(y)\mathcal{H}$ for all $0 \leq m < fd$ with $m \notin \{k_2, k_3\}$.

We immediately deduce (a) for all $0 \leq i, j \leq 1$ and (b) for all $0 \leq m < fd$ with $m \notin \{k_1, k_2, k_3\}$. Finally, (3) implies that the characters $\chi_{k_1,k_3}^{1,2}$ and $\chi_{k_2,k_3}^{1,2}$ are in the same block of $\mathcal{R}_K(y)\mathcal{H}$, whereas by (4), $\chi_{k_1,k_2}^{1,2}$ and $\chi_{k_1,k_3}^{1,2}$ are also in the same block of $\mathcal{R}_K(y)\mathcal{H}$. Thus, the characters $\chi_{k_1,k_2}^{1,2}$ and $\chi_{k_2,k_3}^{1,2}$ belong to the same Rouquier block of $\mathcal{H}$. ■

Theorem 5.5.8. *The Rouquier blocks of $\mathcal{H}$ are stable under the action of $G^\vee$. In particular, the block-idempotents of $\mathcal{R}_K(y)\bar{\mathcal{H}}$ coincide with the block-idempotents of $\mathcal{R}_K(y)\mathcal{H}$.*

Proof. Following Theorem 5.1.1, the Rouquier blocks of $\mathcal{H}$ are unions of the Rouquier blocks associated with all the essential hyperplanes of the form

$$C_{h+md} = C_{h+nd} \ (0 \leq h < d,\ 0 \leq m < n < f).$$

Recall that the hyperplane C_{h+md}=C_{h+nd} is actually essential for $G(2fd,2,2)$ if and only if the element $\zeta_{fd}^{h+md} - \zeta_{fd}^{h+nd}$ belongs to a prime ideal of $\mathbb{Z}[\zeta_{2fd}]$.

Suppose that $f = p_1^{t_1} p_2^{t_2} \cdots p_r^{t_r}$, where the p_i are distinct prime numbers. For $s \in \{1, 2, \ldots, r\}$, we set $h_s := f/p_s^{t_s}$. Then $\gcd(h_s) = 1$ and by Bezout's theorem, there exist integers $(g_s)_{1 \leq s \leq r}$ such that $\sum_{s=1}^r g_s h_s = 1$. The element $1-\zeta_f^{g_s h_s}$ belongs to all the prime ideals of $\mathbb{Z}[\zeta_{2fd}]$ lying over the prime number p_s. Let $h \in \{0, 1, \ldots, d-1\}$ and $m \in \{0, 1, \ldots, f-2\}$ and set

$$l_0 := m \ \text{ and } \ l_s := (l_{s-1} + g_s h_s) \bmod f, \quad \text{for all } s\,(1 \leq s \leq r).$$

We have that the element $\zeta_{fd}^{h+l_{s-1}d} - \zeta_{fd}^{h+l_s d} = \zeta_{fd}^{h+l_{s-1}d}(1 - \zeta_f^{g_s h_s})$ belongs to all the prime ideals of $\mathbb{Z}[\zeta_{2fd}]$ lying over the prime number p_s. Therefore, the hyperplane $C_{h+l_{s-1}d} = C_{h+l_s d}$ is essential for $G(2fd, 2, 2)$ for all s $(1 \leq s \leq r)$. Since $l_0 = m$ and $l_r = m + 1$, Lemma 5.5.7 implies that the Rouquier blocks of $\mathcal{H}$ are unions of the Rouquier blocks associated with the (not necessarily essential) hyperplane

$$C_{h+md} = C_{h+(m+1)d},$$

following their description by Theorem 5.4.2. Since this holds for all $m\,(0 \leq m \leq f-2)$, Lemma 5.5.7 again implies that the Rouquier blocks of $\mathcal{H}$ are unions of the Rouquier blocks associated with all the hyperplanes of the form

$$C_{h+md} = C_{h+nd} \ (0 \leq m < n < f),$$

for all $h\,(0 \leq h < d)$. Consequently, we obtain that

- the characters $(\chi_{i,j,h+md})_{0\leq m<f}$ are in the same block of $\mathcal{R}_K(y)\mathcal{H}$ for all $0 \leq i, j \leq 1$ and $0 \leq h < d$,
- the characters $(\chi^{1,2}_{h+md,h+nd})_{0\leq m<n<f}$ are in the same block of $\mathcal{R}_K(y)\mathcal{H}$ for all $0 \leq h < d$,
- the characters $(\chi^{1,2}_{h+md,h'+nd})_{0\leq m,n<f}$ are in the same block of $\mathcal{R}_K(y)\mathcal{H}$ for all $0 \leq h < h' < d$.

Thus, the blocks of $\mathcal{R}_K(y)\mathcal{H}$ are stable under the action of $G^\vee$. Now, Proposition 5.5.6 implies that the block-idempotents of $\mathcal{R}_K(y)\bar{\mathcal{H}}$ coincide with the block-idempotents of $\mathcal{R}_K(y)\mathcal{H}$. ■

Thanks to the above result, in order to determine the Rouquier blocks of $\bar{\mathcal{H}}$, it suffices to calculate the Rouquier blocks of $\mathcal{H}$: if C is a Rouquier block of $\mathcal{H}$, then $\{\mathrm{Res}^{K(y)\mathcal{H}}_{K(y)\bar{\mathcal{H}}}(\chi) \,|\, \chi \in C\}$ is a Rouquier block of $\bar{\mathcal{H}}$.

