Penalisation of the Standard Random Walk by a Function of the One-sided Maximum, of the Local Time, or of the Duration of the Excursions

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Summary. Call $(\Omega, \mathcal{F}_\infty, \mathbb{P}, X, \mathcal{F})$ the canonical space for the standard random walk on $\mathbb{Z}$. Thus, Ω denotes the set of paths $\phi : \mathbb{N} \to \mathbb{Z}$ such that $|\phi(n+1) - \phi(n)| = 1$, $X = (X_n, n \geqslant 0)$ is the canonical coordinate process on Ω; $\mathcal{F} = (\mathcal{F}_n, n \geqslant 0)$ is the natural filtration of X, $\mathcal{F}_\infty$ the σ-field $\bigvee_{n \geqslant 0} \mathcal{F}_n$, and $\mathbb{P}_0$ the probabilitiy on $(\Omega, \mathcal{F}_\infty)$ such that under $\mathbb{P}_0$, X is the standard random walk started form 0, i.e., $\mathbb{P}_0 \left(X_{n+1} = j \,|\, X_n = i \right) = \frac{1}{2}$ when $|j - i| = 1$.

Let $G : \mathbb{N} \times \Omega \to \mathbb{R}^+$ be a positive, adapted functional. For several types of functionals G, we show the existence of a positive $\mathcal{F}$-martingale $(M_n,\ n \geqslant 0)$ such that, for all n and all $\Lambda_n \in \mathcal{F}_n$,

$$\frac{\mathbb{E}_0[\mathbb{1}_{\Lambda_n} G_p]}{\mathbb{E}_0[G_p]} \quad \longrightarrow \quad \mathbb{E}_0[\mathbb{1}_{\Lambda_n} M_n] \qquad \text{when} \ \ p \to \infty.$$

Thus, there exists a probability Q on $(\Omega, \mathcal{F}_\infty)$ such that $Q(\Lambda_n) = \mathbb{E}_0[\mathbb{1}_{\Lambda_n} M_n]$ for all $\Lambda_n \in \mathcal{F}_n$. We describe the behavior of the process $(\Omega, X, \mathcal{F})$ under Q.

The three sections of the article deal respectively with the three situations when G is a function:

- of the one-sided maximum;
- of the sign of X and of the time spent at zero;
- of the length of the excursions of X.

1 Introduction

Let $\left\{ \Omega, (X_t, \mathcal{F}_t)_{t \geqslant 0}, \mathcal{F}_\infty, \mathbb{P}_x \right\}$ be the canonical one-dimensional Brownian motion. For several types of positive functionals $\Gamma : \mathbb{R}^+ \times \Omega \to \mathbb{R}^+$, B. Roynette, P. Vallois and M. Yor show in [RVY06] that, for fixed s and for all $\Lambda_s \in \mathcal{F}_s$,

$$\lim_{t \to \infty} \frac{\mathbb{E}_x[\mathbb{1}_{\Lambda_s} \Gamma_t]}{\mathbb{E}_x[\Gamma_t]}$$

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exists and has the form $\mathbb{E}_x[\mathbb{1}_{\Lambda_s} M_s^x]$, where $(M_s^x, s \geqslant 0)$ is a positive martingale. This enables them to define a probability Q_x on $(\Omega, \mathcal{F}_\infty)$ by:

$$\forall \Lambda_s \in \mathcal{F}_s \qquad Q_x(\Lambda_s) = \mathbb{E}_x[\mathbb{1}_{\Lambda_s} M_s^x];$$

moreover, they precisely describe the behavior of the canonical process X under Q_x. This they do for numerous functionals Γ, for instance a function of the one-sided maximum, or of the local time, or of the age of the current excursion (cf. [RVY06], [RVY]).

Our purpose is to study a discrete analogue of their results. More precisely, let Ω denote the set of all functions ϕ from $\mathbb{N}$ to $\mathbb{Z}$ such that $|\phi(n+1) - \phi(n)| = 1$, let $X = (X_n, n \geqslant 0)$ be the process of coordinates on that space, $\mathcal{F} = (\mathcal{F}_n, n \geqslant 0)$ the canonical filtration, $\mathcal{F}_\infty$ the σ-field $\bigvee_{n \geqslant 0} \mathcal{F}_n$, and $\mathbb{P}_x$ $(x \in \mathbb{N})$ the family of probabilities on $(\Omega, \mathcal{F}_\infty)$ such that under $\mathbb{P}_x$ X is the standard random walk started at x. For notational simplicity, we often write $\mathbb{P}$ for $\mathbb{P}_0$. Our aim is to establish that for several types of positive, adapted functionals $G : \mathbb{N} \times \Omega \to \mathbb{N}$,
i) for each $n \geqslant 0$ and each $\Lambda_n \in \mathcal{F}_n$,

$$\frac{\mathbb{E}_0[\mathbb{1}_{\Lambda_n} G_p]}{\mathbb{E}_0[G_p]},$$

tends to a limit when p tends to infinity;
ii) this limit is equal to $\mathbb{E}_0[\mathbb{1}_{\Lambda_n} M_n]$, for some $\mathcal{F}$-martingale M such that $M_0 = 1$.

Call $Q(\Lambda_n)$ this limit. Assuming i) and ii), Q is a probability on each σ-field $\mathcal{F}_n$; it extends in a unique way to a probability (still called Q) on the σ-field $\mathcal{F}_\infty$. This can be seen either by applying Kolmogorov's theorem on projective limits (knowing Q on the $\mathcal{F}_n$ amounts to knowing the finite marginal laws of the process X), or directly, since every finitely additive probability on the Boolean algebra $\mathcal{A} = \bigcup_n \mathcal{F}_n$ extends to a σ-additive probability on $\mathcal{F}_\infty$ (a Cantorian diagonal argument shows that every decreasing sequence (A_k) in $\mathcal{A}$ with limit $\bigcap_k A_k = \varnothing$ is stationary; hence every finitely additive probability on $\mathcal{A}$ is σ-additive on $\mathcal{A}$). In short, Q is the unique probability on $(\Omega, \mathcal{F}_\infty)$ such that

$$\forall n \in \mathbb{N} \quad \forall \Lambda_n \in \mathcal{F}_n \qquad Q(\Lambda_n) = \mathbb{E}_0\left[\mathbb{1}_{\Lambda_n} M_n\right].$$

We will also study the process X under Q.
1) In the first section, G is a function of the one-sided maximum, i.e.,

$$G_p = \varphi\left(S_p\right)$$

where $S_p = \sup\{X_k, k \leqslant p\}$ and where φ is a function from $\mathbb{N}$ to $\mathbb{R}^+$ satisfying

$$\sum_{k=0}^{\infty} \varphi(k) = 1$$

We will also need the function $\Phi : \mathbb{N} \longrightarrow \mathbb{R}^+$ given by

$$\Phi(k) := \sum_{j=k}^{\infty} \varphi(j).$$

The results of Section 1 are summarized in the following statement:

Theorem 1. *1. a) For each $n \geqslant 0$ and each $\Lambda_n \in \mathcal{F}_n$, one has*

$$\lim_{p\to\infty} \frac{\mathbb{E}[\mathbb{1}_{\Lambda_n}\varphi(S_p)]}{\mathbb{E}[\varphi(S_p)]} = \mathbb{E}[\mathbb{1}_{\Lambda_n} M_n^\varphi],$$

where $M_n^\varphi := \varphi(S_n)(S_n - X_n) + \Phi(S_n)$.

b) $(M_n^\varphi, n \geqslant 0)$ is a positive martingale, with $M_0^\varphi = 1$, non uniformly integrable; in fact, M_n^φ tends a.s. to 0 when $n \to \infty$.

2. Call Q^φ the probability on $(\Omega, \mathcal{F}_\infty)$ characterized by

$$\forall n \in \mathbb{N}, \Lambda_n \in \mathcal{F}_n, \quad Q^\varphi(\Lambda_n) = \mathbb{E}[\mathbb{1}_{\Lambda_n} M_n^\varphi].$$

Then

a) S_∞ is finite Q^φ-a.s. and satisfies for every $k \in \mathbb{N}$:

$$Q^\varphi(S_\infty = k) = \varphi(k). \tag{1}$$

b) Under Q^φ, the r.v. $T_\infty := \inf\{n \geqslant 0,\ X_n = S_\infty\}$ (which is not a stopping time in general) is a.s. finite and

i. $(X_{n\wedge T_\infty},\ n \geqslant 0)$ and $(S_\infty - X_{T_\infty+n},\ n \geqslant 0)$ are two independent processes;

ii. conditional on the r.v. S_∞, the process $(X_{n\wedge T_\infty},\ n \geqslant 0)$ is a standard random walk stopped when it first hits the level S_∞;

iii. $(S_\infty - X_{T_\infty+n},\ n \geqslant 0)$ is a 3-Bessel walk started from 0.

3. Put $R_n = 2S_n - X_n$. Under Q^φ, $(R_n, n \geqslant 0)$ is a 3-Bessel walk independent of S_∞.

The proofs of the second and third parts of this theorem rest largely upon a theorem due to Pitman (cf. [Pit75]) and on the study of the large p asymptotics of $\mathbb{P}(\Lambda_n | S_p = k)$ for $\Lambda_n \in \mathcal{F}_n$.

We must now explain the precise meaning of the '3-Bessel walk' mentioned in the theorem and further in this article. In fact, two processes, which we call the 3-Bessel walk and the 3-Bessel* walk, will play a role in this work; they are identical up to a one-step space shift.

The 3-Bessel walk is the Markov chain $(R_n, n \geqslant 0)$, with values in $\mathbb{N} = \{0, 1, 2, \ldots\}$, whose transition probabilities from $x \geqslant 0$ are given by

$$\pi(x, x+1) = \frac{x+2}{2x+2}; \qquad \pi(x, x-1) = \frac{x}{2x+2}. \tag{2}$$

The 3-Bessel* walk is the Markov chain $(R_n^*, n \geqslant 0)$, valued in $\mathbb{N}^* = \{1, 2, \ldots\}$, such that $R^* - 1$ is a 3-Bessel walk. So its transition probabilities from $x \geqslant 1$ are

$$\pi^*(x, x+1) = \frac{x+1}{2x}; \qquad \pi^*(x, x-1) = \frac{x-1}{2x}.$$

2) In the second section, the functional G_p will be a function of the local time at 0 of the random walk. The local time is the process $(L_n, n \geqslant 0)$ such that L_n is the number of times that X was null *strictly* before time n. In other words,

$$L_n = \sum_{m \geqslant 0} \mathbb{1}_{m<n} \mathbb{1}_{X_m = 0}.$$

Observe that L_n is also the sum of the number of up-crossings from 0 to 1 and of the number of down-crossings from 0 to -1, up to time n. Given two functions h^+ and h^- from $\mathbb{N}^*$ to $\mathbb{R}^+$ such that

$$\frac{1}{2} \sum_{k=1}^{\infty} \big(h^+(k) + h^-(k)\big) = 1,$$

we consider the penalisation functional

$$G_p := h^+(L_p)\, \mathbb{1}_{X_p > 0} + h^-(L_p)\, \mathbb{1}_{X_p < 0}.$$

Putting

$$\Theta(x) = \frac{1}{2} \sum_{k=x+1}^{\infty} \big(h^+(k) + h^-(k)\big),$$

we obtain the following penalisation theorem.

Theorem 2. *1. a) For each $n \geqslant 0$ and each $\Lambda_n \in \mathcal{F}_n$, one has*

$$\lim_{p \to \infty} \frac{\mathbb{E}[\mathbb{1}_{\Lambda_n} G_p]}{\mathbb{E}[G_p]} = \mathbb{E}[\mathbb{1}_{\Lambda_n} M_n^{h^+, h^-}], \tag{3}$$

where $M_n^{h^+, h^-} := X_n^+ h^+(L_n) + X_n^- h^-(L_n) + \Theta(L_n)$.

b) $M_n^{h^+,h^-}$ *is a positive, non uniformly integrable martingale ; indeed, it tends to* 0 *when* n *tends to infinity.*

2. Call Q^{h^+,h^-} *the probability on* $\mathcal{F}_\infty$ *whose restriction to* $\mathcal{F}_n$ *is given by*

$$\forall \Lambda_n \in \mathcal{F}_n, \qquad Q^{h^+,h^-}(\Lambda_n) = \mathbb{E}[\mathbb{1}_{\Lambda_n} M_n^{h^+,h^-}].$$

This Q^{h^+,h^-} *has the following properties:*

a) L_∞ *is* Q^{h^+,h^-}*-a.s. finite and satisfies*

$$\forall k \in \mathbb{N}^*, \quad Q^{h^+,h^-}(L_\infty = k) = \frac{1}{2}\big(h^+(k) + h^-(k)\big).$$

b) The r.v. $g := \sup\{n \geqslant 0, X_n = 0\}$ *is* Q^{h^+,h^-}*-a.s. finite and, under* Q^{h^+,h^-}*,*

i. The processes $(X_{g+u},\ u \geqslant 0)$ *and* $(X_{u\wedge g},\ u \geqslant 0)$ *are independent.*

ii. With probability $\frac{1}{2}\sum_{k=1}^\infty h^+(k)$*, the process* $(X_{g+u},\ u \geqslant 1)$ *is a 3-Bessel* walk started from* 1*.*
With probability $\frac{1}{2}\sum_{k=1}^\infty h^-(k)$*, the process* $(-X_{g+u},\ u \geqslant 1)$ *is a 3-Bessel* walk started from* 1*.*

iii. Conditional on $L_\infty = l$*, the process* $(X_{u\wedge g}, u \geqslant 0)$ *is a standard random walk stopped at its* l*-th passage at* 0*.*

Our unusual choice for the definition of the local time at 0 will be helpful when proving the first point. The second part of the proof of this theorem rests essentially on an article by Le Gall (cf [LeG85]) which enables us to assess, under specific conditions, that a 3-Bessel* walk for $\mathbb{P}$ is is still a 3-Bessel* walk for Q^{h^+,h^-}.

3) In the third part, the penalisation functional G_p will be a function of the longest excursion completed until time p. Set $g_n := \sup\{k \leqslant n,\ X_k = 0\}$, $d_n := \inf\{k \geqslant n,\ X_k = 0\}$, and $\Sigma_n := \sup\{d_k - g_k,\ d_k \leqslant n\}$; for $n \geqslant 0$, Σ_n is the duration of the longest excursion completed until time n.

