Chapter 5
Local and Global Endoscopy for *GSp*(4)

In this chapter we refine the global description of the endoscopic lift obtained in Corollary 4.2. The main result obtained in this chapter is Theorem 5.2, which is a special case of a global multiplicity formula conjectured by Arthur. We consider the symplectic group $GSp(4)$ over an arbitrary totally real number field F. In the special case $F = \mathbb{Q}$ and for irreducible automorphic representations π with π_∞ in the discrete series the proof of this theorem is given in Sects. 5.2 and 5.3. In Sect. 5.1 we explain how the local endoscopic character lift is constructed for arbitrary local base fields of characteristic zero. In Sects. 5.4 and 5.5 we explain how the Arthur–Selberg trace formula can be used to extend the results obtained in Sects. 5.2 and 5.3 to the case of arbitrary totally real number fields and arbitrary irreducible automorphic not necessarily cohomological representations π. The proof of Theorem 5.2 is based on two ingredients: the principle of exchange and the key formula (stated in Sect. 5.3). The latter can be directly deduced from weak versions of the trace formula (see Chap. 4, Corollary 4.2, or Sect. 5.5). It deals with simultaneous changes of global representations at two places. The principle of exchange, on the other hand, deals with an exchange of the representation at one place exclusively, and its proof boils down to a special case of the Hasse–Brauer–Noether theorem (Lemma 5.4). This is based on an explicit theta lift. At this point we use [56], which unfortunately forces us to make the restrictive assumption that F is a totally real number field. Since we apply results obtained in [56] in a rather specific case, it is very likely that this restriction on F can be removed.

5.1 The Local Endoscopic Lift

In this section we extend the results obtained in Chap. 4 to the case of arbitrary local fields F_v of characteristic zero. We restrict ourselves to the non-Archimedean case. For the Archimedean case see [90, 91].

Let $R_{\mathbb{Z}}[G_v]$ and $R_{\mathbb{Z}}[M_v]$ denote the Grothendieck group of finitely generated, admissible representations of G_v and M_v, respectively. For a finite sum

R. Weissauer, *Endoscopy for GSp(4) and the Cohomology of Siegel Modular Threefolds*, Lecture Notes in Mathematics 1968, DOI: 10.1007/978-3-540-89306-6_5,

$\alpha = \sum n_i \cdot \pi_{v,i} \in R_{\mathbb{Z}}(G_v)$ with integer coefficients n_i and irreducible, admissible representations $\pi_{v,i}$, and a locally constant function f_v with compact support on G_v, put $\alpha(f_v) = \sum n_i \cdot tr\ \pi_{v,i}(f_v)$. Similarly for the group M_v. Irreducible admissible representations σ_v of M_v are considered as a pair $(\sigma_{1v}, \sigma_{2v})$ of irreducible representations of $Gl(2, F_v)$ with equal central characters $\omega_{1v} = \omega_{2v}$. This common central character ω_v will be called the central character of σ_v. For a quasicharacter χ_v, the character twist for G_v is defined via the similitude homomorphisms $G_v \to F_v^*$ (Lemma 4.8), and similarly $\sigma_v \otimes \chi_v = (\sigma_{1,v} \otimes \chi_v, \sigma_{2,v} \otimes \chi_v)$. Recall $\sigma \mapsto \sigma^*$ was defined by $(\sigma_{1,v}, \sigma_{2,v})^* := (\sigma_{2,v}, \sigma_{1,v})$.

Theorem 5.1 (Nonarchimedean Lift). *There exists a unique homomorphism*

$$r = r_{M_v}^{G_v} : R_{\mathbb{Z}}[M_v] \to R_{\mathbb{Z}}[G_v]$$

between the Grothendieck groups with the following properties:

(i) The lift r is endoscopic: For $\alpha \in R_{\mathbb{Z}}[M_v]$

$$\alpha(f_v^{M_v}) \ = \ r(\alpha)(f_v)$$

holds for locally constant functions $f_v, f_v^{M_v}$ with compact support on G_v, and M_v, respectively, and matching orbital integrals.

(ii) The lift r preserves the central character, and commutes with character twists

$$r(\sigma_v \otimes \chi_v) \ = \ r(\sigma_v) \otimes \chi_v .$$

(iii) The lift r commutes with parabolic induction

$$r \circ r_{P_{M_v}}^{M_v}(\rho_v) \ = \ r_{P_v}^{G_v}(\rho_v),$$

where ρ_v is in $R_{\mathbb{Z}}[Gl(2, F_v) \times F_v^*]$, and where P_{M_v} is a maximal proper parabolic of M_v and P_v is the Siegel parabolic of G_v. Similarly for induction from the Borel groups B_{M_v}, B_v of M_v and G_v, we have

$$r \circ r_{B_{M_v}}^{M_v}(\rho_v) \ = \ r_{B_v}^{G_v}(\rho_v).$$

(iv) $r(\sigma^*) \ = \ r(\sigma)$.

(v) r commutes with Galois twists

$$r(\sigma_v^\tau) \ = \ r(\sigma_v)^\tau, \qquad \tau \in Aut(\mathbb{C}/\mathbb{Q}).$$

In addition the refined properties of Sect. 4.11 hold.

Concerning the proof. The above-mentioned properties of the endoscopic lift have already been shown for p-adic fields $\mathbb{Q}_p$. To extend this to the case of a general non-Archimedean local field F_v of characteristic zero, again one reduces the case to that of of irreducible representations σ_v in the discrete series by local arguments. For the global arguments we now replace the topological trace formula by the Arthur–Selberg trace formula. Choose a totally real number field F of degree at least 2 over $\mathbb{Q}$ with completion F_v at the non-Archimedean place v. Suppose one can find a global cuspidal automorphic representation $\sigma \ncong \sigma^*$ extending the given discrete series σ_v, where σ can be chosen to be cuspidal at some additional non-Archimedean place $w \neq v$, belonging to the discrete series at all Archimedean places. Now one stabilizes the Arthur trace formula for the group $GSp(4, A_F)$, using the fundamental lemma for $GSp(4)$ (see Chaps. 6–8). For this it is enough to consider a simple version of the trace formula with a test function of the form $f = \prod_w f_w$, where f_w is a difference of pseudocoefficients of the discrete series representations $\pi_+(\sigma_\infty), \pi_-(\sigma_\infty)$ at all Archimedean places of F. Furthermore, we may assume f_w to be a matrix coefficient of a cuspidal representation $\pi_-(\sigma_w)$ at some auxiliary place w, say, where $F_w = \mathbb{Q}_w$ splits and where σ_w is cuspidal. Then Arthur's trace formula can be stabilized with arguments analogous to those in Chap. 4 starting from [6], Corollaries 7.2 and 7.4. For the details and further technical assumptions, we refer to Sect. 5.5. Once the Arthur trace formula has been stabilized, one can deduce from it the following statement: The endoscopic lift $r(\sigma_v)$ of the character of a discrete series representation σ_v of $M(F_v)$, a priori only defined as a distribution satisfying properties (i) and (iii) from above, can be expressed as a finite integral linear combination $r(\sigma_v) \in R_{\mathbb{Z}}[G_v]$ of characters of irreducible, admissible representations of $G(F_v)$. The construction of this "integral" lift is the crucial step (generalizing Lemma 4.11 or its Corollary 4.5), and it will be explained in more detail later. Once this is known, all properties of the lift are deduced completely analogously to the p-adic case. So, for the proof of the theorem, it remains for us to construct the integral lift, since up to a character twist σ_v can be embedded globally. See the comment on page 182. □

5.1.1 The Integral Character Lift

We want to show the existence of the integral character lift for irreducible, admissible representations σ_v of M_v. It is enough to assume σ_v is in the discrete series. The endoscopic transfer of distributions applied to the character of σ_v defines distributions on G_v, which will be called character lifts of the characters σ_v. We have to show that the character lift is a linear combination of the characters of irreducible representations

$$r(\sigma_v) = \sum_{\pi_v} n(\sigma_v, \pi_v) \cdot \pi_v .$$

This uniquely determines the transfer coefficients $n(\sigma_v, \pi_v)$, once they are known to exist, by linear independence of characters (see [76], Proposition 13.8.1). So it remains for us to show the existence of such an expansion, and integrality of the coefficients.

Archimedean Case. According to Shelstad [90, 91] this holds in the Archimedean case such that, furthermore, the Archimedean transfer coefficients $n(\sigma_\infty, \pi_\infty)$ belong to $\{\pm 1, 0\}$, and only finitely many are nonzero [91], Theorem 4.1.1(i).

Non-Archimedean Case. To obtain a corresponding local character lift for an arbitrary local field of characteristic zero, we copy the arguments used so far. We have to replace the global arguments. Since these arguments involve several steps, we first summarize for the convenience of the reader the different steps involved in the proof of the p-adic case. Afterwards we discuss how this carries over to a non-p-adic local field.

5.1.2 Steps (0)–(xi) of the Proof (in the p-Adic Case)

(A) Global Inputs. The stabilization of the global trace formula in Chaps. 3 and 4 was the source for the desired weak character identities $r(\sigma_v) = \sum n(\sigma_v, \pi_v) \cdot \pi_v$, with coefficients $n(\sigma_v, \pi_v)$ in $\mathbb{C}$. A priori these sums are not finite, but are locally finite and hence absolutely convergent (see Sect. 4.6 [76]). The coefficients $n(\sigma_v, \pi_v)$ are directly related to global multiplicities $m(\pi)$ of automorphic representations π. In fact, the basic result was the identity (Corollary 4.2)

(0) $$\prod_{v\neq\infty} n(\sigma_v, \pi_v) \ = \ m(\pi_\infty^W \otimes \pi_f) - m(\pi_\infty^H \otimes \pi_f),$$

for a global weak endoscopic automorphic lift $\pi_f = \otimes_{v\neq\infty}\pi_v$ of a global irreducible cuspidal automorphic representation $\sigma_f = \otimes_{v\neq\infty}\sigma_v$, for which $\sigma^* \not\cong \sigma$ holds, where $m(\pi)$ denote the multiplicity of π in the discrete spectrum, and $\pi_\infty^H, \pi_\infty^W$ denote the irreducible Archimedean representations defined by the Archimedean character lift $r(\sigma_\infty) = \pi_\infty^W - \pi_\infty^H$ defined by Shelstad [91].

