8

Proof of Theorem 1.3, Part (i)

This Chapter 8 and the next Chapter 9 are devoted to the proof of Theorem 1.3 and Theorem 1.4. In this chapter we prove part (i) of Theorem 1.3. In the proof we make use of Sobolev's imbedding theorems (Theorems 8.1 and 8.2) and a λ-*dependent localization* argument due to Masuda [Ma] (cf. Lemma 8.4) in order to adjust estimate

$$\left\| (A_p - \lambda I)^{-1} \right\| \leq \frac{c_p(\varepsilon)}{|\lambda|} \quad \text{for all } \lambda \in \Sigma_p(\varepsilon) \tag{1.4}$$

to obtain the desired estimate

$$\|(\mathfrak{A} - \lambda I)^{-1}\| \leq \frac{c(\varepsilon)}{|\lambda|} \quad \text{for all } \lambda \in \Sigma(\varepsilon). \tag{1.6}$$

Here we recall that

$$\mathcal{D}(A_p) = \left\{ u \in H^{2,p}(D) = W^{2,p}(D) : Lu = \mu(x') \frac{\partial u}{\partial \mathbf{n}} + \gamma(x')u = 0 \right\}. \tag{1.3}$$

$$\mathcal{D}(\mathfrak{A}) = \left\{ u \in C_0(\overline{D} \setminus M) : Au \in C_0(\overline{D} \setminus M),\ Lu = 0 \right\}. \tag{1.5}$$

8.1 The Space $C_0(\overline{D} \setminus M)$

First, we consider a one-point compactification $K_\partial = K \cup \{\partial\}$ of the space $K = \overline{D} \setminus M$.

We say that two points x and y of $\overline{D}$ are equivalent modulo M if $x = y$ or $x, y \in M$; we then write $x \sim y$. It is easy to verify that this relation $\sim$ enjoys the so-called equivalence laws. We denote by $\overline{D}/M$ the totality of equivalence classes modulo M. On the set $\overline{D}/M$ we define the quotient topology induced by the projection

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$$q : \overline{D} \longrightarrow \overline{D}/M.$$

Namely, a subset O of $\overline{D}/M$ is defined to be open if and only if the inverse image $q^{-1}(O)$ of O is open in $\overline{D}$. It is easy to see that the topological space $\overline{D}/M$ is a *one-point compactification* of the space $\overline{D} \setminus M$ and that the *point at infinity* ∂ corresponds to the set M (see Figure 8.1):

$$\begin{cases} K_\partial := \overline{D}/M, \\ \partial := M. \end{cases}$$

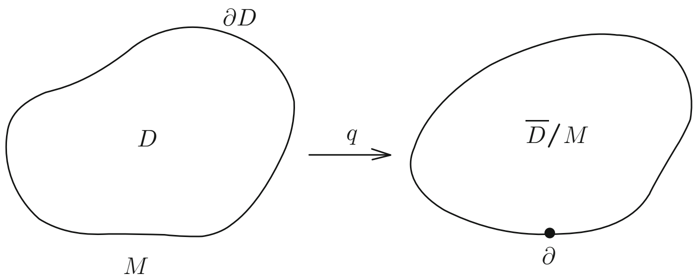

Fig. 8.1.

Furthermore, we obtain the following two assertions:

(i) If $\tilde{u}$ is a continuous function defined on K_∂, then the function $\tilde{u} \circ q$ is continuous on $\overline{D}$ and constant on M.
(ii) Conversely, if u is a continuous function defined on $\overline{D}$ and constant on M, then it defines a continuous function $\tilde{u}$ on K_∂.