Fix an even integer $x \geqslant 0$, and consider the penalisation functional

$$G_p := \mathbb{1}_{\Sigma_p \leqslant x}.$$

To study penalisation by this G, we must also introduce $A_n := n - g_n$, which is the age of the current excursion, and $A_n^* := \sup_{k\leqslant n} A_k$, which is the longest duration of a (complete or incomplete) excursion until n. We also call $\tau = \inf\{n > 0,\ X_n = 0\}$ the first return time to 0, and we put

$$\theta(x) := \mathbb{E}_0\big[\,|X_x|\,\big|\,\tau > x\,\big].$$

Theorem 3. *1. a) For each $n \geqslant 0$ and each $\Lambda_n \in \mathcal{F}_n$:*

$$\lim_{p\to\infty} \frac{\mathbb{E}_0[\mathbb{1}_{\Lambda_n}\mathbb{1}_{\Sigma_p\leqslant x}]}{\mathbb{P}_0[\Sigma_p \leqslant x]} = \mathbb{E}_0[\mathbb{1}_{\Lambda_n} M_n], \tag{4}$$

where

$$M_n := \left\{\frac{|X_n|}{\theta(x)} + \tilde{\mathbb{P}}_{X_n}\left(\tilde{T}_0 \leqslant x - A_n\right)\mathbb{1}_{A_n\leqslant x}\right\}\mathbb{1}_{\Sigma_n\leqslant x}.$$

(In this expression and in similar ones, the meaning of $\tilde{\mathbb{P}}$ and $\tilde{T}_0$ is to be interpreted as follows: $\tilde{\mathbb{P}}_{X_n}\left(\tilde{T}_0 \leqslant x - A_n\right)$ stands for $f(X_n, x-A_n)$, with $f(y,z) = \mathbb{P}_y(T_0 \leqslant z)$.)

b) Moreover, $(M_n, n \geqslant 0)$ is a positive martingale, non uniformly integrable; indeed, $\lim_{n\to\infty} M_n = 0$ $\mathbb{P}$-a.s.

2. Call Q^x the probability on $\mathcal{F}_\infty$ whose restriction to $\mathcal{F}_n$ is defined by

$$\forall \Lambda_n \in \mathcal{F}_n, \qquad Q^x(\Lambda_n) = \mathbb{E}\left[\mathbb{1}_{\Lambda_n} M_n\right].$$

Under Q^x, one has:

a) $\Sigma_\infty \leqslant x$ a.s. and satisfies for all $y \leqslant x$:

$$Q^x(\Sigma_\infty > y) = 1 - \frac{\mathbb{P}(\tau > x)}{\mathbb{P}(\tau > y)}.$$

b) $A^*_\infty = \infty$ a.s.

c) The r.v. $g := \sup\{n \geqslant 0, X_n = 0\}$ is a.s. finite. Moreover, if $p = 2l$ or $2l+1$ with $l \geqslant 0$,

$$Q^x(g > p) = \left(\frac{1}{2}\right)^l \sum_{k=0}^{l\wedge\frac{x}{2}} C_{2l-2k}^{l-k} C_{2k}^{k}\left(1 - \frac{\mathbb{P}(\tau > x)}{\mathbb{P}(\tau > 2k)}\right).$$

d) For y such that $0 \leqslant y \leqslant x$,

i. $\left(A_n, n \leqslant T_y^A\right)$ has the same law under $\mathbb{P}$ and Q^x.

ii. $\left(A_n, n \leqslant T_y^A\right)$ and $X_{T_y^A}$ are independent under $\mathbb{P}$ and under Q^x.

iii. Under Q^x, the law of $X_{T_y^A}$ is given by

$$Q^x\left(X_{T_y^A} = k\right) = \left\{\frac{|k|}{\theta(x)} + \mathbb{P}_k\left(T_0 \leqslant x - y\right)\right\}\mathbb{P}\left(X_y = k \mid \tau > y\right).$$

iv.

$$Q^x\left(g > T_y^A\right) = 1 - \frac{\mathbb{P}\left(\tau > x\right)}{\mathbb{P}\left(\tau > y\right)}.$$

v. Under Q^x, $\left(A_n,\, n \leqslant T_y^A\right)$ *is independent of* $\left\{g > T_y^A\right\}$.

3. Under Q^x,

a) The processes $(X_{n\wedge g},\ n \geqslant 0)$ *and* $(X_{g+n},\ n \geqslant 0)$ *are independent.*

b) With probability $\frac{1}{2}$, *the process* $(X_{g+n}, n \geqslant 0)$ *is a 3-Bessel* walk and with probability* $\frac{1}{2}$, *the process* $(-X_{g+n}, n \geqslant 0)$ *is a 3-Bessel* walk.*

c) Conditional on $L_\infty = l$, *the process* $(X_{n\wedge g},\ n \geqslant 0)$ *is a standard random walk stopped at its* l*-th return time to* 0 *and conditioned by* $\{\Sigma_{\tau_l} \leqslant x\}$, *where* τ_l *is the* l*-th return time to* 0.

The proof of the first point of this theorem rests largely on a Tauberian theorem (cf [Fel50]) which gives the large p asymptotics of $\mathbb{P}\left(\Sigma_p \leqslant x\right)$. And the study of the process X under Q^x rests on arguments similar to those used in the proof of Theorem 2.

2 Principle of Penalisation

Penalisation can intuitively be interpreted as a generalisation of conditioning by a null event.

Consider the event $A_\infty := \{S_\infty \leqslant a\}$, where $a \in \mathbb{N}$. By recurrence of the standard walk, A_∞ is a $\mathbb{P}$-null event. One way of conditioning by A_∞, which involves the filtration $(\mathcal{F}_n)$, is to consider the sequence of events $A_p := \{S_p \leqslant a\}$ and to study the limit

$$\lim_{p\to\infty} \frac{\mathbb{E}\left[\mathbb{1}_{\Lambda_n \cap \{S_p \leqslant a\}}\right]}{\mathbb{E}\left[\mathbb{1}_{\{S_p \leqslant a\}}\right]}, \tag{5}$$

for each $n \in \mathbb{N}$ and each $\Lambda_n \in \mathcal{F}_n$.

Simple arguments show that the limit in (5) exists and equals

$$\mathbb{E}\left[\mathbb{1}_{\{\Lambda_n,\, S_n \leqslant a\}}\ \frac{a+1-X_n}{a+1}\right].$$

Put $M_n := \mathbb{1}_{\{S_n \leqslant a\}} \frac{a+1-X_n}{a+1}$. The process M is the martingale $\frac{a+1-X}{a+1}$ stopped when S first hits $a+1$; so it is a positive $\mathbb{P}_0$-martingale. Since $M_0 = 1$ and $M_\infty = 0$ a.s., M is not uniformly integrable. But a probability $Q_{(n)}$ can be defined on $\mathcal{F}_n$ by

$$\frac{dQ_{(n)}}{d\mathbb{P}_{|\mathcal{F}_n}} = M_n;$$

moreover, for $m < n$, $Q_{(m)}$ and $Q_{(n)}$ agree on $\mathcal{F}_m$. By Kolmogorov's existence theorem (cf [Bil] pp. 430-435), there exists a probability Q on $(\Omega, \mathcal{F}_\infty)$ whose restriction to each $\mathcal{F}_n$ is the corresponding $Q_{(n)}$; in other words, Q is characterized by

$$Q(\Lambda_n) := \mathbb{E}\left[\mathbb{1}_{\{\Lambda_n,\, S_n \leqslant a\}} \frac{a+1-X_n}{a+1}\right]$$

for all $n \in \mathbb{N}$ and $\Lambda_n \in \mathcal{F}_n$.

When studying the behavior of $(X_n,\ n \geqslant 0)$ under the new probability Q, one obtains that S_∞ is a.s. finite and uniformly distributed on $[0, a]$. A more detailed study shows that:

- $(X_{n\wedge T_\infty},\ n \geqslant 0)$ and $(S_\infty - X_{T_\infty+n},\ n \geqslant 0)$ are two independent processes.
- Conditional on $\{S_\infty = k\}$, $(X_{n\wedge T_\infty},\ n \geqslant 0)$ is a standard random walk stopped when it reaches the value k.
- $(S_\infty - X_{T_\infty+n}, n \geqslant 0)$ is a 3-Bessel walk started from 0, independent from (S_∞, T_∞).

This raises several natural questions: What happens when $\mathbb{1}_{\{S_n \leqslant a\}}$ is replaced with a more complicated function of the supremum? In that case, what does the limit (5) become? Can one still define a probability Q, and how is the behavior of $(X_n, n \geqslant 0)$ under Q influenced by this modification?

This simple idea of replacing the indicator by a more complex function is the essence of penalisation. All this is evidently not limited to the case of the one-sided maximum, but extends to many other increasing, adapted functionals tending $\mathbb{P}$-a.s. to $+\infty$. There exist various examples of penalisation, and also a general principle (cf [Deb07]) but this article is only devoted to three examples of penalisation functionals: the one-sided maximum, the local time at 0 and the maximal duration of the completed excursions.

3 Penalisation by a Function of the One-sided Maximum: Proof of Theorem 1

1) We start by recalling a few facts.

The next result is classical (cf. [Fel50] p. 75):

Lemma 1. *For $k \in \mathbb{Z}$ and $n \in \mathbb{N}$,*

$$\mathbb{P}_0(X_n = k) = \left(\frac{1}{2}\right)^n C_n^{\frac{n+k}{2}}.$$

Remark 1. In the sequel, we put

$$p_{n,k} := \mathbb{P}_0(X_n = k);$$

observe that $p_{n,k} \neq 0$ if and only if n and k have the same parity and $|k| \leqslant n$.

Lemma 2. *For k in $\mathbb{Z}$ and n and r in $\mathbb{N}$, one has*

$$\mathbb{P}_0(X_n = k,\ S_n \geqslant r) = \begin{cases} \mathbb{P}(X_n = k) & \text{if } k > r; \\ \mathbb{P}(X_n = 2r - k) & \text{if } k \leqslant r. \end{cases} \tag{6}$$

Proof. This formula is trivial when $k > r$; when $k \leqslant r$, it is Désiré André's well-known reflection principle (see for instance [Fel50] p. 72 and pp. 88-89). □

From Lemma 2 and Remark 1, one easily derives the law of S:

Lemma 3. *For n and r in $\mathbb{N}$, one has*

$$\mathbb{P}_0(S_n = r) = p_{n,r} + p_{n,r+1} = p_{n,r} \vee p_{n,r+1}. \tag{7}$$

Proof. Summing (6) over all $k \in \mathbb{Z}$ gives

$$\mathbb{P}(S_n \geqslant r) = \sum_{k>r} \mathbb{P}(X_n = k) + \sum_{k\leqslant r} \mathbb{P}(X_n = 2r-k) = \mathbb{P}(X_n > r) + \mathbb{P}(X_n \geqslant r).$$

Consequently,

$$\mathbb{P}(S_n = r) = \mathbb{P}(S_n \geqslant r) - \mathbb{P}(S_n \geqslant r+1) = \mathbb{P}(X_n = r+1) + \mathbb{P}(X_n = r),$$

and (7) follows by definition of $p_{n,k}$ and by Remark 1. □

2) We start showing point 1 of Theorem 1.

Lemma 4. *For each $k \geqslant 0$, the ratio*

$$\frac{\mathbb{P}(S_n = k)}{\mathbb{P}(S_n = 0)}$$

is majorized by 1 for all $n \geqslant 0$ and tends to 1 when $n \to +\infty$.

Proof. The denominator is minorated by $\mathbb{P}(X_1 = \ldots = X_n = -1) = 2^{-n}$; so it does not vanish. Observe that, for even n and even $k \geqslant 2$,

$$\frac{\mathbb{P}(S_n = k-1)}{\mathbb{P}(S_n = 0)} = \frac{\mathbb{P}(S_n = k)}{\mathbb{P}(S_n = 0)} = \frac{p_{n,k}}{p_{n,0}} = \left(\frac{n-k+2}{n+2}\right)\left(\frac{n-k+4}{n+4}\right)\cdots\left(\frac{n}{n+k}\right);$$

and for odd n and odd $k \geqslant 1$,

$$\frac{\mathbb{P}(S_n = k-1)}{\mathbb{P}(S_n = 0)} = \frac{\mathbb{P}(S_n = k)}{\mathbb{P}(S_n = 0)} = \frac{p_{n,k}}{p_{n,1}} = \left(\frac{n-k+2}{n+1}\right)\left(\frac{n-k+4}{n+3}\right)\cdots\left(\frac{n+1}{n+k}\right).$$

Clearly, these products are smaller than 1 and tend to 1 when n goes to infinity. □

Lemma 5. *For all $x \in \mathbb{N}$ and $y \in \mathbb{Z}$ such that $y \leqslant x$, the ratio*

$$\frac{\mathbb{E}\big[\varphi\big(x \vee (y+S_n)\big)\big]}{\mathbb{P}(S_n = 0)}$$

is majorized for all $n \in \mathbb{N}$ by $(x-y)\varphi(x)+\Phi(x)$ and tends to $(x-y)\varphi(x)+\Phi(x)$ when n tends to infinity.

Proof. Write

$$\begin{aligned}\frac{\mathbb{E}\big[\varphi\big(x \vee (y+S_n)\big)\big]}{\mathbb{P}(S_n = 0)} &= \varphi(x)\,\frac{\mathbb{P}(y+S_n < x)}{\mathbb{P}(S_n = 0)} + \sum_{k \geqslant x} \varphi(k)\,\frac{\mathbb{P}(y+S_n = k)}{\mathbb{P}(S_n = 0)} \\ &= \varphi(x) \sum_{k<x-y} \frac{\mathbb{P}(S_n = k)}{\mathbb{P}(S_n = 0)} + \sum_{k \geqslant x} \varphi(k)\,\frac{\mathbb{P}(S_n = k-y)}{\mathbb{P}(S_n = 0)}.\end{aligned}$$

By Lemma 4, this sum is majorized by $(x-y)\varphi(x) + \sum_{k\geqslant x} \varphi(k)$ and tends to this value by dominated convergence. □

To establish point 1 of Theorem 1, observe first that

$$M_n^{\varphi} = \varphi(S_n)(S_n - X_n) + \Phi(S_n)$$

is a positive martingale. Positivity is obvious: φ, Φ, and $S - X$ are positive. To see that M^{φ} is a martingale, consider $M_{n+1}^{\varphi} - M_n^{\varphi}$.

If $S_{n+1} = S_n$, the only thing that varies in the expression of M^{φ} when n is changed to $n+1$ is X; so, in that case,

$$M_{n+1}^{\varphi} - M_n^{\varphi} = -\varphi(S_n)(X_{n+1} - X_n).$$

On the other hand, if $S_{n+1} \neq S_n$, one has $S_{n+1} = S_n + 1$ because each step of S is 0 or 1; one also has $X_{n+1} = S_{n+1}$ because S can increase only when pushed up by X, and $X_n = S_n$ because X_n must simultaneously be $\leqslant S_n$ and at distance 1 from X_{n+1}. So $S_{n+1} - X_{n+1} = S_n - X_n = 0$, giving

$$\begin{aligned}M_{n+1}^{\varphi} - M_n^{\varphi} &= \Phi(S_{n+1}) - \Phi(S_n) = \Phi(S_n + 1) - \Phi(S_n) \\ &= -\varphi(S_n) = -\varphi(S_n)(X_{n+1} - X_n).\end{aligned}$$

All in all, the equality $M_{n+1}^{\varphi} - M_n^{\varphi} = -\varphi(S_n)(X_{n+1} - X_n)$ holds everywhere; this entails that M^{φ} is a martingale, verifying

$$|M_n^{\varphi} - M_0^{\varphi}| \leqslant n; \tag{8}$$

and since $M_0^{\varphi} = \Phi(0) = 1$, one has $\mathbb{E}[M_n^{\varphi}] = 1$.

