Introduction

These lecture notes deal with construction methods of Calabi-Yau manifolds with a special arithmetic property. In these methods we use curves with a similar arithmetic property, namely, complex multiplication. In the case of abelian varieties complex multiplication has been well studied by number theorists. The first six chapters describe how this theory for abelian varieties can be applied to the construction of curves with complex multiplication. The remaining five chapters and the appendix are devoted to the construction methods of Calabi-Yau manifolds with a similarly defined arithmetic property.

We give new examples of families of curves with dense sets of complex multiplication fibers and new examples of families of Calabi-Yau manifolds with a dense set of fibers with a similar arithmetic property. Moreover we will acquaint the reader with Mumford-Tate groups, which we use as a main tool for the study of Hodge structures and of variations of Hodge structures. The generic Mumford-Tate groups of families of cyclic covers of the projective line will be computed for a large class of examples.

Let us consider curves and Hodge structures on curves. In particular elliptic curves are both Calabi-Yau manifolds and abelian varieties. In general the points on a curve C of genus g generate a commutative group, which can be endowed with the structure of an abelian variety of dimension g, which is the Jacobian $\mathrm{Jac}(C)$ of C. The curve C can be obtained from $\mathrm{Jac}(C)$ and the principal polarization on $\mathrm{Jac}(C)$. In order to study the curve C and its properties one can also study $\mathrm{Jac}(C)$. Abelian varieties and their arithmetic properties have been well-studied by number theorists.

By Riemann's theorem, a polarized abelian variety with symplectic basis corresponds to a pure polarized integral Hodge structure of weight 1. Thus curves are determined by their Hodge structures. Therefore curves satisfy a Torelli Theorem. For Calabi-Yau manifolds one has also a local Torelli theorem. Thus one can study curves and Calabi-Yau manifolds in terms of their Hodge structures.

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Let $\mathbb{Z} \subseteq R \subseteq \mathbb{R}$ be a ring. Recall that an R-Hodge structure on an R-module V is given by a decomposition of $V_{\mathbb{C}}$ into subvector spaces $V^{p,q}$ with $\overline{V^{p,q}} = V^{q,p}$. We will see that each R-Hodge structure on V can also be given by a corresponding representation

$$h : \mathbb{S} \to \mathrm{GL}(V_{\mathbb{R}})$$

of the Deligne torus $\mathbb{S}$, which is the algebraic subgroup of $\mathrm{GL}(\mathbb{R}^2)$ given by the matrices

$$M(x,y) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}.$$

If V and h yield a $\mathbb{Q}$-Hodge structure, we use the representation h for the definition of the Mumford-Tate group $\mathrm{MT}(V,h)$. The Mumford-Tate group $\mathrm{MT}(V,h)$ is the smallest subgroup of $\mathrm{GL}(V_{\mathbb{R}})$ defined over $\mathbb{Q}$ such that $h(\mathbb{S})$ is contained in $\mathrm{MT}(V,h)$. For a rational Hodge structure (V,h) of weight k one can replace $\mathbb{S}$ by its subgroup S^1 given by the matrices $M(x,y)$ with

$$\det M(x,y) = 1.$$

In this case one can also replace $\mathrm{MT}(V,h)$ by the analogously defined Hodge group $\mathrm{Hg}(V,h)$. The Hodge group $\mathrm{Hg}(V,h)$ coincides with the Zariski connected component of the identity in $\mathrm{MT}(V,h) \cap \mathrm{SL}(V)$. For any field F with $\mathbb{Q} \subseteq F \subseteq \mathbb{R}$ one can also consider F-Hodge structures (V,h) and define $\mathrm{MT}_F(V,h)$ and $\mathrm{Hg}_F(V,h)$ in an analogous way.

Let us consider the information which can be obtained from $\mathrm{MT}(V,h)$: for example one says that an elliptic curve E has complex multiplication, if E has a non-trivial endomorphism. This name is motivated by the fact that in this case the endomorphism ring of E is a CM field. In general an abelian variety X of dimension g is of CM type, if its endomorphism algebra contains a commutative $\mathbb{Q}$-algebra of dimension $2g$. The Mumford-Tate group of the Hodge structure on $H^1(X,\mathbb{Q})$ is a torus, if and only if X is of CM type. We say that a rational Hodge structure (V,h) has complex multiplication (CM), if $\mathrm{MT}(V,h)$ is a torus. For a curve C the Hodge structures on $H^1(C,\mathbb{Q})$ and $H^1(\mathrm{Jac}(C),\mathbb{Q})$ are isomorphic. Hence we say that a curve has CM, if the Mumford-Tate group of the Hodge structure on $H^1(C,\mathbb{Q})$ is a torus algebraic group.

