1

Introduction

1.1 The Sixth Problem of Hilbert

1.1.1 The Mathematical Treatment of the Axioms of Physics

The sixth problem asked by Hilbert in the occasion of the International Congress of Mathematicians held in Paris in 1900 is concerned with the mathematical treatment of the axioms of Physics, by analogy with the axioms of Geometry. Precisely, it states as follows :

"Quant aux principes de la Mécanique, nous possédons déjà au point de vue physique des recherches d'une haute portée; je citerai, par exemple, les écrits de MM. Mach [81], Hertz [64], Boltzmann [14] et Volkmann [107]. Il serait aussi très désirable qu'un examen approfondi des principes de la Mécanique fût alors tenté par les mathématiciens. Ainsi le Livre de M. Boltzmann sur les Principes de la Mécanique nous incite à établir et à discuter au point de vue mathématique d'une manière complète et rigoureuse les méthodes basées sur l'idée de passage à la limite, et qui de la conception atomique nous conduisent aux lois du mouvement des continua. Inversement on pourrait, au moyen de méthodes basées sur l'idée de passage à la limite, chercher à déduire les lois du mouvement des corps rigides d'un système d'axiomes reposant sur la notion d'états d'une matière remplissant tout l'espace d'une manière continue, variant d'une manière continue et que l'on devra définir paramétriquement.

Quoi qu'il en soit, c'est la question de l'équivalence des divers systèmes d'axiomes qui présentera toujours l'intérêt le plus grand quant aux principes."

The problem, suggested by Boltzmann's work on the principles of mechanics, is therefore to develop "mathematically the limiting processes [. . .] which lead from the atomistic view to the laws of motion of continua", namely to obtain a unified description of gas dynamics, including all levels of description. In other words, the challenging question is whether macroscopic concepts such as the viscosity or the nonlinearity can be understood microscopically.

L. Saint-Raymond, *Hydrodynamic Limits of the Boltzmann Equation*,
Lecture Notes in Mathematics 1971, DOI: 10.1007/978-3-540-92847-8_1,

1.1.2 From Microscopic to Macroscopic Equations

Classical dynamics for systems constituted of identical particles are characterized by a Hamiltonian

$$H(x,v) = \frac{1}{2}\sum_{i=1}^{N} |v_i|^2 + \sum_{i\neq j} V(x_i - x_j)$$

with V a two-body potential.

The corresponding Liouville equation is

$$\partial_t f_N(t,x,v) + \mathbf{L} f_N(t,x,v) = 0 \tag{1.1}$$

where f_N is the density with respect to the Lebesgue measure of the system at time t, and the Liouville operator is given by

$$\mathbf{L} = \sum_{i=1}^{N} \left[\frac{\partial H}{\partial v_i}\frac{\partial}{\partial x_i} - \frac{\partial H}{\partial x_i}\frac{\partial}{\partial v_i} \right].$$

For a given configuration $\omega(t) = (x(t), v(t))$ the empirical density and momentum (which rigorously speaking are measures) are then defined by

$$R_\omega(X) = \frac{1}{N}\sum_{i=1}^{N} \delta(X - x_i)$$

$$Q_\omega(X) = \frac{1}{N}\sum_{i=1}^{N} v_i \delta(X - x_i)$$

Macroscopic equations such as the Euler equations or the Navier-Stokes equations (which have been historically derived through a continuum formulation of conservation of mass, momentum and energy) are then expected to be obtained as some asymptotics of the equations governing these observable quantities.

1.2 Formal Study of the Transitions

The microscopic versions of density, velocity, and energy should actually assume their macroscopic, deterministic values through the *law of large numbers*. Therefore, in order the equations describing the evolution of macroscopic quantities to be exact, certain limits have to be taken, with suitably chosen scalings of space, time, and other macroscopic parameters of the systems. So the first step in the derivation of such equations is a choice of scaling.

1.2.1 Scalings

Denote coordinates by (x, t) in the microscopic scale, and by $(\tilde{x}, \tilde{t})$ in the macroscopic scale. Let $\rho = N/L^3$ be the typical density in the microscopic unit, i.e. the number of particles per microscopic unit volume. Then, if ε is the ratio between the microscopic unit and the macroscopic unit, there are typically three choices of scalings :

- the Grad limit $\rho = \varepsilon, \quad (\tilde{x}, \tilde{t}) = (\varepsilon x, \varepsilon t)$;
  (The typical number of collisions per particle is finite.)
- the Euler limit $\rho = 1, \quad (\tilde{x}, \tilde{t}) = (\varepsilon x, \varepsilon t)$;
  (The typical number of collisions per particle is ε^{-1}.)
- the diffusive limit $\rho = 1, \quad (\tilde{x}, \tilde{t}) = (\varepsilon x, \varepsilon^2 t)$;
  (The typical number of collisions per particle is ε^{-2}.)