We now proceed to prove 1.a of Theorem 1. For $0 \leqslant n \leqslant p$, one can write $S_p = S_n \vee (X_n + \widetilde{S}_{p-n})$, where $\widetilde{S}$ is the maximal process of the standard random walk $(X_{n+k} - X_n)_{k\geqslant 0}$, which is independent from $\mathcal{F}_n$. Hence

$$\mathbb{E}[\,\varphi(S_p)\mid\mathcal{F}_n\,]=\widetilde{\mathbb{E}}\big[\varphi\big(S_n\vee(X_n+\widetilde{S}_{p-n})\big)\big],$$

where $\widetilde{\mathbb{E}}$ integrates over $\widetilde{S}_{p-n}$ only, S_n and X_n being kept fixed. So, for $\Lambda_n\in\mathcal{F}_n$,

$$\frac{\mathbb{E}[\mathbb{1}_{\Lambda_n}\,\varphi(S_p)]}{\mathbb{P}(S_{p-n}=0)}=\mathbb{E}\Big[\mathbb{1}_{\Lambda_n}\;\frac{\widetilde{\mathbb{E}}\big[\varphi\big(S_n\vee(X_n+\widetilde{S}_{p-n})\big)\big]}{\widetilde{\mathbb{P}}(\widetilde{S}_{p-n}=0)}\Big].$$

When p tends to infinity, Lemma 5 says that the ratio in the right-hand side tends to M_n^φ and is dominated by M_n^φ, which is integrable by (8). Consequently,

$$\frac{\mathbb{E}[\mathbb{1}_{\Lambda_n}\,\varphi(S_p)]}{\mathbb{P}(S_{p-n}=0)}\quad\begin{cases}\text{is majorated by } \mathbb{E}[\mathbb{1}_{\Lambda_n}\,M_n^\varphi] \text{ for all } p\geqslant n\\ \text{and tends to}\quad \mathbb{E}[\mathbb{1}_{\Lambda_n}\,M_n^\varphi]\quad\text{when } p\to\infty.\end{cases}\tag{9}$$

Taking in particular $\Lambda_n=\Omega$, one also has

$$\frac{\mathbb{E}[\varphi(S_p)]}{\mathbb{P}(S_{p-n}=0)}\to\mathbb{E}[M_n^\varphi]=1\qquad\text{when } p\to\infty,$$

and to establish 1.a of Theorem 1, it suffices to take the ratio of these two limits.

Half of 1.b is already proven: we have seen above that M^φ is a positive martingale, with $M_0^\varphi=1$. The proof that $M_n^\varphi\to0$ a.s. is postponed; we first establish 2.a.

The set-function Q^φ defined on the Boolean algebra $\bigcup_n\mathcal{F}_n$ by $Q^\varphi(\Lambda_n)=\mathbb{E}[\mathbb{1}_{\Lambda_n}M_n^\varphi]$ if $\Lambda_n\in\mathcal{F}_n$, is a probability on each σ-field $\mathcal{F}_n$. As recalled in the introduction, Q^φ automatically extends to a probability (still called Q^φ) on the σ-field $\mathcal{F}_\infty$.

For k and n in $\mathbb{N}$, the event $\{S_n\geqslant k\}$ is equal to $\{T_k\leqslant n\}$, where $T_k=\inf\{m\ :\ X_m\geqslant k\}=\inf\{m\ :\ S_m\geqslant k\}$. Now, by Doob's stopping theorem,

$$\begin{aligned}Q^\varphi(S_n\geqslant k)=Q^\varphi(T_k\leqslant n)&=\mathbb{E}[\mathbb{1}_{T_k\leqslant n}\,M_n^\varphi]\\&=\mathbb{E}[\mathbb{1}_{T_k\leqslant n}\,M_{n\wedge T_k}^\varphi]=\mathbb{E}[\mathbb{1}_{T_k\leqslant n}\,M_{T_k}^\varphi].\end{aligned}$$

But $\mathbb{P}_0$-a.s., $X_{T_k}=S_{T_k}=k$ and $M_{T_k}^\varphi=\Phi(k)$; wherefrom

$$Q^\varphi(S_n\geqslant k)=\Phi(k)\,\mathbb{P}(S_n\geqslant k).$$

Fixing k, let now n tend to infinity. The events $\{S_n\geqslant k\}$ form an increasing sequence, with limit $\{S_\infty\geqslant k\}$; hence

$$Q^\varphi(S_\infty\geqslant k)=\Phi(k)\,\mathbb{P}(S_\infty\geqslant k)=\Phi(k).$$

This implies that S_∞ is Q^φ-a.s. finite, with

$$Q^\varphi(S_\infty=k)=\Phi(k)-\Phi(k+1)=\varphi(k);$$

so 2.a is established.

This also implies that the $\mathbb{P}$-a.s. limit M^φ_∞ of M^φ is null, by the following argument. Using Fatou's lemma, one writes

$$\begin{aligned}\mathbb{E}[\mathbb{1}_{S_\infty \geqslant k}\, M^\varphi_\infty] &= \mathbb{E}\big[\lim_n(\mathbb{1}_{S_n \geqslant k}\, M^\varphi_n)\big] \\ &\leqslant \liminf_n \mathbb{E}[\mathbb{1}_{S_n \geqslant k}\, M^\varphi_n] \\ &= \liminf_n Q^\varphi(S_n \geqslant k) = Q^\varphi(S_\infty \geqslant k) = \Phi(k);\end{aligned}$$

then, by dominated convergence, one has

$$\mathbb{E}[\mathbb{1}_{S_\infty=\infty}\, M^\varphi_\infty] = \mathbb{E}\big[\lim_k(\mathbb{1}_{S_\infty \geqslant k}\, M^\varphi_\infty)\big] = \lim_k \mathbb{E}[\mathbb{1}_{S_\infty \geqslant k}\, M^\varphi_\infty] \leqslant \lim_k \Phi(k) = 0,$$

and $\mathbb{P}(S_\infty = \infty) = 1$ now implies $\mathbb{E}[M^\varphi_\infty] = 0$. Point 1.b is proven.

3) Here are now a few facts on 3-Bessel walks, which will play an important role in the rest of the proof of Theorem 1.

Proposition 1. *Let $(R_n,\ n \geqslant 0)$ be a 3-Bessel walk; put $J_n = \inf_{m\geqslant n} R_m$.*

1. *Conditional on $\mathcal{F}^R_n$, the law of J_n is uniform on $\{0, 1, \dots, R_n\}$.*
2. *Suppose now $R_0 = 0$ (therefore $J_0 = 0$ too).*
   a) *The process $(Z_n,\ n \geqslant 0)$ defined by $Z_n = 2J_n - R_n$ is a standard random walk, and its natural filtration $\mathcal{Z}$ is also the natural filtration of the 2-dimensional process (R, J).*
   b) *If T is a stopping time for $\mathcal{Z}$ such that $R_T = J_T$, then the process $(R_{T+n} - R_T,\ n \geqslant 0)$ is a 3-Bessel walk started from 0 and independent of $\mathcal{Z}_T$.*

Proof. 1. By the Markov property, it suffices to show that if $R_0 = k$, the r.v. J_0 is uniformly distributed on $\{0, \dots, k\}$. The function $f(x) = 1/(1 + x)$ defined for $x \geqslant 0$ is bounded and verifies for $x \geqslant 1$

$$f(x) = \pi(x, x-1)\, f(x-1) + \pi(x, x+1)\, f(x+1),$$

where π is the transition kernel of the 3-Bessel walk, given by (2). Thus f is π-harmonic except at $x = 0$, and $f(R_{n\wedge\sigma_0})$ is a bounded martingale, where σ_x denotes the hitting time of x by R. (This result is due to [LeG85] p. 449.) For $0 \leqslant a \leqslant k$, by stopping, $\mu^a_n = f(R_{n\wedge\sigma_a})$ is also a bounded martingale. A Borel-Cantelli argument shows that the paths of R are a.s. unbounded; hence $\liminf_{n\to\infty} f(R_n) = 0$ and $\mu^a_\infty = f(a)\, \mathbb{1}_{J_0 \leqslant a}$. The martingale equality $f(a)\, \mathbb{P}(J_0 \leqslant a) = \mathbb{E}[\mu^a_\infty] = \mathbb{E}[\mu^a_0] = f(k)$ yields $\mathbb{P}(J_0 \leqslant a) = (a + 1)/(k + 1)$, so the law of J_0 is uniform on $\{0, \dots, k\}$.

Part 2 of Proposition 1 depends only on the law of the process R, so we need not prove it for *all* 3-Bessel walks started at 0, it suffices to prove it for *some particular* 3-Bessel walk started at 0. Given a standard random walk Z' with $Z'_0 = 0$ and its past maximum $S'_n = \sup_{m\leqslant n} Z'_m$, Pitman's theorem [Pit75] says that the process $R = 2S' - Z'$ is a 3-Bessel walk started from 0, with future minimum $J_n = \inf_{m\geqslant n} R_m$ given by $J = S'$. We shall prove 2.a and 2.b for this particular 3-Bessel walk R.

The process $Z = 2J - R$ is also equal to $2S' - R = Z'$, so it is a standard random walk. Both $J = S'$ and $R = 2S' - Z'$ are adapted to the filtration of Z; conversely, $Z = 2J - R$ is adapted to the filtration generated by R and J. This proves 2.a.

To show 2.b, let T be $\mathcal{Z}$-stopping time such that $R_T = J_T$. One has

$$Z'_T = 2J_T - R_T = J_T = S'_T.$$

The process $\widetilde{Z}$ defined by $\widetilde{Z}_n = Z'_{T+n} - Z'_T$ is a standard random walk independent of $\mathcal{Z}_T$, started from 0, with past maximum

$$\widetilde{S}_n = \sup_{m \leqslant n} \widetilde{Z}_m = S'_{T+n} - Z'_T = S'_{T+n} - S'_T.$$

By Pitman's theorem, $\widetilde{R} = 2\widetilde{S} - \widetilde{Z}$ is a 3-Bessel walk, and it is independent of $\mathcal{Z}_T$ because so is $\widetilde{Z}$. Now,

$$\widetilde{R}_n = 2\widetilde{S}_n - \widetilde{Z}_n = 2(S'_{T+n} - S'_T) - (Z'_{T+n} - Z'_T) = R_{T+n} - R_T;$$

thus 2.b holds and Proposition 1 is established. □

4) The next step is the proof of point 3 in Theorem 1. We start with a small computation:

Lemma 6. *Let a r.v. U be uniformly distributed on $\{0, .., r\}$. Then*

$$\mathbb{E}\big[\varphi(U)(r - U) + \Phi(U)\big] = 1$$

Proof. It suffices to write

$$(r+1)\,\mathbb{E}\big[1 - \Phi(U)\big] = \sum_{i=0}^{r}\big(1 - \Phi(i)\big) = \sum_{i=0}^{r}\sum_{j=0}^{i-1}\varphi(j) = \sum_{j=0}^{r-1}\sum_{i=j+1}^{r}\varphi(j)$$
$$= \sum_{j=0}^{r-1}(r - j)\varphi(j) = (r+1)\,\mathbb{E}\big[(r - U)\varphi(U)\big]. \quad \square$$

The next proposition proves the first half of point 3 in Theorem 1.

Proposition 2. *Under Q^φ, the process $(R_n, n \geqslant 0)$ given by $R_n = 2S_n - X_n$ is a 3-Bessel started from* 0.

Proof. According to Pitman's theorem [Pit75], under the probability $\mathbb{P}$, the process $(R_n, n \geqslant 0)$ is a 3-Bessel walk with future infimum $J_n = S_n$. Call $\mathcal{R}$ the natural filtration of R. By Proposition 1.1, the conditional law of S_n given $\mathcal{R}_n$ is uniform on $\{0, \ldots, R_n\}$; consequently Lemma 6 gives

$$\mathbb{E}[M_n^\varphi | \mathcal{R}_n] = \mathbb{E}[\,\varphi(S_n)(R_n - S_n) + \Phi(S_n) \mid \mathcal{R}_n\,] = 1.$$

Now, let f be any bounded function on $\mathbb{N}^{n+1}$. One has

$$\begin{aligned}\mathbb{E}^{Q^\varphi}[f(R_0,\dots,R_n)] &= \mathbb{E}[f(R_0,\dots,R_n)M_n^\varphi] \\ &= \mathbb{E}\big[f(R_0,\dots,R_n)\,\mathbb{E}[M_n^\varphi|\mathcal{R}_n]\big] = \mathbb{E}[f(R_0,\dots,R_n)].\end{aligned}$$

As n and f were arbitrary, R has the same law under Q^φ as under $\mathbb{P}$, that is, Q^φ also makes R a 3-Bessel walk. □

To finish proving point 3, it remains to establish that R is independent of S_∞ under Q^φ. This will easily follow from the next lemma, which decomposes Q^φ as a sum of measures carried by the level sets of S_∞.

Lemma 7. *Call $Q^{(k)}$ the probability Q^φ for $\varphi = \delta_k$, that is, $\varphi(k) = 1$ and $\varphi(x) = 0$ for $x \neq k$. Then $Q^{(k)}$ is supported by the event $\{S_\infty = k\}$, and, for a general φ and for all $\Lambda \in \mathcal{F}_\infty$ one has*

$$Q^\varphi(\Lambda) = \sum_{k\geqslant 0} \varphi(k)\,Q^{(k)}(\Lambda);$$

$$Q^\varphi(\Lambda \mid S_\infty = k) = Q^{(k)}(\Lambda) \qquad \textit{for all } k \textit{ such that } \varphi(k) > 0.$$

Proof. For $\Lambda_n \in \mathcal{F}_n$, one can use formula (9) twice and write

$$\begin{aligned}Q^\varphi(\Lambda_n) &= \lim_p \frac{\mathbb{E}[\mathbb{1}_{\Lambda_n}\varphi(S_p)]}{\mathbb{P}(S_{p-n}=0)} = \lim_p \sum_k \varphi(k)\,\frac{\mathbb{P}(\Lambda_n\cap\{S_p=k\})}{\mathbb{P}(S_{p-n}=0)} \\ &= \sum_k \varphi(k)\ \lim_p \frac{\mathbb{P}(\Lambda_n\cap\{S_p=k\})}{\mathbb{P}(S_{p-n}=0)} = \sum_k \varphi(k)\,Q^{(k)}(\Lambda_n),\end{aligned}$$

where lim and Σ commute by dominated convergence, owing to the majoration in (9). So the probabilities Q^φ and $\sum_k \varphi(k)\,Q^{(k)}$ coincide on $\bigcup_n \mathcal{F}_n$; therefore they also coincide on $\mathcal{F}_\infty$.

Applying now equation (1) with $\varphi = \delta_k$ gives $Q^{(k)}(S_\infty = k) = 1$, that is, $Q^{(k)}$ is supported by $\{S_\infty = k\}$.

