Introduction

Stochastic analysis can be viewed as a branch of infinite-dimensional analysis that stems from a combined use of analytic and probabilistic tools, and is developed in interaction with stochastic processes. In recent decades it has turned into a powerful approach to the treatment of numerous theoretical and applied problems ranging from existence and regularity criteria for densities (Malliavin calculus) to functional and deviation inequalities, mathematical finance, anticipative extensions of stochastic calculus.

The basic tools of stochastic analysis consist in a gradient and a divergence operator which are linked by an integration by parts formula. Such gradient operators can be defined by finite differences or by infinitesimal shifts of the paths of a given stochastic process. Whenever possible, the divergence operator is connected to the stochastic integral with respect to that same underlying process. In this way, deep connections can be established between the algebraic and geometric aspects of differentiation and integration by parts on the one hand, and their probabilistic counterpart on the other hand. Note that the term "stochastic analysis" is also used with somewhat different significations especially in engineering or applied probability; here we refer to stochastic analysis from a functional analytic point of view.

Let us turn to the contents of this monograph. Chapter 1 starts with an elementary exposition in a discrete setting in which most of the basic tools of stochastic analysis can be introduced. The simple setting of the discrete case still captures many important properties of the continuous-time case and provides a simple model for its understanding. It also yields non trivial results such as concentration and deviation inequalities, and logarithmic Sobolev inequalities for Bernoulli measures, as well as hedging formulas for contingent claims in discrete time financial models. In addition, the results obtained in the discrete case are directly suitable for computer implementation. We start by introducing discrete time versions of the gradient and divergence operators, of chaos expansions, and of the predictable representation property. We write the discrete time structure equation satisfied by a sequence $(X_n)_{n\in\mathbb{N}}$ of independent Bernoulli random variables defined on the probability space $\Omega = \{-1,1\}^{\mathbb{N}}$, we construct the associated discrete multiple stochastic integrals and prove the chaos representation property for

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discrete time random walks with independent increments. A gradient operator D acting by finite differences is introduced in connection with the multiple stochastic integrals, and used to state a Clark predictable representation formula. The divergence operator δ, defined as the adjoint of D, turns out to be an extension of the discrete-time stochastic integral, and is used to express the generator of the Ornstein-Uhlenbeck process. The properties of the associated Ornstein-Uhlenbeck process and semi-group are investigated, with applications to covariance identities and deviation inequalities under Bernoulli measures. Covariance identities are stated both from the Clark representation formula and using Ornstein-Uhlenbeck semigroups. Logarithmic Sobolev inequalities are also derived in this framework, with additional applications to deviation inequalities. Finally we prove an Itô type change of variable formula in discrete time and apply it, along with the Clark formula, to option pricing and hedging in the Cox-Ross-Rubinstein discrete-time financial model.

In Chapter 2 we turn to the continuous time case and present an elementary account of continuous time normal martingales. This includes the construction of associated multiple stochastic integrals $I_n(f_n)$ of symmetric deterministic functions f_n of n variables with respect to a normal martingale, and the derivation of structure equations determined by a predictable process $(\phi_t)_{t\in\mathbb{R}_+}$. In case $(\phi_t)_{t\in\mathbb{R}_+}$ is a deterministic function, this family of martingales includes Brownian motion (when ϕ vanishes identically) and the compensated Poisson process (when ϕ is a deterministic constant), which will be considered separately. A basic construction of stochastic integrals and calculus is presented in the framework of normal martingales, with a proof of the Itô formula. In this chapter, the construction of Brownian motion is done via a series of Gaussian random variables and its pathwise properties will not be particularly discussed, as our focus is more on connections with functional analysis. Similarly, the notions of local martingales and semimartingales are not within the scope of this introduction.

Chapter 3 contains a presentation of the continuous time gradient and divergence in an abstract setting. We identify some minimal assumptions to be satisfied by these operators in order to connect them later on to stochastic integration with respect to a given normal martingale. The links between the Clark formula, the predictable representation property and the relation between Skorohod and Itô integrals, as well as covariance identities, are discussed at this level of generality. This general setting gives rise to applications such as the determination of the predictable representation of random variables, and a proof of logarithmic Sobolev inequalities for normal martingales. Generic examples of operators satisfying the hypotheses of Chapter 2 can be constructed by addition of a process with vanishing adapted projection to the gradient operator. Concrete examples of such gradient and divergence operators will be described in the sequel (Chapters 4, 5, 6, and 7), in particular in the Wiener and Poisson cases.