In other words, we have the following isomorphism:

$$C(K_\partial) \cong \left\{ u \in C(\overline{D}) : u(x) \text{ is constant on } M \right\}. \tag{8.1}$$

Now we introduce a closed subspace of $C(K_\partial)$ as in Subsection 2.2.1:

$$C_0(K) = \{ u \in C(K_\partial) : u(\partial) = 0 \}.$$

Then we have, by assertion (8.1),

$$C_0(K) \cong C_0(\overline{D} \setminus M) = \left\{ u \in C(\overline{D}) : u(x) = 0 \text{ on } M \right\}. \tag{8.2}$$

8.2 Sobolev's Imbedding Theorems

It is the imbedding characteristics of Sobolev spaces of L^p type that render these spaces so useful in the study of partial differential equations. We need the following imbedding properties of Sobolev spaces:

Theorem 8.1 (Sobolev). *Let D be a bounded domain in the Euclidean space $\mathbf{R}^N$ with boundary ∂D of class C^2. Then we have the following two assertions:*

(i) If $1 \leq p < N$, we have the continuous injection

$$W^{2,p}(D) \subset W^{1,q}(D) \quad \textit{for} \ \ \tfrac{1}{p} - \tfrac{1}{N} \leq \tfrac{1}{q} \leq \tfrac{1}{p}.$$

(ii) If $N/2 < p < \infty$, $p \neq N$, we have the continuous injection

$$W^{2,p}(D) \subset C^{\nu}(\overline{D}) \quad \textit{for} \ \ 0 < \nu \leq 2 - \tfrac{N}{p}.$$

Theorem 8.2 (Gagliardo–Nirenberg). *Let D be a bounded domain in $\mathbf{R}^N$ with boundary of class C^2, and $1 \leq p, r \leq \infty$. Then we have the following assertions:*

(i) If $p \neq N$ and if

$$\frac{1}{q} = \frac{1}{N} + \theta\left(\frac{1}{p} - \frac{2}{N}\right) + (1-\theta)\frac{1}{r} \quad \textit{for} \ \ \tfrac{1}{2} \leq \theta \leq 1,$$

then we have, for all $u \in W^{2,p}(D) \cap L^r(D)$,

$$\|u\|_{1,q} \leq C_1 \|u\|_{2,p}^{\theta} \|u\|_r^{1-\theta},$$

with a positive constant $C_1 = C_1(D, p, r, \theta)$.

(ii) If $N/2 < p < \infty$, $p \neq N$ and if

$$0 \leq \nu < \theta\left(2 - \frac{N}{p}\right) - (1-\theta)\frac{N}{r},$$

then we have, for all $u \in W^{2,p}(D) \cap L^r(D)$,

$$\|u\|_{C^{\nu}(\overline{D})} \leq C_2 \|u\|_{2,p}^{\theta} \|u\|_r^{1-\theta}, \tag{8.3}$$

with a positive constant $C_2 = C_2(D, p, r, \theta)$.

For a proof of Theorem 8.1, see Adams–Fournier [AF, Theorem 5.4] and for a proof of Theorem 8.2, see Friedman [Fr1, Part I, Theorem 10.1], and also Taira [Ta4].

8.3 Proof of Part (i) of Theorem 1.3

The proof is carried out in a chain of auxiliary lemmas.

Step (I): We begin with a version of estimate (7.1):

Lemma 8.3. *Let $N < p < \infty$. Assume that the following conditions (A) and (B) are satisfied:*

(A) $\mu(x') \geq 0$ *on* ∂D.
(B) $\gamma(x') < 0$ *on* $M = \{x' \in \partial D : \mu(x') = 0\}$.

Then, for every $\varepsilon > 0$, *there exists a positive constant* $r_p(\varepsilon)$ *such that if* $\lambda = r^2 e^{i\theta}$ *with* $r \geq r_p(\varepsilon)$ *and* $-\pi + \varepsilon \leq \theta \leq \pi - \varepsilon$, *we have, for all* $u \in \mathcal{D}(A_p)$,

$$|\lambda|^{1/2}|u|_{C^1(\overline{D})} + |\lambda| \cdot |u|_{C(\overline{D})} \leq C_p(\varepsilon)|\lambda|^{N/2p}\|(A-\lambda)u\|_p, \tag{8.4}$$

with a positive constant $C_p(\varepsilon)$. *Here*

$$\mathcal{D}(A_p) = \left\{ u \in H^{2,p}(D) = W^{2,p}(D) : Lu = \mu(x')\frac{\partial u}{\partial \mathbf{n}} + \gamma(x')u = 0 \right\}.$$