Remark 1. *One can also study families of compact Kähler manifolds and their variations of Hodge structures in terms of Mumford-Tate groups. Let D be a connected complex manifold and $\mathcal{V}$ be a polarized variation of $\mathbb{Q}$-Hodge structures of weight k over D. Then over a dense subset D^0 of D the Mumford-Tate groups of all Hodge structures coincide. Let* $\mathrm{MT}(\mathcal{V})$ *denote the common Mumford-Tate group. The Hodge structures over the points of the complement of D^0 have a Mumford-Tate group contained in* $\mathrm{MT}(\mathcal{V})$. *The group* $\mathrm{MT}(\mathcal{V})$ *is called the generic Mumford-Tate group.*

We will introduce Shimura data, which consist of a reductive $\mathbb{Q}$-algebraic group G and a representation $h : \mathbb{S} \to G_{\mathbb{R}}$ satisfying certain conditions. Again consider an abelian variety X. For example the pair consisting of the Mumford-Tate group of the Hodge structure on $H^1(X, \mathbb{Q})$ and the representation h given by this Hodge structure yields a Shimura datum. By using the conditions which a Shimura datum has to satisfy we obtain:

Theorem 2. *Let (G, h) be a Shimura datum and W be a finite dimensional real vector space. Then the conjugacy class of h in $G_{\mathbb{R}}$ can be endowed with the structure of a complex manifold D. Moreover each closed embedding $G_{\mathbb{R}} \to \mathrm{GL}(W)$ yields a variation of Hodge structures over D such that over a dense set of points $p \in D$ one has Hodge structures with complex multiplication.*

Note that in the case of the Hodge structure on $H^1(X, \mathbb{Q})$ given by h and the closed embedding

$$\mathrm{id} : \mathrm{MT}(H^1(X, \mathbb{Q}), h) \hookrightarrow \mathrm{GL}(H^1(X, \mathbb{Q}))$$

the assumptions of the previous Theorem are satisfied, if X is an abelian variety.

We will give a definition of complex multiplication for arbitrary compact Kähler manifolds. Due to their application in theoretical physics we are especially interested in Calabi-Yau 3-manifolds. In theoretical physics one is also interested in complex multiplication (see [37], [38]).

Here a Calabi-Yau manifold X of dimension n is a compact Kähler manifold of dimension n such that $\Gamma(\Omega^i_X) = 0$ for all $i = 1, \ldots, n-1$ and $\omega_X \cong \mathcal{O}_X$.

For odd dimensional compact Kähler manifolds one has the intermediate Jacobians as a generalization of the Jacobians of curves. In general the intermediate Jacobian J is not an abelian variety, but only a complex torus. In the case of an arbitrary complex torus complex multiplication is defined as for an abelian variety. It can occur that the intermediate Jacobian J is constant for a family of Calabi-Yau 3-manifolds (see Example 1.6.9). Hence one intermediate Jacobian is not sufficient for an accurate description of Calabi-Yau 3-manifolds and their Hodge structures. Nevertheless the intermediate Jacobian of the manifold X of odd dimension k is of CM type, if $\mathrm{Hg}(H^k(X, \mathbb{Q}), h)$ is a torus. Moreover the endomorphism algebra of a Hodge structure (V, h) contains a commutative subalgebra of dimension equal to $\dim V$, if $\mathrm{MT}(V, h)$ is a torus. Thus we say that a compact Kähler manifold X of dimension n has CM over a totally real number field F, if $\mathrm{Hg}_F(H^n(X, F))$ is a torus. It would be very interesting to get mirror pairs of Calabi-Yau 3-manifolds with complex multiplication (see [23]).