Claim 5.1. Formula (0) implies for the local component σ_v of σ:

(i) $r(\sigma_v)$ is an absolutely convergent sum

$$r(\sigma_v) = n(\sigma_v) \cdot \sum_{\pi_v} \varepsilon(\sigma_v, \pi_v) \cdot \pi_v$$

for certain $n(\sigma_v) \in \mathbb{C}^*$ such that the following hold:

(ii) (integrality) $\varepsilon(\sigma_v, \pi_v) \in \mathbb{Z}$ and $\varepsilon(\sigma_v, \pi_v) = 1$ for $\pi_v = \pi_+(\sigma_v)$ (as defined below).

(iii) (product formula) $\prod_v n(\sigma_v) = 1$.

(iv) (Shelstad) $n(\sigma_\infty) = 1$.

Proof of the Claim. Theorem 4.3 provides the existence of a globally generic cuspidal representation $\pi_+ = \pi_+(\sigma)$ such that $(\pi_+)_v = \pi_+(\sigma_v)$ for all v, and $m(\pi_+) = 1$. Being generic implies $(\pi_+)_\infty = \pi_+(\sigma_\infty) = \pi_\infty^W(\sigma_\infty)$. Since

$m(\pi_+) = m(\pi_\infty^W \otimes (\pi_+)_f) = 1$, the principle of exchange (see Lemmas 5.4 and 5.5) implies $m(\pi_\infty^H \otimes (\pi_+)_f) = 0$. Hence, $\prod_{v\neq\infty} n(\sigma_v, \pi_+(\sigma_v)) = 1$ by formula (0). For

$$n(\sigma_v) := n(\sigma_v, \pi_+(\sigma_v))$$

we get $n(\sigma_\infty) = 1$ from [91], Theorem 4.1.1(i), and hence (iv). (iv) together with the product formula just obtained gives (iii). Assertion (i), on the other hand, is a consequence of the global stabilized trace formula. For $\varepsilon(\sigma_v, \pi_v) = n(\sigma_v, \pi_v)/n(\sigma_v, \pi_+(\sigma_v))$ obviously $\varepsilon(\sigma_v, \pi_v) = 1$ for $\pi_v = \pi_+(\sigma_v)$, by definition. So for (ii) only $\varepsilon(\sigma_v, \pi_v) \in \mathbb{Z}$ has to be shown. From [91] we can assume v is non-Archimedean. Put $\tilde{\pi}_f = \pi_v \otimes_{w\neq v,\infty} \pi_+(\sigma_w)$ and $(\pi_+)_f = \otimes_{w\neq\infty} \pi_+(\sigma_w)$. By (iii) and (iv) we have $\prod_{w\neq\infty} n(\sigma_w, \pi_+(\sigma_w)) = 1$; hence,

$$\varepsilon(\sigma_v, \pi_v) = \varepsilon(\sigma_v, \pi_v) \prod_{w\neq\infty} n(\sigma_w, \pi_+(\sigma_w)) =$$

$$\frac{n(\sigma_v, \pi_v)}{n(\sigma_v, \pi_+(\sigma_v))} \prod_{w\neq\infty} n(\sigma_w, \pi_+(\sigma_w)) = \prod_{w\neq\infty} n(\sigma_w, \tilde{\pi}_w).$$

Together with formula (0) applied for the representation $\tilde{\pi}_f$, instead of π_f, this gives

$$\varepsilon(\sigma_v, \pi_v) \;=\; m(\pi_\infty^W \otimes \tilde{\pi}_f) - m(\pi_\infty^H \otimes \tilde{\pi}_f),$$

and hence (ii), since the right side is an integer. This proves the claim. □

(B) Local Inputs. A purely local investigation of the character lift, using compatibility with parabolic induction, gives (Lemmas 4.12 and 4.27):

(v) $n(\sigma_v) = 1$ for generic σ_v not in the discrete series.

For the elliptic scalar product $< \eta_v, \omega_v >_e$ on the elliptic regular locus of G_v write $\|\eta_v\|_e^2 =< \eta_v, \eta_v >_e$. Then (see in particular Sect. 4.5 and Lemma 4.14) we have

(vi) (Weyl integration formula) $\|r(\sigma_v)\|_e^2 = A_v \cdot \|\sigma_v\|_e^2 = A_v$

for irreducible σ_v in the discrete series of M_v, where $A_v = 2$ or 4 depending on whether $\sigma_v^* \not\cong \sigma_v$ or $\sigma_v^* \cong \sigma_v$.

Furthermore (Sects. 4.6, 4.9, Lemma 4.20, and Corollary 4.8), for σ_v in the discrete series the classification of irreducible, admissible representations of G_v implies that there exists a second irreducible representation $\pi_-(\sigma_v)$ such that:

(vii) $\sum_{\pi_v} \varepsilon(\sigma_v, \pi_+(\sigma_v)) \cdot \pi_v = L(\sigma_v) + R(\sigma_v)$ for $L(\sigma_v) = \pi_+(\sigma_v) - \pi_-(\sigma_v)$

and

(viii) $\|L(\sigma_v)\|_e^2 = A_v$ and $< L(\sigma_v), R(\sigma_v) >= 0$

with the same coefficients $A_v = 2, 4$ as above.

Combining (A) and (B). For the finite set S of places, where the (suitable chosen global cuspidal irreducible) representation $\sigma \not\cong \sigma^*$ belongs locally to the discrete series, statements (vi) and (vii) above imply

$$\prod_{v\in S} \|r(\sigma_v)\|_e^2 \;=\; \prod_{v\in S} n(\sigma_v)^2 \cdot \prod_{v\in S} \|\sum_{\pi_v} \varepsilon(\sigma_v, \pi_+(\sigma_v)) \cdot \pi_v\|_e^2 \;\le 1 \cdot \prod_{v\in S} \|L(\sigma_v)\|_e^2,$$

since $\prod_{v\in S} n(\sigma_v)^2 = 1$ by (iii)–(v). The inequality obtained is in fact an equality, since the left and the right sides are equal to $\prod_{v\in S} A_v$ by (vi) and (viii), respectively. That an equality holds forces $\|R(\sigma_v)\|_e^2 = 0$ from (vii) and (viii). Therefore,

(ix) $\sum_{\pi_v} \varepsilon(\sigma_v, \pi_v) \cdot \pi_v \;=\; \pi_+(\sigma_v) - \pi_-(\sigma_v)$

for σ_v in the discrete series (for σ) and $\|r(\sigma_v)\|_e^2 \;=\; \|\sum \varepsilon(\sigma_v, \pi_v)\pi_v\|_e^2$. Hence, $\|r(\sigma_v)\|_e^2 = |n(\sigma_v)|^2 \cdot \|\sum \varepsilon(\sigma_v, \pi_v)\pi_v\|_e^2$ implies $A_v = |n(\sigma_v)|^2 \cdot A_v$, or

(x) $|n(\sigma_v)| = 1$.

On the other hand, the local non-Archimedean theory of Whittaker models implies

(xi) $n(\sigma_v)$ is a positive real number,

since $\pi_-(\sigma_v)$ does not have local Whittaker models (Lemmas 4.35 and 5.6 and Corollary 4.16). This implies $n(\sigma_v) = 1$; hence, we obtain from (x) and (xi) the final formula

$$r(\sigma_v) \;=\; \pi_+(\sigma_v) - \pi_-(\sigma_v)$$

for all σ_v in the discrete series realized in a global representation σ. Since up to a character twist every irreducible representation σ_v in the discrete series of M_v can be realized as the local component of a global cuspidal irreducible representation σ of $M(\mathbb{A}_F)$, which follows from the existence of strong pseudocoefficients for $Gl(2, F_v)$, this completes the proof for σ_v in the discrete series. The general case is reduced to this case by purely local methods (as in Sects. 4.5–4.11).

How this Generalizes. To see how this carries over to the case of an arbitrary non-Archimedean local field of characteristic zero, observe that all the arguments above were of general nature, and hence carry over to a non-Archimedean field of characteristic zero immediately, except that the trace identity (0) needs some equivalent in the general case, which now will be provided by the strong multiplicity 1 theorem for $M(\mathbb{A}_F)$ (or $Gl(2, \mathbb{A}_F)$) and a version of the Arthur trace formula (Lemma 5.8 in Sect. 5.5). Formula (0) then again implies (i)–(iv), from which one deduces (v)–(xi) as above. Finally (0)–(xi) imply all properties of the local endoscopic character lift $r(\sigma_v)$ as in the p-adic case. This includes the statements of Theorem 5.1.

Concerning the Proof of Formula (0). To be accurate, one has to use an avatar of Lemma 5.8 for two reasons. On one hand, in the formulation and proof of Lemma 5.8, the existence and properties of the local character lift will already be used in special cases. Secondly, using an avatar gets rid of twisting by multipliers in the proof of Lemma 5.8. By examining the different steps in the proof of Lemma 5.8, one sees that formula (0) can be proved by making slight changes in the argument. First, it is enough to use the existence of the local character lift only in situations where the formula is already known, say, for the Archimedean places [91] and for places where the number field F splits, i.e., where $F_v \cong \mathbb{Q}_p$ is a p-adic field. Second, for the proof of (0) multipliers are not needed, since in this case it suffices to use a strongly simplified version of the trace formula of Deligne–Kazhdan type. We sketch the main steps of the argument which gives formula (0) at the end of the proof of Lemma 5.8 in Sect. 5.5. For the moment the reader is advised to skip this proof at first, since arguments from Sects. 5.2–5.4 are also used. □

This being said, we return to the representations $\pi_\pm(\sigma_v)$ defining the local L-packets. They can be further described in terms of the general classification of irreducible admissible representations of $GSp(4, F_v)$ as in Sect. 4.11. But they can also be described in terms of a theta lift (Weil representation) in Sect. 4.12.