Proof. First, by applying Theorem 8.2 with $p := r > N$, $\theta := N/p$ and $\nu := 0$ we obtain from the Gagliardo–Nirenberg inequality (8.3) that

$$|u|_{C(\overline{D})} \leq C|u|_{1,p}^{N/p}\|u\|_p^{1-N/p}. \tag{8.5}$$

Here and in the following the letter C denotes a generic positive constant depending on p and ε, but independent of u and λ.

Combining inequality (7.1) with inequality (8.5), we find that

$$\begin{aligned} |u|_{C(\overline{D})} &\leq C\left(|\lambda|^{-1/2}\|(A-\lambda)u\|_p\right)^{N/p}\left(|\lambda|^{-1}\|(A-\lambda)u\|_p\right)^{1-N/p} \\ &= C|\lambda|^{-1+N/2p}\|(A-\lambda)u\|_p, \end{aligned}$$

so that

$$|\lambda| \cdot |u|_{C(\overline{D})} \leq C|\lambda|^{N/2p}\|(A-\lambda)u\|_p \quad \text{for all } u \in \mathcal{D}(A_p). \tag{8.6}$$

Similarly, by applying inequality (8.5) to the functions $D_i u \in W^{1,p}(D)$, $1 \leq i \leq n$, we obtain that

$$\begin{aligned} |D_i u|_{C(\overline{D})} &\leq C|D_i u|_{1,p}^{N/p}\|D_i u\|_p^{1-N/p} \\ &\leq C|u|_{2,p}^{N/p}|u|_{1,p}^{1-N/p} \\ &\leq C\left(\|(A-\lambda)u\|_p\right)^{N/p}\left(|\lambda|^{-1/2}\|(A-\lambda)u\|_p\right)^{1-N/p} \\ &= C|\lambda|^{-1/2+N/2p}\|(A-\lambda)u\|_p. \end{aligned}$$

This proves that

$$|\lambda|^{1/2}|u|_{C^1(\overline{D})} \leq C|\lambda|^{N/2p}\|(A-\lambda)u\|_p \quad \text{for all } u \in \mathcal{D}(A_p). \tag{8.7}$$

Therefore, the desired inequality (8.4) follows by combining inequalities (8.6) and (8.7). □

The next lemma proves estimate (1.6):

Lemma 8.4. *Assume that conditions (A) and (B) are satisfied. Then, for every* $\varepsilon > 0$*, there exists a positive constant* $r(\varepsilon)$ *such that if* $\lambda = r^2 e^{i\theta}$ *with* $r \geq r(\varepsilon)$ *and* $-\pi + \varepsilon \leq \theta \leq \pi - \varepsilon$*, we have, for all* $u \in \mathcal{D}(\mathfrak{A})$*,*

$$|\lambda|^{1/2} |u|_{C^1(\overline{D})} + |\lambda| \cdot |u|_{C(\overline{D})} \leq c(\varepsilon) |(A - \lambda)u|_{C(\overline{D})}, \tag{8.8}$$

with a positive constant $c(\varepsilon)$*. Here*

$$\mathcal{D}(\mathfrak{A}) = \left\{ u \in C_0(\overline{D} \setminus M) : Au \in C_0(\overline{D} \setminus M),\ Lu = 0 \right\}.$$

Proof. We shall make use of a λ-*dependent localization* argument due to Masuda [Ma] in order to adjust the term $\|(A - \lambda)u\|_p$ in inequality (8.4) to obtain inequality (8.8).

First, we remark that

$$\mathfrak{A} \subset A_p \quad \text{for all} \quad 1 < p < \infty.$$

Indeed, since we have, for any $u \in \mathcal{D}(\mathfrak{A})$,

$$u \in C(\overline{D}) \subset L^p(D),\ Au \in C(\overline{D}) \subset L^p(D) \quad \text{and} \quad Lu = 0,$$

it follows from an application of Theorem 4.9 and Lemma 5.1 that

$$u \in W^{2,p}(D).$$

(1) Let x_0 be an arbitrary point of the closure $\overline{D} = D \cup \partial D$.