Consequently, for any $\Lambda \in \mathcal{F}_\infty$, one has $Q^\varphi\big(\Lambda\cap\{S_\infty=k\}\big) = \varphi(k)\,Q^{(k)}(\Lambda)$ because all other terms in the series vanish. Using (1) again, one may replace $\varphi(k)$ with $Q^\varphi(S_\infty = k)$; this proves $Q^\varphi(\,\Lambda \mid S_\infty = k\,) = Q^{(k)}(\Lambda)$ whenever $\varphi(k) > 0$. □

The proof of independence in Theorem 1.3 is now a child's play: Proposition 2 says that the law of R under Q^φ is always the law of the 3-Bessel walk, whatever the choice of φ. We may in particular take $\varphi = \delta_k$, so it is also true under $Q^{(k)}$. Since $Q^{(k)}$ is also the conditioning of Q^φ by $\{S_\infty = k\}$, under Q^φ the law of R conditional on $\{S_\infty = k\}$ does not depend upon k, thus R is independent of S_∞.

5) So far, all of Theorem 1 has been established, except 2.b, to which the rest of the proof will be devoted. Finiteness of T_∞ is due to X being integer-valued and its supremum S_∞ being finite.

Put $U_n = X_{n\wedge T_\infty}$ and $V_n = S_\infty - X_{T_\infty+n}$. To prove 2.b.i and 2.b.iii we have to show that under Q^φ the process V is a 3-Bessel walk independent of the process U. Call ν the law of the 3-Bessel walk. For bounded functionals F and G, we must prove that

$$\mathbb{E}^{Q^\varphi}[F\circ U\, G\circ V] = \mathbb{E}^{Q^\varphi}[F\circ U] \int G(v)\,\nu(\mathrm{d}v).$$

Replacing now Q^φ by $\sum_k \varphi(k)\, Q^{(k)}$ (see Lemma 7), it suffices to show it when $\varphi = \delta_k$. Similarly, 2.b.ii only refers to a conditional law given S_∞; by Lemma 7 again, we may replace Q^φ by $Q^{(k)}$. Finally, *when proving 2.b, we may suppose* $\varphi = \delta_k$ *and* $Q^\varphi = Q^{(k)}$ *for a fixed* $k \geqslant 0$. Hence the random time T_∞ becomes the stopping time $T_k = \inf\{n \geqslant 0,\ X_n = k\}$, and it remains to show that

- $(X_{n\wedge T_k},\ n \geqslant 0)$ is a standard random walk stopped when it first hits the level k;
- $(2k - X_{T_k+n},\ n \geqslant 0)$ is a 3-Bessel walk started at 0;
- These two processes are independent.

By point 3 of Theorem 1, we know that $R = 2S - X$ is a 3-Bessel walk; and as we are now working under $Q^{(k)}$, we have $S_\infty = k$ a.s. Put $J_n = \inf_{m\geqslant n} R_m$.

We shall first show that the processes J and S are equal on the interval $[0, T_k]$. Given n, call τ the first time $p \geqslant n$ when $X_p = S_n$, and observe that on the event $\{T_k \geqslant n\}$, τ is finite because $X_n \leqslant S_n \leqslant k = X_{T_k}$. For all $m \geqslant n$, one has $R_m = S_m + (S_m - X_m) \geqslant S_n + 0$, with equality for $m = \tau$; thus $J_n = S_n$ on $\{\tau < \infty\}$ and a fortiori on $\{T_k \geqslant n\}$.

We shall now apply Proposition 1.2 to the 3-Bessel walk $R = 2S - X$ and its future infimum J. Part 2.a of this proposition says that $Z = 2J - R$ is a standard random walk. We just saw that $J = S$ on the random time-interval $[0, T_k]$; consequently, on this interval, $Z = 2S - R = X$. And as T_k is the first time when $X = k$, it is also the first time when $Z = k$. This proves that $(X_{n\wedge T_k},\ n \geqslant 0)$ is a standard random walk stopped at level k, and also that the $\mathcal{Z}$-stopping time T_k satisfies $\mathcal{Z}_{T_k} = \mathcal{F}_{T_k}$, where $\mathcal{Z}$ is the filtration of Z.

Remarking that $R_{T_k} = J_{T_k} = k$, part 2.b of proposition 1 can be applied to T_k; it says that $(R_{T_k+n} - k,\ n \geqslant 0)$ is a 3-Bessel walk independent of $\mathcal{F}_{T_k}$, and hence also of the process $(X_{n\wedge T_k},\ n \geqslant 0)$. But $R_{T_k+n} = 2S_{T_k+n} - X_{T_k+n} = 2k - X_{T_k+n}$ since $S_{T_k} = k = S_\infty$; so this 3-Bessel walk is nothing but $(k - X_{T_k+n},\ n \geqslant 0)$. This concludes the proof of Theorem 1.

4 Penalisation by a Function of the Local Time: Proof of Theorem 2

Definition 1. *Recall that the 3-Bessel* walk is the Markov chain* $(R^*_n, n \geqslant 0)$, *valued in* $\mathbb{N}^* = \{1, 2, \ldots\}$, *such that* $R^* - 1$ *is a 3-Bessel walk. So its transition probabilities from* $x \geqslant 1$ *are*

$$\pi^*(x, x+1) = \frac{x+1}{2x}; \qquad \pi^*(x, x-1) = \frac{x-1}{2x}.$$

1) We now prove point 1 of Theorem 2. First, $(M_n^{h^+,h^-}, n \geqslant 0)$ is a positive martingale. Positivity is obvious from the definitions of h, h^- and Θ. To see that M^{h^+,h^-} is a martingale, we shall verify that the increment $M_{n+1}^{h^+,h^-} - M_n^{h^+,h^-}$ has the form $(X_{n+1} - X_n)\, K_n$, where K_n is $\mathcal{F}_n$-measurable and $|K_n| \leqslant 1$. There are three cases, depending on the value of X_n.

If $X_n > 0$, then $X_{n+1} \geqslant 0$, so $X_n^+ = X_n$, $X_{n+1}^+ = X_{n+1}$, and $L_{n+1} = L_n$. Consequently, in that case, $M_{n+1}^{h^+,h^-} - M_n^{h^+,h^-} = (X_{n+1} - X_n)\, h^+(L_n)$.

Similarly, if $X_n < 0$, one has $X_n^- = -X_n$, $X_{n+1}^- = -X_{n+1}$, $L_{n+1} = L_n$ and $M_{n+1}^{h^+,h^-} - M_n^{h^+,h^-} = -(X_{n+1} - X_n)\, h^-(L_n)$.

Last, if $X_n = 0$, then $L_{n+1} = L_n + 1$ and $X_{n+1} = \pm 1$. In that case,

$$\begin{aligned} M_{n+1}^{h^+,h^-} - M_n^{h^+,h^-} &= \mathbb{1}_{\{X_{n+1}=1\}} h^+(L_n{+}1) + \mathbb{1}_{\{X_{n+1}=-1\}} h^-(L_n{+}1) \\ &\qquad\qquad\qquad\qquad + \Theta(L_n{+}1) - \Theta(L_n) \\ &= h^{\operatorname{sgn}(X_{n+1}-X_n)}(L_n{+}1) - \frac{1}{2}\big(h^+(L_n{+}1) + h^-(L_n{+}1)\big) \\ &= (X_{n+1} - X_n)\, \frac{1}{2}\big(h^+(L_n{+}1) - h^-(L_n{+}1)\big). \end{aligned}$$

This establishes the claim; consequently, M^{h^+,h^-} is a martingale which satisfies

$$\big|M_n^{h^+,h^-} - M_0^{h^+,h^-}\big| \leqslant n$$

and, as $M_0^{h^+,h^-} = 1$, one has $\mathbb{E}\big[M_n^{h^+,h^-}\big] = 1$.

To finish the proof of point 1 in Theorem 2, it remains to show formula (3). This will use the following lemma.

Lemma 8. *For each integer k such that $0 < k < \lfloor \frac{n}{2} \rfloor$,*

$$\frac{\mathbb{P}\,(L_n = k)}{\mathbb{P}\,(S_n = 0)}$$

is bounded above by 2 and tends to 1 when $n \to \infty$.

Remark 2. In the sequel, for $h : \mathbb{N} \to \mathbb{R}^+$ such that $\sum_{k=1}^{\infty} h(k) < \infty$, we put $M_n^{h,0} = X_n^+ h(L_n) + \Theta(L_n)$ for $n \geqslant 0$. When $\sum_{k=1}^{\infty} h(k) = 1$, this notation is consistent with the one used so far; in general, $M^{h,0}$ is a martingale too, for dividing it by the constant $\Theta(0) = \sum_{k=1}^{\infty} h(k)$ reduces it to the previous case.

Lemma 9. *Let $h : \mathbb{N} \longrightarrow \mathbb{R}^+$ be such that $\displaystyle\sum_{k=1}^{\infty} h(k) < \infty$. For $a \geqslant 0$ and $x \in \mathbb{Z}$,*

$$\frac{\mathbb{E}_x[h(L_n + a)\, \mathbb{1}_{X_n>0}]}{\mathbb{P}\,(S_n = 0)}$$

is bounded above by $2(h(a)x^+ + \frac{1}{2}\sum_{k\geqslant a+1} h(k))$ and converges to $h(a)x^+ + \frac{1}{2}\sum_{k\geqslant a+1} h(k)$ when $n \to \infty$.

Proof of Lemma 8. Call $\gamma_n = |\{p \leqslant n, X_p = 0\}|$ the number of visits to 0 up to time n. Clearly, $\gamma_n = L_{n+1}$ and

$$\mathbb{P}(L_n = k) = \mathbb{P}(\gamma_{n-1} = k).$$

We shall study the law of γ_n. Define a sequence $(V_n,\ n \geqslant 0)$ by

$$\begin{cases} V_0 &= 0 \\ V_{n+1} &= \inf\{k > 0,\ X_{V_n+k} = 0\} \end{cases}$$

and put $(X_n^{(k)})_{n\geqslant 0} = (X_{V_k+n})_{n\geqslant 0}$ and $T_i^{(k)} = \inf\{n \geqslant 0,\ X_n^{(k)} = i\}$.

Owing to the symmetry of the random walk and the Markov property,

$$\forall i \geqslant 1 \qquad \mathbb{P}(V_i = k) = \mathbb{P}(T_1^{(i-1)} = k - 1).$$

So $\forall i \geqslant 1$, $V_i \overset{\mathcal{L}}{=} T_1^{(i-1)} + 1$. Moreover, according to the strong Markov property, $(X_n^{(2)}, n \geqslant 0)$ is independent of $\mathcal{F}_{V_1}$ and hence

$$V_1 + V_2 \overset{\mathcal{L}}{=} T_1^{(0)} + T_1^{(1)} + 2.$$

Wherefrom, by induction,

$$V_1 + V_2 + ... + V_k \overset{\mathcal{L}}{=} T_1^{(0)} + T_1^{(1)} + ... + T_1^{(k-1)} + k.$$

So

$$\begin{aligned} \mathbb{P}(\gamma_n = k) &= \mathbb{P}(V_1 + ... + V_{k-1} \leqslant n < V_1 + ... + V_k) \\ &= \mathbb{P}(T_1^{(0)} + T_1^{(1)} + ... + T_1^{(k-2)} + k - 1 \leqslant n < T_1^{(0)} + T_1^{(1)} + ... + T_1^{(k-1)} + k) \\ &= \mathbb{P}(T_{k-1} + k - 1 \leqslant n < T_k + k) = \mathbb{P}(S_{n-k+1} \geqslant k - 1, S_{n-k} < k) \\ &= \mathbb{P}(S_{n-k+1} = k - 1) + \mathbb{P}(T_k = n - k + 1). \end{aligned}$$

Taking inspiration from the proof of Lemma 4, it is easy to see that

$$\frac{\mathbb{P}(S_{n-k} = k - 1)}{\mathbb{P}(S_n = 0)}$$

is majorated by 1 and tends to 1 when n tends to infinity.

According to [Fel50] p. 89,

$$\mathbb{P}\left(T_r = n\right) = \frac{r}{n}\, C_n^{\frac{n+r}{2}} \left(\frac{1}{2}\right)^n.$$

Appealing again to the proof of Lemma 4, it is easy to show that

$$\frac{\mathbb{P}(T_k = n - k)}{\mathbb{P}\left(S_n = 0\right)}$$

is majorated by 1 and tends to 0 when n goes to infinity. The proof is over. □

Remark 3. From the preceding result, one easily sees that

$$\frac{\mathbb{P}_x\left(L_n = k, X_n > 0\right)}{\mathbb{P}\left(S_n = 0\right)}$$

is majorated by 1 and tends to $\frac{1}{2}$ when $n \to \infty$.

Proof of Lemma 9. Start from

$$\mathbb{E}_x[h(L_n + a)\,\mathbb{1}_{X_n > 0}] = \mathbb{E}_x[h(L_n + a)\mathbb{1}_{X_n > 0}\,(\mathbb{1}_{T_0 > n} + \mathbb{1}_{T_0 \leqslant n})]$$

One has

$$h(L_n + a)\,\mathbb{1}_{X_n > 0}\mathbb{1}_{T_0 > n} = \begin{cases} 0 & si\ x \leqslant 0 \\ h(a)\,\mathbb{1}_{T_0 > n} & si\ x > 0 \end{cases}$$

According to Lemma 4,

$$\frac{h(a)\,\mathbb{1}_{x > 0}\mathbb{P}_x(T_0 > n)}{\mathbb{P}(S_n = 0)}$$

is majorated by $x^+ h(a)$ and converges to $x^+ h(a)$.