5.1.3 Complements on the Local Theta Lift

Extended Jacquet–Langlands Correspondence. Let D_v be the nonsplit quaternion algebra over F_v, and $\check{M}_v = (D_v^* \times D_v^*)/F_v^*$. The quotient is defined by the embedding $t \mapsto (t, t^{-1})$ of F_v^* in $D_v^* \times D_v^*$.

The Jacquet–Langlands lift describes the irreducible discrete series representations of the group $Gl(2, F_v)$ in terms of the irreducible representations of the multiplicative group D_v^* of the quaternion algebra. We can use this to define an extended lift, denoted JL, from the irreducible admissible representations of the group $\check{M}_v$ to the irreducible, admissible, discrete series representations of the group M_v. In fact the description of $\check{M}_v$ and that of M_v as a quotient of two copies of D_v^* and $Gl(2, F_v)$, respectively, allows us to extend the Jacquet–Langlands lift JL in the obvious way. By this the irreducible, admissible discrete series representations σ_v of M_v correspond uniquely to irreducible, admissible representations $\check{\sigma}_v$ of $\check{M}_v$ and vice versa via the correspondence $\sigma_v = JL(\check{\sigma}_v)$.

Quaternary Theta Lifts. M_v can be considered as the group of special orthogonal similitudes $GSO(V)$ of a split quaternary quadratic form over F_v. Similarly $\check{M}_v$ can be considered as the group of special orthogonal similitudes $GSO(V)$ of the quaternary anisotropic quadratic form over F_v. This defines the theta correspondence for the groups $GO(V)$ and $GSp(4)$ as in Chap. 4 for arbitrary number fields F. It relates representations σ_v^{ext} and π_v of $GO(V)$ and $GSp(4)$ if $\sigma_v^{ext} \otimes \pi_v$ is a quotient of the big theta representation $\tilde{w}$ defined in [104], Sect. 4, or in Chap. 4. Although the big theta representation in [104] is defined in a slightly different way, we will use results obtained in [104] whenever they carry over: The groups $GSO(V)$ are subgroups of index 2 in $GO(V)$, so in general irreducible admissible representations σ_v of $GSO(V)$ cannot be extended uniquely to irreducible representations of the group $GO(V)$. If $\sigma_v \cong \sigma_v^*$, there exist possibly two nonisomorphic extensions σ_v^{ext} of the representation σ_v to $GO(V)$. One can consider the representations on $GSp(4, F_v)$ attached to σ_v^{ext} by the theta correspondence.

Notice there is a subtlety present in the definition of the theta correspondence: Either one deals with it on the level of representations of the groups $O(V), Sp(2n)$ or one deals with it on the extended level of the groups $GO(V), GSp(2n)$. Both points of view are of relevance. The representations obtained from the theta lift for $GSp(4, F_v)$, say, with a fixed central character of the center Z_v of $GSp(4, F_v)$, can be obtained – via compact induction from $Sp(4, F_v) \times Z_v \subseteq GSp(4, F_v)$ – from the $Sp(4, F_v)$ representations obtained from the $(O(V), Sp(2n))$ correspondence [104], p. 470. This often allows us to decide whether σ_v^{ext} has a lift to $GSp(4, F_v)$ [104], Proposition 4.11. Inversely, the restriction from $G = GO(V)$ to $O(V)$ is multiplicity free. For this see [41], Lemma 7.2, and the remarks after it on p. 94 for the split and the anisotropic quaternary spaces V.

The Definite Theta Lift. Using the last remarks, the statements, Theorem 9.1, and the results obtained in [41], Sect. 10, carry over from $Sp(4)$ to $GSp(4)$. Hence, the local theta correspondence for $GSp(4, F_v)$ and $GO(V)$ is always nontrivial for anisotropic V. In other words, to every irreducible representation $\check{\sigma}_v$ of $\check{M}_v$ the theta lift $\Theta(\check{\sigma}_v)$ is nonzero for the theta correspondence with respect to some extension of $\check{\sigma}_v$ from $\check{M}_v = GSO(V)$ to $GO(V)$. Let $\mathcal{E}\Theta(\check{\sigma}_v)$ denote the set of isomorphism classes of irreducible admissible representations π_v of $G_v = GSp(4, F_v)$, for which $\check{\sigma}_v \times \pi_v$ is a quotient of the big Weil representation of the pair $GO(V) \times G_v$.

Lemma 5.1 (Density Lemma). *(i)* $\mathcal{E}\Theta(\check{\sigma}_v) = \{\pi_-(\sigma_v)\}$, *and for* $\check{\sigma}_v \ncong \check{\sigma}_v^*$ *or* $dim(\check{\sigma}_v) > 1$ *this is a cuspidal representation.*

(ii) Furthermore, given a finite set S of places w of a number field F and given unitary representations $\check{\sigma}_w$ of the discrete series of M_w and irreducible representations π_w of G_w in $\mathcal{E}\Theta(\check{\sigma}_w)$ for all $w \in S$, suppose the central characters ω_v of $\check{\sigma}_v$ are the components of a grössen character ω of F. Then there exists a corresponding automorphic irreducible cuspidal representation of $M(\mathbb{A}_F)$ and a nontrivial irreducible cuspidal automorphic theta lift π of σ, which realizes the given local representations π_w for all places in S.

Proof. See Corollary 4.17 for (i). For (ii) see page 171, noticing the following □

Comment on Global Embeddings. One can enlarge S by a place w, and choose some $\check{\sigma}_w \ncong \check{\sigma}_w^*$ with central character $\tilde{\omega}_w$ to obtain the situation described in Chap. 4, page 171. From the existence of very strong pseudocoefficients for $Gl(2)$, one can find a global cuspidal σ with central character ω realizing σ_v for $v \in S \cup \{w\}$.

Concerning the Central Character. Let l be the order of the group of roots of unity in F. If we choose the place w outside S of residue characteristic larger than l, we can always extend given unitary central characters $\omega_v, v \in S$ to a grössen character ω of F^*, which is unramified outside $\{w\} \cup S$ and induces the given local character up to an unramified character twist $\omega_v = \tilde{\omega}_v |.|_v^{it_v}$. For this, at one place $v \in S$ we can prescribe $\omega_v = \tilde{\omega}_v$. [We leave this as an exercise for the reader with the following hints: Use [102], p. 342ff. Notice that extending condition (A) of [102] can be solved by choosing a suitable character $\tilde{\omega}_w : \mathfrak{o}_w^* \to \mathbb{C}^*$, which is trivial on $1 + \wp_w$. Then the remaining extension of condition (B) of [102] for the parameters t_v can be solved. This defines an extension ω on the subgroup $(\prod_{v \in \{w\} \cup S} F_v^* \prod_{v\ else} \mathfrak{o}_v^*)/(F_S^*)$ of $(\mathbb{A}_F^1/F^*) \times \mathbb{R}^*$. Since $\mathbb{C}^*$ is divisible, and hence injective, there is no obstruction to extending ω from this subgroup to $\mathbb{A}_F^1/F^*$. Notice there are only finitely many such extensions to a character unramified outside w and S and tame at w, since the S-ideal class group is finite].

Indefinite Theta Lift. If σ_v is an irreducible admissible representation of the group M_v and V is the split quaternary quadratic space, one can define a corresponding set $\mathcal{E}\Theta_+(\sigma_v)$. It consists of the classes of irreducible admissible representations π_v of G_v, for which $\sigma_v \times \pi_v$ is a quotient of the big Weil representation of the pair $GO(V) \times G_v$. From [96], proof of Theorem 3.1 on p. 366, there is at most one generic class of representations in $\mathcal{E}\Theta_+(\sigma_v)$. Otherwise one could easily show that the space of functionals of [96], p. 366, would have dimension 2 or more, which would give a contradiction. A representative of this generic class will be denoted $\theta_+(\sigma_v)$ if it exists. A generic $\theta_+(\sigma_v)$ exists in $\mathcal{E}\Theta_+(\sigma_v)$ if σ_v is in the discrete series of M_v (see Lemma 4.25). Hence, $\mathcal{E}\Theta_+(\sigma_v)$ is nonempty in this case. This argument also gives $\theta_+(\sigma_v) \cong \pi_+(\sigma_v)$ for discrete series representations σ_v of M_v (as in Theorem 4.4).