The Euler and diffusive limits will be referred to as hydrodynamic limits.

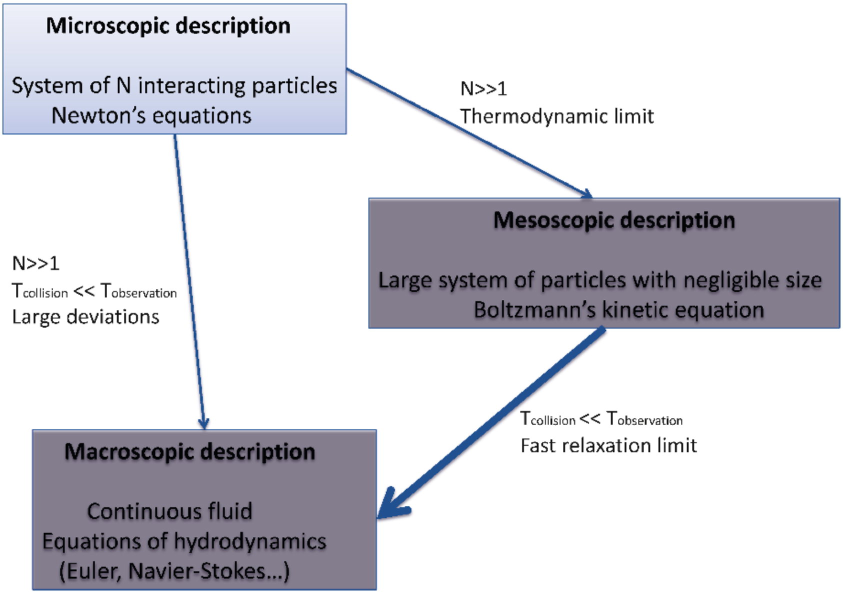

Fig. 1.1. Transitions between the different levels of description

1.2.2 Hydrodynamic Limits

To obtain hydrodynamic equations, we then differentiate the scaled empirical density and momentum and more precisely their integral agasinst any test function φ :

$$\int \varphi(\tilde{x}) R_{\omega(\tilde{t}/\varepsilon),\varepsilon}(\tilde{x}) d\tilde{x} = \frac{1}{N} \sum_{i=1}^{N} \varphi(\varepsilon x_i(\tilde{t}/\varepsilon)),$$

$$\int \varphi(\tilde{x}) Q_{\omega(\tilde{t}/\varepsilon),\varepsilon}(\tilde{x}) d\tilde{x} = \frac{1}{N} \sum_{i=1}^{N} v_i(\tilde{t}/\varepsilon) \varphi(\varepsilon x_i(\tilde{t}/\varepsilon)).$$

We get for instance

$$\frac{d}{d\tilde{t}} \frac{1}{N} \sum_{i=1}^{N} v_i(\tilde{t}/\varepsilon)\varphi(\varepsilon x_i(\tilde{t}/\varepsilon)) = -\frac{1}{N} \sum_{i=1}^{N} \varepsilon^{-1} \varphi(\varepsilon x_i) \frac{\partial H}{\partial x_i} + \frac{1}{N} \sum_{i=1}^{N} v_i \partial_i \varphi(\varepsilon x_i) \frac{\partial H}{\partial v_i}$$

$$= -\frac{1}{2N} \sum_{i=1}^{N} \nabla\varphi(\varepsilon x_i) \sum_{i \neq j} \frac{x_i - x_j}{\varepsilon} \cdot \nabla V\left(\frac{x_i - x_j}{\varepsilon}\right) + \frac{1}{N} \sum_{i=1}^{N} v_i \otimes v_i \nabla\varphi(\varepsilon x_i) + O(\varepsilon)$$

using Taylor's formula for φ, and symmetries to discard the main term.

