Chapter 3
Some preliminaries for families of cyclic covers

In this chapter we collect the remaining preparations for the computations concerning the VHS of our families $\pi : \mathcal{C} \to \mathcal{P}_n$ of cyclic covering of $\mathbb{P}^1$, which we construct in this chapter.

Let $\mathcal{V}$ denote the VHS of the family $\mathcal{X} \to Y$ of curves and $\mathrm{Mon}^0(\mathcal{V})$ denote the identity component of the Zariski closure of the monodromy group of $\mathcal{V}$. In Section 3.1 we introduce the generic Hodge group $\mathrm{Hg}(\mathcal{V})$, which is the maximum of the Hodge groups of all occurring Hodge structures in $\mathcal{V}$. Moreover $\mathrm{Hg}(\mathcal{V})$ coincides with the Hodge groups of the Hodge structures in $\mathcal{V}$ over the complement of a unification of countably many submanifolds of Y. Our families $\pi : \mathcal{C} \to \mathcal{P}_n$ are constructed in Section 3.2. We will also make some general remarks about the monodromy representation of $\mathcal{V}$ including the fact that the Galois group action yields an eigenspace decomposition in Section 3.2. In Section 3.3 we make some explicit computations of the monodromy representations of these eigenspaces. These computations are motivated from the fact that $\mathrm{Mon}^0(\mathcal{V})$ is a normal subgroup of the derived group $\mathrm{Hg}^{\mathrm{der}}(\mathcal{V})$ of the generic Hodge group! as we see in Section 3.1.

3.1 The generic Hodge group

We want to study the variations of Hodge structures (VHS) of the families of cyclic covers of $\mathbb{P}^1$, which will be constructed in the next section. Hence let us first make some general observations about the relation between their monodromy groups and Hodge groups resp., Mumford-Tate groups. These observation lead to the definition of the generic Hodge group defined below.

Proposition 3.1.1. *Let W be a connected complex manifold and $\mathcal{V}$ be a polarized variation of rational Hodge structures of weight k over W. Then there is a countable union $W' \subset W$ of submanifolds such that all $\mathrm{MT}(\mathcal{V}_p)$ coincide (up to conjugation by integral matrices) for all $p \in W \setminus W'$. Moreover one has $\mathrm{MT}(\mathcal{V}_{p'}) \subset \mathrm{MT}(\mathcal{V}_p)$ for all $p' \in W'$ and $p \in W \setminus W'$.*

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Proof. (see [43], Subsection 1.2) □

Remark 3.1.2. There exist the following versions of the previous proposition:

If one replaces W by a connected complex algebraic manifold in the previous proposition, the submanifolds $W' \subset W$ of the previous proposition are algebraic, too (see also [43], Subsection 1.2).

Now let F be a totally real number field, W be a complex connected algebraic manifold, $\mathcal{A} \to W$ be a family of abelian varieties and $\mathcal{V}$ be its polarized variation of F-Hodge structures of weight 1 over W. Then there is a countable union $W' \subset W$ of subvarieties such that all $\mathrm{MT}(\mathcal{V}_p)$ coincide (up to conjugation by integral matrices) for all closed $p \in W \setminus W'$ (see [42], Subsection 1.2).

The previous remark motivates the definition of the generic Mumford-Tate group $\mathrm{MT}_F(\mathcal{V})$ of a polarized variation $\mathcal{V}$ of F-Hodge structures of weight 1 of a family of abelian varieties over a connected complex algebraic manifold W. Moreover the preceding proposition motivates the definition of the generic Mumford-Tate group $\mathrm{MT}(\mathcal{V})$ of a polarized variation $\mathcal{V}$ of $\mathbb{Q}$-Hodge structures of weight k on a connected complex manifold. The generic Mumford-Tate group is given by $\mathrm{MT}_F(\mathcal{V}) = \mathrm{MT}_F(\mathcal{V}_p)$ resp., $\mathrm{MT}(\mathcal{V}) = \mathrm{MT}(\mathcal{V}_p)$ for all closed $p \in W \setminus W'$.

