Chapter 8
Other topics related with the Dirac spectrum

We outline the main topics in relation with the spectrum of Dirac operators that have been left aside in this overview.

8.1 Other eigenvalue estimates

As we have seen in Section 2.2, the Dirac operator on homogeneous spaces can be described as a family of matrices using the decomposition of the space of L^2-sections of ΣM into irreducible components. What happens if the homogeneity assumption is slightly weakened? This question has first been addressed by M. Kraus in the cases of isometric SO_n-actions and warped products over $\mathbb{S}^1$ respectively. Although the explicit knowledge of the Dirac spectrum becomes out of reach, the eigenvalues can still be approximated in a reasonable way.

Theorem 8.1.1 (M. Kraus [165, 166]) *For $n \geq 2$, let g be any Riemannian metric on $\mathbb{S}^n$ such that SO_n acts isometrically on (S^n, g). Write $f_{\max}^{n-1} \cdot \mathrm{Vol}(S^{n-1}, \mathrm{can})$ for the maximal volume of the orbits of the SO_n-action. Then the Dirac spectrum of (S^n, g) is symmetric about the origin,*

$$\lambda_1(D^2_{\mathbb{S}^n,g}) \geq \frac{(n-1)^2}{4f_{\max}^2}$$

and there are at most $2^{[\frac{n}{2}]} \cdot \binom{n-1+k}{k}$ eigenvalues of $D^2_{\mathbb{S}^n,g}$ in the interval $[\frac{(\frac{n-1}{2}+k)^2}{f_{\max}^2}, \frac{(\frac{n-1}{2}+k+1)^2}{f_{\max}^2}[$, for every nonnegative integer k.

The proof of Theorem 8.1.1 relies on the following arguments: the SO_n-action allows a dense part of $(\mathbb{S}^n, g)$ to be written as a warped product of $\mathbb{S}^{n-1}$ with an interval. On this dense part the eigenvalue problem on $(\mathbb{S}^n, g)$ translates into a singular nonlinear differential equation of first order with boundary conditions at both ends. The rest of the proof involves Sturm-Liouville theory, we refer to [166] for details.

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Note that the inequality in Theorem 8.1.1 is not sharp for the standard metric on $\mathbb{S}^n$ since $\lambda_1(D^2_{\mathbb{S}^n,\mathrm{can}}) = \frac{n^2}{4}$ (see Theorem 2.1.3). However Theorem 8.1.1 provides sharp asymptotical eigenvalue estimates in the two following situations. First consider the cylinder $C^n(L) :=]0, L[\times\mathbb{S}^{n-1}$ with half n-dimensional spheres glued at both ends. Obviously $C^n(L)$ admits an isometric SO_n-action for which $f_{\max} = 1$, in particular Theorem 8.1.1 implies that $\lambda_1(D^2_{C^n(L)}) \geq \frac{(n-1)^2}{4}$. On the other hand, $C^n(L)$ sits in $\mathbb{R}^{n+1}$ by construction; now C. Bär's upper bound (5.19) in terms of the averaged total squared mean curvature is not greater than

$$\frac{(n-1)^2 L\mathrm{Vol}(\mathbb{S}^{n-1},\mathrm{can}) + n^2\mathrm{Vol}(\mathbb{S}^n,\mathrm{can})}{4(L\mathrm{Vol}(\mathbb{S}^{n-1},\mathrm{can}) + \mathrm{Vol}(\mathbb{S}^n,\mathrm{can}))},$$

so that [166]

$$\lim_{L\to\infty} \lambda_1(D^2_{C^n(L)}) = \frac{(n-1)^2}{4}.$$

For the 2-dimensional ellipsoid $M_a := \{x \in \mathbb{R}^3 \,|\, x_1^2 + x_2^2 + \frac{x_3^2}{a^2} = 1\}$ (where $a > 0$) the maximal length of $\mathbb{S}^1$-orbits is 2π, so that by Theorem 8.1.1 the inequality $\lambda_1(D^2_{M_a}) \geq \frac{1}{4}$ holds. Combining this with the upper bound (5.8) provides [165]

$$\lim_{a\to\infty} \lambda_1(D^2_{M_a}) = \frac{1}{4}.$$

The technique of separation of variables used in the proof of Theorem 8.1.1 also provides a lower eigenvalue bound on warped product fibrations over $\mathbb{S}^1$ in terms of the Dirac eigenvalues of the fibres, see [167, Thm. 2]. As for the case of higher dimensional fibres over arbitrary base manifolds, the only family which has been considered so far is that of warped products with fibre $\mathbb{S}^k$ with $k \geq 2$, where decomposing the Dirac operator into block operator matrices provides similar results to those of Theorem 8.1.1, see [169].

Another natural but completely different way to study the Dirac eigenvalues consists in comparing them with those of other geometric operators. Hijazi's inequality (3.18) is already of that kind since μ_1 is the smallest eigenvalue of the conformal Laplace operator. As for spectral comparison results between the Dirac and the scalar Laplace operators, the first ones were proved by M. Bordoni. They rely on a very nice general comparison principle between two operators satisfying some kind of Kato-type inequality. The estimate which can be deduced reads as follows.

Theorem 8.1.2 (M. Bordoni [60]) *Let* $0 = \lambda_0(\Delta) < \lambda_1(\Delta) \leq \lambda_2(\Delta) \leq \ldots$ *be the spectrum of the scalar Laplace operator* Δ *on a closed* $n(\geq 2)$*-dimensional Riemannian spin manifold* (M^n, g)*. Then for any positive integer* N [60, Prop. 4.20]

$$\lambda_{2N}(D^2) \geq \frac{n}{4(n-1)}\Big(\inf_M(S) + \frac{\lambda_k(\Delta)}{2(2^{[\frac{n}{2}]}+1)^2}\Big), \tag{8.1}$$

where $k = [\frac{N}{2^{[\frac{n}{2}]}+1}]$.

In particular Bordoni's inequality (8.1) implies Friedrich's inequality (3.1) as well as the presence of at most $2^{[\frac{n}{2}]}$ eigenvalues of D^2 in the interval

$$[\frac{n}{4(n-1)}\inf_M(S), \frac{n}{4(n-1)}\Big(\inf_M(S) + \frac{\lambda_1(\Delta)}{2(2^{[\frac{n}{2}]}+1)^2}\Big)[,$$

see Section 8.2 for further results on the spectral gap.