One can also consider the Hodge groups of the Hodge structures $H^k(X, \mathbb{Q})$ for some $k \neq \dim X$. In the case of a Calabi-Yau manifold X of dimension $n > 3$, it may occur that the Hodge structure on $H^n(X, \mathbb{Q})$ has CM and the Hodge structure on $H^{n-1}(X, \mathbb{Q})$ has not CM for example. By considering the Hodge diamond of a Calabi-Yau manifold X of dimension $n \leq 3$, one concludes that this can not occur for $\dim X \leq 3$. In this case the condition of complex multiplication is equivalent to the property that for all k the Hodge group of $H^k(X, \mathbb{C})$ is commutative. We will call any family of Calabi-Yau n-manifolds, which has a dense set of fibers X satisfying the property that for all k the Hodge group of the Hodge structure on $H^k(X, \mathbb{Q})$ is commutative, a *CMCY* family of n-manifolds. Here we will give some examples of *CMCY* families of 3-manifolds and explain how to construct *CMCY* families of n-manifolds in an arbitrarily high dimension. Moreover we will explicitly determine some fibers with complex multiplication (see Example 7.3.1, Section 7.4, Remark 8.3.6, Remark 9.4.1 and Remark 11.3.13).

Example 3. *The first example of a CMCY family of 3-manifolds was given by C. Borcea [8]. This example uses the family $\mathcal{E}$ of elliptic curves given by*

$$\mathbb{P}^2 \supset V(y^2x_0 + x_1(x_1 - x_0)(x_1 - \lambda x_0)) \to \lambda \in \mathbb{A}^1 \setminus \{0, 1\}.$$

By $y \to -y$, one has a global involution ι on $\mathcal{E}$. Now let $\mathcal{E}_i$ with involution ι_i be a copy of $\mathcal{E}$ for $i = 1, 2, 3$. We construct the family

$$\mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3 / \langle (\iota_1, \iota_2), (\iota_2, \iota_3) \rangle \to (\mathbb{A}^1 \setminus \{0, 1\})^3.$$

By blowing up the singular sections, we obtain a CMCY family of Calabi-Yau 3-manifolds.

In a similar way one can use n copies of $\mathcal{E}$ and construct a *CMCY* family of n-manifolds (see [56]). Similar to the previous example, we will use involutions on *CMCY* families to obtain new *CMCY* families of manifolds in higher dimension. The other main tool of construction which we use is motivated by the following example:

Example 4. *Starting with a family of cyclic covers of $\mathbb{P}^1$ with a dense set of CM fibers, E. Viehweg and K. Zuo [58] have constructed a CMCY family of 3-manifolds. This construction is given by a tower of projective algebraic manifolds starting with a family $\mathcal{F}_1$ of cyclic covers of $\mathbb{P}^1$ given by*

$$\mathbb{P}^2 \supset V(y_1^5 + x_1(x_1 - x_0)(x_1 - \alpha x_0)(x_1 - \beta x_0)x_0) \to (\alpha, \beta) \in \mathcal{M}_2,$$

which has a dense set of CM fibers. Since each of these covers given by the fibers of the family can be embedded into $\mathbb{P}^2$, the fibers of $\mathcal{F}_1$ are the branch loci of the fibers of a family $\mathcal{F}_2$ of cyclic covers of $\mathbb{P}^2$ of degree 5. Moreover the fibers of $\mathcal{F}_2$, which can be embedded into $\mathbb{P}^3$, are the branch loci of the fibers of a family $\mathcal{F}_3$ of cyclic covers of $\mathbb{P}^3$, which can be embedded into $\mathbb{P}^4$. The family $\mathcal{F}_3$ is given by

$$\mathbb{P}^4 \supset V(y_3^5 + y_2^5 + y_1^5 + x_1(x_1 - x_0)(x_1 - \alpha x_0)(x_1 - \beta x_0)x_0) \to (\alpha, \beta) \in \mathcal{M}_2.$$

By the adjunction formula, the fibers of $\mathcal{F}_3$ are Calabi-Yau 3-manifolds.

Let $q \in \mathcal{M}_2$. The fiber $(\mathcal{F}_3)_q$ has CM, if $(\mathcal{F}_2)_q$ has CM and $(\mathcal{F}_2)_q$ has CM, if $(\mathcal{F}_1)_q$ has CM. Because of this argument, the family $\mathcal{F}_3$ has a dense set of CM fibers which lie over the same points as the CM fibers of the family of curves we have started with.

The previous example contains a deformation of the Fermat quintic in $\mathbb{P}^4$, which is a well-studied example of a Calabi-Yau manifold with complex multiplication (see [38]). In the appendix we will give some examples of Calabi-Yau 3-manifolds which are not necessarily a fiber of a family with infinitely many CM fibers.

