Chapter I
Preliminaries

In this part we review some necessary concepts and results from ergodic theory, which will be frequently used in this monograph.

Throughout this book, M is an m_0-dimensional, smooth, compact and connected Riemannian manifold without boundary. We use $f \in C^r(O,M)$ to denote a C^r map from O to M, where O is an open subset of M, and we call f a C^r *endomorphism* on M if $f \in C^r(M,M)$. We use Tf to denote the tangent map induced by f when $r \geq 1$.

For any compact metrizable space X and continuous map $T : X \to X$, We use $\mathscr{M}_T(X)$ to denote the set of T-invariant Borel probability measures on X.

I.1 Metric Entropy

Let X be a compact metrizable space, $T : X \to X$ a continuous map on X, and μ a T-invariant Borel probability measure on X.

For any finite partition $\eta = \{C_i\}$ of X, define the *entropy* of η by

$$H_\mu(\eta) = -\sum_i \mu(C_i) \log \mu(C_i).$$

Let

$$h_\mu(T,\eta) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\eta \wedge T^{-1}\eta \wedge \cdots \wedge T^{-n+1}\eta).$$

Then define the *metric entropy* of T with respect to μ as

$$h_\mu(T) = \sup\{ h_\mu(T,\eta) : \eta \text{ is a finite partition of } X\}.$$

For properties of the metric entropy, we refer the reader to [92].

M. Qian et al., *Smooth Ergodic Theory for Endomorphisms*, Lecture Notes in Mathematics 1978, DOI 10.1007/978-3-642-01954-8_I,

I.2 Multiplicative Ergodic Theorem

From Oseledec's theorem we have the following version of Multiplicative Ergodic Theorem for differentiable maps [92].

Theorem I.2.1 *Let f be a C^1 endomorphism on M. Then there exists a Borel subset $\Gamma \subset M$ with $f\Gamma \subset \Gamma$ and $\mu(\Gamma) = 1$ for any $\mu \in \mathscr{M}_f(M)$. Moreover, the following properties hold.*

(1) *There is a measurable integer function $r : \Gamma \to \mathbb{Z}^+$ with $r \circ f = r$.*
(2) *For any $x \in \Gamma$, there are real numbers*

$$+\infty > \lambda_1(x) > \lambda_2(x) > \cdots > \lambda_{r(x)}(x) \geq -\infty,$$

where $\lambda_{r(x)}(x)$ could be $-\infty$.
(3) *If $x \in \Gamma$, there are linear subspaces*

$$V^{(0)}(x) = T_x M \supset V^{(1)}(x) \supset \cdots \supset V^{(r(x))}(x) = \{0\}$$

of $T_x M$.
(4) *If $x \in \Gamma$ and $1 \leq i \leq r(x)$, then*

$$\lim_{n\to\infty} \frac{1}{n} \log\left|T_x f^n \xi\right| = \lambda_i(x)$$

for all $\xi \in V^{(i-1)}(x) \backslash V^{(i)}(x)$. Moreover,

$$\lim_{n\to\infty} \frac{1}{n} \log\left|\det(T_x f^n)\right| = \sum_{i=1}^{r(x)} \lambda_i(x) m_i(x),$$

where $m_i(x) = \dim V^{(i-1)}(x) - \dim V^{(i)}(x)$ for all $1 \leq i \leq r(x)$.
(5) *$\lambda_i(x)$ is measurably defined on $\left\{x \in \Gamma \,\middle|\, r(x) \geq i\right\}$ and f-invariant, i.e. $\lambda_i(fx) = \lambda_i(x)$.*
(6) *$T_x f\left(V^{(i)}(x)\right) \subset V^{(i)}(fx)$ if $i \geq 0$.*

The numbers $\left\{\lambda_i(x)\right\}_{i=1}^{r(x)}$, given by Theorem I.2.1 are called the *Lyapunov exponents* of f at x, and $m_i(x)$ is called the *multiplicity* of $\lambda_i(x)$.