Bordoni's results were generalized by M. Bordoni and O. Hijazi in the Kähler setting [61], where essentially the Friedrich-like term in the lower bound must be replaced by the Kirchberg-type one of inequality (3.10) in odd complex dimension.

Comparisons between Dirac and Laplace eigenvalues which go the other way round can be obtained in particular situations. In the case of surfaces, J.-F. Grosjean and E. Humbert proved the following.

Theorem 8.1.3 (J.-F. Grosjean and E. Humbert [113]) *Let* $[g]$ *be a conformal class on a closed orientable surface* M^2 *with fixed spin structure, then* [113, Cor. 1.2]

$$\inf_{\overline{g}\in[g]}\Big(\frac{\lambda_1(D^2_{\overline{g}})}{\lambda_1(\Delta_{\overline{g}})}\Big) \leq \frac{1}{2}, \tag{8.2}$$

where here $\lambda_1(D^2_{\overline{g}})$ *denotes the smallest* positive *eigenvalue of* $D^2_{\overline{g}}$.

Inequality (8.2) is optimal and sharp for $M^2 = \mathbb{S}^2$: indeed for any Riemannian metric g one has $\lambda_1(D^2_{\mathbb{S}^2,g}) \geq \frac{\lambda_1(\Delta_{\mathbb{S}^2,g})}{2}$ as a straightforward consequence of Bär's inequality (3.17) and Hersch's inequality (3.22). Moreover, (8.2) completes [1] where I. Agricola, B. Ammann and T. Friedrich prove the existence of a 1-parameter family $(g_t)_{t\geq 0}$ of $\mathbb{S}^1$-invariant Riemannian metrics on $\mathbb{T}^2$ for which, in the same notations as just above, $\lambda_1(\Delta_{\mathbb{T}^2,g_t}) < \lambda_1(D^2_{\mathbb{T}^2,g_t})$ for any $t \geq 0$, where $\mathbb{T}^2$ is endowed with its trivial spin structure. The inequality $\lambda_k(\Delta_{\mathbb{T}^2,g}) \geq \lambda_k(D^2_{\mathbb{T}^2,g})$ for k large enough and for particular metrics g on $\mathbb{T}^2$ with trivial spin structure has been proved independently by M. Kraus [168].

In the case where the manifold sits as a hypersurface in some spaceform, the best known result is the following.

Theorem 8.1.4 (C. Bär [40]) *Let* (M^n, g) *be isometrically immersed into* $\mathbb{R}^{n+1}$ *or* $\mathbb{S}^{n+1}$ *and carry the induced spin structure, then* [40, Thm. 5.1]

$$\lambda_N(D^2) \leq \frac{n^2}{4}\Big(\sup_M(H^2) + \kappa\Big) + \lambda_{[\frac{N-1}{2^\mu}]}(\Delta) \tag{8.3}$$

for every positive $N \in \mathbb{N}$*, where* $\kappa \in \{0,1\}$ *denotes the sectional curvature of the ambient space,* H *denotes the mean curvature of* M^n *and* μ *is the integer defined by* $\mu := [\frac{n+1}{2}] - n \bmod 2$.

Inequality (8.3) follows from the min-max principle and from (5.16) where one chooses f to be an eigenfunction of Δ and ψ to be the restriction of a non-zero Killing spinor.

8.2 Spectral gap

Another method to obtain information on the eigenvalues consists in estimating their difference, which is called the *spectral gap*. Initiated by H.C. Yang (see reference in [76]) for the scalar Laplacian, this approach turns out to provide similar results for the Dirac operator. The proof of the following theorem relies on the min-max principle and a clever input of coordinate functions of the immersion into the Rayleigh quotient, see [76] for details.

Theorem 8.2.1 (D. Chen [76]) *Let* (M^n, g) *be any* n*-dimensional closed immersed Riemannian spin submanifold of* $\mathbb{R}^N$ *for some* $N \geq n+1$*. Denote the spectrum of* D^2 *by* $\{\lambda_j(D^2)\}_{j\geq 1}$ *and set, for every* $j \geq 1$,

$$\mu_j := \lambda_j(D^2) + \frac{1}{4}\big(n^2 \sup_M(H^2) - \inf_M(S)\big),$$

where H *and* S *are the mean and the scalar curvature of* M *respectively. Then for any* $k \geq 1$

$$\sum_{j=1}^{k}(\mu_{k+1} - \mu_j)(\mu_{k+1} - (1 + \frac{4}{n})\mu_j) \leq 0. \tag{8.4}$$

Note that the codimension of M is arbitrary and that no compatibility condition between the spin structure of M and that of $\mathbb{R}^N$ is required. Elementary computations show that inequality (8.4) implies

$$\mu_{k+1} \leq \frac{1}{k}(1 + \frac{4}{n})\sum_{j=1}^{k}\mu_j,$$

which itself provides

$$\mu_{k+1} - \mu_k \leq \frac{4}{nk}\sum_{j=1}^{k}\mu_j,$$

which had been shown independently by N. Anghel [26]. In particular precise estimates on the growth rate of the Dirac eigenvalues can be deduced.

Theorem 8.2.1 has been extended by D. Chen and H. Sun to holomorphically immersed submanifolds of the complex projective space [77, Thm. 3.2].

8.3 Pinching Dirac eigenvalues

If Friedrich's inequality (3.1) is an equality for the smallest eigenvalue $\lambda_1(D^2)$, then from Theorem 3.1.1 and Proposition A.4.1 the underlying Riemannian manifold must be Einstein, which is a quite rigid geometric condition. Does the manifold remain "near to" Einstein if $\lambda_1(D^2)$ - or at least some lower eigenvalue - is close enough to Friedrich's lower bound? This kind of issue is designed under the name *eigenvalue pinching*. It addresses the continuous dependence of the geometry on the spectrum, in a sense that must be precised. We denote in the rest of this section by K_{sec}, diam and S the sectional curvature, diameter and scalar curvature of a given Riemannian manifold respectively. We also call two spin manifolds *spin diffeomorphic* if there exists a spin-structure-preserving diffeomorphism between them.