Since the image of the embedding $\mathrm{SL}(\mathcal{V}_{F,p}) \hookrightarrow \mathrm{GL}(\mathcal{V}_{F,p})$ is independent with respect to the chosen coordinates on $\mathcal{V}_{F,p}$, Lemma 1.3.17 allows us to define the generic Hodge group $Hg_F(\mathcal{V}) := (\mathrm{MT}_F(\mathcal{V}) \cap \mathrm{SL}_F(\mathcal{V}))^0$ such that $\mathrm{Hg}_F(\mathcal{V}) = \mathrm{Hg}_F(\mathcal{V}_p)$ for all (closed) $p \in W \setminus W'$.

Definition 3.1.3. Let $\mathbb{Q} \subseteq K \subseteq \mathbb{R}$ be a field and $\mathcal{V} = (\mathcal{V}_K, \mathcal{F}^\bullet, Q)$ be a polarized variation of K Hodge structures on a connected complex manifold D. Then $\mathrm{Mon}^0_K(\mathcal{V})_p$ denotes the connected component of identity of the Zariski closure of the monodromy group in $\mathrm{GL}((\mathcal{V}_K)_p)$ for some $p \in D$. For simplicity we write $\mathrm{Mon}^0(\mathcal{V})_p$ instead of $\mathrm{Mon}^0_{\mathbb{Q}}(\mathcal{V})_p$.

Theorem 3.1.4. *Keep the assumptions and notations of Proposition* 3.1.1. *One has that* $\mathrm{Mon}^0_F(\mathcal{V})_p$ *is a subgroup of* $\mathrm{MT}^{\mathrm{der}}_F(\mathcal{V}_p)$ *for all* $p \in W \setminus W'$. *Moreover for a variation of* $\mathbb{Q}$ *Hodge structures one has that* $\mathrm{Mon}^0(\mathcal{V})_p$ *is a normal subgroup of* $\mathrm{MT}^{\mathrm{der}}(\mathcal{V}_p)$ *and*

$$\mathrm{Mon}^0(\mathcal{V})_p = \mathrm{MT}^{\mathrm{der}}(\mathcal{V}_p)$$

for all $p \in W \setminus W'$, *if* $\mathcal{V}_{\mathbb{Q}}$ *has a CM point.*

Proof. (see [43], Theorem 1.4 for the statement about the variations of $\mathbb{Q}$ Hodge structures and [42], Properties 7.14 for the statement about the variations of F Hodge structures) □

Corollary 3.1.5. *Keep the assumptions of Proposition* 3.1.1. *Then the group* $\mathrm{Mon}^0(\mathcal{V})$ *is semisimple.*

Proof. By Theorem 3.1.4, the Lie subalgebra $Lie(\mathrm{Mon}^0_{\mathbb{Q}}(\mathcal{V})_{\mathbb{R}})$ of $Lie(\mathrm{MT}^{\mathrm{der}}_{\mathbb{Q}}(\mathcal{V})_{\mathbb{R}})$ is an ideal. Recall that $\mathrm{MT}^{\mathrm{der}}_{\mathbb{Q}}(\mathcal{V})_{\mathbb{R}}$ is semisimple. Hence the algebra $Lie(\mathrm{Mon}^0_{\mathbb{Q}}(\mathcal{V})_{\mathbb{R}})$ consists of the direct sum of simple subalgebras of $Lie(\mathrm{MT}^{\mathrm{der}}_{\mathbb{Q}}(\mathcal{V})_{\mathbb{R}})$. Thus $\mathrm{Mon}^0_{\mathbb{Q}}(\mathcal{V})_{\mathbb{R}}$ and $\mathrm{Mon}^0(\mathcal{V})$ are semisimple. □

3.2 Families of covers of the projective line

Let S be some $\mathbb{C}$-scheme. Recall that the covers $c_1 : V_1 \to \mathbb{P}^1_S$ and $c_2 : V_2 \to \mathbb{P}^1_S$ are equivalent, if there is a S-isomorphism $j : V_1 \to V_2$ such that $c_1 = c_2 \circ j$.