In many cases, we require that system (M, f, μ) satisfies the following integrability condition

$$\log\left|\det(T_x f)\right| \in L^1(M, \mu). \tag{I.1}$$

By Multiplicative Ergodic Theorem, under condition (I.1) we have

$$\int_M \log\left|\det(T_x f)\right| d\mu(x) = \int_\Gamma \sum_{i=1}^{r(x)} \lambda_i(x) m_i(x)\, d\mu(x). \tag{I.2}$$

Define

$$\Gamma_\infty = \left\{ x \in \Gamma \;\middle|\; T_x f \text{ is degenerate or } \lambda_{r(x)}(x) = -\infty \right\}.$$

The integrability condition (I.1) and identity (I.2) imply that

$$\mu(\Gamma_\infty) = 0. \tag{I.3}$$

Let

$$\Gamma' = \Gamma \setminus \bigcup_{n=0}^{\infty} f^{-n}(\Gamma_\infty). \tag{I.4}$$

It is easy to see that $f(\Gamma') \subset \Gamma'$ and for any $x \in \Gamma'$, $T_x f$ is an isomorphism and $\lambda_{r(x)}(x) > -\infty$. From (I.3) we have $\mu(\Gamma') = 1$.

For $x \in M$ and $1 \le k \le m_0$, let $(T_xM)^{\wedge k}$ be the k^{th}-*exterior power space* of T_xM, namely, $(T_xM)^{\wedge k}$ is the linear space of all linear combinations of elements in $\left\{\xi_1 \wedge \ldots \wedge \xi_k : \xi_i \in T_xM, 1 \le i \le k\right\}$ in which the following relations hold:

(1) for all $\alpha, \beta \in \mathbb{R}$ and $1 \le i \le k$,

$$\begin{aligned}\xi_1 \wedge \cdots \wedge (\alpha \xi_i + \beta \xi_i') \wedge \cdots \wedge \xi_k = {} & \alpha \xi_1 \wedge \cdots \wedge \xi_i \wedge \cdots \wedge \xi_k \\ & + \beta \xi_1 \wedge \cdots \wedge \xi_i' \wedge \cdots \wedge \xi_k\end{aligned}$$

(2) for all $1 \le i, j \le k$,

$$\xi_1 \wedge \cdots \wedge \xi_i \wedge \cdots \wedge \xi_j \wedge \cdots \wedge \xi_k = -\xi_1 \wedge \cdots \wedge \xi_j \wedge \cdots \wedge \xi_i \wedge \cdots \wedge \xi_k$$

Obviously, if $\left\{\xi_i : 1 \le i \le m_0\right\}$ is a basis of T_xM, then $\left\{\xi_{i_1} \wedge \cdots \wedge \xi_{i_k} : 1 \le i_1 \le \cdots \le i_k \le m_0\right\}$ is a basis of $(T_xM)^{\wedge k}$. Now, if $\left\{e_i : 1 \le i \le m_0\right\}$ is an orthonormal basis of T_xM, then by letting

$$< e_{i_1} \wedge \cdots \wedge e_{i_k}, e_{j_1} \wedge \cdots \wedge e_{j_k} > \stackrel{\text{def}}{=} \begin{cases} 1 \text{ if } (i_1, \cdots, i_k) = (j_1, \cdots, j_k) \\ 0 \qquad\qquad \text{otherwise} \end{cases}$$

we can define an inner product $< \cdot, \cdot >$ on $(T_xM)^{\wedge k}$, and it is clearly independent of the choice of the orthonormal basis $\left\{e_i : 1 \le i \le m_0\right\}$. We shall denote also by $|\cdot|$ the norm on $(T_xM)^{\wedge k}$ induced by this inner product.

If $f : M \to M$ is a C^1 map, we define for $x \in M$ and $1 \le k \le m_0$

$$\begin{aligned}(T_xf)^{\wedge k} : {} & (T_xM)^{\wedge k} \to (T_{fx}M)^{\wedge k} \\ & \xi_1 \wedge \cdots \wedge \xi_k \mapsto (T_xf\xi_1) \wedge \cdots \wedge (T_xf\xi_k)\end{aligned}$$

and define

$$|(T_xf)^{\wedge}| \stackrel{\text{def}}{=} 1 + \sum_{k=1}^{m_0} |(T_xf)^{\wedge k}|.$$