If x_0' is a *boundary* point and if χ is a smooth coordinate transformation such that χ maps $B(x_0, \eta_0) \cap D$ into $B(0, \delta) \cap \mathbf{R}^N_+$ and flattens a part of the boundary ∂D into the plane $x_N = 0$ (see Figure 8.2), then we let

$$\begin{aligned} G_0 &= B(x_0', \eta_0) \cap D, \\ G' &= B(x_0', \eta) \cap D,\ 0 < \eta < \eta_0, \\ G'' &= B(x_0', \eta/2) \cap D,\ 0 < \eta < \eta_0. \end{aligned}$$

Here $B(x, \eta)$ denotes the open ball of radius η about x.

Similarly, if x_0 is an *interior* point and if χ is a smooth coordinate transformation such that χ maps $B(x_0, \eta_0)$ into $B(0, \delta)$, then we let (see Figure 8.3)

$$\begin{aligned} G_0 &= B(x_0, \eta_0), \\ G' &= B(x_0, \eta),\ 0 < \eta < \eta_0, \\ G'' &= B(x_0, \eta/2),\ 0 < \eta < \eta_0. \end{aligned}$$

(2) Now we take a function $\theta(t)$ in $C_0^\infty(\mathbf{R})$ such that $\theta(t)$ equals one near the origin, and define

$$\varphi(x) = \theta(|x'|^2)\, \theta(x_N), \quad x = (x', x_N).$$

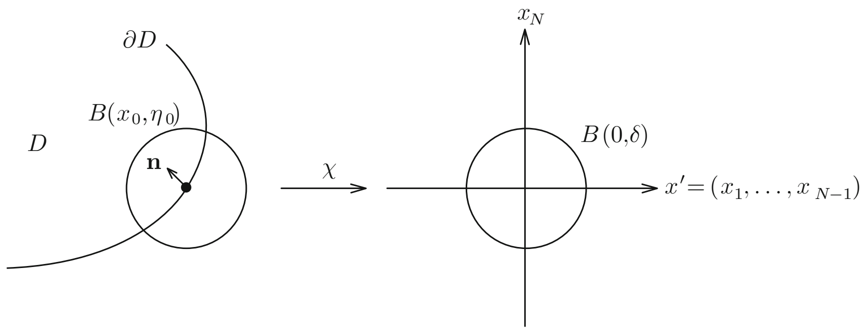

Fig. 8.2.

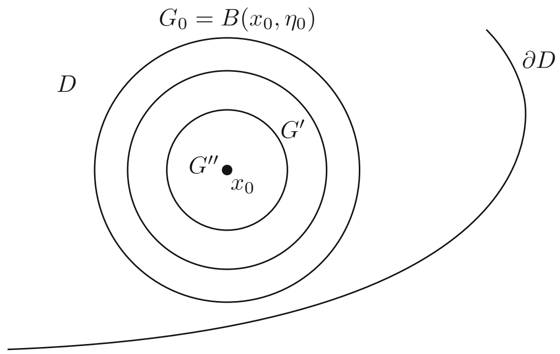

Fig. 8.3.

Here we may assume that the function $\varphi(x)$ is chosen so that

$$\begin{cases} \operatorname{supp} \varphi \subset B(0,1), \\ \varphi(x) = 1 \text{ on } B(0,1/2). \end{cases}$$

We introduce a localizing function

$$\varphi_0(x,\eta) \equiv \varphi\left(\frac{x - x_0}{\eta}\right) = \theta\left(\frac{|x' - x_0'|^2}{\eta^2}\right)\theta\left(\frac{x_N - t}{\eta}\right), \quad x_0 = (x_0', t) \in \overline{D}.$$