The first pinching result for Dirac eigenvalues is due to B. Ammann and C. Sprouse. It deals with the case where the scalar curvature almost vanishes. Tori with flat metric and trivial spin structure carry a maximal number of linearly independent parallel (hence harmonic) spinors. Theorem 8.3.1 below states that, under boundedness assumptions for the diameter and the sectional and scalar curvatures, one stays near to a flat torus in case some lower Dirac eigenvalue is not too far away from 0. Recall that a nilmanifold is the (left or right) quotient of a nilpotent Lie group by a cocompact lattice. If a (left or right) invariant metric is fixed on the nilmanifold, then the trivial lift of the lattice to the spin group provides a spin structure called the trivial one, see Proposition 1.4.2 for spin structures on coverings.

Theorem 8.3.1 (B. Ammann and C. Sprouse [25]) *Let K, d be positive real constants, $n \geq 2$ be an integer, $r := 1$ if $n = 2, 3$ and $r := 2^{[\frac{n}{2}]-1}+1$ if $n \geq 4$. Then there exists an $\varepsilon = \varepsilon(n, K, d) > 0$ such that every n-dimensional closed Riemannian spin manifold (M^n, g) with*

$$|K_{\mathrm{sec}}(M^n, g)| < K, \quad \mathrm{diam}(M^n, g) < d, \quad S(M^n, g) > -\varepsilon \ \text{ and } \ \lambda_r(D^2_{M^n,g}) < \varepsilon$$

is spin diffeomorphic to a nilmanifold with trivial spin structure.

Theorem 8.3.1 implies the existence of a uniform lower eigenvalue bound for the Dirac operator in the following family: there exists an $\varepsilon = \varepsilon(n, K, d) > 0$ such that on every n-dimensional closed Riemannian spin manifold (M^n, g) with $|K_{\mathrm{sec}}(M^n, g)| < K$, $\mathrm{diam}(M^n, g) < d$, $S(M^n, g) > -\varepsilon$ and which is not spin diffeomorphic to a nilmanifold with trivial spin structure the r^{th} eigenvalue of D^2 satisfies

$$\lambda_r(D^2) \geq \varepsilon.$$

The choice for r, which looks *a priori* curious, is actually optimal since the product of a so-called K3-surface with a torus carries exactly $r-1$ linearly independent parallel spinors, see [25, Ex. (2) p.411]. The proof of Theorem 8.3.1 makes use of an approximation result by U. Abresch (see reference in [25]) in an essential way, we refer to [25, Sec. 7] for details.

Under the supplementary assumption of a lower bound on the volume, the metric can even be shown to stay near to some with parallel spinors.

Theorem 8.3.2 (B. Ammann and C. Sprouse [25]) *Let K, d, V, δ be positive real constants, $n \geq 2$ be an integer, $r := 1$ if $n = 2, 3$ and $r := 2^{[\frac{n}{2}]-1} + 1$ if $n \geq 4$. Then there exists an $\varepsilon = \varepsilon(n, K, d, V, \delta) > 0$ such that for every n-dimensional closed Riemannian spin manifold (M^n, g) with*

$$|K_{\mathrm{sec}}(M^n, g)| < K, \quad \mathrm{diam}(M^n, g) < d, \quad S(M^n, g) > -\varepsilon, \quad \mathrm{Vol}(M^n, g) > V$$

and $\lambda_r(D^2_{M^n,g}) < \varepsilon$, the metric g is at $C^{1,\alpha}$-distance at most δ to a metric admitting a non-zero parallel spinor.

The proof of Theorem 8.3.2 relies on a similar general eigenvalue pinching valid for arbitrary rough Laplacians on arbitrary vector bundles due to P. Petersen (see reference in [25]) and on the Schrödinger-Lichnerowicz formula (3.2). Petersen's method can also be applied to the rough Laplacian associated to the deformed covariant derivative $X \mapsto \nabla_X + \rho X \cdot$ and in this case it provides the following:

Theorem 8.3.3 (B. Ammann and C. Sprouse [25]) *Let K, d, V, ρ, δ be positive real constants, $n \geq 2$ be an integer, $r := 1$ if $n = 2, 3$ and $r := 2^{[\frac{n}{2}]-1} + 1$ if $n \geq 4$. Then there exists an $\varepsilon = \varepsilon(n, K, d, V, \rho, \delta) > 0$ such that for every n-dimensional closed Riemannian spin manifold (M^n, g) with*

$$|K_{\mathrm{sec}}(M^n, g)| < K,\ \mathrm{diam}(M^n, g) < d,\ \mathrm{Vol}(M^n, g) > V,\ S(M^n, g) \geq n(n-1)\rho^2$$

and $\lambda_r(D^2_{M^n,g}) < \frac{n^2\rho^2}{4} + \varepsilon$, the metric g is at $C^{1,\alpha}$-distance at most δ to a metric with constant sectional curvature ρ^2.

Note that the bound on the sectional curvature is necessary because of Bär-Dahl's result [46] discussed in Section 3.2. However the minimal number r necessary for the result to hold can be enhanced.

Theorem 8.3.4 (A. Vargas [231]) *The conclusion of* Theorem 8.3.3 *holds with*

$$r := \begin{cases} 3 & \text{if } n = 6 \text{ or } n \equiv 1\ (4) \\ \frac{n+9}{4} & \text{if } n \equiv 3\ (4). \end{cases}$$

8.4 Spectrum of other Dirac-type operators

Up to now we have concentrated onto the fundamental (or spin) Dirac operator on a spin manifold. As already mentioned at the beginning of Chapter 1, Dirac-type operators may be defined in the more general context where a so-called Clifford bundle [173, Sec. II.3] is at hand. Roughly speaking, a Clifford bundle is given by a Hermitian vector bundle together with a covariant derivative and on which the tangent bundle acts by Clifford multiplication such that all three objects (Hermitian metric, covariant derivative and Clifford multiplication) are compatible with each other in the sense of Definition 1.2.2 and Proposition 1.2.3. The associated Dirac operator is defined as the Clifford multiplication applied to the covariant derivative. One may add a zero-order term and obtain a so-called Dirac-Schrödinger operator. In this section we discuss spectral results in relation with the spin^c Dirac operator, with twisted Dirac-Schrödinger operators, with Dirac operators associated to particular geometrically relevant connections, with the basic Dirac operator and in the pseudo-Riemannian setting.