Write

$$\frac{\mathbb{E}_x[h(L_n + a)\,\mathbb{1}_{\{X_n > 0, T_0 \leqslant n\}}]}{\mathbb{P}(S_n = 0)} = \sum_{k \geqslant 1} \frac{\mathbb{P}_x\left(L_n = k,\ X_n > 0\right)}{\mathbb{P}\left(S_n = 0\right)}\ h(k + a)$$

By Lemma 8,this sum is majorated by $\sum_{k \geqslant 1} h(k + a)$ and converges to $\frac{1}{2}\sum_{k \geqslant 1} h(k + a)$ when $n \to \infty$. □

We shall now prove point 1.a in Theorem 2. For each $0 \leqslant n \leqslant p$, one has $L_p = L_n + \tilde{L}_{p-n}$ where $\tilde{L}$ is the local time at 0 of the standard random walk $(X_{n+k})_{k \geqslant 0}$ which, given X_n, is independent of $\mathcal{F}_n$. So

$$\mathbb{E}\left[\, h(L_p)\,\mathbb{1}_{X_p > 0} \,\middle|\, \mathcal{F}_n \,\right] = \tilde{\mathbb{E}}_{X_n}\left[h(L_n + \tilde{L}_{p-n})\,\mathbb{1}_{\tilde{X}_{p-n} > 0}\right]$$

where $\tilde{\mathbb{E}}$ integrates only $\tilde{L}_{p-n}$ and $\tilde{X}_{p-n}$ and where L_n and X_n are fixed. Then, for all $\Lambda_n \in \mathcal{F}_n$,

$$\frac{\mathbb{E}\left[h(L_p)\,\mathbb{1}_{X_p > 0, \Lambda_n}\right]}{\mathbb{P}(S_{p-n} = 0)} = \mathbb{E}\left[\mathbb{1}_{\Lambda_n} \frac{\tilde{\mathbb{E}}_{X_n}\left[h(L_n + \tilde{L}_{p-n})\,\mathbb{1}_{\tilde{X}_{p-n} > 0}\right]}{\mathbb{P}(S_{p-n} = 0)}\right]$$

When $p \to \infty$, Lemma 9 says that the ratio in the right-hand side tends to $M_n^{h,0}$ and is dominated by $2M_n^{h,0}$, which is integrable. Consequently, when $p \to \infty$,

$$\frac{\mathbb{E}[h(L_p)\,\mathbb{1}_{X_p > 0, \Lambda_n}]}{\mathbb{P}\left(S_{p-n} = 0\right)} \to \mathbb{E}[\mathbb{1}_{\Lambda_n} M_n^{h,0}],$$

and taking $\Lambda_n = \Omega$, one has

$$\frac{\mathbb{E}[h(L_p)\,\mathbb{1}_{X_p>0}]}{\mathbb{P}(S_{p-n}=0)} \to \mathbb{E}[M_n^{h,0}].$$

Taking the ratio of these two limits yields

$$\frac{\mathbb{E}[h(L_p)\,\mathbb{1}_{X_p>0,\Lambda_n}]}{\mathbb{E}[h(L_p)\,\mathbb{1}_{X_p>0}]} \to \frac{\mathbb{E}[\Lambda_n M_n^{h,0}]}{\mathbb{E}[M_n^{h,0}]}.$$

To finalize the proof of point 1.a, it now suffices to use the symmetry of the standard random walk and the fact that $\mathbb{E}[M_n^{h^+,h^-}] = 1$.

2) Let us now show point 2 in Theorem 2. Put $\tau_l = \inf\{k \geqslant 0, \gamma_k = l\}$. Then

$$\begin{aligned} Q^{h^+,h^-}(L_n \geqslant l) &= Q^{h^+,h^-}(\tau_l \leqslant n-1) \\ &= \mathbb{E}[\mathbb{1}_{\tau_l \leqslant n-1} M_{\tau_l}^{h^+,h^-}] = \Theta(l-1)\mathbb{P}(\tau_l \leqslant n-1). \end{aligned}$$

For fixed l, the sequence of events $\{L_n \geqslant l\}$ is increasing and tends to $\{L_\infty \geqslant l\}$; so

$$Q^{h^+,h^-}(L_\infty \geqslant l) = \Theta(l-1)\mathbb{P}(\tau_l \leqslant \infty) = \Theta(l-1).$$

Hence L_∞ is Q^{h^+,h^-}-a.s. finite, with

$$Q^{h^+,h^-}(L_\infty = l) = \Theta(l-1) - \Theta(l) = \frac{1}{2}\big(h^+(l) + h^-(l)\big)$$

and 2.a is established.

To show that the $\mathbb{P}$-a.s. limit $M_\infty^{h^+,h^-}$ of M^{h^+,h^-} is null, it suffices to apply the same method as for M^φ, with L instead of S and M^{h^+,h^-} instead of M^φ.

The study of the process $(X_n,\ n \geqslant 0)$ under Q^{h^+,h^-} starts with the next three lemmas.

Lemma 10. *Under $\mathbb{P}_1$ and conditional on the event $\{T_p < T_0\}$, the process $(X_n,\ 0 \leqslant n \leqslant T_p)$ is a 3-Bessel* walk started from 1 and stopped when it first hits the level p (cf. [LeG85]).*

For typographical simplicity, call $T_{p,n} := \inf\{k > n,\ X_k = p\}$ the time of the first visit to p after n, and $\mathcal{H}_l := \{T_{p,\tau_l} < \tau_{l+1}, X_{\tau_l+1}=1\}$, the event that the l-th excursion is positive and reaches level p.

Lemma 11. *Under the law Q^{h^+,h^-} and conditional on the event $\mathcal{H}_l$, the process $(X_{n+\tau_l},\ 1 \leqslant n \leqslant T_{p,\tau_l} - \tau_l)$ is a 3-Bessel* walk started from 1 and stopped when it first hits the level p.*

Lemma 12. *Put $\Gamma^+ := \{X_{n+g} > 0, \forall n > 0\}$ and $\Gamma^- := \{X_{n+g} < 0, \forall n > 0\}$. Then:*

$$Q^{h^+,h^-}(\Gamma^+) = 1 - Q^{h^+,h^-}(\Gamma^-) = \frac{1}{2}\sum_{k=1}^{\infty} h^+(k)$$

Proof of Lemma 11. Let G be a function from $\mathbb{Z}^n$ to $\mathbb{R}^+$. Then, according to the definition of the probability Q^{h^+,h^-} and owing to Doob's stopping theorem,

$$\begin{aligned}\mathcal{K} &:= Q^{h^+,h^-}\big[G(X_{\tau_l+1},\dots,X_{\tau_l+n})\,\mathbb{1}_{n+\tau_l\leqslant T_{p,\tau_l}}\;\big|\;\mathcal{H}_l\big]\\ &= \frac{Q^{h^+,h^-}\big[G(X_{\tau_l+1},\dots,X_{\tau_l+n})\,\mathbb{1}_{\tau_l+n\leqslant T_{p,\tau_l}<\tau_{l+1},X_{\tau_l+1}=1}\big]}{Q^{h^+,h^-}(\mathcal{H}_l)}\\ &= \frac{\mathbb{E}\big[G(X_{\tau_l+1},\dots,X_{\tau_l+n})\,\mathbb{1}_{\tau_l+n\leqslant T_{p,\tau_l}<\tau_{l+1},X_{\tau_l+1}=1}M^{h^+,h^-}_{\tau_{l+1}}\big]}{\mathbb{E}[\mathbb{1}_{\mathcal{H}_l}M^{h^+,h^-}_{\tau_{l+1}}]}.\end{aligned}$$

Replacing $M^{h^+,h^-}_{\tau_{l+1}}$ by the constant $\Theta(l)$ and using the Markov property, one gets

$$\begin{aligned}\mathcal{K} &= \frac{\mathbb{E}[G(X_{\tau_l+1},\dots,X_{\tau_l+n})\,\mathbb{1}_{\tau_l+n\leqslant T_{p,\tau_l}<\tau_{l+1},X_{\tau_l+1}=1}\big]}{\mathbb{P}(\mathcal{H}_l)}\\ &= \frac{\mathbb{E}_1[G(X_0,\dots,X_{n-1})\,\mathbb{1}_{n-1\leqslant T_p<T_0}\big]}{\mathbb{P}_1\,(T_p<T_0)}\\ &= \mathbb{E}_1[G(X_0,\dots,X_{n-1})\,\mathbb{1}_{n-1\leqslant T_p}\mid T_p<T_0]. \qquad \square\end{aligned}$$

Remark 4. By letting $p\to\infty$, one deduces therefrom that, conditional on $\{g=\tau_l, X_{\tau_l+1}=1\}$, $(X_{n+g}, n\geqslant 1)$ is a 3-Bessel* walk under Q^{h^+,h^-}.

Proof of Lemma 12. As g is Q^{h^+,h^-}-a.s. finite and as $X_n\neq 0$ for $n>g$, one has

$$Q^{h^+,h^-}(\Gamma^+) = \lim_{n\to\infty} Q^{h^+,h^-}(X_n>0). \tag{10}$$

Now,

$$Q^{h^+,h^-}(X_n>0) = \mathbb{E}[\mathbb{1}_{X_n>0}M_n^{h^+,h^-}] = \mathbb{E}[\mathbb{1}_{X_n>0}\Theta(L_n) + X_n^+h^+(L_n)].$$

Since $\mathbb{1}_{X_n>0}\,\Theta(L_n)\leqslant\Theta(L_n)\leqslant 1$, the dominated convergence theorem gives

$$\mathbb{E}[\mathbb{1}_{X_n>0}\Theta(L_n)] \stackrel{n\to\infty}{\longrightarrow} 0.$$

We already know that $M^{h^+,0}$ is a martingale. Consequently,

$$\mathbb{E}[M_n^{h^+,0}] = \mathbb{E}[M_0^{h^+,0}] = \frac{1}{2}\sum_{k=1}^{\infty} h^+(k),$$

wherefrom

$$\mathbb{E}[X_n^+h^+(L_n)] = \frac{1}{2}\mathbb{E}\Big[\sum_{k=1}^{L_n} h^+(k)\Big] \leqslant \frac{1}{2}\sum_{k=1}^{\infty} h^+(k).$$

By dominated convergence again,

$$\lim_{n\to\infty} \mathbb{E}[X_n^+ h^+(L_n)] = \frac{1}{2}\sum_{k=1}^{\infty} h^+(k),$$

and so, according to (10), $Q^{h^+,h^-}(\Gamma^+) = \frac{1}{2}\sum_{k=1}^{\infty} h^+(k)$. □

For $F : \mathbb{Z}^n \to \mathbb{R}^+$,

$$\begin{aligned}
&\mathbb{E}^Q\big[F(X_{g+1},\dots,X_{g+n})\,\mathbb{1}_{X_{g+1}=1}\big] = \sum_{l\geqslant 1}\mathbb{E}^Q\big[F(X_{g+1},\dots,X_{g+n})\,\mathbb{1}_{g=\tau_l,X_{g+1}=1}\big]\\
&= \sum_{l\geqslant 1}\mathbb{E}^Q\big[F(X_{g+1},\dots,X_{g+n})\,\big|\, g=\tau_l, X_{\tau_l+1}=1\big]\,Q^{h^+,h^-}(g=\tau_l, X_{\tau_l+1}=1)\\
&= \mathbb{E}_1\big[F(X_0,\dots,X_{n-1})\,\big|\,T_0=\infty\big]\,\sum_{l\geqslant 1} Q^{h^+,h^-}(g=\tau_l, X_{\tau_l+1}=1)\\
&= \mathbb{E}_1\big[F(X_0,\dots,X_{n-1})\,\big|\,T_0=\infty\big]\,Q^{h^+,h^-}(\Gamma^+).
\end{aligned}$$

This shows half of point 2.b.ii. The other half, when $X_{g+1} = -1$, is easily obtained using the symmetry of the walk.

To end of the proof of Theorem 2, we shall show that, conditional on $\{L_\infty = l\}$ and under the law Q^{h^+,h^-}, the process $(X_u,\ u < g)$ is a standard random walk stopped at its l-th passage at 0.

Let F be a function from $\mathbb{Z}^n$ to $\mathbb{R}^+$ and l an element of $\mathbb{N}^*$. From the definition of Q^{h^+,h^-} and the optional stopping theorem,

$$\begin{aligned}
&\mathbb{E}^Q\big[F(X_1,\dots,X_n)\,\mathbb{1}_{n<\tau_l}\,\big|\,L_\infty = l\big] = \frac{\mathbb{E}^Q\big[F(X_1,\dots,X_n)\,\mathbb{1}_{n<\tau_l<\infty}\mathbb{1}_{\tau_{l+1}=\infty}\big]}{Q^{h^+,h^-}(L_\infty=l)}\\
&= \frac{\mathbb{E}^Q\big[F(X_1,\dots,X_n)\,\mathbb{1}_{n<\tau_l<\infty}\big] - \mathbb{E}^Q\big[F(X_1,\dots,X_n)\,\mathbb{1}_{n<\tau_l<\tau_{l+1}<\infty}\big]}{Q^{h^+,h^-}(L_\infty=l)}\\
&= \frac{\mathbb{E}\big[F(X_1,\dots,X_n)\,\mathbb{1}_{n<\tau_l}M_{\tau_l}\big] - \mathbb{E}^Q\big[F(X_1,\dots,X_n)\,\mathbb{1}_{n<\tau_l}M_{\tau_{l+1}}\big]}{Q^{h^+,h^-}(L_\infty=l)}\\
&= \frac{\mathbb{E}\big[F(X_1,\dots,X_n)\,\mathbb{1}_{n<\tau_l}\big]\,(\Theta(l-1)-\Theta(l))}{\frac{1}{2}(h^+(l)+h^-(l))} = \mathbb{E}\big[F(X_1,\dots,X_n)\,\mathbb{1}_{n<\tau_l}\big].
\end{aligned}$$
□

5 Penalisation by the Length of the Excursions

5.1 Notation

For $n \geqslant 0$, call g_n (respectively d_n) the last zero before n (respectively after n):

$$g_n := \sup\{k \leqslant n,\ X_k = 0\}$$
$$d_n := \inf\{k > n,\ X_k = 0\}$$

Thus $d_n - g_n$ is the duration of the excursion that straddles n. Put

$$\Sigma_n = \sup\{d_k - g_k,\ d_k \leqslant n\},$$

so Σ_n is the longest excursion before g_n; remark that

$$\Sigma_n = \Sigma_{g_n}. \tag{11}$$

Define $(A_n,\ n \geqslant 0)$, the "age process", by

$$A_n = n - g_n,$$

and call $\mathcal{A}_n = \sigma(A_n,\ n \geqslant 0)$ the filtration generated by A. Set

$$A_n^* = \sup_{k \leqslant n} A_k, \tag{12}$$

and observe that

$$A_n^* = (\Sigma_n - 1) \vee (n - g_n),$$

wherefrom

$$A_{g_n}^* = \Sigma_{g_n} - 1. \tag{13}$$

In the sequel, $\gamma_l := \sum_{k=0}^{n} \mathbb{1}_{\{X_k=0\}}$ is the number of passage times at 0 up to time n, $\tau = \inf\{n > 0,\ X_n = 0\}$ is the first return time to 0 and a function θ is defined by

$$\mathbb{E}\left[|X_x| \mid \tau > x\right] =: \theta(x).$$

5.2 Proof of Theorem 3

1) We start with point 1 of Theorem 3. To show formula (4), we need:

Proposition 3.

$$\mathbb{P}(\Sigma_k \leqslant x) \underset{k\to\infty}{\sim} \left(\frac{2}{\pi k}\right)^{\frac{1}{2}} \theta(x).$$

To establish this Proposition, we will use the following lemma:

Lemma 13. *For every $f : \mathbb{Z} \to \mathbb{R}^+$, every $n \geqslant 0$ and every $k > 0$,*

$$\mathbb{E}\left[f(X_n) \mid A_n = k\right] = \mathbb{E}\left[f(X_k) \mid \tau > k\right].$$

and a Tauberian Theorem:

Theorem 4 (Cf. [Fel71] p. 447). *Given $q_n \geqslant 0$, suppose that the series*

$$S(s) = \sum_{n=0}^{\infty} q_n s^n$$

converges for $0 \leqslant s < 1$. If $0 < p < \infty$ and if the sequence $\{q_n\}$ is monotone, then the two relations:

$$S(s) \underset{s\to 1^-}{\sim} \frac{1}{(1-s)^p} C$$

and

$$q_n \underset{n\to\infty}{\sim} \frac{1}{\Gamma(p)} n^{p-1} C$$

where $0 < C < \infty$, *are equivalent.*

Proof of Lemma 13. By the Markov property,

$$\begin{aligned} &\mathbb{E}\left[f(X_n) \mid A_n = k\right] = \mathbb{E}\left[f(X_n) \mid n - g_n = k\right] \\ &= \mathbb{E}\left[f(X_n) \mid X_{n-k} = 0, X_{n-k+1} \neq 0, \dots, X_n \neq 0\right] = \mathbb{E}\left[f(X_k) \mid \tau > k\right]. \quad \square \end{aligned}$$