By the previous example, we are led to be interested in the examples of families of curves with a dense set of CM fibers for our search for $CMCY$ families of n-manifolds. There is an other motivation given by an open question in the theory of curves, too. In [11] R. Coleman formulated the following conjecture:

Conjecture 5. *Fix an integer $g \geq 4$. Then there are only finitely many complex algebraic curves C of genus g such that $\mathrm{Jac}(C)$ is of CM type.*

Let $\mathcal{P}_n$ denote the configuration space of $n+3$ points in $\mathbb{P}^1$. One can endow these $n+3$ points in $\mathbb{P}^1$ with local monodromy data and use these data for the construction of a family $\mathcal{C} \to \mathcal{P}_n$ of cyclic covers of $\mathbb{P}^1$ (see Construction 3.2.1).

The action of $\mathrm{PGL}_2(\mathbb{C})$ on $\mathbb{P}^1$ yields a quotient $\mathcal{M}_n = \mathcal{P}_n/\mathrm{PGL}_2(\mathbb{C})$. By fixing 3 points on $\mathbb{P}^1$, the quotient $\mathcal{M}_n$ can also be considered as a subspace of $\mathcal{P}_n$.

Remark 6. *In [29] J. de Jong and R. Noot gave counterexamples for $g = 4$ and $g = 6$ to the conjecture above. In [58] E. Viehweg and K. Zuo gave an additional counterexample for $g = 6$. The counterexamples are given by families $\mathcal{C} \to \mathcal{P}_n$ of cyclic covers of $\mathbb{P}^1$ with dense sets of CM fibers. Here we will find additional families $\mathcal{C} \to \mathcal{P}_n$ of cyclic genus 5 and genus 7 covers of $\mathbb{P}^1$ with dense sets of complex multiplication fibers, too.*

All new examples $\mathcal{C} \to \mathcal{P}_n$ of the preceding remark have a variation $\mathcal{V}$ of Hodge structures similar to the examples of J. de Jong and R. Noot [29], and of E. Viehweg and K. Zuo [58], which we call pure $(1,n) - VHS$. Let $\mathrm{Hg}(\mathcal{V})$ denote the generic Hodge group of $\mathcal{V}$ and let K denote an arbitrary maximal compact subgroup of $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{V})(\mathbb{R})$. In Section 4.4 we prove that a pure $(1,n) - VHS$ induces an open (multivalued) period map to the symmetric domain associated with $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{V})(\mathbb{R})/K$, which yields the dense sets of complex multiplication fibers. We obtain the following result in Chapter 6:

Theorem 7. *There are exactly 19 families $\mathcal{C} \to \mathcal{P}_n$ of cyclic covers of $\mathbb{P}^1$ which have a pure $(1,n) - VHS$ (including all known and new examples).*

We will use the fact that the monodromy group $\mathrm{Mon}^0(\mathcal{V})$ is a subgroup of the derived group $\mathrm{Hg}^{\mathrm{der}}(\mathcal{V})$ and we will study $\mathrm{Mon}^0(\mathcal{V})$. Let ψ be a generator of the Galois group of $\mathcal{C} \to \mathcal{P}_n$ and $C(\psi)$ be the centralizer of ψ in the symplectic group with respect to the intersection pairing on an arbitrary fiber of $\mathcal{C}$. In Chapter 4 we obtain the result, which will be useful for our study of $\mathrm{Hg}^{\mathrm{der}}(\mathcal{V})$ and $\mathrm{Mon}^0(\mathcal{V})$:

Lemma 8. *The monodromy group* $\mathrm{Mon}^0(\mathcal{V})$ *and the Hodge group* $\mathrm{Hg}(\mathcal{V})$ *are contained in* $C(\psi)$.

We will not be able to determine $\mathrm{Mon}^0(\mathcal{V})$ for all families $\mathcal{C} \to \mathcal{P}_n$ of cyclic covers of $\mathbb{P}^1$. But we will obtain for example the following results in Chapter 5:

Proposition 9. *Let $\mathcal{C} \to \mathcal{P}_n$ be a family of cyclic covers of degree m onto $\mathbb{P}^1$. Then one has:*

1. *If the degree m is a prime number ≥ 3, the algebraic groups* $C^{\mathrm{der}}(\psi)$, $\mathrm{Mon}^0(\mathcal{V})$ *and* $\mathrm{Hg}^{\mathrm{der}}(\mathcal{V})$ *coincide.*
2. *If $\mathcal{C} \to \mathcal{P}_{2g+2}$ is a family of hyperelliptic curves, one obtains*

$$\mathrm{Mon}^0(\mathcal{V}) = \mathrm{Hg}(\mathcal{V}) \cong \mathrm{Sp}_{\mathbb{Q}}(2g).$$

3. *In the case of a family of covers of $\mathbb{P}^1$ with 4 branch points, we need a pure $(1,1)-VHS$ to obtain an open period map to the symmetric domain associated with* $\mathrm{Hg}^{\mathrm{ad}}(\mathcal{V})(\mathbb{R})/K$.