First, the concept of spin structure may be weakened to that of spin^c structure, whose structure group is the spin^c group $\mathrm{Spin}_n^c := \mathrm{Spin}_n \times \mathbb{S}^1/_{\mathbb{Z}_2}$. Such a structure comes along with a $\mathbb{S}^1$-principal bundle, or equivalently with a complex line bundle $\mathcal{L}$. We do not want to define spin^c structures more precisely but mention that all spin manifolds are spin^c and that all almost-Hermitian manifolds have a canonical spin^c structure [173, App. D]. Moreover the choice of a covariant derivative on the line bundle induces a covariant derivative and hence a Dirac operator on the associated spinor bundle over the underlying manifold. In that case it can be expected that most of the results valid for the spin Dirac operator remain valid for the spin^c one, except that the curvature of the line bundle must in some situations be taken into account. For example, M. Herzlich and A. Moroianu proved the analog of Hijazi's inequality (3.18) in the spin^c context: denote by ω the curvature form of the line bundle $\mathcal{L}$ and by μ_1 the smallest eigenvalue of the scalar operator $L_\omega := 4\frac{n-1}{n-2}\Delta + S - 2[\frac{n}{2}]^{\frac{1}{2}}|\omega|$, then any eigenvalue λ of the spin^c Dirac operator satisfies [126, Thm. 1.2]

$$\lambda^2 \geq \frac{n}{4(n-1)}\mu_1.$$

We note however that little has been done in the spin^c context in comparison with the spin one.

If the underlying space is again our familiar spin manifold (M^n, g) and if we choose an arbitrary Riemannian or Hermitian vector bundle E over M, then the tensor product bundle $\Sigma M \otimes E$ carries a canonical Clifford multiplication (extend the Clifford multiplication by the identity on the second factor). If we endow E with a metric covariant derivative, then we obtain a structure of Clifford bundle and an associated Dirac operator called Dirac operator of

M twisted with E. This operator is usually denoted by D_M^E. For example, the Euler operator $d+\delta$ can be seen as the Dirac operator of M twisted with ΣM: this follows essentially from (1.2) and may actually be stated without any spin structure on M [173, Sec. II.6]. Another prominent example is the Dirac operator of a spin submanifold twisted with the spinor bundle of its normal bundle (where the latter is assumed to be spin). Various studies have been devoted to the spectrum of twisted Dirac operators, therefore we restrict ourselves to a few ones which we hope to be representative. We include all that concerns Dirac-Schrödinger operators, since in that case the zero order term mainly translates the upper or lower bounds by a constant.

Let first E be as above, M be closed and f be a smooth real function on M. Denote by κ_1 the smallest eigenvalue on M of the pointwise linear operator $\sum_{k,l=1}^n e_k \cdot e_l \cdot R^E_{e_k,e_l}$, where R^E is the curvature tensor of the chosen covariant derivative on E (and $(e_k)_{1\leq k\leq n}$ is a local o.n.b. of TM). If the inequalities $n(S+\kappa_1) > (n-1)f^2 > 0$ hold on M, then any eigenvalue λ of the Dirac-Schrödinger operator $D_M^E - f$ acting on $\Gamma(\Sigma M \otimes E)$ satisfies [105, Prop. 4.1]

$$\lambda^2 \geq \frac{1}{4}\inf_M \left(\sqrt{\frac{n}{n-1}(S+\kappa_1)} - |f|\right)^2. \tag{8.5}$$

Inequality (8.5), which can be deduced from a clever choice of modified covariant derivative, stands for the analog of Friedrich's inequality in this context, see [105] for other kinds of estimates and references to earlier works on that topic (such as [196]). In the particular case where $n=4$, $f=0$, E is arbitrary and carries a selfdual covariant derivative, the estimate (8.5) can be enhanced using the decomposition $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$ and the vanishing of one half of the auxiliary curvature term computed from R^E: H. Baum proved [52, Thm. 2] that

$$\lambda^2 \geq \frac{1}{3}\inf_M(S)$$

for any eigenvalue λ of D_M^E, which is exactly Friedrich's inequality (3.1) for the eigenvalues of the spin Dirac operator.

Staying in dimension 4, if the spin manifold (M^4, g) carries a Hermitian structure J (i.e., an orthogonal complex structure on TM) then one is led to the Dirac operator twisted with $E = \Sigma M = \Lambda^{0,*}TM$ which is nothing else than the Dolbeault operator $\sqrt{2}(\partial + \overline{\partial})$. Although Kirchberg's inequality (3.10) does not apply, sharp lower bounds for the eigenvalues of the Dolbeault operator are still available: B. Alexandrov, G. Grantcharov and S. Ivanov proved [6, Thm. 2] that

$$\lambda^2 \geq \frac{1}{6}\inf_M(S)$$

for any eigenvalue λ of $\sqrt{2}(\partial + \overline{\partial})$. Beware that equality cannot occur for a non-flat Kähler metric because of (3.10). The proof of that inequality relies on Weitzenböck formulas and the clever choice of twistor operators associated to a canonical one-parameter-family of connections, we refer to [6] for details. Besides, we mention that upper eigenvalue bounds for particular twisted Dirac operators have been obtained in [52], [40] and [101].

From the point of view of geometers investigating the integrability of particular G-structures, there exists another interesting family of Dirac-type operators which are usually denoted by $D^{\frac{1}{3}}$ and defined by $D^{\frac{1}{3}} := D_g + \frac{T}{4}\cdot$, where T is some given 3-form and D_g is the spin Dirac operator on the Riemannian spin manifold (M^n, g). For example if (M^n, g) is a so-called reductive homogeneous space then $D^{\frac{1}{3}}$ is the so-called Kostant Dirac operator (see reference in [3]); if (M^n, g) is a Hermitian manifold then $D^{\frac{1}{3}}$ coincides with the Dolbeault-operator defined just above. In case T is the characteristic torsion of a 5-dimensional closed spin Sasaki manifold with scalar curvature bounded from below, the use of suitable deformations of the connection by polynomials of the torsion form allowed I. Agricola, T. Friedrich and M. Kassuba to prove the following estimates of any eigenvalue λ of $(D^{\frac{1}{3}})^2$ [3, Thm. 4.1]:

$$\lambda \geq \left| \begin{array}{ll} \frac{1}{16}(1 + \frac{1}{4}\inf_M(S))^2 & \text{if } -4 < S \leq 4(9+4\sqrt{5}) \\ \\ \frac{5}{16}\inf_M(S) & \text{if } S \geq 4(9+4\sqrt{5}). \end{array} \right.$$

Equality holds if (M^5, g) is η-Einstein (see [3] for a definition). Surprisingly enough the first lower bound depends quadratically on the scalar curvature, which makes the estimate better for small S. We refer to [3] for the proof. We also note that in the context of contact metric manifolds (which have a canonical spin^c structure) Weitzenböck formulas for the Dirac operator associated to the so-called Tanaka-Webster connection have also been produced in order to prove vanishing theorems [205], however no study of the spectrum is still available.