Definite vs. Indefinite Theta Lift. For σ_v in the discrete series of M_v, there exists a unique $\check{\sigma}_v$ for which for $\sigma_v = JL(\check{\sigma}_v)$. Then we have defined $\theta_+(\sigma_v)$ (indefinite theta lift) and we define $\theta_-(\sigma_v)$ to be the anisotropic theta lift $\theta_-(\check{\sigma}_v)$ attached to $\check{\sigma}_v$.

$$\begin{array}{llll} \sigma_v & \stackrel{\Theta}{\longmapsto} & \theta_+(\sigma_v) & \text{unique generic class in } \mathcal{E}\Theta_+(\sigma_v) \\ \uparrow JL & & & \\ \check{\sigma}_v & \stackrel{\Theta}{\longmapsto} & \theta_-(\sigma_v) & \text{defining } \mathcal{E}\Theta_-(\sigma_v) := \{\theta_-(\sigma_v)\} \end{array}$$

Proposition 5.1. *For irreducible, admissible representations σ_v in the discrete series of M_v the set $\mathcal{E}\Theta_-(\sigma_v)$ has cardinality 1. Furthermore, there is a unique generic representation in $\mathcal{E}\Theta_+(\sigma_v)$, and the endoscopic lift is given by these Weil representations*

$$r(\sigma_v) = \theta_+(\sigma_v) - \theta_-(\sigma_v).$$

Proof. The proof for arbitrary F_v remains the same as that for Corollary 4.16. However, recall that the proof of Corollary 4.16 depended on Lemma 5.6, which is proved later in this chapter. □

Remark 5.1. $\mathcal{E}\Theta_+(\sigma_v)$ presumably has cardinality 1, although we have not been able to show this. Notice since we want to include places of residue characteristic 2, and since we consider the theta lift for the groups of similitudes, the result obtained by Waldspurger for the Howe duality cannot be directly applied in this situation. To overcome this – for the global applications later – we introduce the subset

$$\mathcal{E}\Theta_\pm^{glr}(\sigma_v) \ \subseteq \ \mathcal{E}\Theta_\pm(\sigma_v)$$

of "globally relevant" representations. It consists of all classes of representations π_v in $\mathcal{E}\Theta_\pm(\sigma_v)$ for which there exists a global irreducible cuspidal automorphic representation $\sigma \ncong \sigma^*$ (or $\check{\sigma} \ncong \check{\sigma}^*$), for which π_v is the local component of a weak lift π attached to σ. The notion of weak lift will be explained in Sect. 5.2. The density lemma, stated above, implies

$$\mathcal{E}\Theta_-^{glr}(\sigma_v) = \mathcal{E}\Theta_-(\sigma_v),$$

in the analogous sense. Later, in Corollary 5.1, with proof in Sect. 5.3, and more generally for $F \neq \mathbb{Q}$ in Sect. 5.4, it is shown that for discrete series representations σ_v

$$\mathcal{E}\Theta_+^{glr}(\sigma_v) = \{\theta_+(\sigma_v)\}.$$

So $\mathcal{E}\Theta_+^{glr}(\sigma_v)$ has cardinality 1.

5.2 The Global Situation

General Assumptions. Let F be a number field, and $\mathbb{A}_F$ its ring of adeles. Since we apply the findings of [56] (in the argument preceding Lemma 5.3) we assume F to be totally real. This restriction will not be needed otherwise (and therefore is most likely unnecessary). Let $G = GSp(4)$ and let $M = Gl(2) \times Gl(2)/\mathbb{G}_m^*$ be its proper elliptic endoscopic group. Let $\pi = \otimes_v \pi_v$ be an irreducible, unitary cuspidal automorphic representation of the group $G(\mathbb{A}_F)$. Recall an irreducible automorphic cuspidal representation π is a cuspidal representation associated with a parabolic subgroup (CAP representation) if it is weakly equivalent to an automorphic representation associated with an Eisenstein series. Two irreducible automorphic representations are called weakly equivalent if their local components are isomorphic at almost all places. In the following we assume π is not a CAP representation. Then, by the Langlands theory of Eisenstein series, π only contributes to the cuspidal part of the discrete spectrum

$$m_{cusp}(\pi) \ = \ m_{disc}(\pi).$$

Definition 5.1. An irreducible, unitary cuspidal automorphic representation π of the group $G(\mathbb{A}_F)$ is called a weak endoscopic lift (from $M(\mathbb{A}_F)$) if there exist two automorphic representations

$$\sigma_1 \ , \ \sigma_2$$

of $Gl(2, \mathbb{A}_F)$ such that σ_i are either induced from a pair of global grössen characters or are irreducible cuspidal such that the representations σ_i have the same central character $\omega_{\sigma_1} = \omega_{\sigma_2}$, and such that the spinor L-series of π satisfies

$$L_v(\pi_v, s) \;=\; L_v(\sigma_{1,v}, s) \cdot L_v(\sigma_{2,v}, s)$$

for almost all places v of F. We also call π a weak lift of the irreducible automorphic representation $\sigma = (\sigma_1, \sigma_2)$ of $M(\mathbb{A}_F)$ and $Gl(2, \mathbb{A}_F) \times Gl(2, \mathbb{A}_F)$ in this situation.

Suppose π is a weak endoscopic cuspidal lift as in the definition above, which is not CAP. Let σ be the corresponding automorphic representation of $Gl(2, \mathbb{A}_F) \times Gl(2, \mathbb{A}_F)$. Then $\sigma = (\sigma_1, \sigma_2)$ has to be a cuspidal representation. Otherwise either σ_1 or σ_2, say, σ_1, has L-factors of the form $L_v(\sigma_1, s) = L_v(\lambda, s) L_v(\lambda^{-1}\omega_{\sigma_2}, s)$ for an idele class character λ for $v \notin S$. Put $\tau = \sigma_2 \otimes \lambda^{-1}$. Then $L^S(\pi, s) = L^S(\lambda, s) L(\lambda\omega_\tau, s) L^S(\tau \otimes \lambda, s)$. If $\tau^S \cong \nu^S \times \mu^S$ (locally induced from the unramified characters ν_v, μ_v), then this is the L-series with the local parameters $\lambda_v, \lambda_v\nu_v, \lambda_v\mu_v, \lambda_v\nu_v\mu_v$ or from [97], p. 85, the L-series of the induced representation $\nu^S \times \mu^S \triangleleft \lambda^S$ of $GSp(4, \mathbb{A}^S)$ in the short notation used in Chap. 3, page 121, and induced from a Borel subgroup. Since $\nu^S \times \mu^S \triangleleft \lambda^S = \tau^S \triangleleft \lambda^S$ (induction in steps), we see that $L^S(\pi, s)$ would be the partial L-series of the automorphic form $\tau \boxtimes \lambda$ of the Levi component of the Siegel parabolic subgroup. Hence, the weak lift would be associated with an automorphic Eisenstein representation, which is weakly equivalent to π. Hence, π is CAP of Saito–Kurokawa type. This implies

$$\pi \text{ not CAP} \Rightarrow \sigma \text{ cuspidal.}$$

Notice that σ is not uniquely defined by π. If π is a weak lift of $\sigma = (\sigma_1, \sigma_2)$, then it is also a weak lift of $\sigma^* = (\sigma_2, \sigma_1)$. These are the only possibilities.

Proposition 5.2. *Suppose π is an irreducible cuspidal automorphic representation of the group $G(\mathbb{A}_F)$, which is not a CAP representation. If π is a weak endoscopic lift of the representations σ and $\tilde{\sigma}$ of $M(\mathbb{A}_F)$, then $\tilde{\sigma}$ is isomorphic to either σ or σ^* and σ is cuspidal.*

Proof. Write $\sigma = (\sigma_1, \sigma_2)$ and $\tilde{\sigma} = (\sigma_3, \sigma_4)$. Then $\sigma, \tilde{\sigma}$; hence, $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ are cuspidal representations, as explained above. Then for a cuspidal representation ρ of $Gl(2, \mathbb{A})$ there exists a finite set S of places of F outside of which we have an equality of partial L-series

$$L^S(\sigma_1 \times \rho, s) \cdot L^S(\sigma_2 \times \rho, s) \;=\; L^S(\sigma_3 \times \rho, s) \cdot L^S(\sigma_4 \times \rho, s).$$

Since the σ_i are cuspidal, there exist complex numbers s_i for which $|.|^{s_i} \otimes \sigma_i$ becomes unitary. By a suitable (re)indexing, we may suppose $Re(s_1) = max_i(Re(s_i))$. For the contragredient ρ of σ_1 the function $L^S(\sigma_1 \times \rho, s)$ has a simple pole at $s = 1$, and $L^S(\sigma_2 \times \rho, s)$ does not vanish at $s = 1$ by statements (2.2) and (2.3) in [8], p. 200. Furthermore, if σ_1 is neither isomorphic to σ_3 nor isomorphic to σ_4, the right side is holomorphic at $s = 1$. This gives a contradiction. So, possibly by switching σ_3 and σ_4, we may assume $\sigma_1 \cong \sigma_3$. But then $L^S(\sigma_2, s) = L^S(\sigma_4, s)$, and the strong multiplicity 1 theorem for $Gl(2)$ implies $\sigma_2 \cong \sigma_4$. $\square$

Definition 5.2. Let π be an irreducible (unitary) cuspidal automorphic representation of the group $G(\mathbb{A}_F)$, which is not a CAP representation. Let π be a weak endoscopic lift. Suppose σ is a corresponding cuspidal automorphic representation of $M(\mathbb{A}_F)$. Then the set of equivalence classes of irreducible, automorphic representations π' of $G(\mathbb{A}_F)$, which are weakly equivalent to π, is called the *global L-packet* of π.

Since this global L-packet consists of all the weak endoscopic lifts attached to the given cuspidal representation σ, it will also be called the global L-packet attached to σ. Since π was assumed not to be CAP, all representations in the global L-packet are cuspidal.