Then an important conclusion from [77] gives

Proposition I.2.2 *Let (M, f, μ) be given. Then we have*

$$\lim_{n\to+\infty} \frac{1}{n} \log |(T_x f^n)^\wedge| = \sum_i \lambda_i(x)^+ m_i(x), \quad \mu - \ a.e.$$

and

$$\lim_{n\to+\infty} \frac{1}{n} \int \log |(T_x f^n)^\wedge| \, d\mu = \int \sum_i \lambda_i(x)^+ m_i(x) \, d\mu.$$

I.3 Inverse Limit Space

Let X be a compact metric space. For any continuous map T on X, let X^T denote the subset of $X^{\mathbb{Z}}$ consisting of all full orbits, i.e.,

$$X^T = \left\{ \tilde{x} = \{x_i\}_{i\in\mathbb{Z}} \;\middle|\; x_i \in X,\ Tx_i = x_{i+1},\ \forall i \in \mathbb{Z} \right\}.$$

Obviously, X^T is a closed subset of $X^{\mathbb{Z}}$ (endowed with the product topology and the metric $d(\tilde{x}, \tilde{y}) = \sum_{i=-\infty}^{+\infty} 2^{-|i|} d(x_i, y_i)$ for $\tilde{x} = \{x_i\}_{i\in\mathbb{Z}}, \tilde{y} = \{y_i\}_{i\in\mathbb{Z}} \in X^{\mathbb{Z}}$). X^T is called the *inverse limit space* of system (X, T). Let p denote the natural projection from X^T to X, i.e.,

$$p(\tilde{x}) = x_0, \quad \forall \tilde{x} \in X^T,$$

and $\theta : X^T \to X^T$ as the shift homeomorphism. Clearly the following diagram commutes,

$$\begin{tikzcd} X^T \arrow[r, "\theta"] \arrow[d, "p"'] & X^T \arrow[d, "p"] \\ X \arrow[r, "T"'] & X \end{tikzcd}$$

i.e. $p \circ \theta = T \circ p$.

It is a basic knowledge that p induces a continuous map from $\mathscr{M}_\theta(X^T)$ to $\mathscr{M}_T(X)$, usually still denoted by p, i.e. for any θ-invariant Borel probability measure $\tilde{\mu}$ on X^T, p maps it to a T-invariant Borel probability measure $p\tilde{\mu}$ on X defined by

$$p\tilde{\mu}(\varphi) = \tilde{\mu}(\varphi \circ p), \quad \forall \varphi \in C(X).$$

The following proposition guarantees that p is a bijection between $\mathscr{M}_\theta(X^T)$ and $\mathscr{M}_T(X)$.

Proposition I.3.1 *Let T be a continuous map on X. For any T-invariant Borel probability measure μ on X, there exists a unique θ-invariant Borel probability measure $\tilde{\mu}$ on X^T such that $p\tilde{\mu} = \mu$.*

Before providing the proof of the above proposition, we first introduce two elementary lemmas.

Lemma I.3.2 *Let X and Y be two compact metrizable spaces, and $h : X \to Y$ a continuous surjective map. Then for any Borel probability measure μ on Y, there exists a Borel probability measure ν on X such that $h\nu = \mu$.*

Proof. Let

$$W = \{\psi \in C(X) \mid \exists \varphi \in C(Y) \text{ such that } \psi = \varphi \circ h\}.$$

Obviously W is a linear subspace of $C(X)$. Define a bounded linear functional L on W as follows,

$$L\psi = \mu(\varphi), \quad \text{where} \quad \varphi \in C(Y) \text{ such that } \psi = \varphi \circ h.$$

It is easy to see that L is a positive bounded linear functional with $L1 = 1$. By a modification of the Hahn-Banach Theorem L can be extended to a positive bounded linear functional on $C(X)$ preserving the property $L1 = 1$. Then Rieze Representation Theorem implies that there is a Borel probability measure ν on X such that $L\psi = \nu(\psi)$ for all $\psi \in C(X)$. It is easy to verify that $h\nu = \mu$. □

Lemma I.3.3 *Let X and Y be two compact metrizable spaces, and $T : X \to X$ and $S : Y \to Y$ measurable mappings on corresponding spaces. Suppose there is a continuous surjective map $h : X \to Y$ such that $S \circ h = h \circ T$. Then for any S-invariant Borel probability measure μ on Y, there is a T-invariant Borel probability measure ν on X such that $h\nu = \mu$.*

Proof. From Lemma I.3.2, there is a Borel probability measure ν_0 on X such that $h\nu_0 = \mu$. Let

$$\nu_n = \frac{1}{n}\sum_{i=0}^{n-1} T^i \nu_0,$$

and suppose that $\nu_{n_k} \to \nu$ as $n_k \to +\infty$. It is then easy to see that $\nu \in \mathscr{M}_T(X)$ and $h\nu = \mu$. □

We are now ready to prove Proposition I.3.1.