Sasaki manifolds can also be viewed as particular foliated Riemannian manifolds. Spin structures can be defined on Riemannian foliations in much the same way as on the tangent bundle and an associated covariant derivative and Dirac operator may be defined which are called the transversal covariant derivative and transversal Dirac operator respectively. The transversal Dirac operator, which acts on the space of basic spinors (spinors whose transversal covariant derivatives vanish along all directions normal to the leaves), is in general not formally self-adjoint, therefore one considers the symmetrized operator called *basic* Dirac operator of the foliation and denoted by D_b. It is a not-so-straightforward adaptation of the proof of Friedrich's inequality by G. Habib and K. Richardson to show that any eigenvalue λ of D_b on a closed underlying manifold (M^n, g) satisfies [123, Eq. (1.1)]

$$\lambda^2 \geq \frac{q}{4(q-1)} \inf_M(S^{\mathrm{tr}}),$$

where $q \geq 2$ stands for the codimension of the foliation and S^{tr} for its transversal scalar curvature. In case the normal bundle of the foliation carries a Kähler or a quaternionic Kähler structure, analogs of Kirchberg's inequality (3.10) and of (3.15) can also be derived [151, 152, 121, 120, 119].

To close this section we mention the only result known to us about the spectrum of the Dirac operator in the *pseudo-Riemannian* (non-Riemannian) setting. First spin structures require the pseudo-Riemannian manifold to be simultaneously space- and time-oriented in order to be well-defined, see [49] or [53, Sec. 2]. In that case the choice of a maximal timelike subbundle induces an L^2-Hermitian inner product on the space of spinors. Unlike its Riemannian version the associated (spin) Dirac operator is neither formally self-adjoint w.r.t. that inner product nor elliptic. However H. Baum could show with the help of suitable endomorphisms of the spinor bundle commuting or anti-commuting with the Dirac operator that the point spectrum, the continuous spectrum and the residual spectrum of the Dirac operator on any even-dimensional pseudo-Riemannian manifold are symmetric w.r.t. the real and imaginary axes. We refer to [53] for further statements and the proof.

8.5 Conformal spectral invariants

In this section we are interested in two invariants associated to the Dirac spectrum. A good reference for the whole section is [149]. Given a closed spin manifold M^n with fixed conformal class $[g]$ and spin structure denoted by ϵ, let $\lambda_1(D^2_{M,g})$ be the smallest eigenvalue of the square of the Dirac operator of (M^n, g). The *Bär-Hijazi-Lott invariant* [13, eq. (2.4.1) p.12] of $(M^n, [g], \epsilon)$ is the nonnegative real number $\lambda_{\min}(M^n, [g], \epsilon)$ defined by

$$\lambda_{\min}(M^n, [g], \epsilon) := \inf_{\overline{g}\in[g]} \left(\sqrt{\lambda_1(D^2_{M,\overline{g}})} \cdot \mathrm{Vol}(M, \overline{g})^{\frac{1}{n}} \right).$$

Of course the expression on the r.h.s. is chosen so as to remain scaling-invariant. By definition $\lambda_{\min}(M^n, [g], \epsilon)$ is a conformal invariant. The Bär-Hijazi-Lott invariant is tightly connected to and behaves much like the Yamabe invariant. Indeed, it already follows from Bär's inequality (3.17) and from Hijazi's inequality (3.20) that

$$\lambda_{\min}(M^2, [g], \epsilon)^2 \geq 2\pi\chi(M^2) \quad \text{and} \quad \lambda_{\min}(M^n, [g], \epsilon)^2 \geq \frac{n}{4(n-1)} Y(M, [g]) \tag{8.6}$$

for every $n \geq 3$, where $\chi(M^2)$ and $Y(M, [g])$ are the Euler characteristic and the Yamabe invariant respectively. For $M^2 = \mathbb{S}^2$ this implies that the Bär-Hijazi-Lott invariant is positive. More generally, as a consequence of J. Lott's estimate (3.21), the Bär-Hijazi-Lott invariant is positive as soon as the Dirac operator is invertible for some - hence any - metric in the conformal class. In particular $\lambda_{\min}(M^n, [g], \epsilon)$ vanishes if and only if (M^n, g)

admits non-zero harmonic spinors. Generalizing J. Lott's Sobolev-embedding techniques [181] to the case where the Dirac kernel is possibly non-trivial, B. Ammann showed the positivity of $\inf\limits_{\overline{g}\in[g]}\left(\sqrt{\lambda^+(D^2_{M,\overline{g}})}\cdot\mathrm{Vol}(M,\overline{g})^{\frac{1}{n}}\right)$ to hold true in general [12, Thm. 2.3], where $\lambda^+(D^2_{M,\overline{g}})$ denotes the smallest positive eigenvalue of $D^2_{M,\overline{g}}$. As an example, the Bär-Hijazi-Lott invariant of $\mathbb{S}^n$ $(n\geq 2)$ with standard conformal class [can] and canonical spin structure is given by $\frac{n}{2}\omega_n^{\frac{1}{n}}$, where ω_n is the volume of $\mathbb{S}^n$ carrying the metric of sectional curvature 1 (denoted by "can"): this follows from Corollaries 3.3.2 and 3.3.3 together with $\lambda_1(D^2_{\mathbb{S}^n,\mathrm{can}})=\frac{n^2}{4}$ and $Y(\mathbb{S}^n,[\mathrm{can}])=n(n-1)\omega_n^{\frac{2}{n}}$ if $n\geq 3$.