Chapter 4 introduces a first example of a pair of gradient and divergence operators satisfying the hypotheses of Chapter 3, based on the notion of multiple stochastic integral $I_n(f_n)$ of a symmetric function f_n on $\mathbb{R}_+^n$ with respect to a normal martingale. Here the gradient operator D is defined by lowering the degree of multiple stochastic integrals (i.e. as an annihilation operator), while its adjoint δ is defined by raising that degree (i.e. as a creation operator). We give particular attention to the class of normal martingales which can be used to expand any square-integrable random variable into a series of multiple stochastic integrals. This property, called the chaos representation property, is stronger than the predictable representation property and plays a key role in the representation of functionals as stochastic integrals. Note that here the words “chaos” and “chaotic” are not taken in the sense of dynamical systems theory and rather refer to the notion of chaos introduced by N. Wiener [148]. We also present an application to deviation and concentration inequalities in the case of deterministic structure equations. The family of normal martingales having the chaos representation property, includes Brownian motion and the compensated Poisson process, which will be dealt with separately cases in the following sections.

The general results developed in Chapter 3 are detailed in Chapter 5 in the particular case of Brownian motion on the Wiener space. Here the gradient operator has the derivation property and the multiple stochastic integrals can be expressed using Hermite polynomials, cf. Section 5.1. We state the expression of the Ornstein-Uhlenbeck semi-group and the associated covariance identities and Gaussian deviation inequalities obtained. A differential calculus is presented for time changes on Brownian motion, and more generally for random transformations on the Wiener space, with application to Brownian motion on Riemannian path space in Section 5.7.

In Chapter 6 we introduce the main tools of stochastic analysis under Poisson measures on the space of configurations of a metric space X. We review the connection between Poisson multiple stochastic integrals and Charlier polynomials, gradient and divergence operators, and the Ornstein-Uhlenbeck semi-group. In this setting the annihilation operator defined on multiple Poisson stochastic integrals is a difference operator that can be used to formulate the Clark predictable representation formula. It also turns out that the integration by parts formula can be used to characterize Poisson measure. We also derive some deviation and concentration results for random vectors and infinitely divisible random variables.

In Chapter 7 we study a class of local gradient operators on the Poisson space that can also be used to characterize the Poisson measure. Unlike the finite difference gradients considered in Chapter 6, these operators do satisfy the chain rule of derivation. In the case of the standard Poisson process on the real line, they provide another instance of an integration by parts setting that fits into the general framework of Chapter 3. In particular this operator can be used in a Clark predictable representation formula and it is closely connected to the stochastic integral with respect to the compensated Poisson process via its associated divergence operator. The chain rule of derivation, which is not satisfied by the difference operators considered in Chapter 6, turns out to be necessary in a number of application such as deviation inequalities, chaos expansions, or sensitivity analysis.

Chapter 8 is devoted to applications in mathematical finance. We use normal martingales to extend the classical Black-Scholes theory and to construct complete market models with jumps. The results of previous chapters are applied to the pricing and hedging of contingent claims in complete markets driven by normal martingales. Normal martingales play only a modest role in the modeling of financial markets. Nevertheless, in addition to Brownian and Poisson models, they provide examples of complete markets with jumps.

To close this introduction we turn to some informal remarks on the Clark formula and predictable representation in connection with classical tools of finite dimensional analysis. This simple example shows how analytic arguments and stochastic calculus can be used in stochastic analysis. The classical "fundamental theorem of calculus" can be written using entire series as

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \alpha_n x^n \\ &= \alpha_0 + \sum_{n=1}^{\infty} n\alpha_n \int_0^x y^{n-1} dy \\ &= f(0) + \int_0^x f'(y) dy, \end{aligned}$$

and commonly relies on the identity

$$x^n = n \int_0^x y^{n-1} dy, \qquad x \in \mathbb{R}_+. \tag{0.1}$$

Replacing the monomial x^n with the Hermite polynomial $H_n(x,t)$ with parameter $t > 0$, we do obtain an analog of (0.1) as

$$\frac{\partial}{\partial x} H_n(x,t) = n H_{n-1}(x,t),$$

however the argument contained in (0.1) is no longer valid since $H_{2n}(0,t) \neq 0$, $n \geq 1$. The question of whether there exists a simple analog of (0.1) for the Hermite polynomials can be positively answered using stochastic calculus with respect to Brownian motion $(B_t)_{t\in\mathbb{R}_+}$ which provides a way to write $H_n(B_t,t)$ as a stochastic integral of $nH_{n-1}(B_t,t)$, i.e.