In this section we construct a family of cyclic covers of $\mathbb{P}^1$ such that all equivalence classes of covers with a fixed number of branch points with fixed branch indices are represented by some of its fibers. For us it is sufficient to start with a space, which is not a moduli scheme, but whose closed points "hit" all equivalence classes of covers of $\mathbb{P}^1$ with Galois group $G = (\mathbb{Z}/m, +)$ and a fixed number of branch points with fixed branch indices.

We start with the space

$$(\mathbb{P}^1)^{n+3} \supset \mathcal{P}_n := (\mathbb{P}^1)^{n+3} \setminus \{z_i = z_j | i \neq j\},$$

which parametrizes the injective maps $\phi : N \to \mathbb{P}^1$, where $N := \{s_1, \ldots, s_{n+3}\}$. Thus a point $q \in \mathcal{P}_n$ corresponds to an injective map $\phi_q : N \to \mathbb{P}^1$.[1] One can consider $\mathcal{P}_n$ as configuration space of $n+3$ ordered points, too.

We endow the points $s_k \in N$ with some local monodromy data $\alpha_k = e^{2\pi i \mu_k}$, where

$$\mu_k \in \mathbb{Q}, \quad 0 < \mu_k < 1 \quad \text{and} \quad \sum_{k=1}^{n+3} \mu_k \in \mathbb{N}.$$

Now we construct a family of covers of $\mathbb{P}^1$ by these local monodromy data:

Construction 3.2.1. Let m be the smallest integer such that $m\mu_k \in \mathbb{N}$ for $k = 1, \ldots, n+3$, and $D_k \subset \mathbb{P}_{\mathcal{P}_n} := \mathbb{P}^1 \times \mathcal{P}_n$ be the prime divisor given by

$$D_k = \{(a_k, a_1, \ldots, a_k, \ldots, a_{n+3})\}.$$

[1] The set N is some arbitrary finite set, where the set S of the preceding chapter is a concrete set $S \subset \mathbb{P}^1$ given by $S = \phi_q(N)$ for some $q \in \mathcal{P}_n$.

Let D be the divisor

$$D := \sum_{k=1}^{n+3} m\mu_k D_k \sim mD_0 \quad \text{with} \quad D_0 := (\sum_{k=1}^{n+3} \mu_k) \cdot (\{0\} \times \mathcal{P}_n).$$

By the sheaf $\mathcal{L} := \mathcal{O}_{\mathbb{P}_{\mathcal{P}_n}}(D_0)$ and the divisor D, we obtain an irreducible cyclic cover $\mathcal{C}$ of degree m onto $\mathbb{P}_{\mathcal{P}_n}$ as in [20], §3 (where irreducible means that the covering variety is irreducible). By $\pi : \mathcal{C} \to \mathbb{P}^1 \times \mathcal{P}_n \overset{pr_2}{\to} \mathcal{P}_n$, this cyclic cover yields a family of irreducible cyclic covers of degree m onto $\mathbb{P}^1$.