By our new examples of Viehweg-Zuo towers, we will only obtain $CMCY$ families of 2-manifolds. C. Voisin [60] has described a method to obtain Calabi-Yau 3-manifolds by using involutions on $K3$ surfaces. C. Borcea [9] has independently arrived at a more general version of the latter method, which allows to construct Calabi-Yau manifolds in arbitrary dimension. By using this method, we obtain in Section 7.2:

Proposition 10. *For $i = 1,2$ assume that $\mathcal{C}^{(i)} \to V_i$ is a $CMCY$ family of n_i-manifolds endowed with the V_i-involution ι_i such that for all $p \in V_i$ the ramification locus $(R_i)_p$ of $\mathcal{C}_p^{(i)} \to \mathcal{C}_p^{(i)}/\iota_i$ consists of smooth disjoint hypersurfaces. In addition assume that V_i has a dense set of points $p \in V_i$ such that for all k the Hodge groups* $\mathrm{Hg}(H^k(\mathcal{C}_p^{(i)}, \mathbb{Q}))$ *and* $\mathrm{Hg}(H^k((R_i)_p, \mathbb{Q}))$ *are commutative. By blowing up the singular locus of the family $\mathcal{C}^{(1)} \times \mathcal{C}^{(2)}/\langle(\iota_1, \iota_2)\rangle$, one obtains a $CMCY$ family of n_1+n_2-manifolds over $V_1 \times V_2$ endowed with an involution satisfying the same assumptions as ι_1 and ι_2.*

Remark 11. *By the preceding proposition, one can apply the construction of C. Borcea and C. Voisin for families to obtain an infinite tower of $CMCY$ families of n-manifolds, which we call a Borcea-Voisin tower.*

Example 12. *The family $\mathcal{C} \to \mathcal{M}_1$ given by*

$$\mathbb{P}^2 \supset V(y_1^4 - x_1(x_1 - x_0)(x_1 - \lambda x_0)x_0) \to \lambda \in \mathcal{M}_1$$

has a pure $(1,1)-VHS$*. Hence by the construction of Viehweg and Zuo [58], one concludes that the family* $\mathcal{C}_2$ *given by*

$$\mathbb{P}^3 \supset V(y_2^4 + y_1^4 - x_1(x_1 - x_0)(x_1 - \lambda x_0)x_0) \to \lambda \in \mathcal{M}_1 \tag{1}$$

is a $CMCY$ *family of 2-manifolds.*

This family has many $\mathcal{M}_1$*-automorphisms. The quotients by some of these automorphisms yield new examples of* $CMCY$ *families of 2-manifolds. Moreover there are some involutions on* $\mathcal{C}_2$ *which make this family and its quotient families of* $K3$*-surfaces suitable for the construction of a Borcea-Voisin tower (see Section* 7.4 *for the construction of* $\mathcal{C}_2$*, and for the automorphism group and the quotient families of* $\mathcal{C}_2$ *see Section* 9.3*, Section* 9.4 *and Section* 9.5*).*

Example 13. *The family* $\mathcal{C} \to \mathcal{M}_3$ *given by*

$$\mathbb{P}(2,1,1) \supset V(y_1^3 - x_1(x_1 - x_0)(x_1 - ax_0)(x_1 - bx_0)(x_1 - cx_0)x_0) \to (a,b,c) \in \mathcal{M}_3$$

has a pure $(1,3)-VHS$*. The desingularization* $\tilde{\mathbb{P}}(2,2,1,1)$ *of the weighted projective space* $\mathbb{P}(2,2,1,1)$ *is given by blowing up the singular locus. By a modification of the construction of Viehweg and Zuo, the family* $\mathcal{W}$ *given by*

$$\begin{aligned}\tilde{\mathbb{P}}(2,2,1,1) \supset \tilde{V}(y_2^3 + y_1^3 - x_1(x_1 - x_0)(x_1 - ax_0)(x_1 - bx_0)(x_1 - cx_0)x_0)\\ \to (a,b,c) \in \mathcal{M}_3\end{aligned} \tag{2}$$

is a $CMCY$ *family of 2-manifolds. The family* $\mathcal{W}$ *has a degree 3 quotient, which yields a* $CMCY$ *family of 2-manifolds. Moreover it has an involution, which makes it and its degree 3 quotient suitable for the construction of a Borcea-Voisin tower (see Chapter 8 for the construction of* $\mathcal{W}$ *and Section* 9.1 *for its degree 3 quotient).*