Theorem 5.2 (Main Theorem). *Suppose π is an irreducible, cuspidal automorphic representation of $GSp(4, \mathbb{A}_F)$, and suppose π is not CAP. Suppose the global L-packet attached to the cuspidal irreducible automorphic representation σ of $Gl(2, \mathbb{A}_F)^2/\mathbb{A}_F^*$ is nonempty and contains π as a weak endoscopic lift. Then:*

1. *σ_1 and σ_2 are not isomorphic as representations of $Gl(2, \mathbb{A}_F)$ for $\sigma = (\sigma_1, \sigma_2)$.*
2. *The restriction of π to $Sp(4, \mathbb{A}_F)$ is obtained as a Weil representation from the orthogonal group attached to a quaternary quadratic form of a square discriminant.*
3. *All local representations π_v of $\pi = \otimes' \pi_v$ belong to the local L-packets attached to σ_v. Hence,*
$$\pi_v \in \{\pi_+(\sigma_v), \pi_-(\sigma_v)\}$$
if σ_v belongs to the discrete series, and $\pi_v = \pi_+(\sigma_v)$ otherwise.
4. *The multiplicity of any irreducible representation $\pi' = \otimes \pi'_v$ weakly equivalent to π is*
$$m(\pi') = \frac{1}{2}(1 + (-1)^{e(\pi')}),$$
where $e(\pi')$ denotes the (finite) number of representations π'_v which do not have a local Whittaker model.
5. *Let $d(\sigma)$ be the number of local components σ_v of σ in the discrete series. The global L-packet attached to σ contains a single representation if $d(\sigma) < 2$ and contains $2^{d(\sigma)-1}$ representations, each occurring with multiplicity 1 otherwise.*

In Addition. *For any cuspidal irreducible automorphic representation $\sigma = (\sigma_1, \sigma_2)$ with $\sigma_1 \not\cong \sigma_2$ the global L-packet attached to σ is nonempty.*

Remark 5.2. For CAP representations a result analogous to statement (5) of the last theorem is Theorem 2.6 in [69].

Proof. The proof of the main theorem has several steps, and it is carried out in the remaining sections of this chapter. We deal separately with the case where π_∞ belongs to the discrete series and F is the field of rational numbers. This situation is easier to begin with since we can directly refer to the multiplicity formula of Corollary 4.2 in Chap. 4. For the general case we have to use the Arthur trace formula to prove an analogous multiplicity formula, and some of the arguments therefore become more complicated. See Sects. 5.4 and 5.5 for this. Although the arguments are primarily global, they simultaneously provide us with the necessary local information, e.g., on local Whittaker models. The relevant local information will be derived in the subsequent Lemmas 5.4–5.6. By some kind of bootstrap, we then improve this to complete the proof for general representations π_∞ and for general (totally real) number fields F in Sects. 5.4 and 5.5. □

We start with the proof of statements (1) and (2) of the main theorem.

Lemma 5.2 (Global Endoscopic = Theta Lift). *Suppose π is an irreducible cuspidal automorphic representation of $GSp(4,\mathbb{A}_F)$, which is a weak lift in the global L-packet attached to an irreducible automorphic representation $\sigma = (\sigma_1,\sigma_2)$ of $M(\mathbb{A}_F)$.*

(A) Suppose π is not CAP. Then:

1. *σ is cuspidal and*
$$\sigma_1 \ncong \sigma_2 \quad \text{for } \sigma = (\sigma_1,\sigma_2).$$
2. *Furthermore, all irreducible constituents of the restriction of π to $Sp(4,\mathbb{A}_F)$ are contained in the image of theta lifts of the form $\Theta : \mathcal{A}_{cusp}(H(\mathbb{A}_F)) \to \mathcal{A}(Sp(4,\mathbb{A}_F))$, where H are orthogonal groups of similitudes over F, which are inner forms of $Gl(2,F)^2/F^*$. Hence, $H(F) = D^* \times D^*/\{(x,x^{-1})|x \in F^*\}$, where D^* is either the multiplicative group of a quaternion algebra over F or $Gl(2,F)$.*
3. *For each local component π_v of π there exists a quadratic character μ_v such that $\pi_v \otimes \mu_v$ is contained in*
$$\mathcal{E}\Theta_+^{glr}(\sigma_v) \cup \mathcal{E}\Theta_-^{glr}(\sigma_v).$$

(B) Conversely, suppose σ is cuspidal and $\sigma_1 \ncong \sigma_2$. Then the weak lift π of σ is not CAP.

Concerning the notation. $\mathcal{A}$ will denote packets of irreducible automorphic representations.

Proof. Concerning (A) it is enough to prove the global statements (1) and (2). The local statement (3) then follows by Frobenius reciprocity [104], Proposition 4.12(c), using compact induction. So let us prove (A)(1) and (2). Let $\omega = \omega_\pi$ denote the central character of the irreducible representation π of $GSp(4,\mathbb{A}_F)$. The standard L-function of π corresponds to the four-dimensional standard representation st of the L-group ${}^LG$ of G. The alternating square of the standard representation is a six-dimensional representation $\bigwedge^2(st)$ of the L-group ${}^LG$. The corresponding L-series of π outside a finite set S of ramified places is a product $\zeta^S(\pi,\bigwedge^2(st),s) = L^S(\omega,s)\cdot\zeta^S(\pi,\omega,s)$ of the Dirichlet L-series $L(\omega,s)$ for the character ω and a twist of the degree 5 standard L-series $\zeta(\pi,\omega,s)$ of the restriction of π to $Sp(4,\mathbb{A}_F)$. Here, by abuse of notation, we again let π denote any of the irreducible constituents of its restriction to $Sp(4,\mathbb{A}_F)$. Then in fact, the degree 5 L-series $\zeta(\pi,\omega,s)$ is the standard L-series attached to the representation $\pi\times\omega$ of $Sp(4,\mathbb{A}_F)\times Gl(1,\mathbb{A}_F)$, at least if we consider partial L-series at almost all unramified places. Since π is a weak lift, we get from this

The Partial *L*-series. If we omit a suitable chosen finite set of places S depending on σ and π, then

$$\zeta^S(\pi,\bigwedge^2(st),s) \;=\; L^S(\omega_{\sigma_1},s)\cdot L^S(\omega_{\sigma_2},s)\cdot L^S(\sigma_1\otimes\sigma_2,s).$$

Hence, from $\omega_{\sigma_1}=\omega_{\sigma_2}$

$$\begin{aligned} L^S(\omega_\pi\omega_{\sigma_2}^{-1},s)\zeta^S(\pi,\omega_\pi\omega_{\sigma_2}^{-1},s) &= \zeta^S(s)^2\cdot L^S(\sigma_1\otimes\sigma_2\otimes\omega_{\sigma_2}^{-1},s)\\ &= \zeta^S(s)^2\cdot L^S(\sigma_1\otimes\sigma_2\check{},s) \end{aligned}$$

for the contragredient representation $\sigma_2\check{}$ of σ_2. For cuspidal σ_i the L-series $L^S(\sigma_1\otimes\sigma_2\check{},s)$ does not vanish at $s=1$. By a character twist the σ_i become unitary. A twist of the cuspidal representations σ_i becomes unitary if and only if their central character ω_{σ_i} becomes unitary. Since this is a common character, one can take the same character twist in both cases. So without restriction of generality, we can assume that σ_1 and σ_2 are both unitary. Notice the representations have the same central character. If $\omega_\Pi\neq\omega_{\sigma_i}$, this implies that $\zeta^S(\pi,\omega_\pi\omega_{\sigma_2}^{-1},s)$ has a pole of order 2 at $s=2$. This is impossible, as will be explained below. Therefore,

$$\omega_\pi=\omega_{\sigma_2}$$

holds. Then the partial degree 5 standard L-function $\zeta^S(\pi,s)=\zeta(\pi,1,s)$ of π is given by the formula

$$\zeta^S(\pi,s) \;=\; \zeta^S(s)\cdot L^S(\sigma_1\otimes\sigma_2\otimes\omega_{\sigma_2}^{-1},s) \;=\; \zeta^S(s)\cdot L^S(\sigma_1\otimes\sigma_2\check{},s).$$

Then $L^S(\sigma_1\otimes\sigma_2\check{},1)\neq 0$ holds at $s=1$ for cuspidal unitary representations σ_i. Hence,

$$ord_{s=1}\zeta^S(\pi,s)\geq 1.$$

For $\sigma_1\cong\sigma_2$ the pole order would be 2 or more. Hence, the first assertion of Lemma 5.2 comes from the next result obtained by Soudry.

Review of a Result Obtained by Soudry.

$\zeta^S(\pi,\chi,s)$ has at most a simple pole at $s=1$ for unitary cuspidal π.

In fact, this is proved in [97] for a special character $\chi=\chi_T$. Indeed, this is the most difficult case. For other characters one can use the same approach. Then again as in [97], the order of the pole at $s=1$ can be estimated by the pole order of an Eisenstein series. Up to a factor $L^S(\chi^2, 2s)L^S(\chi, s+1)$, which is irrelevant at $s = 1$, the L-series $\zeta^S(\pi, \chi_T\chi, s)$ is (except for an unimportant function $A(s)$ with $A(1) \neq 0, \infty$) an integral $\int \varphi(g)\theta^{\phi}_{T,\psi}(g,1)E(f_{s,\chi}; g)dg$ by (2.2) and (2.5) in [97]. The poles of $E(f_{s,\chi}; g)$, therefore, give an upper bound for the poles of $\zeta^S(\pi, \chi\chi_T, s)$. From [97], Theorem 2.4, the poles of these Eisenstein series are contained in the poles of $L(\chi^2, 2s)L(\chi, s+1)$, $L(\chi^2, 2s)L(\chi, s)$, $L(\chi^2, 2s-1)L(\chi, s)$, $L(\chi^2, 2s-1)L(\chi, s-1)$ counted with multiplicity. There is no pole at $s = 1$ for $\chi^2 \neq 1$ and at most a simple pole for $\chi \neq 1$. The difficult case is $\chi = 1$. In this case there is no pole according to [97], Theorems 3.1 and 4.4(b). Therefore, the claim follows.

Now Apply Kudla–Rallis–Soudry Theorem 7.1 [56]. This is now the point where we have to restrict ourselves to the case of totally real number fields. This assumption regarding the number field is made in [56]. Under this assumption the existence of a pole for the partial degree 5 L-series $\zeta^S(\pi, s)$ at $s = 1$ implies that π – or more precisely any irreducible constituent of π as a representation of $Sp(4, \mathbb{A}_F)$ – is a constituent of a suitable theta lift

$$\Theta(\mathcal{A}(O(V', \mathbb{A}_F))$$

attached to a quadratic space V' of discriminant 1 over F:

$$\Delta(V') = \Delta(V_T) \cdot \Delta(V_{T'}) = 1.$$

This formula for the discriminants follows from $\chi_T\chi_{T'} = 1 = (\ .\ , \Delta(V_T)) \cdot (\ .\ , \Delta(V_{T'}))$ using Theorem 7.1(ii) in [56] if $\chi_T \neq 1$ or Theorem 7.1(i) if $\chi_T = 1$.