In order to obtain the conservation of momentum in the **Euler equations** we then need to show that the microscopic current

$$-\frac{1}{2N} \sum_{i=1}^{N} \nabla\varphi(\varepsilon x_i) \sum_{i \neq j} \frac{x_i - x_j}{\varepsilon} \cdot \nabla V\left(\frac{x_i - x_j}{\varepsilon}\right)$$

converges to some macroscopic current $P = P(R, Q, E)$ depending on the macroscopic density, momentum and internal energy, in the limit $\varepsilon \to 0$. This convergence has to be understood in the sense of law of large numbers with respect to the density f_N (solution to the Liouville equation)

$$\frac{1}{N} \int f_N(t, \omega) \left| \sum_i \nabla\varphi(\varepsilon x_i) \left[\sum_{i \neq j} \frac{x_i - x_j}{\varepsilon} \cdot \nabla V\left(\frac{x_i - x_j}{\varepsilon}\right) - P(R, Q, E) \right] \right| d\omega \to 0 \tag{1.2}$$

The key observation, due to Morrey [86], is that (1.2) holds if we replace f_N by any Gibbs measure with Hamiltonian H, or more generally if "locally" f_N is a Gibbs measure of the Hamiltonian H.

The point is therefore to establish that "locally" $f_N(t)$ is a equilibrium measure with finite specific entropy. The conclusion follows then from the *ergodicity* of the infinite system of interacting particles : the translation invariant stationary measures of the dynamics such that the entropy per microscopic unit of volume is finite are Gibbs ($\exp(-\beta H)$).

The **Navier-Stokes equations** are the next order corrections to the Euler equations. In order to derive them one needs to show that the microscopic current is well approximated up to order ε by the sum of the macroscopic current $P = P(R, Q, E)$ and a viscosity term $\varepsilon\nu\nabla Q$ (in the sense of law of large numbers).

Since there is an ε appearing in the viscosity term, proving such an asymptotics requires to understand the next order correction to Boltzmann's hypothesis. This difficulty, recognized long time ago by Dobrushin, Lebowitz and Spohn, has been overcome recently for simplified particle dynamics : the mathematical interpretation is indeed given by the fluctuation-dissipation equation which states

$$\begin{aligned}-\frac{1}{2N}\sum_{i=1}^{N}\nabla\varphi(\varepsilon x_i)\sum_{i\neq j}\frac{x_i-x_j}{2\varepsilon}\cdot\nabla V\left(\frac{x_i-x_j}{\varepsilon}\right)\\ = P(R_{\omega,\varepsilon},Q_{\omega,\varepsilon},E_{\omega,\varepsilon})+\varepsilon\nu\nabla Q_{\omega,\varepsilon}+\varepsilon\mathbf{L}g_{\omega,\varepsilon}+o(\varepsilon)\end{aligned} \tag{1.3}$$

for some function $g_{\omega,\varepsilon}$, where $\mathbf{L}$ is the Liouville operator. In other words, the expected asymptotics is correct only up to a quotient of the image of the Liouville operator. The image of the Liouville operator is understood as a fluctuation, negligible in the relevant scale *after time average* : for any bounded function g

$$\varepsilon\int_0^t dsf_N(s,\omega)(\varepsilon\mathbf{L}g)(\omega)d\omega=\varepsilon^2(f_N(t,\omega)-f_N(s,\omega))g(\omega)d\omega=O(\varepsilon^2)$$

and is thus negligible to the first order in ε.

In order to avoid the difficulties of the multiscale asymptotics, we may turn to the **incompressible Navier-Stokes equations** which are invariant under the incompressible scaling

$$(x,t,u,p)\mapsto(\lambda x,\lambda^2 t,\lambda^{-1}u,\lambda^2 p)$$

under which the fluctuation-dissipation equation becomes

$$\begin{aligned}-\frac{1}{2N}\sum_{i=1}^{N}\nabla\varphi(\varepsilon x_i)\sum_{i\neq j}\frac{x_i-x_j}{\varepsilon}\cdot\nabla V\left(\frac{x_i-x_j}{\varepsilon}\right)\\ = P(R_{\omega,\varepsilon},Q_{\omega,\varepsilon},E_{\omega,\varepsilon})+\nu\nabla Q_{\omega,\varepsilon}+\mathbf{L}g_{\omega,\varepsilon}+o(\varepsilon)\end{aligned} \tag{1.4}$$

where both the viscosity ν and the functions g are unknown. Notice that the solution to the fluctuation-dissipation equation requires inversion of the Liouville operator.