We remark that

$$\begin{cases} \operatorname{supp} \varphi_0 \subset B(x_0,\eta), \\ \varphi_0(x,\eta) = 1 \text{ on } B(x_0,\eta/2). \end{cases}$$

Then we have the following:

Claim 8.5. *If $u \in \mathcal{D}(\mathfrak{A})$, then it follows that $\varphi_0(x,\eta)u \in \mathcal{D}(A_p)$ for all $1 < p < \infty$.*

Proof. (i) First, we recall that

$$u \in W^{2,p}(D) \quad \text{for all} \quad 1 < p < \infty.$$

Hence we have the assertion

$$\varphi_0(x, \eta)u \in W^{2,p}(D).$$

(ii) Secondly, it is easy to verify (see Figure 8.4) that the function $\varphi_0(x, \eta)u$, $x \in \overline{D}$, satisfies the boundary condition

$$L(\varphi_0(x, \eta)u) = 0 \quad \text{on } \partial D.$$

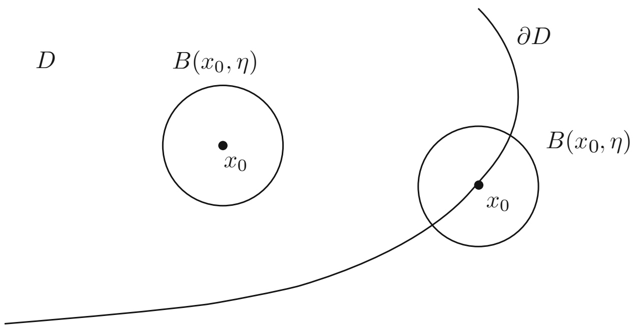

Fig. 8.4.

Indeed, this is obvious if we have the condition

$$\operatorname{supp}(\varphi_0(x, \eta)u) \subset B(x_0, \eta), \quad x_0 \in D.$$

Moreover, if we have the condition

$$\operatorname{supp}(\varphi_0(x, \eta)u) \subset B(x_0, \eta) \cap \overline{D}, \quad x_0 \in \partial D,$$

then it follows that

$$\frac{\partial}{\partial x_N}(\varphi_0(x, \eta))\bigg|_{x_N=0} = \frac{1}{\eta}\,\theta'(0) \cdot \theta\left(\frac{|x' - x_0'|^2}{\eta^2}\right) = 0,$$

since $\theta'(0) = 0$. This proves that

$$\frac{\partial}{\partial \mathbf{n}}(\varphi_0(x, \eta)) = 0 \quad \text{on } \partial D.$$

Therefore, we have the assertion

$$\begin{aligned} L(\varphi_0(x,\eta)u) &= \mu(x')\frac{\partial}{\partial \mathbf{n}}(\varphi_0(x,\eta)u) + \gamma(x')\varphi_0(x,\eta)u \\ &= \varphi_0(x,\eta)(Lu) + \mu(x')\left(\frac{\partial}{\partial \mathbf{n}}(\varphi_0(x,\eta))\right)u \\ &= 0 \quad \text{on } \partial D, \end{aligned}$$

since $Lu = 0$ on ∂D.

Summing up, we have proved that

$$\varphi_0(x,\eta)u \in \mathcal{D}(A_p) \quad \text{for all } 1 < p < \infty.$$

The proof of Claim 8.5 is complete. $\square$

(3) Now we take a positive number p such that

$$N < p < \infty.$$

Then, by Claim 8.5 we can apply inequality (8.4) to the function $\varphi_0(x,\eta)u$, $u \in \mathcal{D}(\mathfrak{A})$, to obtain that