In a similar way as for the Yamabe invariant, the Bär-Hijazi-Lott invariant cannot be greater than that of the sphere: if $n\geq 3$ or $M^2=\mathbb{S}^2$ then B. Ammann [12, Thm. 3.1 & 3.2] proved that $\inf\limits_{\overline{g}\in[g]}\left(\sqrt{\lambda^+(D^2_{M,\overline{g}})}\cdot\mathrm{Vol}(M,\overline{g})^{\frac{1}{n}}\right)\leq\lambda_{\min}(\mathbb{S}^n,[\mathrm{can}])$, in particular

$$\lambda_{\min}(M^n,[g],\epsilon)\leq\lambda_{\min}(\mathbb{S}^n,[\mathrm{can}]). \tag{8.7}$$

The proof relies on a suitable cut-off argument performed on Dirac eigenvectors on the gluing of a sphere with large radius to the manifold, see [12, Sec. 3] for the details.

The next step would consist in showing that (8.7) is a strict inequality if $(M^n,[g])$ is not conformally equivalent to $(\mathbb{S}^n,[\mathrm{can}])$. This has been done by B. Ammann, E. Humbert and B. Morel in the conformally flat setting where one introduces a further datum, namely the so-called *mass endomorphism*. The mass endomorphism of a locally conformally flat Riemannian spin manifold is a self-adjoint endomorphism field of its spinor bundle and can be locally defined out of the difference between the Green's operators for the Dirac operators associated to the original metric and to the Euclidean one in suitable coordinates, see [23, Def. 2.10] for a precise definition. The name comes from the corresponding term for the Yamabe operator and which is known to provide the mass of an asymptotically flat Riemannian spin manifold. Moreover, the mass endomorphism is "well-behaved" regarding conformal changes of metric [23, Prop. 2.9]. In case the locally conformally flat manifold $(M^n,[g])$ has an invertible Dirac operator (for some hence any metric in the conformal class) and if its mass endomorphism has a non-zero eigenvalue somewhere on M^n, then [23, Thm. 1.2]

$$\lambda_{\min}(M^n,[g],\epsilon)<\lambda_{\min}(\mathbb{S}^n,[\mathrm{can}]). \tag{8.8}$$

At this point one should beware that the mass endomorphism of $(\mathbb{S}^n,[\mathrm{can}])$ vanishes and that this does not characterize the round sphere since flat tori also have vanishing mass endomorphism. We refer to [23] for the details. For a generalization of the Bär-Hijazi-Lott invariant to manifolds with non-empty boundary we refer to [212, 214].

We also mention that the Bär-Hijazi-Lott invariant has been generalized to the noncompact setting, where it provides an obstruction to the existence of conformal spin compactifications of the manifold [114]. More precisely, let M^n be any n-dimensional manifold with conformal class $[g]$ and spin structure ϵ and define $\lambda_{\min}^+(M^n, [g], \epsilon)$ as in Section 7.3. If

$$\lim_{r\to\infty} \lambda_{\min}^+(M^n \setminus \overline{B}_r(p), [g], \epsilon) < \lambda_{\min}(\mathbb{S}^n, [\mathrm{can}]),$$

where $p \in M$ is arbitrary, then $(M^n, [g])$ is not conformal to a subdomain with induced spin structure of a closed spin manifold [116, Thm. 3.0.1] (see also [114, Thm. 1.4]). The vanishing of $\lambda_{\min}^+(M^n, [g], \epsilon)$ also prevents the existence of conformal spin compactifications of M, since $\lambda_{\min}^+(M^n, [g], \epsilon) > 0$ on closed manifolds [12, Thm. 2.3] and a monotonicity principle holds for $\lambda_{\min}^+$ [116, Lemma 2.0.3], see [116, Rem. 3.0.4].

The Green's operators for the Dirac operator have also revealed as a powerful tool in general problems from geometric analysis such as the classical Yamabe conjecture (*"find a metric with constant scalar curvature in a fixed conformal class"*). As shown by R. Schoen, the Yamabe conjecture is implied by the positive mass theorem through the fact that the constant term in the asymptotic expansion in inverted normal coordinates of the Green's operator for the conformal Laplace operator is proportional to the mass of the conformal blow-up. Furthermore but independently, E. Witten [235] showed that in the spin setting the positive mass theorem is in turn implied by the existence of spinor field which is harmonic and asymptotically constant on the conformal blow-up. Now it is a striking result by B. Ammann and E. Humbert [21] that the Green's operators for the Dirac operator provide such a spinor. More precisely, if (M^n, g) is closed Riemannian spin manifold with positive Yamabe invariant and which is locally conformally flat if $n \geq 6$, then its conformal blow-up has positive mass. For positive mass theorems we refer to Section 8.8.

If one lets the conformal class vary on the closed manifold M^n, then one is led to the so-called τ-invariant of M^n with spin structure ϵ and which is defined by

$$\tau(M^n, \epsilon) := \sup_{[g]} \big(\lambda_{\min}(M^n, [g], \epsilon)\big).$$

The introduction of the spinorial invariant τ is inspired from that of R. Schoen's σ-invariant which is defined in dimension 2 by $\sigma(M^2) := 4\pi\chi(M^2)$ and in dimension $n \geq 3$ by

$$\sigma(M^n) := \sup_{[g]} \big(Y(M^n, [g])\big),$$

where $Y(M^n, [g])$ denotes the Yamabe invariant on $(M^n, [g])$. There are at least two motivations for the study of the τ-invariant. First, the τ-invariant bounds the σ-invariant from above since it follows from (8.6) that, in every dimension $n \geq 2$,

$$\tau(M^n,\epsilon)^2 \geq \frac{n}{4(n-1)}\sigma(M^n),$$

with equality for $\mathbb{S}^n$. Therefore upper bounds for $\tau(M^n,\epsilon)$ provide upper bounds for $\sigma(M^n)$, on which little is known. In an independent context, the inequality $\lambda_{\min}(M^2,[g],\epsilon) < 2\sqrt{\pi} = \tau(\mathbb{S}^2)$ guarantees the existence of a metric $\overline{g} \in [g]$ for which any simply-connected open subset of $(M^2,\overline{g})$ can be isometrically embedded with constant mean curvature into $\mathbb{R}^3$ [13, Sec. 5.4]. Hence it is of geometric interest to know when the inequality $\tau(M^2,\epsilon) < 2\sqrt{\pi}$ holds. In case $M^2 = \mathbb{T}^2$ B. Ammann and E. Humbert have shown [22, Thm. 1.1] that

$$\tau(\mathbb{T}^2,\epsilon) = 2\sqrt{\pi}$$

for any of its non-trivial spin structures ϵ (obviously $\tau(\mathbb{T}^2,\epsilon_0) = 0$ for the trivial spin structure ϵ_0). Note that this neither proves nor contradicts the existence of immersed constant mean curvature tori in $\mathbb{R}^3$. As a generalization, the τ-invariant of $\mathbb{S}^{n-1}\times\mathbb{S}^1$ is equal to zero if $n = 2$ and $\mathbb{S}^1$ carries the trivial spin structure and to $\tau(\mathbb{S}^n)$ otherwise [22, Thm. 1.2].