Proof of Proposition I.3.1. Let $X_0 = \bigcap_{n=0}^{\infty} T^n(X)$. Obviously X_0 is a compact subset of X, and $T(X_0) = X_0$, $\mu(X_0) = 1$ for any $\mu \in \mathscr{M}_T(X)$. Therefore $X^T = X_0^T$ and $p : X_0^T \to X_0$ is continuous and surjective. As a consequence of Lemma I.3.3, there is $\tilde{\mu} \in \mathscr{M}_\theta(X^T)$ such that $p\tilde{\mu} = \mu$. Since X^T is a compact subset of $X^{\mathbb{Z}}$, $\tilde{\mu}$ can be uniquely determined by its values on all cylinder sets. For any Borel subsets A_0, $A_1, \dots, A_n \subset M$, we have

$$\tilde{\mu}\left([A_0, A_1, \dots, A_n]\right) = \mu(A_0 \bigcap T^{-1}A_1 \bigcap \cdots \bigcap T^{-n}A_n),$$

where

$$[A_0, A_1, \dots, A_n] = \left\{\tilde{x} \in X^T \mid x_i \in A_i, i = 0, 1, \dots, n\right\}$$

is a cylinder set in X^T. This ensures that $\tilde{\mu}$ is uniquely determined by μ. The proof is completed. □

Remark I.1. In the circumstances of Proposition I.3.1, it is not hard to see that $(X^T, \theta, \tilde{\mu})$ is ergodic if and only if (X, T, μ) is ergodic.

The following proposition provides the relationship between the entropies of these two systems.

Proposition I.3.4 *Let $T : X \to X$ be a continuous map on the compact metric space X with an invariant Borel probability measure μ. Let X^T be the inverse limit space of (X, T), θ the shift homeomorphism and $\tilde{\mu}$ the θ-invariant Borel probability measure on X^T such that $p\tilde{\mu} = \mu$. Then*

$$h_\mu(T) = h_{\tilde{\mu}}(\theta). \tag{I.5}$$

Proof. For each $n \in \mathbb{N}$, take a maximal $1/n$-separated set E_n of X. (Recall that a subset E of a metric space (X, d) is an ε-*separated set* of X iff $d(x,y) \geq \varepsilon$ for any distinct points $x, y \in E$. It is called a *maximal ε-separated set* of X if in addition E is maximal, i.e., for any point $x \notin E$ and $y \in E$, $d(x,y) < \varepsilon$. Given a transform $T : X \hookleftarrow$ and a positive integer n, one can define a new metric d_n as

$$d_n(x,y) := \max\left\{d(T^k x, T^k y) : 0 \leq k \leq n\right\}.$$

Then an ε-separated set of (X, d_n) is called an (n, ε)-*separating set* of X.) We define a measurable finite partition $\xi_n = \left\{\xi_n(x) \mid x \in E_n\right\}$ of X such that $\xi_n(x) \subset \overline{\mathrm{Int}(\xi_n(x))}$ and $\mathrm{Int}(\xi_n(x)) = \left\{y \in X \mid d(y,x) < d(y,x_i) \text{ if } x \neq x_i \in E_n\right\}$ for every $x \in E_n$. Clearly $\mathrm{Diam}\xi_n \leq 1/n$. By Theorem 8.3 of [92],

$$h_\mu(T) = \lim_{n\to\infty} h_\mu(T, \xi_n). \tag{I.6}$$

Using ξ_n, we may construct a measurable finite partition η_n of X^T by

$$\eta_n = \bigvee_{i=-n}^{n} \theta^i(p^{-1}\xi_n).$$

It is easy to see $\mathrm{Diam}\eta_n \to 0$ as $n \to \infty$, thus

$$h_{\tilde{\mu}}(\theta) = \lim_{n\to\infty} h_{\tilde{\mu}}(\theta, \eta_n). \tag{I.7}$$