$$\begin{aligned} &|\lambda|^{1/2}|u|_{C^1(\overline{G''})} + |\lambda|\cdot|u|_{C(\overline{G''})} \\ &\le |\lambda|^{1/2}\left|\varphi_0(x,\eta)u\right|_{C^1(\overline{G'})} + |\lambda|\cdot|\varphi_0(x,\eta)u|_{C(\overline{G'})} \\ &= |\lambda|^{1/2}\left|\varphi_0(x,\eta)u\right|_{C^1(\overline{D})} + |\lambda|\cdot|\varphi_0(x,\eta)u|_{C(\overline{D})} \\ &\le C|\lambda|^{N/2p}\left\|(A-\lambda)(\varphi_0(x,\eta)u)\right\|_{L^p(D)} \\ &= C|\lambda|^{N/2p}\left\|(A-\lambda)(\varphi_0(x,\eta)u)\right\|_{L^p(G')}, \quad 0 < \eta < \eta_0, \end{aligned} \tag{8.9}$$

since we have the assertions

$$\begin{cases} \varphi_0(x,\eta) = 1 \quad \text{on } G'', \\ \operatorname{supp}(\varphi_0(x,\eta)u) \subset \overline{G'}. \end{cases}$$

However, we have the formula

$$(A-\lambda)(\varphi_0(x,\eta)u) = \varphi_0(x,\eta)\left((A-\lambda)u\right) + [A,\varphi_0(x,\eta)]u, \tag{8.10}$$

where $[A,\varphi_0(x,\eta)]$ is the commutator of A and $\varphi_0(x,\eta)$ defined by the formula

$$\begin{aligned} [A,\varphi_0(x,\eta)]u &= A(\varphi_0(x,\eta)u) - \varphi_0(x,\eta)Au \\ &= 2\sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial \varphi_0}{\partial x_i}\frac{\partial u}{\partial x_j} \\ &\quad + \left(\sum_{i,j=1}^{N} a^{ij}(x)\frac{\partial^2 \varphi_0}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b^i(x)\frac{\partial \varphi_0}{\partial x_i}\right)u. \end{aligned} \tag{8.11}$$

Here we need the following elementary inequality:

Claim 8.6. *We have, for all* $v \in C^j(\overline{G'})$, $j = 0, 1, 2,$

$$\|v\|_{W^{j,p}(G')} \leq |G'|^{1/p} \|v\|_{C^j(\overline{G'})},$$

where $|G'|$ *denotes the measure of* G'.

Proof. It suffices to note that we have, for all $w \in C(\overline{G'})$,

$$\int_{G'} |w(x)|^p \, dx \leq |G'| \, |w|^p_{C(\overline{G'})}.$$

This proves Claim 8.6. □

Since we have (see Figure 8.3), for some positive constant c,

$$|G'| \leq |B(x_0, \eta)| \leq c\eta^N,$$

it follows from an application of Claim 8.6 that

$$\|\varphi_0(x,\eta)((A-\lambda)u)\|_{L^p(G')} \leq c^{1/p}\eta^{N/p}|(A-\lambda)u|_{C(\overline{G'})}, \quad 0 < \eta < \eta_0. \tag{8.12}$$

Furthermore, we remark that

$$|D^\alpha \varphi_0(x,\eta)| = O\left(\eta^{-|\alpha|}\right) \quad \text{as } \eta \downarrow 0.$$

Hence it follows from an application of Claim 8.6 that

$$\left\| \frac{\partial \varphi_0}{\partial x_i} \frac{\partial u}{\partial x_j} \right\|_{L^p(G')} \leq \frac{C}{\eta} |u|_{1,p,G'} \leq C\eta^{-1+N/p} |u|_{C^1(\overline{G'})}, \quad 0 < \eta < \eta_0, \tag{8.13}$$

$$\left\| \frac{\partial^2 \varphi_0}{\partial x_i \partial x_j} u \right\|_{L^p(G')} \leq \frac{C}{\eta^2} |u|_{L^p(G')} \leq C\eta^{-2+N/p} |u|_{C(\overline{G'})}, \quad 0 < \eta < \eta_0, \tag{8.14}$$

$$\left\| \frac{\partial \varphi_0}{\partial x_i} u \right\|_{L^p(G')} \leq \frac{C}{\eta} |u|_{L^p(G')} \leq C\eta^{-1+N/p} |u|_{C(\overline{G'})}, \quad 0 < \eta < \eta_0. \tag{8.15}$$