8.6 Convergence of eigenvalues

Given a converging sequence of closed Riemannian spin manifolds, does their Dirac spectrum have to converge to that of the limit? Three very different contexts have up to now been considered where this question can be given sense and answered. The simplest and historically the first one deals with the behaviour of the Dirac spectrum of $\mathbb{S}^1$-bundles under collapse. In that case the behaviour depends sensitively of the spin structure as shown by B. Ammann and C. Bär [15]. Let M denote the total space of an $\mathbb{S}^1$-bundle which is simultaneously a Riemannian submersion with totally geodesic fibres over a base manifold B. Two kinds of spin structures can be defined on M according to whether the $\mathbb{S}^1$-action can be lifted to the spin level or not; in the former case the spin structure is called projectable and in the latter it is called non-projectable. Projectable spin structures on M stand in one-to-one correspondence with spin structures on B. The main result of [15] states the following about the convergence of the Dirac spectrum of M as the fibre-length goes to 0: either the spin structure of M is projectable and there exist Dirac eigenvalues of M converging to those of B or it is non-projectable and all Dirac eigenvalues of M tend to ∞ or $-\infty$ [15, Thm. 4.1 & 4.5]. As an interesting application, the Dirac spectrum of all complex odd-dimensional complex projective spaces can be deduced from that of the Berger spheres (Theorem 2.2.2). Parts of those results have been generalized by B. Ammann to $\mathbb{S}^1$-bundles with non-geodesic fibres [7, 8].

The second natural context deals with hyperbolic degenerations, i.e., with sequences of closed hyperbolic spin manifolds $(M_j)_{j\in\mathbb{N}}$ converging to a non-compact complete hyperbolic spin manifold M (here a hyperbolic metric is a metric with constant sectional curvature -1). Those sequences only exist in dimensions 2 and 3 and, provided the convergence respects the spin structures in some sense, the limit manifold must have discrete Dirac spectrum in dimension 3 whereas it may have continuous spectrum in dimension 2, see references in [208] where a precise description of hyperbolic degenerations is recalled. In case the limit manifold M is assumed to have discrete Dirac spectrum, F. Pfäffle proved the convergence of the Dirac spectrum of $(M_j)_{j\in\mathbb{N}}$ in the following sense [208, Thm. 1.2] (see also [207]): For all $\varepsilon > 0$ and $\Lambda \geq 0$, there exists an $N \in \mathbb{N}$ such that for all $j \geq N$ the real number Λ lies neither in the spectrum of D nor in that of D_{M_j}, both Dirac operators D_{M_j} and D have only discrete eigenvalues and no other spectrum in $[-\Lambda, \Lambda]$, they have the same number m of eigenvalues in $[-\Lambda, \Lambda]$ which can be ordered so that $|\lambda_k^{(j)} - \lambda_k| \leq \varepsilon$ holds for all $1 \leq k \leq m$.

The diameter of the converging sequence of degenerating hyperbolic manifolds cannot be controlled since the limit-manifold must have a finite number of so-called cusps, which by definition are unbounded. The third context to have been considered precisely deals with the situation where both the diameter and the sectional curvature of the converging sequence are assumed to remain bounded. In that case J. Lott proved the following very general result [183]. Consider a sequence $(g_j)_{j\in\mathbb{N}}$ of bundle metrics on the total space of a spin fibre bundle M over a base spin manifold B. Assume the fibre length to go to 0 as j tends to ∞ while both the diameter and the sectional curvature of (M, g_j) remain bounded. Then the Dirac spectrum of (M, g_j) converges in the sense just above to that of some differential operator of first order on B which can be explicitly constructed. Since a precise formulation and the discussion of the results would require too many details we recommend the introduction of [183].

8.7 Eta-invariants

As we have seen in Theorem 1.3.7, the Dirac spectrum of any closed n-dimensional Riemannian spin manifold is symmetric w.r.t. the origin in dimension $n \not\equiv 3\ (4)$. To measure the asymmetry of the Dirac spectrum in case $n \equiv 3\ (4)$, Atiyah, Patodi and Singer introduced [27] the so-called η-invariant of D which is defined by $\eta(D) := \eta(0, D)$, where, for every $s \in \mathbb{C}$ with $\Re e(s) > n$,

$$\eta(s, D) := \sum_{\lambda_j \neq 0} \frac{\mathrm{sgn}(\lambda_j)}{|\lambda_j|^s}.$$

The λ_j's denote the eigenvalues of D. It is already a non-trivial statement that $s \mapsto \eta(s, D)$ can be meromorphically extended onto $\mathbb{C}$ and is regular at $s = 0$, see [27]. Originally the η-invariant was introduced to describe some boundary term in the Atiyah-Patodi-Singer index theorem [27]. In a simple-minded way, the η-invariant of D can be thought of as the difference between the number of positive and that of negative Dirac eigenvalues (of course this has no sense since both numbers are infinite). In particular the η-invariant of D vanishes as soon as the Dirac spectrum is symmetric.

Few η-invariants are known explicitly. One of the first computations of η-invariant goes back to Hitchin [148], where the explicit knowledge of the Dirac spectrum on the Berger sphere $\mathbb{S}^3$ allows the η-invariant to be explicited. This was generalized onto all Berger spheres by D. Koh [162]. In the flat setting, the η-invariant can also be deduced from the Dirac spectrum in dimension $n = 3$ [206] and for particular holonomies in dimension $n \geq 4$ [188]. Theorem 2.2.3 provides the η-invariant on particular closed 3-dimensional hyperbolic manifolds [218]. The most general formula allowing the determination of the η-invariant has been proved by S. Goette [108, Thm. 2.33] on homogeneous spaces, where $\eta(D)$ arises as the sum of three terms: a representation-theoretical expression, an index-theoretical one and so-called equivariant η-invariants, which can themselves be deduced from finer representation-theoretical data [106, 107].