In the following two sections, we intend to describe briefly the different mathematical approaches which allow to obtain rigorous convergence results for these asymptotics. These results will be stated in a rather unformal way in order to avoid definitions and notations. We refer to the quoted publications for precise statements and proofs.

1.3 The Probabilistic Approach

The most natural approach for the mathematical understanding of hydrodynamic limits consists in using probabilistic tools such as the law of large numbers and some large deviations principle. Nevertheless the complexity of the problem is such that there is still no complete derivation of fluid models starting from the full deterministic Hamiltonian dynamics.

1.3.1 The Euler Limit

Concerning the derivation of the Euler equations, what has been proved by Olla, Varadhan and Yau [89] is the following result.

Theorem 1.3.1 *Consider a general Hamiltonian system with superstable pairwise potential, and the corresponding stochastic dynamics obtained by adding a noise term which exchanges the momenta of nearby particles. Suppose the Euler equation has a smooth solution in* $[0,T]$*. Then the empirical density, velocity and energy converge to the solution of the Euler equations in* $[0,T]$ *with probability one.*

The strength of the noise term is of course chosen to be very small so that it disappears in the scaling limit.

The proof consists of two main ingredients. The first point is to establish the ergodicity of the system, and more precisely the following statement : if, under a stationary measure, the distribution of velocities conditioned to the positions is a convex combinations of gaussians, then the stationary measure is a convex combination of Gibbs. Noise is therefore added to the system in order to guarantee such information on the distributions. The second point is to prove that there is no spatial or temporal meso-scale fluctuation to prevent the convergence (1.2).

It is based on the *relative entropy method*, so-called because the fundamental quantity to be considered is the relative entropy defined by

$$H(f|g) = \int f \log(f/g) d\omega$$

for any two probability densities f and g.

If f_N is the solution to the Liouville equation (1.1) and ψ_t is any density, we have the following identity

$$\partial_t H(f_N(t)|\psi_t) = -\int f_N(t) \left(\psi_t^{-1}(\mathbf{L} - \partial_t)\psi_t\right) d\omega\,.$$

From Jensen's inequality, we then deduce that

$$\partial_t H(f_N(t)|\psi_t) \leq H(f_N(t)|\psi_t) + \log \int \psi_t \left(\psi_t^{-1}(\mathbf{L} - \partial_t)\psi_t\right) d\omega\,.$$

Thus, if we have

$$\frac{1}{N} \log \int \psi_t \left(\psi_t^{-1}(\mathbf{L} - \partial_t)\psi_t\right) d\omega \to 0 \tag{1.5}$$

the relative entropy can be controlled on the relevant time scale. The remaining argument can be summarized as showing that a weak version of (1.5) holds if and only if ψ_t is a local Gibbs state with density, velocity and energy chosen according to the Euler equations :

$$\begin{aligned} &\partial_t R + \nabla_x \cdot (RU) = 0, \\ &\partial_t (RU) + \nabla_x \cdot (RU \otimes U + P) = 0, \\ &\partial_t (RE) + \nabla_x \cdot (REU - UP) = 0. \end{aligned}$$

This is therefore a dynamical variational approach because the problem is solved by guessing a good test function.

1.3.2 The Incompressible Navier-Stokes Equations

Equation (1.4) is very difficult to solve as it requires inversion of the Liouville operator. It has been first studied by Landim and Yau [68] for the asymmetric exclusion process.

The rigorous derivation of the incompressible Navier-Stokes equations from particle systems has then been obtained in the framework of *stochastic lattice models* which are more manageable. Esposito, Marra and Yau [46] have established the convergence when the target equations have smooth solutions :

Theorem 1.3.2 *Consider a 3D lattice system of particles evolving by random walks and binary collisions, with "good" ergodic and symmetry properties. Suppose the incompressible Navier-Stokes equations have a smooth solution u in $[0, t^*]$. Then the rescaled empirical velocity densities u_ε converge to that solution u.*

Quastel and Yau [91] have then been able to remove the regularity assumption :

Theorem 1.3.3 *Consider a 3D lattice system of particles evolving by random walks and binary collisions, with "good" ergodic and symmetry properties. Let u_ε be the distributions of the empirical velocity densities. Then u_ε are precompact as a set of probability measures with respect to a suitable topology, and any weak limit is entirely supported on weak solutions of the incompressible Navier-Stokes equations satisfying the energy inequality.*

The method used to prove this last result differs from the relative entropy method, insofar as it considers more general solutions to the target equations, but - as a counterpart - gives a weaker form of convergence. One main step of the proof is to obtain the energy estimate for the incompressible Navier-Stokes equations directly from the lattice gas dynamics by implementing a renormalization group. A difficult point is to control the large fluctuation using the entropy method and logarithmic Sobolev inequalities.