Suppose that r divides m. By taking the quotient of the subgroup of order r of the Galois group of the cyclic cover $\mathcal{C} \to \mathbb{P}^1 \times \mathcal{P}_n$, one gets a family $\pi_r : \mathcal{C}_r \to \mathcal{P}_n$ of cyclic covers of degree $\frac{m}{r}$ onto $\mathbb{P}^1$. Let $\phi_r : \mathcal{C} \to \mathcal{C}_r$ denote the quotient map. One has

$$\pi = \pi_r \circ \phi_r.$$

Remark 3.2.2. Without loss of generality one may assume that $q := (a_1, \ldots, a_{n+3}) \in \mathcal{P}_n$ is contained in $\mathbb{A}^{n+3}$, too. Thus the fiber $\mathcal{C}_q$ is given by the equation

$$y^m = (x - a_1)^{d_1} \cdot \ldots \cdot (x - a_{n+3})^{d_{n+3}}$$

with $d_k = m\mu_k$. By Remark 2.1.5, the local monodromy datum α_k describes the lifting of a path γ_k around $a_k \in \mathbb{P}^1$.[2] One checks easily that each equivalence class of cyclic covers of degree m with $n+3$ branch points and fixed branch indexes $d_1, \ldots, d_{n+3}$ is represented by some fibers of $\mathcal{C}$. Moreover for $C = \mathcal{C}_q$ the quotient C_r of Remark 2.2.10 is given by the fiber $(\mathcal{C}_r)_q$.

A family of smooth algebraic curves over $\mathbb{C}$ determines a proper submersion $\tau : X \to Y$ in the category of differentiable manifolds ([61], Proposition 9.5). By the Ehresmann theorem, we obtain that over any contractible submanifold W of Y the family is diffeomorphic to $X_0 \times W$, where X_0 is the fiber of some point $0 \in W$. This fact has some consequences for the monodromy representation of the variation of integral Hodge structures.

Recall that $R^1\tau_*(\mathbb{Z})$ is the sheaf associated to the presheaf given by

$$V \to H^1(\tau^{-1}(V), \mathbb{Z}|_{\pi^{-1}(V)})$$

for all open subsets $V \subset \mathcal{P}_n$. Moreover we have

$$H^1(X_0, \mathbb{Z}) = H^1(X_W, \mathbb{Z}) = (R^1\tau_*(\mathbb{Z}))(W)$$

for some contractible $W \subset \mathcal{P}_n$ with $0 \in W$, which implies that $R^1\tau_*(\mathbb{Z})$ is a local system (see [61], 9.2.1).

[2] This circumstance explains the term "local monodromy datum".

By using these facts, one can easily ensure that the monodromy group of the VHS of a family of curves can be calculated over any arbitrary field $\mathbb{Q} \subseteq K \subseteq \mathbb{C}$:

Lemma 3.2.3. *Let K be a field with $char(K) = 0$. Moreover let $\tau : X \to Y$ be a holomorphic family of curves. Then we obtain*

$$R^1\tau_*(K) = R^1\tau_*(\mathbb{Z}) \otimes_{\mathbb{Z}} K.$$

Proof. The sheaf $R^1\tau_*(K)$ is given by the sheafification of the presheaf

$$V \to H^1(\tau^{-1}(V), K|_{\tau^{-1}(V)}).$$

Hence by the description of the cohomology by Čech complexes, this presheaf is given by

$$V \to H^1(\tau^{-1}(V), \mathbb{Z}|_{\tau^{-1}(V)}) \otimes_{\mathbb{Z}} K.$$

By the fact that a local section of $\mathbb{Z}$ or K on a connected component of V resp., $\tau^{-1}(V)$ is constant, one does not need to differ between the locally constant sheaves given by $\mathbb{Z}$ resp., K on X or Y for the computation of $R^1\tau_*(K)$. This yields the desired identification. □

By the fact that the integral cohomology of a curve does not have torsion, one concludes:

Corollary 3.2.4. *Keep the assumptions of Lemma* 3.2.3. *Then the monodromy representations of $R^1\tau_*(\mathbb{Z})$ and $R^1\tau_*(K)$ coincide.*

Remark 3.2.5. Recall that we have an eigenspace decomposition of

$$H^1(\mathcal{C}_0, \mathbb{C}) = H^1(\mathcal{C}_0, \mathbb{Z}) \otimes \mathbb{C}$$

with respect to the Galois group action. By $H^1(\mathcal{C}_0, \mathbb{C}) = (R^1\pi_*(\mathbb{C}))(W)$ for some contractible $W \subset \mathcal{P}_n$ with $0 \in W$, we obtain an eigenspace decomposition of $(R^1\pi_*(\mathbb{C}))(W)$. Since we have this decomposition over all contractible $W \subset \mathcal{P}_n$, we can glue these eigenspaces, which yields a decomposition of the whole sheaf $R^1\pi_*(\mathbb{C})$ into eigenspaces with respect to the Galois group action.