Remark 5.3. The assumption F is totally real is most likely unnecessary since it is needed only for the case of split discriminant. In fact, looking at the integral representation in [97], formula (2.2), it would suffice to show that the residues of the nonholomorphic Siegel Eisenstein series $E(f_s; g)$ at $s = 1$ ([97], 2(b)) are binary theta series. A partial result in this direction – namely, the local case of this statement – is Lemma 3.3 in [97].

Structure of V'. The quaternary quadratic F-spaces V' of discriminant $\Delta(V') = 1$ are classified by their local Hasse invariants, or in this case alternatively by the structure of their orthogonal groups of similitudes

$$GSO(V')(F) \cong D^* \times D^*/F^*.$$

Here D is the corresponding quaternion algebra or $Gl(2, F)$, having invariants determined by the signs of the local Hasse invariants of V'. This proves part (A) of Lemma 5.2, and it also proves statements (1) and (2) of the main theorem.

For the converse part (B) of Lemma 5.2 we may assume σ has unitary central character. Notice $L^S(\pi, s) = L^S(\sigma_1, s)L^S(\sigma_2, s)$ and $\zeta^S(\pi \otimes \chi, s) = L^S(\chi, s)L^S(\sigma_1 \otimes \sigma_2\check{} \otimes \chi, s)$ outside a suitable finite set S. The known CAP criteria for π in terms of poles of these L-series (Soudry and Piatetski-Shapiro) exclude the possibility that π is CAP for cuspidal σ, since the known analytic behavior of L-series of cuspidal forms for $Gl(2)$ or $Gl(2) \times Gl(2)$ excludes the existence of poles at $s = 3/2$ for the degree 4 L-series and poles at $s = 2$ for degree 5 L-series (for unitary central characters and unitary χ). This proves part (B), and completes the proof of Lemma 5.2. □

Lemma 5.3. *We have*

$$\bigcup_{\sigma_v} \mathcal{E}\Theta_+(\sigma_v) \ \cap \ \bigcup_{\sigma_v} \mathcal{E}\Theta_-(\sigma_v) = \emptyset,$$

where the union is over all generic irreducible admissible representations σ_v of M_v on the left, and over all generic irreducible admissible discrete series representations σ_v of M_v on the right. In other words, the two theta lifts attached to M_v (split case) and $\check{M}_v$ (anisotropic case) have no irreducible representation of $GSp(4, F_v)$ in common. The same statement also holds globally or after restriction to the subgroup $Sp(4, F_v)$.

Proof. It is enough to consider the restrictions to $Sp(4, F_v)$. Here we refer to Howe and Piatetski-Shapiro [41], Theorem 9.4, where it is shown that there is at most one representation in common. In the proof of Lemma 4.26 we saw that the common representation corresponds to a one-dimensional representation σ_v. However, this is not a generic representation, and therefore it is excluded in the statement above. This proves Lemma 5.3. □

Lemma 5.4 (Principle of Exchange). *Suppose the cuspidal irreducible automorphic representation $\pi = \otimes \pi_v$ of $GSp(4, \mathbb{A}_F)$ is a weak endoscopic lift, but is not CAP. Let σ denote the corresponding cuspidal representation of $M(\mathbb{A}_F)$:*

1. *Fix some place v_0. Then π_{v_0} is in $\mathcal{E}\Theta_\varepsilon^{glr}(\sigma_{v_0} \otimes \mu_{v_0})$ for some quadratic character μ_v and some $\varepsilon \in \{\pm\}$. Suppose $\pi'_{v_0} \not\cong \pi_{v_0}$ is an irreducible representation of $GSp(4, F_{v_0})$ and consider*

$$\pi' = \pi'_{v_0} \otimes \bigotimes_{v \neq v_0} \pi_v.$$

   Make the assumption*: $\pi'_{v_0} \notin \mathcal{E}\Theta_\varepsilon^{glr}(\sigma_{v_0} \otimes \mu'_{v_0})$ for all quadratic characters μ'_v. Under this assumption π' has global multiplicity*

$$m_{disc}(\pi') = 0.$$

2. *If $d(\sigma) < 2$ ($d(\sigma)$ is defined in Theorem 5.2(5), then $\pi_v = \theta_+(\sigma_v \otimes \mu_v)$ for all v and certain local quadratic characters μ_v. In particular, π_v is generic.*

Proof. Suppose that to the contrary $m(\pi') \geq 1$ holds. Then π' is in the global L-packet of π, and hence it is a weak endoscopic lift attached to the same irreducible representation σ of $M(\mathbb{A})$ as π. According to Lemma 5.2 all constituents of π'

and π, after restriction to the group $Sp(4,\mathbb{A}_F)$, are in the image of some theta lifts Θ. These theta correspondences lift automorphic forms from orthogonal groups arising from global simple algebras of rank 4 to automorphic forms on $GSp(4)$. In our case let D and D' be the corresponding algebras, respectively. From the known local properties of the theta lift it follows that

$$D_v \cong D'_v \qquad (\forall v \neq v_0),$$

since by Lemma 5.3 we would otherwise get a contradiction to the fact that

$$\pi_v \cong \pi'_v \qquad (\forall v \neq v_0).$$

But then $D_{v_0} \cong D'_{v_0}$, as a consequence of the theorem of Hasse–Brauer–Noether. $D_{v_0} \cong D'_{v_0}$ implies that the constituents of both π_v and π'_v (after restriction to $Sp(4,F_v)$) either both belong or both do not belong to the local theta lift $\mathcal{E}\Theta_\varepsilon^{glr}(\sigma_v)$. Since this contradicts the assumptions, the proof of the first part of Lemma 5.4 is complete.

The second part is shown similarly. Now D splits globally. Then [41], Theorem 5.7b or 8.1b, implies that the restriction of π_v to $Sp(4,F_v)$ is generic. By compact induction also $\pi_v \otimes \mu_v$, and hence π_v is generic. This implies $\pi_v \cong \theta_+(\sigma_v \otimes \mu_v) = \theta_+(\sigma_v) \otimes \mu_v$ for some quadratic character. This proves part (2) of the lemma. □

The principle of exchange can be applied for an Archimedean place in the situation of Lemma 5.5, since in this case the assumption made in Lemma 5.4(1) is satisfied. In fact, assume $F_{v_0} = \mathbb{R}$ and let π_{v_0} and π'_{v_0} be representations in the local Archimedean L-packet $\{\pi_\infty^W, \pi_\infty^H\} = \{\pi_+(\sigma_\infty), \pi_-(\sigma_\infty)\}$ attached to a discrete series representation σ_∞ of M_∞. By Lemma 5.5 $\pi^W \in \mathcal{E}\Theta_+(\sigma_\infty)$ and $\pi^H \in \mathcal{E}\Theta_-(\sigma_\infty \otimes \mu_\infty)$; hence, $\pi^H \notin \mathcal{E}\Theta_+(\sigma_\infty \otimes \mu_\infty)$ for all μ_∞ by Lemma 5.3. Hence, the principle of exchange can be applied.

Lemma 5.5. *Let $F_v = \mathbb{R}$ and $\pi'_v = \pi_\infty^W$ and $\pi_v = \pi_\infty^H$, then the constituents of π'_v restricted to $Sp(4,\mathbb{R})$ have Whittaker models, whereas the constituents of π_v restricted to $Sp(4,\mathbb{R})$ do not have a Whittaker model. Furthermore, $\pi^W \otimes \mu_v \cong \pi^W, \pi^H \otimes \mu_v \cong \pi^H$ for all quadratic characters μ_∞, and also $\pi^W \in \mathcal{E}\Theta_+(\sigma_\infty \otimes \mu_\infty)$ and $\pi^H \in \mathcal{E}\Theta_-(\sigma_\infty \otimes \mu_\infty)$ for all quadratic characters μ_∞.*

Proof. Well known. □

5.3 The Multiplicity Formula

The main results obtained in this section are the following: the local–global principle (Lemma 5.6) and the key formula. Under the assumption $F = \mathbb{Q}$ and the assumption that in the global L-packet of the representation π a representation exists for which the Archimedean component π_∞ belongs to the discrete series, Lemma 5.6 and the key formula are proven in this section. This of course suffices for the applications in

Chap. 2. The proof of Lemma 5.6 and key formula for general totally real number fields and arbitrary cusp forms π is contained in Sects. 5.4 and 5.5. In this section we also show that the local–global principle (Lemma 5.6) and the key formula imply for arbitrary F and π certain corollaries (Corollaries 5.1–5.5), from which the main Theorem 5.2 follows.

Let π be a cuspidal irreducible weak endoscopic lift attached to some irreducible cuspidal representation σ of $M(\mathbb{A}_F)$. Write $\pi = \pi_\infty \otimes \pi_f$, where π_∞ denotes the tensor product over all Archimedean components. Assume π is not CAP. Then σ is cuspidal and $\sigma \not\cong \sigma^*$ (Lemma 5.2(1)). For irreducible automorphic representations of the group $Gl(2,\mathbb{A}_F)$ cuspidal implies generic. So the same is true for the group $M(\mathbb{A}_F)$.

So Lemma 5.6 proves assertion (3) of the main theorem.