Proof of Proposition 3. Let δ_β be a geometric r.v. with parameter β, where $0 < \beta < 1$, and such that δ_β is independent of the walk X. Then

$$\mathbb{P}\left(\Sigma_{\delta_\beta} \leqslant x\right) = \sum_{k=1}^{\infty} \mathbb{P}\left(\delta_\beta = k\right) \, \mathbb{P}\left(\Sigma_k \leqslant x\right) = \sum_{k=1}^{\infty} (1-\beta)^{k-1} \beta \, \mathbb{P}\left(\Sigma_k \leqslant x\right).$$

Now, from (11) and (13),

$$\begin{aligned} \mathbb{P}(\Sigma_{\delta_\beta} \leqslant x) &= \mathbb{P}(\Sigma_{g_{\delta_\beta}} \leqslant x) = \mathbb{P}(A^*_{g_{\delta_\beta}} \leqslant x) = \mathbb{P}(T_x^A \geqslant g_{\delta_\beta}) \\ &= \mathbb{P}(\delta_\beta \leqslant d_{T_x^A}) = 1 - \mathbb{P}(\delta_\beta > d_{T_x^A}) = 1 - \mathbb{E}\left[(1-\beta)^{d_{T_x^A}}\right] \\ &= 1 - \mathbb{E}\left[(1-\beta)^{T_x^A} (1-\beta)^{T_0 \circ \theta_{T_x^A}}\right] \\ &= 1 - \mathbb{E}\left[(1-\beta)^{T_x^A} \, \mathbb{E}_{X_{T_x^A}}\left[(1-\beta)^{T_0}\right]\right]. \quad \square \end{aligned} \tag{14}$$

Definition 2. *A stopping time T is said to be X-standard if T is a.s. finite and if the stopped process $(X_{n\wedge T}, n \geqslant 0)$ is uniformly integrable.*

According to [ALR04], if T is X-standard and if T is independent of X_T, then

$$\forall \alpha \in \mathbb{R} \qquad \mathbb{E}\left[\mathrm{ch}(\alpha)^{-T}\right] = \mathbb{E}\left[\exp(\alpha X_T)\right]^{-1}. \tag{15}$$

Recall that $\mathrm{Arg\,ch}(\alpha) = \ln\left(\alpha + \sqrt{\alpha^2 - 1}\right)$. When $\mathrm{ch}\,\alpha = (1-\beta)^{-1}$,

$$\alpha = \mathrm{Arg\,ch}\left(\frac{1}{1-\beta}\right) = \ln\left(\frac{1}{1-\beta} + \sqrt{\frac{1}{(1-\beta)^2} - 1}\right) = \ln\left(\frac{1+\sqrt{2\beta - \beta^2}}{1-\beta}\right).$$

According to [ALR04], T_k and T_x^A satisfy these properties, hence

$$\mathbb{E}_k\left[(1-\beta)^{T_0}\right] = \mathbb{E}_0\left[(1-\beta)^{T_k}\right] = \left(\frac{1+\sqrt{2\beta-\beta^2}}{1-\beta}\right)^{-k}$$

$$\mathbb{E}\left[(1-\beta)^{T_x^A}\right] = \mathbb{E}\left[\left(\frac{1+\sqrt{2\beta-\beta^2}}{1-\beta}\right)^{X_{T_x^A}}\right]^{-1}$$

So, owing to the independence of T_x^A et $X_{T_x^A}$ and the above formulae,

$$\mathbb{P}\left(\Sigma_{\delta_\beta} \leqslant x\right) = 1 - \mathbb{E}\left[(1-\beta)^{T_x^A}\right]\mathbb{E}\left[\left(\frac{1+\sqrt{2\beta-\beta^2}}{1-\beta}\right)^{-|X_{T_x^A}|}\right]$$

$$= \frac{\frac{1}{2}\left[\mathbb{E}\left[\left(\frac{1+\sqrt{2\beta-\beta^2}}{1-\beta}\right)^{|X_{T_x^A}|}\right] - \mathbb{E}\left[\left(\frac{1+\sqrt{2\beta-\beta^2}}{1-\beta}\right)^{-|X_{T_x^A}|}\right]\right]}{\mathbb{E}\left[\left(\frac{1+\sqrt{2\beta-\beta^2}}{1-\beta}\right)^{X_{T_x^A}}\right]}.$$

For all $k \in \mathbb{N}$,

$$\left[\frac{1+\sqrt{2\beta-\beta^2}}{1-\beta}\right]^k \underset{\beta\to 0^+}{\sim} 1 + k\sqrt{2\beta},$$

and consequently $\mathbb{P}\left(\Sigma_{\delta_\beta} \leqslant x\right) \underset{\beta\to 0^+}{\sim} \mathbb{E}\left[|X_{T_x^A}|\right]\sqrt{2\beta}$.

Thus we have obtained

$$\sum_{k=1}^{\infty}(1-\beta)^k \mathbb{P}\left(\Sigma_k \leqslant x\right) \underset{\beta\to 0^+}{\sim} \sqrt{\frac{2}{\beta}}\,(1-\beta)\,\mathbb{E}\left[|X_{T_x^A}|\right].$$

In order to apply Theorem 4, put $\alpha = 1-\beta$. This gives

$$\sum_{k=1}^{\infty}\alpha^k \mathbb{P}\left(\Sigma_k \leqslant x\right) \underset{\alpha\to 1^-}{\sim} \frac{\sqrt{2}}{\sqrt{1-\alpha}}\mathbb{E}\left[|X_{T_x^A}|\right],$$

and now Theorem 4 with $p = \frac{1}{2}$ and $C = \sqrt{2}\mathbb{E}\left[|X_{T_x^A}|\right]$ gives

$$\mathbb{P}(\Sigma_k \leqslant x) \underset{\alpha\to 1^-}{\sim} \frac{1}{\Gamma\left(\frac{1}{2}\right)}k^{\frac{1}{2}-1}C = \left(\frac{2}{\pi k}\right)^{\frac{1}{2}}\mathbb{E}\left[|X_{T_x^A}|\right].$$

By Lemma 13,

$$\mathbb{E}\left[|X_{T_x^A}|\right] = \mathbb{E}\big[\,|X_{T_x^A}|\;\big|\;A_{T_x^A} = x\,\big] = \mathbb{E}\big[\,|X_x|\;\big|\;\tau > x\,\big] = \theta(x).$$

It is now possible to finalise the proof of point 1.a. Let $\tilde{T}_0$ be the hitting time of 0 by the walk $(X_{n+k})_{k\geqslant 0}$, and Σ' be the maximal length of the excursions of the walk $(X_{k+n+\tilde{T}_0})_{k\geqslant 0}$.

$$\mathbb{E}\left[\mathbb{1}_{\Lambda_n,\Sigma_p\leqslant x}\right] = \mathbb{E}\left[\mathbb{1}_{\Lambda_n,\Sigma_n\leqslant x,T_0\circ\theta_n>p-n}\right] + \mathbb{E}\left[\mathbb{1}_{\Lambda_n,\Sigma_n\leqslant x,T_0\circ\theta_n\leqslant(p-n)\wedge(x-A_n),{\Sigma'}_{p-n-T_0\circ\theta_n}\leqslant x}\right] = (1)+(2)$$

Call $\tilde{\mathbb{P}}$ the measure associated to the walk $(X_{n+k})_{k\geqslant 0}$, X_n and A_n being fixed. Then

$$(1) = \mathbb{E}\left[\mathbb{1}_{\Lambda_n,\Sigma_n\leqslant x}\tilde{\mathbb{P}}_{X_n}\left(\tilde{T}_0 > p-n\right)\right] \underset{p\to\infty}{\sim} \mathbb{E}\left[\mathbb{1}_{\Lambda_n,\Sigma_n\leqslant x}\left(\frac{2}{\pi p}\right)^{\frac{1}{2}}|X_n|\right]$$

Call also $\mathbb{P}'$ the measure associated to the walk $(X_{k+n+\tilde{T}_0})_{k\geqslant 0}$, $\tilde{T}_0$ being fixed. For $p > n+x$, $(p-n)\wedge x - A_n = x - A_n$; consequently

$$\begin{aligned}(2) &\underset{p\to\infty}{\sim} \mathbb{E}\left[\mathbb{1}_{\Lambda_n,\Sigma_n\leqslant x,A_n\leqslant x}\tilde{\mathbb{P}}_{X_n}(\tilde{T}_0\leqslant x-A_n)\mathbb{P}'\left(\Sigma'_{p-n-\tilde{T}_0}\leqslant x\right)\right]\\ &\underset{p\to\infty}{\sim} \mathbb{E}\left[\mathbb{1}_{\Lambda_n,\Sigma_n\leqslant x,A_n\leqslant x}\tilde{\mathbb{P}}_{X_n}(\tilde{T}_0\leqslant x-A_n)\left(\frac{2}{\pi p}\right)^{\frac{1}{2}}\theta(x)\right].\end{aligned}$$

One derives therefrom

$$\lim_{p\to\infty}\frac{\mathbb{E}[\mathbb{1}_{\Lambda_n}\mathbb{1}_{\Sigma_p\geqslant x}]}{\mathbb{E}[\mathbb{1}_{\Sigma_p\geqslant x}]} = \mathbb{E}\left[\mathbb{1}_{\Lambda_n}\left\{\frac{|X_n|}{\theta(x)} + \tilde{\mathbb{P}}_{X_n}(\tilde{T}_0\leqslant x-A_n)\,\mathbb{1}_{A_n\leqslant x}\right\}\mathbb{1}_{\Sigma_n\leqslant x}\right].$$

Remark 5. These notations $\tilde{\mathbb{P}}$ et $\tilde{T}_0$, or similar ones, will frequently occur in the sequel. We have not been completely rigorous when defining them; a rigorous definition is possible as follows: $\tilde{\mathbb{P}}_{\tilde{X}_n}(\tilde{T}_0\leqslant x-A_n)$ stands for $f(X_n, x-A_n)$ where $f(y,z) = \mathbb{P}_y(T_0\leqslant z)$.

We shall now see that $(M_n,\ n\geqslant 0)$ is indeed a martingale. The parity of $n+1$ comes into play, so we shall consider two cases.

Suppose first that $n+1$ is odd. In that case, $\Sigma_{n+1} = \Sigma_n$ and $A_{n+1} = A_n+1$. Recall that $x\to|x|$ is harmonic except at 0 for the symmetric random walk. Hence, on the event $\{X_n\neq 0\}$, the only relevant term is

$$\mathcal{C}_{n+1} := \mathbb{1}_{\{A_{n+1}\leqslant x,\Sigma_n\leqslant x\}}\,\tilde{\mathbb{P}}_{X_{n+1}}(\tilde{T}_0\leqslant x-A_{n+1}),$$

and on $X_n = 0$, it sufices to verify that, when conditioned by $\mathcal{F}_n$, this quantity equals $\left(1-\frac{1}{\theta}\right)\mathbb{1}_{\Sigma_n\leqslant x}$.

By the Markov property, if $X_n\neq 0$,

$$\tilde{\mathbb{P}}_{X_n}(\tilde{T}_0\leqslant x-A_n) = \frac{1}{2}(\tilde{\mathbb{P}}_{X_{n+1}}(\tilde{T}_0\leqslant x-A_n-1) + \tilde{\mathbb{P}}_{X_{n-1}}(\tilde{T}_0\leqslant x-A_n-1)).$$

So

$$\begin{aligned}\mathbb{E}[\mathbb{1}_{X_n\neq 0}\mathcal{C}_{n+1}|\mathcal{F}_n] &= \mathbb{E}[\mathbb{1}_{X_n\neq 0}(\mathbb{1}_{X_{n+1}=X_n+1}+\mathbb{1}_{X_{n+1}=X_n-1})\mathcal{C}_{n+1}|\mathcal{F}_n]\\ = \mathbb{1}_{X_n\neq 0,\Sigma_n\leqslant x,A_n\leqslant x-1}\frac{1}{2}&[\tilde{\mathbb{P}}_{X_n+1}(\tilde{T}_0\leqslant x-A_n-1)+\tilde{\mathbb{P}}_{X_n-1}(\tilde{T}_0\leqslant x-A_n-1)]\\ &= \mathbb{1}_{X_n\neq 0,\Sigma_n\leqslant x,A_n\leqslant x-1}\tilde{\mathbb{P}}_{X_n}(\tilde{T}_0\leqslant x-A_n)\end{aligned}$$

And, as $\mathbb{1}_{X_n\neq 0,\, A_n=x}\tilde{\mathbb{P}}_{X_n}(\tilde{T}_0 \leqslant x - A_n) = 0$, one has

$$\mathbb{E}[\mathbb{1}_{X_n\neq 0}\,\mathcal{C}_{n+1}|\mathcal{F}_n] = \mathbb{1}_{X_n\neq 0}\,\mathcal{C}_n.$$

It remains to show that

$$\mathbb{E}[\mathbb{1}_{X_n=0}\mathcal{C}_{n+1}|\mathcal{F}_n] = \mathbb{1}_{X_n=0,\Sigma_n\leqslant x}\left(1-\frac{1}{\theta}\right). \tag{16}$$

This will use the classical result ([Fel50] pp. 73-77)

$$\mathbb{P}\left(X_1>0,\ldots,X_{2n-1}>0,X_{2n}=2r\right) = \frac{1}{2}\left(p_{2n-1,2r-1}-p_{2n-1,2r+1}\right). \tag{17}$$

where $p_{n,r} = \frac{1}{2^n}C_n^{\frac{n+r}{2}}$.