By using inequalities (8.13), (8.14) and (8.15), we obtain from formula (8.11) that

$$\begin{aligned} &\|[A, \varphi_0(x,\eta)]u\|_{L^p(G')} \\ &\leq C\left(\eta^{-1+N/p}|u|_{C^1(\overline{G'})} + \eta^{-2+N/p}|u|_{C(\overline{G'})} + \eta^{-1+N/p}|u|_{C(\overline{G'})}\right) \\ &\leq C\left(\eta^{-1+N/p}|u|_{C^1(\overline{D})} + \eta^{-2+N/p}|u|_{C(\overline{D})}\right), \quad 0 < \eta < \eta_0. \end{aligned} \tag{8.16}$$

In view of formula (8.10), it follows from inequalities (8.12) and (8.16) that

$$\begin{aligned} &\|(A-\lambda)(\varphi_0(x,\eta)u)\|_{L^p(G')} \\ &\leq \|\varphi_0(x,\eta)((A-\lambda)u)\|_{L^p(G')} + \|[A, \varphi_0(x,\eta)]u\|_{L^p(G')} \\ &\leq C\eta^{N/p}\left(|(A-\lambda)u|_{C(\overline{G'})} + \eta^{-1}|u|_{C^1(\overline{D})} + \eta^{-2}|u|_{C(\overline{D})}\right), \\ &\qquad 0 < \eta < \eta_0. \end{aligned} \tag{8.17}$$

Therefore, by combining inequalities (8.9) and (8.17) we obtain that

$$\begin{aligned} &|\lambda|^{1/2}|u|_{C^1(\overline{G''})} + |\lambda| \cdot |u|_{C(\overline{G''})} \\ &\leq C|\lambda|^{N/2p} \, \|(A-\lambda)(\varphi_0(x,\eta)u)\|_{L^p(G')} \\ &\leq C|\lambda|^{N/2p} \, \eta^{N/p} \left(|(A-\lambda)u|_{C(\overline{G'})} + \eta^{-1}|u|_{C^1(\overline{G'})} + \eta^{-2}|u|_{C(\overline{G'})} \right) \\ &\leq C|\lambda|^{N/2p} \, \eta^{N/p} \left(|(A-\lambda)u|_{C(\overline{D})} + \eta^{-1}|u|_{C^1(\overline{D})} + \eta^{-2}|u|_{C(\overline{D})} \right), \\ &\qquad 0 < \eta < \eta_0. \end{aligned} \tag{8.18}$$

We remark (see Figure 8.5) that the closure $\overline{D} = D \cup \partial D$ can be covered by a finite number of sets of the forms

$$B(x_0', \eta/2) \cap \overline{D}, \quad x_0' \in \partial D,$$

and

$$B(x_0, \eta/2), \quad x_0 \in D.$$

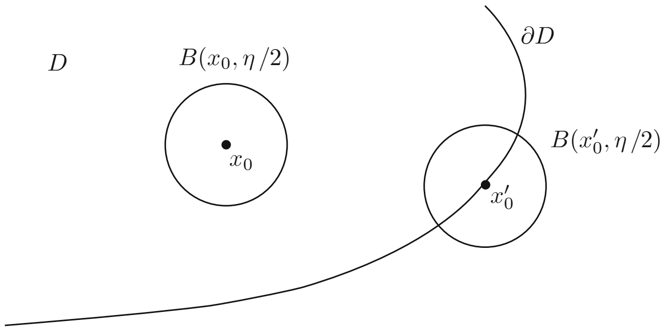

Fig. 8.5.