$$H_n(B_t,t) = n \int_0^t H_{n-1}(B_s,s) dB_s. \tag{0.2}$$

Consequently $H_n(B_t, t)$ can be written as an n-fold iterated stochastic integral with respect to Brownian motion $(B_t)_{t\in\mathbb{R}_+}$, which is denoted by $I_n(1_{[0,t]^n})$. This allows us to write down the following expansion of a function f depending on the parameter t into a series of Hermite polynomials, as follows:

$$\begin{aligned} f(B_t, t) &= \sum_{n=0}^{\infty} \beta_n H_n(B_t, t) \\ &= \beta_0 + \sum_{n=1}^{\infty} n\beta_n \int_0^t H_{n-1}(B_s, s) dB_s, \end{aligned}$$

$\beta_n \in \mathbb{R}_+$, $n \in \mathbb{N}$. Using the relation $H_n'(x, t) = nH_{n-1}(x, t)$, this series can be written as

$$f(B_t, t) = \mathbb{E}[f(B_t, t)] + \int_0^t \mathbb{E}\left[\frac{\partial f}{\partial x}(B_t, t)\Big|\mathcal{F}_s\right] dB_s, \tag{0.3}$$

since, by the martingale property of (0.2), $H_{n-1}(B_s, s)$ coincides with the conditional expectation $\mathbb{E}[H_{n-1}(B_t, t) \mid \mathcal{F}_s]$; $s < t$, where $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is the filtration generated by $(B_t)_{t\in\mathbb{R}_+}$.

It turns out that the above argument can be extended to general functionals of the Brownian path $(B_t)_{t\in\mathbb{R}_+}$ to prove that the square integrable functionals of $(B_t)_{t\in\mathbb{R}_+}$ have the following expansion in series of multiple stochastic integrals $I_n(f_n)$ of symmetric functions $f_n \in L^2(\mathbb{R}_+^n)$:

$$\begin{aligned} F &= \mathbb{E}[F] + \sum_{n=1}^{\infty} I_n(f_n) \\ &= \mathbb{E}[F] + \sum_{n=1}^{\infty} n \int_0^{\infty} I_{n-1}(f_n(*, t)1_{\{*\le t\}}) dB_t. \end{aligned}$$

Using again stochastic calculus in a way similar to the above argument will show that this relation can be written under the form

$$F = \mathbb{E}[F] + \int_0^{\infty} \mathbb{E}[D_t F \mid \mathcal{F}_t] dB_t, \tag{0.4}$$

where D is a gradient acting on Brownian functionals and $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is the filtration generated by $(B_t)_{t\in\mathbb{R}_+}$. Relation (0.4) is a generalization of (0.3) to arbitrary dimensions which does not require the use of Hermite polynomials, and can be adapted to other processes such as the compensated Poisson process, and more generally to the larger class of normal martingales.

Classical Taylor expansions for functions of one or several variables can also be interpreted in a stochastic analysis framework, in relation to the explicit determination of chaos expansions of random functionals. Consider for instance the classical formula

$$a_n = \frac{\partial^n f}{\partial x^n}(x)_{|x=0}$$

for the coefficients in the entire series

$$f(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}.$$

In the general setting of normal martingales having the chaos representation property, one can similarly compute the function f_n in the development of

$$F = \sum_{n=0}^{\infty} \frac{1}{n!} I_n(f_n)$$

as

$$f_n(t_1, \ldots, t_n) = \mathbb{E}[D_{t_1} \cdots D_{t_n} F], \quad a.e.\ t_1, \ldots, t_n \in \mathbb{R}_+, \qquad (0.5)$$

cf. [66], [138]. This identity holds in particular for Brownian motion and the compensated Poisson process. However, the probabilistic interpretation of $D_t F$ can be difficult to find except in the Wiener and Poisson cases, i.e. in the case of deterministic structure equations.

Our aim in the next chapters will be in particular to investigate to which extent these techniques remain valid in the general framework of normal martingales and other processes with jumps.