Using formula (17) with $x = 2n$, one can write

$$\begin{aligned}
\mathbb{P}\left(\tau>x\right)\theta(x) &= \mathbb{P}\left(\tau>x\right)\mathbb{E}\left[|X_x|\mid\tau>x\right] = \mathbb{E}\left[|X_x|\mathbb{1}_{\{\tau>x\}}\right] \\
&= \mathbb{E}\left[X_x\mathbb{1}_{\{\tau>x,X_x>0\}}\right] - \mathbb{E}\left[X_x\mathbb{1}_{\{\tau>x,X_x<0\}}\right] = 2\,\mathbb{E}\left[X_x\mathbb{1}_{\{\tau>x,X_x>0\}}\right] \\
&= 2\sum_{k>0,k\text{ even}}^{x} k\,\mathbb{P}\left(X_x=k,\tau>x\right) = 4\sum_{\ell>0}^{n}\ell\,\mathbb{P}\left(X_{2n}=2\ell,\tau>2n\right) \\
&= 2\sum_{\ell>0}^{n}\ell\left(p_{2n-1,2\ell-1}-p_{2n-1,2\ell+1}\right) = \left(\frac{1}{2}\right)^{2n-2}\sum_{\ell>0}^{n}\ell\left(C_{2n-1}^{n+\ell-1}-C_{2n-1}^{n+\ell}\right).
\end{aligned}$$

Now, $\sum_{\ell=1}^{n}\ell\left(C_{2n-1}^{n+\ell-1}-C_{2n-1}^{n+\ell}\right) = \sum_{\ell=0}^{n-1}C_{2n-1}^{n+\ell} = 2^{2n-2}$; so we obtain

$$\theta(x)\,\mathbb{P}\left(\tau>x\right) = 1. \tag{18}$$

On the other hand,

$$\begin{aligned}
\mathbb{E}[\mathbb{1}_{X_n=0}\mathcal{C}_{n+1}|\mathcal{F}_n] &= \mathbb{1}_{X_n=0,\Sigma_n\leqslant x}\frac{1}{2}\left(\mathbb{P}_{1}\left(T_0\leqslant x-1\right)+\mathbb{P}_{-1}\left(T_0\leqslant x-1\right)\right) \\
&= \mathbb{1}_{X_n=0,\Sigma_n\leqslant x}\mathbb{P}(\tau\leqslant x) = \mathbb{1}_{X_n=0,\Sigma_n\leqslant x}\left(1-\mathbb{P}(\tau>x)\right);
\end{aligned} \tag{19}$$

hence, considering (18) and (19), formula (16) is established.

We now consider the case that $n+1$ is even. In that case, $\{A_n\leqslant x\} = \{A_n\leqslant x-1\}$. Indeed, $A_n = n - g_n$ is odd and x is even by hypothesis, so the event $\{A_n = x\}$ is null. Moreover, if $|X_n|\geqslant 3$, on a $\Sigma_{n+1}=\Sigma_n$.
Last, if $|X_n| = 1$, there are two cases. Either $X_{n+1}\neq 0$ and one always has $\Sigma_{n+1}=\Sigma_n$, or $X_{n+1}=0$ and we must see that in that case

$$\{\Sigma_{n+1}\leqslant x\} = \{\Sigma_n\leqslant x, n+1-g_n\leqslant x\} = \{\Sigma_n\leqslant x, A_n\leqslant x-1\}.$$

So, one is always on the event $\{\Sigma_n\leqslant x, A_n\leqslant x-1\}$, and the same argument as when $n+1$ was odd and $X_n\neq 0$ shows that, conditional on $\mathcal{F}_n$, M_{n+1} is equal to M_n. This shows that M is a martingale; positivity is immediate. The proof that M is not uniformly integrable is postponed until later in this section.

2) We now start studying the process Σ under Q^x. We shall first show that, for all $y \leqslant x$, $Q^x(\Sigma_\infty > y) = 1 - \frac{\mathbb{P}(\tau>x)}{\mathbb{P}(\tau>y)}$.

Put $T_y^\Sigma := \inf\{n \geqslant 0,\ \Sigma_n > y\}$. Clearly, $X_{T_y^\Sigma} = 0$ and hence

$$Q^x(\Sigma_p > y) = Q^x\left(T_y^\Sigma \leqslant p\right) = \mathbb{E}\left[1\!\!1_{T_y^\Sigma \leqslant p} M_{T_y^\Sigma}\right]$$
$$= \mathbb{E}\left[1\!\!1_{T_y^\Sigma \leqslant p}\left\{\frac{|X_{T_y^\Sigma}|}{\theta(x)} + \tilde{\mathbb{P}}_{X_{T_y^\Sigma}}\left(\tilde{T}_0 \leqslant x - A_{T_y^\Sigma}\right) 1\!\!1_{A_{T_y^\Sigma} \leqslant x}\right\} 1\!\!1_{\Sigma_{T_y^\Sigma} \leqslant x}\right]$$
$$= \mathbb{P}\left[T_y^\Sigma \leqslant p, \Sigma_{T_y^\Sigma} \leqslant x\right].$$

Letting p go to infinity, we obtain that $Q^x(\Sigma_\infty > y) = \mathbb{P}(\Sigma_{T_y^\Sigma} \leqslant x)$. For $y \leqslant x$, $\left\{\Sigma_{T_y^A} \leqslant x\right\}$ is a full event; so

$$\left\{\Sigma_{T_y^\Sigma} \leqslant x\right\} = \left\{\Sigma_{T_y^A} \leqslant x\right\} \cap \left\{T_0 \circ \theta_{T_y^A} + y \leqslant x\right\} = \left\{T_0 \circ \theta_{T_y^A} + y \leqslant x\right\}.$$

By the Markov property and Lemma 13,

$$Q^x(\Sigma_\infty > y) = \mathbb{E}\left[\mathbb{E}\left[1\!\!1_{T_0 \circ \theta_{T_y^A} + y \leqslant x} \mid \mathcal{A}_{T_y^A}\right]\right] = \mathbb{E}\left[\tilde{\mathbb{P}}_{X_{T_y^A}}\left(\tilde{T}_0 \leqslant x - y\right)\right]$$
$$= \mathbb{E}\left[\tilde{\mathbb{P}}_{X_y}\left(\tilde{T}_0 \leqslant x - y\right) \mid \tau > y\right] = 1 - \frac{\mathbb{E}\left[\tilde{\mathbb{P}}_{X_y}\left(\tilde{T}_0 > x - y\right) 1\!\!1_{\tau > y}\right]}{\mathbb{P}(\tau > y)}$$
$$= 1 - \frac{\mathbb{E}\left[1\!\!1_{T_0 \circ \theta_y > x - y, \tau > y}\right]}{\mathbb{P}(\tau > y)} = 1 - \frac{\mathbb{P}(\tau > x)}{\mathbb{P}(\tau > y)}.$$

On the other hand, for all $n \geqslant 0$, one has $Q^x(\Sigma_n \leqslant x) = 1$. According to the definition of the probability Q^x,

$$Q^x(\Sigma_n \leqslant x) = \lim_{p\to\infty} \frac{\mathbb{P}(\Sigma_n \leqslant x, \Sigma_p \leqslant x)}{\mathbb{P}(\Sigma_p \leqslant x)} = \lim_{p\to\infty} \frac{\mathbb{P}(\Sigma_p \leqslant x)}{\mathbb{P}(\Sigma_p \leqslant x)} = 1.$$

3) We shall now describe several properties of g and $(A_n, n \geqslant 0)$ under Q^x.

a) We first show that g is Q^x-a.s. finite; this implies that $A_\infty = \infty$ Q^x-a.s.

Lemma 14. *For all $n \geqslant 0$ and $k \geqslant 0$,*

$$\mathbb{P}(A_{2n} = 2k) = \mathbb{P}(A_{2n+1} = 2k+1) = C_{2n-2k}^{n-k} C_{2k}^{k} \left(\frac{1}{2}\right)^n.$$

Proof. According to [Fel50] p. 79, "Arcsin law for last visit",

$$\mathbb{P}(g_{2n} = 2k) = C_{2n-2k}^{n-k}\, C_{2k}^{k} \left(\frac{1}{2}\right)^n.$$

Therefore

$$\mathbb{P}(A_{2n} = 2k) = \mathbb{P}(2n - g_{2n} = 2k) = \mathbb{P}(g_{2n} = 2n - 2k) = C_{2n-2k}^{n-k} C_{2k}^{k} \left(\frac{1}{2}\right)^n;$$

and as $A_{2n+1} = A_{2n} + 1$, the proof is over. □

The next lemma is instrumental in the sequel.

Lemma 15. *For each $p > 0$,*

$$Q^x\left(g > p \mid \mathcal{F}_p\right) = \tilde{\mathbb{P}}_{X_p}(\tilde{\tau} \leqslant x - A_p)\,\frac{1}{M_p}.$$

Proof. Recall that $T_{0,p} := \inf\{n > p, X_n = 0\}$ is the first zero after p, and remark that $\Sigma_{T_{0,p}} = \Sigma_p \vee \{A_p + \tau \circ \theta_p\}$. Recall also that under Q^x, the event $\{\Sigma_p \leqslant x\}$ is almost sure. So, for every $\Lambda_p \in \mathcal{F}_p$,

$$\begin{aligned}
Q^x\big(\{\Lambda_p\} \cap \{g > p\}\big) &= Q^x\big(\{\Lambda_p\} \cap \{T_{0,p} < \infty\}\big) \\
&= \mathbb{E}\big[\mathbb{1}_{\Lambda_p} M_{T_{0,p}}\big] = \mathbb{E}\big[\mathbb{1}_{\Lambda_p}\mathbb{1}_{\Sigma_{T_{0,p}} \leqslant x}\big] = \mathbb{E}\big[\mathbb{1}_{\Lambda_p,\,\Sigma_p \leqslant x}\tilde{\mathbb{P}}_{X_p}\left[\tilde{\tau} \leqslant x - A_p\right]\big] \\
&= \mathbb{E}\left[\mathbb{1}_{\Lambda_p}\frac{\tilde{\mathbb{P}}_{X_p}\left[\tilde{\tau} \leqslant x - A_p\right]}{M_p} M_p\right] = \mathbb{E}^{Q^x}\left[\mathbb{1}_{\Lambda_p}\frac{\tilde{\mathbb{P}}_{X_p}\left[\tilde{\tau} \leqslant x - A_p\right]}{M_p}\right],
\end{aligned}$$

and consequently one has

$$Q^x\left(g > p \mid \mathcal{F}_p\right) = \tilde{\mathbb{P}}_{X_p}(\tilde{\tau} \leqslant x - A_p)\,\frac{1}{M_p}. \qquad \square$$

We now suppose that $p = 2l$ where $l \geqslant 0$; when $p = 2l+1$ the computation is similar, we won't give it (see Lemma 14). According to Lemma 15,

$$\begin{aligned}
Q^x(g > p) &= \mathbb{E}^{Q^x}\left[\mathbb{E}^{Q^x}\left[\mathbb{1}_{g>p} \mid \mathcal{F}_p\right]\right] = \mathbb{E}^{Q^x}\left[\tilde{\mathbb{P}}_{X_p}(\tilde{\tau} \leqslant x - A_p)\,\frac{1}{M_p}\right] \\
&= \mathbb{E}\big[\tilde{\mathbb{P}}_{X_p}\left(\tilde{\tau} \leqslant x - A_p\right)\big] = \sum_{k=0}^{l \wedge \frac{x}{2}} \mathbb{E}\big[\tilde{\mathbb{P}}_{X_p}(\tilde{\tau} \leqslant x - A_p)\,\mathbb{1}_{A_p = 2k}\big] \\
&= \sum_{k=0}^{l \wedge \frac{x}{2}} \mathbb{E}\big[\tilde{\mathbb{P}}_{X_p}(\tilde{\tau} \leqslant x - A_p) \,\big|\, A_p = 2k\big]\,\mathbb{P}(A_p = 2k) \\
&= \sum_{k=0}^{l \wedge \frac{x}{2}} \mathbb{E}\big[\tilde{\mathbb{P}}_{X_{2k}}(\tilde{\tau} \leqslant x - 2k) \,\big|\, \tau > 2k\big]\,\mathbb{P}(A_p = 2k) \\
&= \sum_{k=0}^{l \wedge \frac{x}{2}} \frac{\mathbb{E}\big[\tilde{\mathbb{P}}_{X_{2k}}(\tilde{\tau} \leqslant x - 2k)\,\mathbb{1}_{\tau > 2k}\big]}{\mathbb{P}\left(\tau > 2k\right)}\,\mathbb{P}(A_p = 2k) \\
&= \sum_{k=0}^{l \wedge \frac{x}{2}} \left[1 - \frac{\mathbb{P}(\tau > x)}{\mathbb{P}(\tau > 2k)}\right]\mathbb{P}(A_p = 2k) \\
&= \sum_{k=0}^{l \wedge \frac{x}{2}} C_{2l-2k}^{l-k}\,C_{2k}^{k}\left(\frac{1}{2}\right)^l\left(1 - \frac{\mathbb{P}(\tau > x)}{\mathbb{P}(\tau > 2k)}\right).
\end{aligned}$$

This gives the law of g under Q^x. Then, for $p > 2$, $Q^x(g > p) \leqslant \mathbb{E}[\mathbb{1}_{A_p} \leqslant x]$. Now, A_p tends to infinity $\mathbb{P}$-a.s.; consequently,

$$Q^x(g = \infty) = \lim_{p \to \infty} Q^x(g > p) \leqslant \lim_{p \to \infty} \mathbb{P}(A_p \leqslant x) = 0,$$

and g is Q^x-a.s. finite.

Remark 6. It is now easy to see that M is not uniformly integrable. Indeed, as g is finite, so is also L_∞, and the argument given earlier for M^φ and S immediately adapts to M and L.

b) To establish 2.d.i et 2.d.ii., we shall need:

Lemma 16. *For all $y \leqslant x$, one has*

$$\mathbb{E}\big[M_{T_y^A}\big] = 1$$

Proof of Lemma 16. Recall that the event $\big\{\Sigma_{T_y^A} \leqslant x\big\}$ has probability 1. By formula (18) and the proof of point 2.a,

$$\begin{aligned}\mathbb{E}\big[M_{T_y^A}\big] &= \mathbb{E}\Big[\frac{|X_{T_y^A}|}{\theta(x)} + \tilde{\mathbb{P}}_{X_{T_y^A}}(\tilde{T}_0 \leqslant x - y)\Big] \\ &= \frac{\theta(y)}{\theta(x)} + \mathbb{E}\big[\tilde{\mathbb{P}}_{X_{T_y^A}}(\tilde{T}_0 \leqslant x - y)\big] = \frac{\mathbb{P}\,(\tau > x)}{\mathbb{P}\,(\tau > y)} + 1 - \frac{\mathbb{P}(\tau > x)}{\mathbb{P}(\tau > y)}.\end{aligned}$$

Let F be a positive functional and $G : \mathbb{R} \to \mathbb{R}^+$. Recall that after [ALR04], $X_{T_y^A}$ and $\mathcal{A}_{T_y^A}$ are independent under $\mathbb{P}$. On the other hand, as $M_{T_y^A}$ is a function of $X_{T_y^A}$, one has

$$\begin{aligned}\mathbb{E}^{Q^x}\big[F\big(A_n,\, n \leqslant T_y^A\big)\, G\big(X_{T_y^A}\big)\big] &= \mathbb{E}\big[F\big(A_n,\, n \leqslant T_y^A\big)\, G\big(X_{T_y^A}\big) M_{T_y^A}\big] \\ &= \mathbb{E}\big[F\big(A_n,\, n \leqslant T_y^A\big)\big]\, \mathbb{E}\big[G\big(X_{T_y^A}\big) M_{T_y^A}\big]. \quad (20)\end{aligned}$$