By using the preceding example, we will obtain (see Section 9.2 for the construction and Section 10.3 for the maximality):

Theorem 14. *Let* $\mathbb{F}_3$ *be the Fermat curve of degree 3 and* $\alpha_{\mathbb{F}_3}$ *denote a generator of the Galois group of the degree 3 cover* $\mathbb{F}_3 \to \mathbb{P}^1$*. The family* $\mathcal{W}$ *has two* $\mathcal{M}_3$*-automorphism* α' *and* α'' *of order 3 such that the quotients* $\mathcal{W}\times\mathbb{F}_3/\langle(\alpha', \alpha_{\mathbb{F}_3})\rangle$ *and* $\mathcal{W}\times\mathbb{F}_3/\langle(\alpha'', \alpha_{\mathbb{F}_3})\rangle$ *have desingularizations, which are* $CMCY$ *families of 3-manifolds. Moreover one of these families is maximal.*

By using the V. V. Nikulins classification of involutions on $K3$ surfaces [51] and the construction of C. Voisin [60], we obtain in Chapter 11:

Theorem 15. *For each integer* $1 \leq r \leq 11$ *there exists a maximal holomorphic* $CMCY$ *family of algebraic 3-manifolds with Hodge number* $h^{2,1} = r$*.*

This book is organized as follows. The first three chapters explain well-known facts and yield an introduction of the notations. Chapter 1 is an introduction to Hodge Theory and Shimura varieties with a special view towards complex multiplication. We consider cyclic covers of $\mathbb{P}^1$ in Chapter 2. Moreover Chapter 3 introduces the remaining facts, which we need for the description of families of cyclic covers of $\mathbb{P}^1$ and their variations of Hodge structures.

In Chapter 4 we consider the Galois group action of a cyclic cover of $\mathbb{P}^1$ and we state first results for the generic Hodge group of a family $\mathcal{C} \to \mathcal{P}_n$. Moreover we will give a sufficient criterion for the existence of a dense set of CM fibers given by the pure $(1,n) - VHS$. In Chapter 5 we compute $\mathrm{Mon}^0(\mathcal{V})$, which provides much information about $\mathrm{Hg}(\mathcal{V})$. We will see that $\mathrm{Mon}^0(\mathcal{V})$ coincides with $C^{\mathrm{der}}(\psi)$ in infinitely many cases. In Chapter 6 we classify the examples of families of cyclic covers of $\mathbb{P}^1$ providing a pure $(1,n) - VHS$.

The basic methods of the construction of $CMCY$-families in higher dimension are explained in Chapter 7. We introduce the Borcea-Voisin tower and the Viehweg-Zuo tower and realize that only a small number of families of cyclic covers of $\mathbb{P}^1$ are suitable to start the construction of a Viehweg-Zuo tower. We will also discuss some methods to find concrete CM fibers at the end of this chapter. In Chapter 8 we will give a modified version of the method of E. Viehweg and K. Zuo to construct the $CMCY$ family of 2-manifolds given by (2). We consider the automorphism groups of our examples given by (1) and (2) in Chapter 9. This yields the further quotients of the families given by (1) and (2) which are $CMCY$ families of 2-manifolds. We will see that these quotients are endowed with involutions, which make them suitable for the construction of a Borcea-Voisin tower. Moreover we will construct the families $\mathcal{Q}$ and $\mathcal{R}$ of Theorem 14 in Chapter 9. The next chapter is devoted to the *length* of the Yukawa coupling of our examples families (motivated by the question of rigidity) and the Hodge numbers of their fibers. We finish this chapter with an outlook onto the possibilities to construct $CMCY$ families of 3-manifolds by quotients of higher order. In Chapter 11 we use directly the mirror construction of C. Voisin to obtain maximal holomorphic $CMCY$ families of 2-manifolds, which are suitable for the construction of a holomorphic Borcea-Voisin tower.

Throughout this book we use the conventions of Algebraic Geometry as in [26]. Most of the results and conventions about Hodge theory which we need can be found in [61].

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