Recall that we have an identification between the characters of the Galois group of some fiber and the elements $j \in \mathbb{Z}/(m)$. This identification allows a compatible identification between the characters of the Galois group of the family and the elements $j \in \mathbb{Z}/(m)$. Let $\mathcal{L}_j$ denote the eigenspace of $R^1\pi_*(\mathbb{C})$ with respect to the character j.

Remark 3.2.6. Let $0 \in \mathcal{P}_n$. We have a monodromy action $\rho_{\mathcal{C}}$ by diffeomorphisms on the fiber $\mathcal{C}_0$, which is induced by the gluing diffeomorphisms of the locally constant family of manifolds given by $\mathcal{C}$. Since these gluing diffeomorphisms induce the gluing homomorphisms of $R^1\pi_*(\mathbb{Z})$ in the obvious natural way, the monodromy representation ρ of $R^1\pi_*(\mathbb{Z})$ is given by

$$\rho(\gamma)(\eta) = (\rho_{\mathcal{C}}(\gamma))_*(\eta) \quad (\forall \quad \eta \in H^1(\mathcal{C}_0, \mathbb{Z})).$$

Remark 3.2.7. Since each gluing diffeomorphism of the locally constant family of manifolds corresponding to $\mathcal{C}$ respects intersection form, Remark 3.2.6 implies that the monodromy group of $R^1\pi_*(\mathbb{C})$ respects the polarization of the Hodge structures. Assume that $H^1_j(\mathcal{C}_q, \mathbb{C}) = (\mathcal{L}_j)_q$ satisfies that $H^{1,0}_j(\mathcal{C}_q) = n_1$ and $H^{0,1}_j(\mathcal{C}_q)_2 = n_2$. This means that the polarized variation of integral Hodge structure endows $(\mathcal{L}_j)_q$ with an Hermitian form with signature (n_1, n_2). Hence the monodromy group of this eigenspace is contained in $\mathrm{U}(n_1, n_2)$. In this sense we say that $\mathcal{L}_j$ is of type (n_1, n_2).

3.3 The homology and the monodromy representation

In this section we study the monodromy representation of $\pi_1(\mathcal{P}_n)$ on the dual of $R^1\pi_*(\mathbb{C})$ given by the complex homology. This will yield corresponding statements for the monodromy representation of $R^1\pi_*(\mathbb{C})$.

In the case of the configuration space $\mathcal{P}_n$ of $n+3$ points, we make a difference between these different points. One says that the points are "colored" by different "colors". Moreover one can identify its fundamental group with the subgroup of the braid group on $n+3$ strands in $\mathbb{P}^1$, which is given by the braids leaving the strands invariant (see [24], Chapter **I**. 3.). This subgroup of the braid group is called the colored braid group. An element of this group is for example given by the Dehn twist T_{k_1,k_2} with $1 \leq k_1 < k_2 \leq n+3$. The Dehn twist T_{k_1,k_2} is given by leaving a_{k_2} "run" counterclockwise around a_{k_1}.

Now we consider a fiber $C = \mathcal{C}_q$ of $\mathcal{C}$. Recall that C is a cyclic cover of $\mathbb{P}^1$ described in Chapter 2. Let ψ denote the generator of the Galois group as in Section 2.2. We keep the notation of Chapter 2.