So, taking $G \equiv 1$ and using Lemma 16, one has

$$\mathbb{E}^{Q^x}\big[F\big(A_n,\, n \leqslant T_y^A\big)\big] = \mathbb{E}\big[F\big(A_n,\, n \leqslant T_y^A\big)\big],$$

which shows that $(A_n,\, n \leqslant T_y^A)$ has the same law under $\mathbb{P}$ and Q^x. Using again formula (20), one obtains

$$\mathbb{E}^{Q^x}\big[F\big(A_n,\, n \leqslant T_y^A\big)\, G\big(X_{T_y^A}\big)\big] = \mathbb{E}^{Q^x}\big[F\big(A_n,\, n \leqslant T_y^A\big)\big]\, \mathbb{E}^{Q^x}\big[G\big(X_{T_y^A}\big)\big];$$

this shows that $(A_n,\, n \leqslant T_y^A)$ and $X_{T_y^A}$ are independent under Q^x.

c) The rest of the proof of point 2 is quite easy, taking into account what has already been done:

$$\begin{aligned}
\mathbb{E}^{Q^x}\left[G\big(X_{T_y^A}\big)\right] &= \mathbb{E}\big[G\big(X_{T_y^A}\big)M_{T_y^A}\big] = \mathbb{E}\big[\mathbb{E}\big[G\big(X_{T_y^A}\big)M_{T_y^A} \mid \mathcal{A}_{T_y^A}\big]\big] \\
&= \mathbb{E}\Big[\mathbb{E}\Big[G(X_y)\left\{\frac{|X_y|}{\theta(x)} + \tilde{\mathbb{P}}_{X_y}(\tilde{T}_0 \leqslant x-y)\right\} \Big| \, \tau > y\Big]\Big] \\
&= \mathbb{E}\left[G\left(X_y\right)\left\{\frac{|X_y|}{\theta(x)} + \tilde{\mathbb{P}}_{X_y}\left(\tilde{T}_0 \leqslant x-y\right)\right\} \mid \tau > y\right] \\
&= \mathbb{E}\Big[G(X_y)\left\{\frac{|X_y|}{\theta(x)} + \tilde{\mathbb{P}}_{X_y}(\tilde{T}_0 \leqslant x-y)\right\} \Big| \, \tau > y\Big] \\
&= \sum_k G(k)\Big\{\frac{|k|}{\theta(x)} + \mathbb{P}_k\left(T_0 \leqslant x-y\right)\Big\}\, \mathbb{P}(X_y = k \mid \tau > y).
\end{aligned}$$

Consequently, the law of $X_{T_y^A}$ under Q^x satisfies

$$Q^x\big(X_{T_y^A} = k\big) = \left\{\frac{|k|}{\theta(x)} + \mathbb{P}_k\left(T_0 \leqslant x-y\right)\right\} \mathbb{P}(X_y = k \mid \tau > y).$$

(The quantity $\mathbb{P}(X_y = k \mid \tau > y)$ is explicitly given in [Fel50] p. 77).

We now compute $Q^x(g > T_y^A)$:

$$Q^x(g > T_y^A) = \mathbb{E}^{Q^x}\left[\mathbb{E}^{Q^x}\left[1_{g > T_y^A} \;\middle|\; \mathcal{F}_{T_y^A}\right]\right] = \mathbb{E}^{Q^x}\left[\frac{\tilde{\mathbb{P}}_{X_{T_y^A}}(\tilde{\tau} \leqslant x-y)}{M_{T_y^A}}\right]$$

$$= \mathbb{E}\big[\tilde{\mathbb{P}}_{X_{T_y^A}}(\tilde{\tau} \leqslant x-y)\big] = \mathbb{E}\big[\tilde{\mathbb{P}}_{X_y}(\tilde{\tau} \leqslant x-y) \bigm| \tau > y\big] = 1 - \frac{\mathbb{P}\left(\tau > x\right)}{\mathbb{P}\left(\tau > y\right)}.$$

Last, we now show that $\left(A_n,\, n \leqslant T_y^A\right)$ and $\left\{g > T_y^A\right\}$ are independent under Q^x; we use again the independence of $X_{T_y^A}$ and $A_{T_y^A}$ under $\mathbb{P}$.

$$\begin{aligned}
\mathbb{E}^{Q^x}\left[F(A_n,\, n \leqslant T_y^A)1_{g > T_y^A}\right] &= \mathbb{E}^{Q^x}\left[F(A_n,\, n \leqslant T_y^A)\,\mathbb{E}^{Q^x}\left[1_{g > T_y^A} \;\middle|\; \mathcal{A}_{T_y^A}\right]\right] \\
&= \mathbb{E}^{Q^x}\left[\frac{F(A_n,\, n \leqslant T_y^A)\,\tilde{\mathbb{P}}_{X_{T_y^A}}(\tilde{\tau} \leqslant x-y)}{M_{T_y^A}}\right] \\
&= \mathbb{E}\left[F\left(A_n,\, n \leqslant T_y^A\right)\right]\, \mathbb{E}\big[\tilde{\mathbb{P}}_{X_{T_y^A}}\left(\tilde{\tau} \leqslant x-y\right)\big] \\
&= \mathbb{E}^{Q^x}\left[F(A_n,\, n \leqslant T_y^A)\right] Q^x(g > T_y^A).
\end{aligned}$$

4) To study the process $(X_n,\ n \geqslant 0)$ under Q^x, we start with the law of the process $(X_n,\ n \geqslant g)$. Recall that $\Gamma^+ = \{X_n > 0,\ n > g\}$ and $\Gamma^- = \{X_n < 0,\ n > g\}$; these events Γ^+ and Γ^- are symmetric under Q_0^x:

Lemma 17.

$$Q^x\left(\Gamma^+\right) = Q^x\left(\Gamma^-\right) = \frac{1}{2}.$$

Proof. First remark that

$$Q^x(\Gamma^+) = \lim_{n\to\infty} Q^x(X_n > 0), \qquad Q^x(\Gamma^-) = \lim_{n\to\infty} Q^x(X_n < 0).$$

By definition of Q^x,

$$Q^x(X_n > 0) = \mathbb{E}\Big[\mathbb{1}_{X_n>0}\Big\{\frac{|X_n|}{\theta(x)} + \tilde{\mathbb{P}}_{X_n}\big(\tilde{T}_0 \leqslant x - A_n\big)\,\mathbb{1}_{A_n\leqslant x}\Big\}\mathbb{1}_{\Sigma_n\leqslant x}\Big].$$

Owing to the symmetry of the walk under $\mathbb{P}$, one has

$$\begin{aligned} Q^x(X_n > 0) &= \mathbb{E}\Big[\mathbb{1}_{X_n<0}\Big\{\frac{|X_n|}{\theta(x)} + \tilde{\mathbb{P}}_{X_n}\big(\tilde{T}_0 \leqslant x - A_n\big)\,\mathbb{1}_{A_n\leqslant x}\Big\}\mathbb{1}_{\Sigma_n\leqslant x}\Big] \\ &= Q^x(X_n < 0). \end{aligned}$$

One also has $\lim_{n\to\infty} Q^x(X_n = 0) = 0$ because g is Q^x-a.s. finite; and as

$$Q^x(X_n > 0) + Q^x(X_n < 0) + Q^x(X_n = 0) = 2Q^x(X_n > 0) + Q^x(X_n = 0) = 1,$$

taking limits when n tends to infinity, on obtains

$$Q^x(\Gamma^+) + Q^x(\Gamma^-) = 2Q^x(\Gamma^+) = 1. \qquad \square$$

We now describe the behavior of $(X_{n+g},\ n > 0)$ under Q^x on Γ^+ (the other case is completely similar). Take $a \in \mathbb{N}^*$ and $p \geqslant x$, and set $q_{a,a+1} := Q(X_{n+1} = a+1 | X_n = a,\, n > g)$.

$$\begin{aligned} q_{a,a+1} &= Q(X_{n+1} = a+1 | X_n = a,\, \forall i \leqslant p\ X_{n+i} > 0) \\ &= \frac{Q(X_{n+1} = a+1,\, X_n = a,\, \forall i \leqslant p\ X_{n+i} > 0)}{Q(X_n = a,\, \forall i \leqslant p\ X_{n+i} > 0)} \\ &= \frac{\mathbb{E}\left[\mathbb{1}_{X_{n+1}=a+1,\, X_n=a,\, \forall i\leqslant p\ X_{n+i}>0}\, M_{p+n}\right]}{\mathbb{E}\left[\mathbb{1}_{X_n=a,\, \forall i\leqslant p\ X_{n+i}>0}\, M_{p+n}\right]}. \end{aligned}$$

Here $M_{p+n} = \frac{X_{p+n}}{\Theta(x)}\,\mathbb{1}_{\Sigma_n\leqslant x}$; hence we can condition the numerator (resp. the denominator) by $\mathcal{F}_{n+1}$ (resp. $\mathcal{F}_n$). The Markov property gives

$$q_{a,a+1} = \frac{\mathbb{E}\left[\mathbb{1}_{X_{n+1}=a+1,\, X_n=a,\, \Sigma_n\geqslant x}\mathbb{E}_{a+1}\left[X_p\mathbb{1}_{X_i>0,\forall i\leqslant p-1}\right]\right]}{\mathbb{E}\left[\mathbb{1}_{X_n=a,\, \Sigma_n\geqslant x}\mathbb{E}_a\left[X_p\mathbb{1}_{X_i>0,\forall i\leqslant p}\right]\right]}.$$

Clearly, $(X_p\mathbb{1}_{X_i>0,\forall i\leqslant p})_{p\geqslant 0}$ is a martingale, wherefrom

$$q_{a,a+1} = \frac{(a+1)\mathbb{E}\left[\mathbb{1}_{X_{n+1}=a+1,\, X_n=a,\, \Sigma_n\geqslant x}\right]}{a\mathbb{E}\left[\mathbb{1}_{X_n=a,\, \Sigma_n\geqslant x}\right]}.$$

Last, conditioning the numerator by $\mathcal{F}_n$ one gets

$$q_{a,a+1} = \frac{a+1}{2a},$$

the transition probability of a 3-Bessel* walk.

Recall the following notation:

$$\gamma_n := |\{k \leqslant n, X_k = 0\}|\ ,\ \gamma_\infty := \lim_{n\to\infty} \gamma_n$$
$$\tau_1 := T_0\ ,\ \forall n \geqslant 2,\ \tau_n := \inf\{k > \tau_{n-1},\ X_k = 0\}$$

It remains to show that, conditional on $\{\gamma_\infty = l\}$, $(X_u,\ u \leqslant g)$ is a standard random walk stopped at τ_l and conditioned by $\Sigma_{\tau_l} \leqslant x$.

Let F be a functional on $\mathbb{Z}^n$.

$$\begin{aligned}
\mathbb{E}^{Q^x}\left[F(X_1,\ldots,X_n)\,\mathbb{1}_{n\leqslant\tau_l} \,\middle|\, \gamma_\infty = l\right] &= \frac{\mathbb{E}^{Q^x}\left[F(X_1,\ldots,X_n)\,\mathbb{1}_{n\leqslant\tau_l}\mathbb{1}_{\gamma_\infty=l}\right]}{\mathbb{E}^{Q^x}[\gamma_\infty = l]}\\
&= \frac{\mathbb{E}^{Q^x}\left[F(X_1,\ldots,X_n)\,\mathbb{1}_{n\leqslant\tau_l<\infty}\right] - \mathbb{E}^{Q^x}\left[F(X_1,\ldots,X_n)\mathbb{1}_{n\leqslant\tau_l<\tau_{l+1}<\infty}\right]}{\mathbb{E}^{Q^x}\left[\mathbb{1}_{\tau_l<\infty}\mathbb{1}_{\tau_{l+1}=\infty}\right]}\\
&= \frac{\mathbb{E}^{Q^x}\left[F(X_1,\ldots,X_n)\,\mathbb{1}_{n\leqslant\tau_l<\infty}\right] - \mathbb{E}^{Q^x}\left[F(X_1,\ldots,X_n)\mathbb{1}_{n\leqslant\tau_l<\tau_{l+1}<\infty}\right]}{\mathbb{E}^{Q^x}\left[\mathbb{1}_{\tau_l<\infty}\right] - \mathbb{E}^{Q^x}\left[\mathbb{1}_{\tau_{l+1}<\infty}\right]}\\
&= \frac{\mathbb{E}\left[F(X_1,\ldots,X_n)\,\mathbb{1}_{n\leqslant\tau_l<\infty}\,M_{\tau_l}\right] - \mathbb{E}\left[F(X_1,\ldots,X_n)\mathbb{1}_{n<\tau_{l+1}<\infty}\,M_{\tau_{l+1}}\right]}{\mathbb{E}\left[\mathbb{1}_{\tau_l<\infty}\,M_{\tau_l}\right] - \mathbb{E}\left[\mathbb{1}_{\tau_{l+1}<\infty}\,M_{\tau_{l+1}}\right]}.
\end{aligned}$$

Under $\mathbb{P}$, $\{\tau_l < \infty\}$ has probability 1, and so

$$M_{\tau_l} - M_{\tau_{l+1}} = \mathbb{1}_{\Sigma_{\tau_l}\leqslant x}(1 - \mathbb{1}_{\tau_{l+1}-\tau_l\leqslant x}) = \mathbb{1}_{\Sigma_{\tau_l}\leqslant x,\,\tau_{l+1}-\tau_l>x}.$$

As $\tau_{l+1} - \tau_l$ is independent of $\mathcal{F}_{\tau_l}$, one gets

$$\begin{aligned}
\mathbb{E}^{Q^x}\left[F(X_1,\ldots,X_n)\,\mathbb{1}_{n\leqslant\tau_l} \mid \gamma_\infty = l\right] &= \frac{\mathbb{E}\left[F(X_1,\ldots,X_n)\,\mathbb{1}_{n\leqslant\tau_l}\left(M_{\tau_l} - M_{\tau_{l+1}}\right)\right]}{\mathbb{E}\left[M_{\tau_l} - M_{\tau_{l+1}}\right]}\\
&= \frac{\mathbb{E}\left[F(X_1,\ldots,X_n)\,\mathbb{1}_{\{n\leqslant\tau_l,\Sigma_{\tau_l}\leqslant x,\tau_{l+1}-\tau_l>x\}}\right]}{\mathbb{E}\left[\mathbb{1}_{\{\Sigma_{\tau_l}\leqslant x,\tau_{l+1}-\tau_l>x\}}\right]}\\
&= \frac{\mathbb{E}\left[F(X_1,\ldots,X_n)\,\mathbb{1}_{n\leqslant\tau_l,\Sigma_{\tau_l}\leqslant x}\right]\mathbb{E}\left[\mathbb{1}_{\{\tau_{l+1}-\tau_l>x\}}\right]}{\mathbb{E}\left[\mathbb{1}_{\Sigma_{\tau_l}\leqslant x}\right]\mathbb{E}\left[\mathbb{1}_{\tau_{l+1}-\tau_l>x}\right]}\\
&= \frac{\mathbb{E}\left[F(X_1,\ldots,X_n)\,\mathbb{1}_{\{n\leqslant\tau_l,\Sigma_{\tau_l}\leqslant x\}}\right]}{\mathbb{E}\left[\mathbb{1}_{\Sigma_{\tau_l}\leqslant x}\right]} = \mathbb{E}\left[F(X_1,\ldots,X_n)\,\mathbb{1}_{n\leqslant\tau_l} \mid \Sigma_{\tau_l} \leqslant x\right].
\end{aligned}$$

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