Notice that θ is invertible, by Theorem 4.12 (vii) of [92] we have

$$h_{\tilde{\mu}}(\theta, \eta_n) = h_{\tilde{\mu}}(\theta, p^{-1}\xi_n) = h_\mu(T, \xi_n).$$

This together with (I.6) and (I.7) yields that identity (I.5) holds. □

In the previous proposition, we see that the entropies of these two systems are in fact identical. Now we consider the relationship between the Lyapunov exponents of these two systems.

For any continuous endomorphism f on the manifold M, let M^f denote the inverse limit space of system (M, f). We still use p to denote the natural projection from M^f to M, and θ to denote the shift homeomorphism. For any f-invariant Borel probability measure μ on M, we still use $\tilde{\mu}$ to denote the θ-invariant Borel probability measure on M^f such that $p\tilde{\mu} = \mu$.

Let $E = p^*TM$ for the pull back bundle of the tangent bundle TM by the projection $p : M^f \to M$, and

$$E_{\tilde{x}} = p^*_{\tilde{x}}TM \underset{p^*_{\tilde{x}}}{\overset{p_*}{\rightleftarrows}} T_{x_0}M$$

for the natural isomorphisms between fibers $E_{\tilde{x}}$ and $T_{x_0}M$:

$$\xi = (\tilde{x}, v) \underset{p^*_{\tilde{x}}}{\overset{p_*}{\rightleftarrows}} v, \quad \forall v \in T_{x_0}M, \tilde{x} \in M^f.$$

When f is C^1, a fiber preserving map on E, with respect to θ, can be defined as

$$p^*_{\theta\tilde{x}} \circ Tf \circ p_* : E_{\tilde{x}} \to E_{\theta\tilde{x}}, \quad \text{for each} \quad \tilde{x} \in M^f.$$

Since it is equivalent to Tf on the fibers, we still denote it as Tf,

$$\begin{tikzcd}
TM \arrow[d, "Tf"'] & E \arrow[l, "p_*"'] \arrow[r] \arrow[d, "Tf"'] & M^f \arrow[d, "\theta"] \\
TM \arrow[r, "p^*"'] & E \arrow[r] & M^f
\end{tikzcd}$$

i.e. Tf is a continuous bundle endomorphism covering the homeomorphism θ of the compact base M^f, so Tf is a linear map on each fiber and there is a constant $K > 0$ such that $\|Tf\| \le K$ for any $\tilde{x} \in M^f$.

Let

$$\Delta = M^f \setminus \bigcup_{n=-\infty}^{+\infty} \theta^n \left(p^{-1}(\Gamma_\infty)\right).$$

Obviously $\theta(\Delta) = \Delta$ and for any $\tilde{x} = \{x_n\}_{n\in\mathbb{Z}} \in \Delta$ we have $x_n \in M \setminus \Gamma_\infty$.

When the integrability condition (I.1) is satisfied, we have $\tilde{\mu}(\Delta) = 1$. By Theorem 5.2 of [69], (1)-(3) of [83] and the Oseledec's theorem in the Appendix of [33], we have the following fundamental result.

Proposition I.3.5 *There exists a Borel set $\tilde{\Delta} \subset \Delta$, such that $\theta\tilde{\Delta} = \tilde{\Delta}$ and $\tilde{\mu}(\tilde{\Delta}) = 1$. Furthermore, for every $\tilde{x} = \{x_n\}_{n\in\mathbb{Z}} \in \tilde{\Delta}$, there is a splitting of the tangent space $T_{x_0}M$*

$$T_{x_0}M = E_1(\tilde{x}) \oplus E_2(\tilde{x}) \oplus \cdots \oplus E_{r(\tilde{x})}(\tilde{x})$$

and numbers $+\infty > \lambda_1(\tilde{x}) > \lambda_2(\tilde{x}) > \cdots > \lambda_{r(\tilde{x})}(\tilde{x}) > -\infty$ and $m_i(\tilde{x})$ $(i = 1, 2, \ldots, r(\tilde{x}))$, such that