Hence, by taking the supremum of inequality (8.18) over $x \in \overline{D}$ we find that

$$\begin{aligned} &|\lambda|^{1/2}|u|_{C^1(\overline{D})} + |\lambda| \cdot |u|_{C(\overline{D})} \\ &\leq C|\lambda|^{N/2p}\eta^{N/p} \left(|(A-\lambda)u|_{C(\overline{D})} + \eta^{-1}|u|_{C^1(\overline{D})} + \eta^{-2}|u|_{C(\overline{D})} \right), \\ &\qquad 0 < \eta < \eta_0. \end{aligned} \tag{8.19}$$

(4) We now choose the localization parameter η. We let

$$\eta = \frac{\eta_0}{|\lambda|^{1/2}} K,$$

where K is a positive constant (to be chosen later) satisfying the condition

$$0 < \eta = \frac{\eta_0}{|\lambda|^{1/2}} K < \eta_0,$$

that is,

$$0 < K < |\lambda|^{1/2}.$$

Then it follows from inequality (8.19) that

$$\begin{aligned} &|\lambda|^{1/2} |u|_{C^1(\overline{D})} + |\lambda| \cdot |u|_{C(\overline{D})} \\ &\leq C\, \eta_0^{N/p} K^{N/p} |(A-\lambda)u|_{C(\overline{D})} + \left(C\, \eta_0^{N/p-1} K^{-1+N/p} \right) |\lambda|^{1/2} \cdot |u|_{C^1(\overline{D})} \\ &\quad + \left(C\, \eta_0^{N/p-2} K^{-2+N/p} \right) |\lambda| \cdot |u|_{C(\overline{D})} \quad \text{for all } u \in \mathcal{D}(\mathfrak{A}). \end{aligned} \tag{8.20}$$

However, since the exponents $-1 + N/p$ and $-2 + N/p$ are negative for $N < p < \infty$, we can choose the constant K so large that

$$C\, \eta_0^{N/p-1} K^{-1+N/p} < 1,$$

and

$$C\, \eta_0^{N/p-2} K^{-2+N/p} < 1.$$

Then the desired inequality (8.8) follows from inequality (8.20).

The proof of Lemma 8.4 is complete. □

Step (II): The next lemma, together with Lemma 8.4, proves that the resolvent set of $\mathfrak{A}$ contains the set

$$\Sigma(\varepsilon) = \left\{ \lambda = r^2 e^{i\theta} : r \geq r(\varepsilon),\ \ -\pi + \varepsilon \leq \theta \leq \pi - \varepsilon \right\},$$

that is, the resolvent $(\mathfrak{A} - \lambda I)^{-1}$ exists for all $\lambda \in \Sigma(\varepsilon)$.

Lemma 8.7. *If $\lambda \in \Sigma(\varepsilon)$, then, for any $f \in C_0(\overline{D} \setminus M)$, there exists a unique function $u \in \mathcal{D}(\mathfrak{A})$ such that $(\mathfrak{A} - \lambda I)u = f$.*

Proof. Since we have the assertion

$$f \in C_0(\overline{D} \setminus M) \subset L^p(D) \quad \text{for all} \quad 1 < p < \infty,$$

it follows from an application of Theorem 1.2 that if $\lambda \in \Sigma(\varepsilon)$ there exists a unique function $u \in W^{2,p}(D)$ such that

$$(A - \lambda)u = f \quad \text{in } D, \tag{8.21}$$

and

$$Lu = \mu(x') \frac{\partial u}{\partial \mathbf{n}} + \gamma(x')u = 0 \quad \text{on } \partial D. \tag{8.22}$$

However, part (ii) of Theorem 8.1 asserts that

$$u \in W^{2,p}(D) \subset C^{2-N/p}(\overline{D}) \subset C^1(\overline{D}) \quad \text{if} \quad N < p < \infty.$$

Hence we have, by formula (8.22) and condition (B),

$$u = 0 \text{ on } M = \{x' \in \partial D : \mu(x') = 0\},$$

so that

$$u \in C_0(\overline{D} \setminus M).$$

Furthermore, in view of formula (8.21) it follows that

$$Au = f + \lambda u \in C_0(\overline{D} \setminus M).$$

Summing up, we have proved that

$$\begin{cases} u \in \mathcal{D}(\mathfrak{A}), \\ (\mathfrak{A} - \lambda I)u = f. \end{cases}$$

Now the proof of part (i) of Theorem 1.3 is complete. □