Lemma 5.6 (Local–Global Principle). *Let π be a weak endoscopic lift contained in the global L-packet attached to some generic, irreducible cuspidal automorphic representation σ of $M(\mathbb{A})$. Suppose π is cuspidal but not CAP. Then for all places v of F the local components π_v of π belong to the local L-packets $\{\pi_\pm(\sigma_v)\}$ and $\{\pi_+(\sigma_v)\}$ of σ_v.*

Proof of Lemma 5.6 for $F = \mathbb{Q}$ and π_∞ in the discrete series (for the general case see Sects. 5.4 and 5.5). By assumption π_∞ is contained in some local Archimedean L-packet $\{\pi^W, \pi^H\}$ attached to a discrete series representation of M_∞. From the topological trace formula we have the

Weak Multiplicity Formula (Corollary 4.2). For $F = \mathbb{Q}$ and π_f as above

$$m(\pi_\infty^W \otimes \pi_f) - m(\pi_\infty^H \otimes \pi_f) \;=\; n(\pi_f),$$

where $n(\pi_f)$ is zero if there exists a non-Archimedean place v for which π_v is not contained in the local L-packet of σ_v, and where $n(\pi_f)$ is equal to $(-1)^{e(\pi_f)}$ otherwise. Here $e(\pi_f)$ denotes the number of non-Archimedean places v for which $\pi_v \cong \pi_-(\sigma_v)$ holds. Since $e(\pi_f) < d(\sigma)$, this number is finite. *Some comments*: The multiplicity $m(\pi)$ is the multiplicity of π in the discrete spectrum of $G(\mathbb{A}_\mathbb{Q})$ or equivalently the multiplicity in the cuspidal spectrum since π is not CAP. We remark that the multiplicity formula above is a reformulation of Corollary 4.2. In fact we used Corollary 4.4 to identify the multiplicities $m_1(\pi_f), m_2(\pi_f)$ used in Corollary 4.2 with the multiplicities $m_1(\pi_f) = m(\pi_\infty^H \otimes \pi_f)$ and $m_2(\pi_f) = m(\pi_\infty^W \otimes \pi_f)$. We also used the property that the normalizing sign ε from Corollary 4.2 is $\varepsilon = -1$ (Lemma 8.4). Also recall that in Corollary 4.2 the value $n(\pi_f) = \prod_{v \neq \infty} n(\pi_v)$ is defined in terms of the multiplicities $n(\pi_v)$ of π_v in the local character expansion $r(\sigma(\sigma_v)) = \sum n(\pi_v) \cdot \pi_v$ for the local endoscopic lift $r : R_\mathbb{Z}(M_v) \to R_\mathbb{Z}(G_v)$. The expression for $n(\pi_f)$ in the form stated above follows from the structure of the local L-packets of the local endoscopic lift $r(\sigma_v)$ of a generic representation σ_v of M_v. This was described in Sect. 4.11 and also in Sect. 5.1 of this chapter in the non-Archimedean case and in Lemma 5.5 in the Archimedean case. The L-packets consist of two representations $\pi_+(\sigma_v), \pi_-(\sigma_v)$ if σ_v belongs to the discrete series, and of one representation $\pi_+(\sigma_v)$ if σ_v does not belong to the discrete series. Among these only the representations $\pi_-(\sigma_v)$ do not have Whittaker models. Hence, $n(\pi_v) = 0$ unless π_v is in the local L-packet of σ_v. Furthermore, $n(\pi_v)$ is $+1$ or -1 otherwise, and the sign depends on the existence of a Whittaker model.

If π_v is not in the local L-packet of σ_v, then v is non-Archimedean and the weak multiplicity formula implies $m(\pi_\infty^W \otimes \pi_f) = m(\pi_\infty^H \otimes \pi_f)$. But then $m(\pi) > 0$ and $m(\pi') > 0$ for $\pi = \pi_\infty^W \otimes \pi_f$ and $\pi' = \pi_\infty^H \otimes \pi_f$. But this contradicts the principle of exchange. By Lemma 5.5 the assumptions of Lemma 5.4 are satisfied for the place $v_0 = \infty$. Hence, Lemma 5.6 follows under the assumptions $F = \mathbb{Q}$ and π_∞ is in the discrete series.

Now let us return to the general case. In fact, suppose that Lemma 5.6 holds assuming the results obtained in Sects. 5.4 and 5.5. From Lemma 5.6 we now deduce the next three corollaries. First recall from Sect. 5.1 the global $(GSp(4), GO(V))$-theta correspondence: For discrete series representations σ_v we constructed a global irreducible automorphic cuspidal representation σ of $M(\mathbb{A}_F)$ and a nonvanishing global theta lift π of σ with local components π_v such that $\pi_v \in \mathcal{E}\Theta_-(\sigma_v)$ which do not have local Whittaker models. Hence, $\pi_v \ncong \pi_+(\sigma_v)$ since $\pi_+(\sigma_v)$ has Whittaker models. This implies $\pi_v = \pi_-(\sigma_v)$ (Lemma 5.6). In particular, $\pi_-(\sigma_v)$ is in $\mathcal{E}\Theta_-(\sigma_v)$ and does not have local Whittaker models. Since we also know that $\pi_+(\sigma_v) = \theta_+(\sigma_v)$ is generic, we obtain from Lemma 5.6 □

Corollary 5.1 (Local Endoscopic Lift = Theta Lift). *Under the assumptions of Lemma 5.6 the representations $\pi_\pm(\sigma_v)$ of the local L-packet attached to a discrete series representation σ_v of M_v are the two Weil representations $\theta_\pm(\sigma_v)$:*

(a) $\Theta_+^{glr}(\sigma_v) = \{\pi_+(\sigma_v)\} = \{\theta_+(\sigma_v)\}$ *for split D.*
(b) $\Theta_-(\sigma_v) = \{\pi_-(\sigma_v)\} = \{\theta_-(\sigma_v)\}$ *for nonsplit D.*

In particular, $\pi_+(\sigma_v)$ has a local Whittaker model, whereas $\pi_-(\sigma_v)$ does not have local Whittaker models.

Notice Lemma 5.6 and its Corollary 5.1 imply Proposition 5.1.

Corollary 5.2 (Whittaker Models). *Suppose the assumptions of Lemma 5.6 are satisfied. Let v be a place where σ_v belongs to the discrete series. Then $\pi_+(\sigma_v) \in \Theta_+(\sigma_v)$ has local Whittaker models, whereas $\pi_-(\sigma_v) \otimes \mu_v \notin \Theta_+(\sigma)$ for all (quadratic) characters μ_v, and these representations are cuspidal and do not have local Whittaker models.*

Proof. Follows from Corollary 5.2 and Lemmas 4.17 and 5.3. □

Corollary 5.3 (Refined Principle of Exchange). *Suppose the assumptions of Lemma 5.6 are satisfied. For the weak automorphic lift $\pi = \otimes_v \pi_v$ of σ put $\pi' = \pi'_v \otimes_{v \neq v_0} \pi_v$ for an irreducible admissible representation π'_v of G_v. Then for $\pi'_v \ncong \pi_v$*

$$m(\pi') = 0.$$

Proof. Suppose $m(\pi') > 0$ and $\pi'_v \not\cong \pi_v$. Then both π and π' are weak automorphic endoscopic lifts of σ. Hence, $\pi_v, \pi'_v \in \{\pi_+(\sigma_v), \pi_-(\sigma_v)\}$ by Lemma 5.6. Since they are not isomorphic, we can apply the principle of exchange (Lemma 5.4), because Corollaries 5.1 and 5.2 imply that all the assumptions of Lemma 5.4 are now satisfied. Hence, $m(\pi') = 0$ (Lemma 5.4), which completes the proof of Corollary 5.3. □

Proof of the Remaining Assertions (4) and (5) of the Main Theorem. Suppose π is an automorphic representation of $G(\mathbb{A}_F)$ as in the main theorem. π is irreducible, cuspidal automorphic but not CAP, and π is a weak lift attached to some irreducible automorphic representation σ of $M(\mathbb{A}_F)$. Then, as already shown, σ is cuspidal and its local components σ_v are generic. □

Conversely for an irreducible cuspidal automorphic representation $\sigma = (\sigma_1, \sigma_2)$ of $M(\mathbb{A}_F)$ consider the global L-packet of all weak endoscopic lifts attached to it. Since we are mainly interested in cuspidal automorphic representations, assume

$$\sigma_1 \not\cong \sigma_2$$

because otherwise the global L-packet attached to σ does not contain a cuspidal representation (Lemma 5.2). See also Corollary 5.5. In fact, recall from Theorem 4.3

The Base Point of the *L*-packet. *From a result obtained by Soudry this global L-packet is nonempty and contains the representation $\pi_+(\sigma) = \prod_v \pi_+(\sigma_v)$ as a cuspidal automorphic representation with multiplicity $m(\pi_+(\sigma)) = 1$.*

In fact the situation in Theorem 4.3 is slightly more restrictive. So for the convenience of the reader we sketch the argument.

Sketch of Proof. $\pi_+(\sigma)$ is the theta lift $\mathcal{E}\Theta_+(\sigma)$ (for the split case). By the assumptions regarding σ this theta lift does not vanish, and it is cuspidal since $\sigma_1 \not\cong \sigma_2$. Hence, $m(\pi_+(\sigma)) > 0$. Then from a result obtained by Soudry

$$m(\pi_+(\sigma)) = 1,$$

since every cuspidal representation which is isomorphic to $\pi_+(\sigma)$ has a global Whittaker model. This follows from another result obtained by Soudry, namely, since by construction $\pi_+(\sigma)$ is locally generic at all places v, it is enough to show the nonvanishing of $\zeta^S(\pi_+(\sigma), s)$ at $Re(s) = 1$ (if ω is normalized to be unitary). But this nonvanishing follows immediately from the computations made in the proof of Lemma 5.2, since under our assumptions the L-series $L^S(\sigma_1 \times \hat{\sigma}_2, s)$ does not vanish on the line $Re(s) = 1$. This completes the proof of the claim. □

Corollary 5.1 now implies that for every representation π in the global L-packet there exists a finite set of places S of F where π_v does not have a Whittaker model and such that

$$\pi \;=\; \bigotimes_{v\in S} \pi_-(\sigma_v) \;\otimes\; \bigotimes_{v\notin S} \pi_+(\sigma_v).$$

Let $e(\pi) = \#S$ be cardinality of this uniquely defined set

$$S = S(\pi).$$

This is a subset of the finite set of places, where σ_v belongs to the discrete series.