Example 5.5.9. Let $f := 2$ and $d := 1$. The group $G(4, 4, 2)$ is isomorphic to the group $G(2, 1, 2)$. The generic Hecke algebra $\mathcal{H}_{4,2,2}$ of $G(4, 4, 2)$ is generated over the Laurent polynomial ring in 4 indeterminates

$$\mathbb{Z}[x_0, x_0^{-1}, x_1, x_1^{-1}, y_0, y_0^{-1}, y_1, y_1^{-1}]$$

by the elements S and T satisfying the relations:

- $(S - x_0)(S - x_1) = (T - y_0)(T - y_1) = 0$.
- $STST = TSTS$.

Let

$$\phi : \begin{cases} x_i \mapsto (-1)^i q^{a_i} & (0 \leq i \leq 1), \\ y_j \mapsto (-1)^j q^{b_j} & (0 \leq j \leq 1) \end{cases}$$

be a cyclotomic specialization of $\mathcal{H}_{4,2,2}$. Since $G(4, 4, 2) \cong G(2, 1, 2)$, we can use the results on the Ariki-Koike algebras in order to determine the Rouquier blocks of $(\mathcal{H}_{4,2,2})_\phi$. However, here we will demonstrate how we can apply Theorem 5.5.8 and obtain the Rouquier blocks of $(\mathcal{H}_{4,2,2})_\phi$ from the Rouquier blocks of $(\mathcal{H}_4)_\vartheta$, where $\mathcal{H}_4$ is the generic Hecke algebra associated to $G(4, 2, 2)$ and

$$\vartheta : \begin{cases} x_i \mapsto (-1)^i q^{a_i} & (0 \leq i \leq 1), \\ y_j \mapsto (-1)^j q^{b_j} & (0 \leq j \leq 1), \\ z_k \mapsto (-1)^k & (0 \leq k \leq 1). \end{cases}$$

Set $\mathcal{H} := (\mathcal{H}_4)_\vartheta$, $\bar{\mathcal{H}} := (\mathcal{H}_{4,4,2})_\phi$, $K := \mathbb{Q}(i)$ and $y^{|\mu(K)|} := q$. We have that

$$\mathrm{Irr}(K(y)\bar{\mathcal{H}}) = \{\chi_{((2),\emptyset)}, \chi_{(\emptyset,(2))}, \chi_{((1,1),\emptyset)}, \chi_{(\emptyset,(1,1))}, \chi_{((1),(1))}\},$$

where

- $\chi_{((2),\emptyset)} = \mathrm{Res}^{K(y)\mathcal{H}}_{K(y)\bar{\mathcal{H}}}(\chi_{000}) = \mathrm{Res}^{K(y)\mathcal{H}}_{K(y)\bar{\mathcal{H}}}(\chi_{001})$,
- $\chi_{(\emptyset,(2))} = \mathrm{Res}^{K(y)\mathcal{H}}_{K(y)\bar{\mathcal{H}}}(\chi_{010}) = \mathrm{Res}^{K(y)\mathcal{H}}_{K(y)\bar{\mathcal{H}}}(\chi_{011})$,

- $\chi_{((1,1),\emptyset)} = \mathrm{Res}^{K(y)\mathcal{H}}_{K(y)\bar{\mathcal{H}}}(\chi_{100}) = \mathrm{Res}^{K(y)\mathcal{H}}_{K(y)\bar{\mathcal{H}}}(\chi_{101})$,
- $\chi_{(\emptyset,(1,1))} = \mathrm{Res}^{K(y)\mathcal{H}}_{K(y)\bar{\mathcal{H}}}(\chi_{110}) = \mathrm{Res}^{K(y)\mathcal{H}}_{K(y)\bar{\mathcal{H}}}(\chi_{111})$,
- $\chi_{((1),(1))} = \mathrm{Res}^{K(y)\mathcal{H}}_{K(y)\bar{\mathcal{H}}}(\chi^1_{01}) = \mathrm{Res}^{K(y)\mathcal{H}}_{K(y)\bar{\mathcal{H}}}(\chi^2_{01})$.

Following Theorem 5.5.8, the Rouquier blocks associated with no essential hyperplane for $G(4,4,2)$ are trivial (as expected). For $a_0 = 2$, $a_1 = 4$, $b_0 = 3$ and $b_1 = 1$, the Rouquier blocks of $\mathcal{H}$ have been calculated in Example 5.4.3 and are:

$$\{\chi_{000}, \chi_{001}, \chi_{110}, \chi_{111}, \chi^1_{01}, \chi^2_{01}\},\ \{\chi_{010}, \chi_{011}\},\ \{\chi_{100}, \chi_{101}\}.$$

Thanks to Theorem 5.5.8, we deduce that the Rouquier blocks of $\bar{\mathcal{H}}$ are:

$$\{\chi_{((2),\emptyset)}, \chi_{(\emptyset,(1,1))}, \chi_{((1),(1))}\},\ \{\chi_{(\emptyset,(2))}\},\ \{\chi_{((1,1),\emptyset)}\}.$$

We can verify the above result with the use of Proposition 5.3.16, which yields that two irreducible characters $(\chi_\lambda)_\phi$ and $(\chi_\mu)_\phi$ are in the same Rouquier block of $\bar{\mathcal{H}}$ if and only if $\mathrm{Contc}_\lambda = \mathrm{Contc}_\mu$ with respect to the weight system $(0,1)$.