It is important to note that such a derivation fails if the dimension of the physical space is less than three, meaning in particular that the 2D Navier-Stokes equations should be relevant only for 3D flows having some translation invariance.

1.4 The Analytic Approach

Here we will adopt a slightly different approach since our starting point will be the Boltzmann equation, which is the master equation of collisional kinetic theory. In other words, we will focus on the transition from the mesoscopic level of description to fluid mechanics indicated by the boldtype arrow in Figure 1.1.

Note that this will give a partial answer to Hilbert's problem insofar as Lanford [69] has proved the convergence of the hard core billiards to the Boltzmann equation in the Grad limit. Lanford's result, which is the only rigorous result on the scaling limits of many-body Hamiltonian systems with no unproven assumption, is however restrictive as it considers only short times (which will be not uniform in the hydrodynamic scalings) and perfect gases (low density limit).

For the sake of simplicity, we will consider in this section the only case when the microscopic interaction between particles is that of a hard sphere gas. We refer to the next chapter for a discussion on collision cross-sections.

1.4.1 Formal Derivations

The first mathematical studies of hydrodynamic limits of the Boltzmnn equation are due to Hilbert [65] on the one hand, and to Chapman and Enskog [33] on the other hand. Note that, in both cases, the derivations are purely formal.

Hilbert's method consists in seeking a formal solution to the scaled Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} Q(f, f)$$

with small variable Knudsen number ε, in the form

$$f(t, x, v, \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n f_n(t, x, v).$$

Identifying the coefficients of the different powers of ε, we then obtain systems of equations for the successive approximations f_0, $f_0 + \varepsilon f_1$, ...

Chapman-Enskog's method is a variant of the previous asymptotic expansion, in which the coefficients f_n are functions of the velocity v and of the hydrodynamic fields, namely the macroscopic density $R(t, x, \varepsilon)$, the bulk velocity $U(t, x, \varepsilon)$ and the temperature $T(t, x, \varepsilon)$ associated to f. For details, we refer to the next chapter.

Both methods allow to derive formally the Euler equations, as well as the weakly viscous Navier-Stokes equations. Let us mention however that, at higher order with respect to ε, one obtains systems of equations such as the Burnett model, the physical relevance of which is not clear. Moreover, these asymptotic expansions do not converge in general for fixed ε, and thus can represent only a very restricted class of solutions to the Boltzmann equation.

Grad [59] has proposed another, much simpler, method to derive formally hydrodynamic limits of the Boltzmann equation. This method, also called *moment method* can be actually compared to Morrey's analysis in the framework of particle systems. The first step consists in writing the local conservation laws for the hydrodynamic fields, namely the macroscopic density $R(t, x, \varepsilon)$, the bulk velocity $U(t, x, \varepsilon)$ and the temperature $T(t, x, \varepsilon)$ associated to f. The problem is then to get a closure for this system of equations, i.e. a state relation based on the hypothesis of local thermodynamic equilibrium.

1.4.2 Convergence Proofs Based on Asymptotic Expansions

Many of the early justifications of hydrodynamic limits of the Boltzmann equation are based on truncated asymptotic expansions. For instance, Caflisch [24] gave a rigorous justification of the compressible Euler limit up to the first singular time for the solution of the Euler system, which is the counterpart of the result in [89] for particle systems :

Theorem 1.4.1 *Suppose the Euler equations have a smooth solution* (R, U, T) *in* $[0, t^*]$. *Then there exists a sequence* (f_ε) *of Boltzmann solutions*

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} Q(f, f)$$

the moments $(R_\varepsilon, U_\varepsilon, T_\varepsilon)$ *of which tend to* (R, U, T) *as the mean free path* ε *tends to zero.*

Later Lachowicz [66] completed Caflisch's analysis by including initial layers in the asymptotic expansion, thereby dealing with more general initial data than in Caflish's original paper.

By the same method, DeMasi, Esposito and Lebowitz [42] justified the hydrodynamic limit of the Boltzmann equation leading to the incompressible Navier-Stokes equations. Like Caflisch's, their proof holds for as long as the solution of the Navier-Stokes equations is smooth, which is also reminiscent of the difficulty encountered in the framework of particle systems [46].