Consider the eigenspace $\mathbb{L}_j$, which can be extended from a local system on $\mathbb{P}^1 \setminus S$ to a local system on $\mathbb{P}^1 \setminus S_j$ with $S_j = \{a_1, \ldots, a_{n_j+3}\}$. For simplicity one may without loss of generality assume that $a_{n_j+3} = \infty$ and $a_k \in \mathbb{R}$ such that $a_k < a_{k+1}$ for all $k = 1, \ldots, n_{j+2}$. Here we assume that δ_k is the oriented path from a_k to a_{k+1} given by the straight line.

Construction 3.3.1. Let ζ be a path on $\mathbb{P}^1$. Assume without loss of generality that $\zeta((0,1))$ is contained in a simply connected open subset U of $\mathbb{P}^1 \setminus S$. Otherwise we decompose ζ into such paths. It has m liftings $\zeta^{(0)}, \ldots, \zeta^{(m-1)}$ to C such that $\psi(\zeta^{(\ell)}) = \zeta^{([\ell-1]_m)}$. By the tensorproduct of $\mathbb{C}$ with the free abelian group generated by the paths on C, one obtains the vector space of $\mathbb{C}$-valued paths on C. Now let $c \in \mathbb{C}$ and take the linear combination of $\mathbb{C}$-valued paths on C given by

$$\hat{\zeta} = c\zeta^{(0)} + \ldots + ce^{2\pi i \frac{jr}{m}} \zeta^{(r)} + \ldots + ce^{2\pi i \frac{j(m-1)}{m}} \zeta^{(m-1)}.$$

By the assumptions, one verifies easily that $\psi(\hat{\zeta}) = e^{2\pi i \frac{j}{m}} \hat{\zeta}$. Moreover by the local sections given by $c, \ldots, ce^{2\pi i \frac{jr}{m}}, \ldots, ce^{2\pi i \frac{j(m-1)}{m}}$ on the corresponding sheets over U containing the different $\zeta^{(\ell)}((0,1))$, one obtains a corresponding section $\tilde{c} \in \mathbb{L}_j(U)$. In this sense we have a $\mathbb{L}_j$-valued path $\tilde{c} \cdot \zeta$ on $\mathbb{P}^1$.

Remark 3.3.2. Consider the (oriented) path δ_k from the branch point a_k to the branch point a_{k+1}. Let e_k be a non-zero local section of $\mathbb{L}_j$ defined over an open set containing $\delta_k((0,1))$. The $\mathbb{L}_j$-valued path $e_k \cdot \delta_k$ yields an element $[e_k \cdot \delta_k]$ of the homology group of $H_1(C, \mathbb{C})$, which is represented by the corresponding linear combination of paths in C lying over δ_k. It has the character j with respect to the Galois group representation. Let $H_1(C, \mathbb{C})_j$ denote the corresponding eigenspace.

Definition 3.3.3. A partition of S_j into some disjoint sets $S^{(1)} \cup \ldots \cup S^{(\ell)} = S_j$ is stable with respect to the local monodromy data μ_k of $\mathbb{L}_j$, if

$$\sum_{a_k \in S^{(1)}} \mu_k \notin \mathbb{N}, \ldots, \sum_{a_k \in S^{(\ell)}} \mu_k \notin \mathbb{N}.$$

Theorem 3.3.4. *Assume that* $S_j = \{a_i : i = 1, \ldots, n_j + 3\}$ *has the stable partition* $\{a_1, \ldots, a_{\ell+1}\}, \{a_{\ell+2}, \ldots, a_{n_j+3}\}$ *for some* $1 \leq \ell \leq n_j + 1$. *Then the eigenspace* $H_1(C, \mathbb{C})_j$ *of the complex homology group of* C *has a basis given by*

$$\mathcal{B} = \{[e_k \delta_k] : k = 1, \ldots, \ell\} \cup \{[e_k \delta_k] : k = \ell + 2, \ldots, n_j + 2\}.$$

Proof. By [36], Lemma 4.5, one has that $\{[e_k \delta_k] : k = 1, \ldots, n_j + 1\}$ is a basis of $H_1(C, \mathbb{C})_j$. Hence $\{[e_k \delta_k] : k = 1, \ldots, n_j + 2\}$ is not linearly independent.