Though unknown in most cases, the η-invariant behaves nicely under connected sums: roughly speaking, if a closed Riemannian spin manifold is separated in two pieces M_1, M_2 by a closed hypersurface N, about which both the metric and the Dirac operator split as on a Riemannian product, then the η-invariant of D consists of the sum of the η-invariants of D_{M_1} and D_{M_2} plus the so-called Maslov-index of a pair of Lagrangian subspaces of $\mathrm{Ker}(D_N)$ making D_{M_j} self-adjoint, plus some index-theoretical integers (U. Bunke [67, Thm 1.9]). We refer to [67] for an overview of η-invariants of general Dirac-type operators and numerous useful references.

We also mention that some kind of η-invariant can be defined in the non-compact setting, see [118] and references therein.

8.8 Positive mass theorems

Although this section has more to do with physics as with the Dirac spectrum, we include it because on the one hand the proofs of the results presented involve simple spinorial techniques as already used above, and on the other hand positive mass theorems nowadays play a central role in many other topics of global analysis such as the Yamabe problem. A good but not up-to-date reference for that topic is [125].

A positive mass theorem (sometimes called positive energy theorem) is a two-fold statement reading roughly as follows: Let (M^n, g) be a Riemannian manifold which is asymptotic to a model manifold (in a sense that must be precised) and some of which curvature invariant satisfies a pointwise inequality, then some asymptotic geometric invariant called its mass also satisfies a similar inequality and, if this latter inequality is an equality, then the whole manifold is globally isometric to the original model manifold. To fix the ideas we concentrate from now on onto the original positive mass theorem as proved by R. Schoen and S.-T. Yau [215, 216] and independently by E. Witten [235] in the spinorial setting, in particular we leave aside all recent developments in what has become a whole field of research at the intersection between mathematics and general relativity, see e.g. [236] for references.

Let (M^n, g) be a Riemannian manifold of dimension $n \geq 3$. Call it asymptotically flat of order $\tau \in \mathbb{R}$ if there exists a compact subset $K \subset M$, a positive real number R and a diffeomorphism $M \setminus K \longrightarrow \{x \in \mathbb{R}^n, |x| > R\}$ such that the pushed-out metric fulfills: $g_{ij} - \delta_{ij} = O(|x|^{-\tau})$, $\frac{\partial g_{ij}}{\partial x_k} = O(|x|^{-\tau-1})$ and $\frac{\partial^2 g_{ij}}{\partial x_k \partial x_l} = O(|x|^{-\tau-2})$ as $|x| \to \infty$, for all $1 \leq i, j, k, l \leq n$. Given such a manifold (M^n, g), set

$$m(g) := \frac{1}{16\pi} \cdot \lim_{r \to \infty} \int_{S_r} \sum_{i,j=1}^{n} (\frac{\partial g_{ij}}{\partial x_i} - \frac{\partial g_{ii}}{\partial x_j}) \nu_j dA,$$

where S_r denotes the Euclidean sphere of radius r about $0 \in \mathbb{R}^n$ with outside unit normal ν and dA its canonical measure. Beware here that in general $m(g)$ does not make any sense: the integral need not converge, and even if it converges it depends on the choice of asymptotic coordinates. If however $\tau > \frac{n-2}{2}$ and the scalar curvature of (M^n, g) is integrable, then a highly non-trivial theorem of R. Bartnik (see reference in [125]) ensures $m(g)$ to be well-defined. In that case it is called the ADM-mass of (M^n, g). The canonical example of asymptotically flat manifold (of any order) is $(\mathbb{R}^n, \mathrm{can})$, whose ADM-mass vanishes. The positive mass theorem states that, with the assumptions above and if the scalar curvature S of (M^n, g) is non-negative, then $m(g) \geq 0$ with equality if and only if $(M^n, g) = (\mathbb{R}^n, \mathrm{can})$. This is a very deep statement since it establishes a direct relationship between the geometry at infinity and the global geometry of M. For example, as a consequence, any Riemannian metric on $\mathbb{R}^n$ with $S \geq 0$ and which is flat outside a compact subset must be flat. Surprisingly enough, the positive mass theorem follows from relatively simple considerations involving some kind of boundary value problem for the Dirac operator, at least in case M is spin, as shown by E. Witten [235]. Let us sketch his idea.

The first and main step in Witten's proof consists in choosing any non-zero "constant" spinor field ψ_0 at infinity and exhibiting a sufficiently regular non-zero spinor field ψ lying in the kernel of D^2 and being asymptotic to ψ_0. This can be done by showing the invertibility of D^2 between suitable Hölder spaces. Applying Schrödinger-Lichnerowicz' formula (1.15), integrating on a Euclidean ball of (sufficiently large) radius r and using (3.29) together with Green's formula one obtains

$$0 = \int_{B_r} \langle D^2\psi, \psi \rangle v_g = \int_{B_r} (|\nabla \psi|^2 + \frac{S}{4}|\psi|^2) v_g - \int_{S_r} \langle \nabla_\nu \psi, \psi \rangle dA,$$

where ν denotes here the outer unit normal to $S_r = \partial B_r$. The miracle in Witten's proof happens here: it can be easily shown that the boundary term $\int_{S_r} \langle \nabla_\nu \psi, \psi \rangle dA$ is asymptotic to $m(g)$ times some finite positive constant c as r goes to ∞. After passing to the limit one is left with $m(g) = c(\int_M (|\nabla\psi|^2 + \frac{S}{4}|\psi|^2) v_g$, which implies $m(g) \geq 0$. The equality $m(g) = 0$ requires ψ to be parallel for any ψ constructed this way, in particular the spinor bundle of (M^n, g) must be trivialized by parallel spinors, from which the identity $(M^n, g) = (\mathbb{R}^n, \mathrm{can})$ can be deduced. An alternative spinorial proof but with supplementary assumptions on the dimension or the Weyl tensor has been given by B. Ammann and E. Humbert [21] using the Green's operators associated to the Dirac operator, see Section 8.5.