Theorem 1.4.2 *Suppose the incompressible Navier-Stokes equations have a smooth solution* u *in* $[0, t^*]$. *Then there exists a sequence* (f_ε) *of Boltzmann solutions*

$$\partial_t f + \frac{1}{\varepsilon} v \cdot \nabla_x f = \frac{1}{\varepsilon^2} Q(f, f)$$

which is close to the Maxwellian $\mathcal{M}_{(1,\epsilon u,1)}$ *with unit density and temperature, and bulk velocity* εu, *in some appropriate function space.*

Besides the solution of the Boltzmann equation so constructed that converges to a local equilibrium governed by the Navier-Stokes equation fail to be nonnegative. It could be that this problem can be solved by the same method as in Lachowicz's paper; however there is no written account of this so far.

1.4.3 Convergence Proofs Based on Spectral Results

Many other rigorous results have been obtained in a perturbative framework, using the spectral properties of the linearized collision operator at some Maxwellian equilibrium. Let us mention for instance the result by Nishida [88] which was the first mathematical proof of the compressible Euler limit of the Boltzmann equation. His argument used the description of the spectrum of the linearized Boltzmann equation by Ellis and Pinski [45], together with an abstract variant of the Cauchy-Kovalevski theorem due to Nirenberg and Ovsyannikov.

The more striking result based on such a spectral analysis is probably the one by Bardos and Ukai [7] concerning the incompressible Navier-Stokes limit of the Boltzmann equation. Although in the same spirit as Nishida's result, it puts less severe restrictions on the regularity of the target hydrodynamic solutions. Indeed Nishida's analysis considered analytic solutions of the compressible Euler system, and therefore was only local in time; on the contrary, the work of Bardos and Ukai considered global solutions to the Navier-Stokes equations, corresponding to initial velocity fields that are small in some appropriate Sobolev norm.

Theorem 1.4.3 *Let M be a global thermodynamic equilibrium (for instance the reduced centered Gaussian), and g_0 be some fluctuation of small norm in some appropriate weighted Sobolev space.*

Then, for any $\varepsilon \in]0,1]$ there exists a unique global solution $f_\varepsilon = M(1+\varepsilon g_\varepsilon)$ to the scaled Boltzmann equation

$$\begin{aligned} &\partial_t f_\varepsilon + \frac{1}{\varepsilon} v \cdot \nabla_x f_\varepsilon = \frac{1}{\varepsilon^2} Q(f_\varepsilon, f_\varepsilon), \\ &f_{\varepsilon|t=0} = M(1+\varepsilon g_0). \end{aligned}$$

Furthermore the bulk velocity $\int M g_\varepsilon v dv$ converges uniformly to the unique strong solution of the incompressible Navier-Stokes equations.

The perturbative method employed to prove that result uses the existence of classical solutions for the incompressible Navier-Stokes equations in the Sobolev space H^l for $l > \frac{3}{2}$ with initial data small enough. The main idea by Ukai [103] is to prove that a similar theory holds for the Boltzmann equation in diffusive regime. The derivation of the Navier-Stokes limit relies then on a rigorous proof of the relation between these two theories. The point to be stressed is that exactly the same type of assumptions are made on the initial data. The Bardos-Ukai statement results then from sharp bounds on the linearized collision operator.

1.4.4 A Program of Deriving Weak Solutions

The main restrictions in the previous results are the regularity and smallness conditions on the initial data (the second assumption being possibly replaced by some restriction on the time interval on which one can prove the validity of the approximation). Such assumptions are not expected to be necessary, working with Leray solutions of the incompressible Navier-Stokes equations and with renormalized solutions to the Boltzmann equation.

That is why Bardos, Golse and Levermore [4, 5] have proposed - at the beginning of the nineties - a program of deriving weak solutions of fluid models from the DiPerna-Lions solutions of the Boltzmann equation. Their ultimate goal was to obtain a theorem of hydrodynamic limits that should need only a priori estimates coming from physics, i.e. from mass, energy and entropy bounds. In spite of significant difficulties linked to our poor understanding of renormalized solutions, this program has achieved important successes, especially in the diffusive scaling limit for which a complete convergence result is now established.

The goal of the present volume is to present an overview of these relatively recent results, and some challenging questions that remain open in that field.