One can compute a non-trivial linear combination, which yields 0, in the following way: Choose a non-zero section of $\mathbb{L}_j$ over

$$U = \mathbb{P}^1 \setminus (\bigcup_{k=1}^{n_j+2} \delta_k).$$

This yields a linear combination of the sheets over U, on which ψ acts by j. By the boundary operator ∂, one gets the desired non-trivial linear combination of $\mathbb{L}_j$-valued paths, which is equal to 0. Now let α_k denote the local monodromy datum of $\mathbb{L}_j$ around $a_k \in S_j$ for all $k = 1, \ldots, n_j + 3$. By the local monodromy data, one can easily compute this linear combination. This computation yields that $\{\delta_1, \ldots, \delta_\ell\} \cup \{\delta_{\ell+2}, \ldots, \delta_{n_j+2}\}$ is linearly independent, if and only if $\{a_1, \ldots, a_{\ell+1}\}, \{a_{\ell+2}, \ldots, a_{n_j+3}\}$ is a stable partition. □

Let α_k denote the local monodromy datum of $\mathbb{L}_j$ around $a_k \in S_j$ for all $k = 1, \ldots, n_j + 3$. One has up to a certain normalization of the basis vectors $[e_1\delta_1], \ldots [e_1\delta_{n_j+1}]$ the following description of the monodromy representation:

The Dehn twist $T_{k,k+1}$ leaves obviously δ_ℓ invariant for all $|k - \ell| > 1$. Moreover by following a path representing $T_{k,k+1}$, one sees that the monodromy action of $T_{k,k+1}$ on $H_1(C, \mathbb{C})_j$ (induced by push-forward) is given by

$$\begin{aligned} [e_{k-1}\delta_{k-1}] &\to [e_{k-1}\delta_{k-1}] + \alpha_k(1 - \alpha_{k+1})[e_k\delta_k], \\ [e_k\delta_k] &\to \alpha_k\alpha_{k+1}[e_k\delta_k] \\ \text{and} \quad [e_{k+1}\delta_{k+1}] &\to [e_{k+1}\delta_{k+1}] + (1 - \alpha_k)[e_k\delta_k]. \end{aligned}$$

Hence the monodromy representation is given by:

Proposition 3.3.5. *The monodromy representation of $T_{\ell,\ell+1}$ on $H_1(C, \mathbb{C})_j$ is given with respect to the basis $\{[e_k\delta_k] | k = 1, \ldots n_j + 1\}$ of $H_1(C, \mathbb{C})_j$ by the matrix with the entries:*

$$M_{\ell,\ell+1}(a, b) = \begin{cases} 1 & : \quad a = b \quad \textit{and} \quad a \neq \ell \\ \alpha_\ell\alpha_{\ell+1} & : \quad a = b = \ell \\ \alpha_\ell(1 - \alpha_{\ell+1}) & : \quad a = \ell \quad \textit{and} \quad b = \ell - 1 \\ 1 - \alpha_\ell & : \quad a = \ell \quad \textit{and} \quad b = \ell + 1 \\ 0 & : \quad \textit{elsewhere} \end{cases}$$

Remark 3.3.6. The monodromy representation of Proposition 3.3.5 corresponds to an eigenspace in the local system given by the direct image of the complex homology. By integration over $\mathbb{C}$-valued paths, this eigenspace is the dual local system of $\mathcal{L}_{m-j}$. By the polarization, $\mathcal{L}_j$ is the dual of $\mathcal{L}_{m-j}$, too. Hence Proposition 3.3.5 yields the monodromy representation of $\mathcal{L}_j$.