1) $T_{x_n}f : T_{x_n}M \to T_{x_{n+1}}M$ *is an isomorphism,* $\forall n \in \mathbb{Z}$.
2) $r(\cdot)$, $\lambda_i(\cdot)$ *and* $m_i(\cdot)$ *are* θ*-invariant, i.e.,*

$$r(\theta\tilde{x}) = r(\tilde{x}), \quad \lambda_i(\theta\tilde{x}) = \lambda_i(\tilde{x}) \quad \text{and} \quad m_i(\theta\tilde{x}) = m_i(\tilde{x})$$

for each $i = 1, \ldots, r(\tilde{x})$.
3) $\dim E_i(\tilde{x}) = m_i(\tilde{x})$ *for all* $n \in \mathbb{Z}$ *and* $1 \le i \le r(\tilde{x})$.
4) *The splitting is invariant under* Tf*, i.e.,*

$$T_{x_n}fE_i(\theta^n\tilde{x}) = E_i(\theta^{n+1}\tilde{x})$$

for all $n \in \mathbb{Z}$ *and* $1 \le i \le r(\tilde{x})$.
5) *For any* $n, m \in \mathbb{Z}$*, let*

$$T_n^m(\tilde{x}) = \begin{cases} T_{x_n}f^m, & \text{if } m > 0, \\ id, & \text{if } m = 0, \\ (T_{n+m}^{-m})^{-1}, & \text{if } m < 0. \end{cases}$$

Then,

$$\lim_{m\to\pm\infty} \frac{1}{m} \log\left|T_n^m(\tilde{x})\xi\right| = \lambda_i(\tilde{x}),$$

for all $0 \ne \xi \in E_i(\theta^n\tilde{x})$, $1 \le i \le r(\tilde{x})$.
6) *We introduce*

$$\rho^{(1)}(\tilde{x}) \ge \rho^{(2)}(\tilde{x}) \ge \cdots \ge \rho^{(m_0)}(\tilde{x})$$

to denote $\lambda_1(\tilde{x}), \ldots, \lambda_1(\tilde{x}), \ldots, \lambda_i(\tilde{x}), \ldots, \lambda_i(\tilde{x}), \ldots, \lambda_{r(\tilde{x})}(\tilde{x}), \ldots, \lambda_{r(\tilde{x})}(\tilde{x})$ *with* $\lambda_i(\tilde{x})$ *being repeated* $m_i(\tilde{x})$ *times. Now if* $\{\xi_1, \ldots, \xi_{m_0}\}$ *is a basis of* $T_{x_0}M$ *which satisfies*

$$\lim_{m\to\pm\infty} \frac{1}{m} \log\left|T_0^m(\tilde{x})\xi_i\right| = \rho^{(i)}(\tilde{x})$$

for every $1 \le i \le m_0$*, then for any two non-empty disjoint subsets* $P, Q \subset \{1, \cdots, m_0\}$ *we have*

$$\lim_{m\to\pm\infty} \frac{1}{m} \log\gamma\big(T_0^m(\tilde{x})E_P, T_0^m(\tilde{x})E_Q\big) = 0$$

where E_P *and* E_Q *denote the subspaces of* $T_{x_0}M$ *spanned by the vectors* $\{\xi_i\}_{i\in P}$ *and* $\{\xi_j\}_{j\in Q}$ *respectively and* $\gamma(\cdot,\cdot)$ *denotes the angle between the two associated subspaces.*
7) $x_0 \in \Gamma'$ *and* $r(\tilde{x}) = r(x_0)$, $\lambda_i(\tilde{x}) = \lambda_i(x_0)$ *and* $m_i(\tilde{x}) = m_i(x_0)$ *for all* $i = 1, 2, \ldots, r(\tilde{x})$*, where* $r(x_0)$, $\lambda_i(x_0)$ *and* $m_i(x_0)$ *are as in Theorem I.2.1.*

Definition I.3.1 The numbers $\{\lambda_i(\tilde{x})\}_{i=1}^{r(\tilde{x})}$, given by Proposition I.3.5 are called the *Lyapunov exponents* of $(M^f, \theta, \tilde{\mu})$ at $\tilde{x}$, and $m_i(\tilde{x})$ is called the *multiplicity* of $\lambda_i(\tilde{x})$.