Corollary 5.4. *Suppose the assumptions of Lemma 5.6 are satisfied. Also suppose the key formula (stated below) holds. Then for any π in the global L-packet of a cuspidal irreducible representation σ with $\sigma \not\cong \sigma^*$ the multiplicity $m_{cusp}(\pi) = m_{disc}(\pi)$ is*

$$m(\pi) \;=\; \frac{1}{2}(1 + (-1)^{e(\pi)}), \qquad e(\pi) = \#S(\pi).$$

In Other Words: $m(\pi) = 1$ *for even* $e(\pi)$*, and* $m(\pi) = 0$ *otherwise.*

Proof. As in the proof of Lemma 5.4(2) this is true for $e(\pi) < 2$. So we may assume $e(\pi) \geq 2$. Pick two places $v_1 \neq v_2$ which contribute to $e(\pi)$. $\square$

If some Archimedean π_v belongs to the discrete series we assume v_1 to be Archimedean. More generally, let S be the set of places v where π_v is in the discrete series. Then we may assume v_1 to be some fixed place of S. Now consider the representation π' of $G(\mathbb{A}_F)$, which is obtained from π by replacing the two local representations π_{v_1} and π_{v_2} within their local L-packets. Since π_v is in the discrete series for $v = v_1, v_2$, the local L-packets both have cardinality 2 at v_1 and at v_2. With this notation we formulate the

Key Formula. *Let π be a weak endoscopic lift of a cuspidal automorphic representation σ with $\sigma^* \not\cong \sigma$. Suppose π is not a CAP representation and suppose $e(\pi) \geq 2$. Let π' be defined as above by replacements at two places v_1, v_2, where π belongs to the discrete series. Then $m(\pi) + m(\pi') > 0$ implies that $e(\pi)$ is even, and implies*

$$m(\pi) + m(\pi') \;=\; 2.$$

Continuation of the Proof of Corollary 5.4. It is clear that with this key formula we can prove Corollary 5.4 by induction on $e(\pi)$, by reduction to the known cases $e(\pi) = 0, 1$, since every set S' of even cardinality of the set of places where σ_v belongs to the discrete series can be obtained by a finite number of exchanges at two places of this set. It is of no harm to assume, in addition, that at one of the places, say, v_1 (for $F = \mathbb{Q}$, e.g., the Archimedean place, if π_∞ belongs to the discrete series), σ_{v_1} remains unchanged. This implies $m(\pi') = 1$ or $m(\pi') = 0$ by induction, depending on whether $e(\pi')$ is even or odd. This proves Corollary 5.4. $\square$

Proof of the Key Formula. We now prove it for $F = \mathbb{Q}$ and π_∞ in the discrete series. The general case is done in Sect. 5.4. Put $\pi = \pi_{\varepsilon_2}(\sigma_{v_2}) \otimes \pi_{\varepsilon_1}(\sigma_{v_1}) \otimes \pi^{v_1,v_2}$. Then by the weak multiplicity formula, applied twice,

$$\big(m(\pi_{\varepsilon_2}(\sigma_{v_2}) \otimes \pi_{\varepsilon_1}(\sigma_{v_1}) \otimes \pi^{v_1,v_2}) - m(\pi_{\varepsilon_2}(\sigma_{v_2}) \otimes \pi_{-\varepsilon_1}(\sigma_{v_1}) \otimes \pi^{v_1,v_2})\big)$$

$$-\big(m(\pi_{-\varepsilon_2}(\sigma_{v_2}) \otimes \pi_{\varepsilon_1}(\sigma_{v_1}) \otimes \pi^{v_1,v_2}) - m(\pi_{-\varepsilon_2}(\sigma_{v_2}) \otimes \pi_{-\varepsilon_1}(\sigma_{v_1}) \otimes \pi^{v_1,v_2})\big)$$

$$= (-1)^{e(\pi)} - (-1)^{e(\pi)\pm 1} = 2 \cdot (-1)^{e(\pi)} = 2 \cdot (-1)^{e(\pi)}.$$

By the principle of exchange, two of these multiplicities always vanish. If $m(\pi_{v_1 v_2}) > 0$ or $m(\pi) > 0$, then $m(\pi_{-\varepsilon_2}(\sigma_{v_2}) \otimes \pi_{\varepsilon_1}(\sigma_{v_1}) \otimes \pi^{v_1,v_2}) = m(\pi_{\varepsilon_2}(\sigma_{v_2}) \otimes \pi_{-\varepsilon_1}(\sigma_{v_1}) \otimes \pi^{v_1,v_2}) = 0$ vanishes (Corollary 5.3). Hence,

$$m\big(\pi_{\varepsilon_2}(\sigma_{v_2}) \otimes \pi_{\varepsilon_1}(\sigma_{v_1}) \otimes \pi^{v_1,v_2}\big) + m\big(\pi_{-\varepsilon_2}(\sigma_{v_2}) \otimes \pi_{-\varepsilon_1}(\sigma_{v_1}) \otimes \pi^{v_1,v_2}\big) = 2 \cdot (-1)^{e(\pi)}.$$

The left side is nonnegative by assumption, so $e(\pi)$ is even and $m(\pi) + m(\pi_{\infty,v}) = 2$. This proves the key formula, and hence Corollary 5.4 and the main theorem (in the special case). □

Corollary 5.5. *Suppose that σ is a (generic) irreducible cuspidal representation of $M(\mathbb{A}_F) = Gl(2, \mathbb{A}_F) \times Gl(2, \mathbb{A}_F)/A_F^*$. Suppose $\sigma = JL(\check{\sigma})$ is the Jacquet–Langlands lift of an irreducible representation $\check{\sigma}$ of some inner form $D^*(\mathbb{A}) \times D^*(\mathbb{A})/\mathbb{A}^*$ of $M(\mathbb{A})$, where D is a quaternion algebra. Then the corresponding theta lift $\Theta_D(\check{\sigma})$ does not vanish.*

Remark 5.4. The corresponding statement for $D^* = Gl(2)$ is easier and follows from the existence of Whittaker models. See [41] or Chap. 4. See also the discussion preceding Corollary 5.4.

Proof of Corollary 5.5. Suppose $\sigma_1 \cong \sigma_2$. Consider the theta lift $\Theta_D(\sigma)$ and its zero Fourier coefficient with respect to the maximal parabolic Q, which is not the Siegel parabolic. In the classical theory of Siegel modular forms this corresponds to considering the Siegel ϕ-operator. As a representation of the $Gl(2, \mathbb{A}_F)$-factor of the Levi component, this Fourier coefficient essentially defines the Jacquet–Langlands lift attached to the automorphic representation $\check{\sigma}_1 \cong \check{\sigma}_2$ of $D^*(\mathbb{A}_F)$ (for details we refer to [41] and Chap. 4, Corollary 4.13). Since the Jacquet–Langlands lift is always nontrivial, this implies that the theta lift $\Theta_D(\sigma)$ is not trivial. In particular, it is not cuspidal under the assumption $\sigma_1 \cong \sigma_2$.

To prove the first assertion we can now assume $\sigma_1 \ncong \sigma_2$. We will then show that the theta lift $\Theta_D(\sigma)$ does not vanish. Let S be the set of nonsplit places of the quaternion algebra D. As the cardinality of S is even, the multiplicity of the representation $\pi = \prod_{v \in S} \pi_-(\sigma_v) \otimes \prod_{v \notin S} \pi_+(\sigma_v)$ is 1 by Corollary 5.4. Hence, in particular, it is not zero, and π is therefore a cuspidal automorphic representation. As π is a weak endoscopic lift by definition, Proposition 5.1 implies that π is a theta lift. The set S of places where π_v does not have a Whittaker model is the set of nonsplit places of the corresponding algebra by Corollary 5.1. Hence, this algebra is D. But then by Proposition 5.2 π is a theta lift of type $\Theta_D(\sigma)$ or $\Theta_D(\sigma^*)$. But $\Theta_D(\sigma) = \Theta_D(\sigma^*)$ since the theta lift was originally defined by passing from $M = GSO(4)$ to $GO(4)$ (we refer to the remarks at the end of Sect. 5.1). Hence the claim follows. This proves Corollary 5.5. □

5.4 Local and Global Trace Identities

We say a locally constant function f_v on G_v with compact support on G_v satisfies the *condition (RS)*, if its support is contained in the locus of regular points. A semisimple element of G_v is called elliptic if it is an element of an elliptic Cartan subgroup. The set of regular elliptic points is open in G_v. For locally constant functions f_v on G_v with compact support we say *condition (ES)* holds if the support is contained in the regular elliptic locus. Condition (ES) implies that all the orbital integrals $O^{G_v}_{\gamma_v}(f_v)$ of f_v vanish for G_v-regular nonelliptic elements $\gamma_v \in G_v = GSp(4, F_v)$

$$O^{G_v}_{\gamma_v}(f_v) = 0 \quad \text{for } \gamma_v \text{ regular and not elliptic.}$$

Hence, $f_v \in A(G_v)$ in the sense of [44]. And $f_v \in A(G_v)$ implies that f_v is *cuspidal* in the sense of [6], p. 538. By definition, this means $(f_v)_{M_v} = 0$ for all Levi subgroups $M_v \neq G_v$ of G_v, or equivalently by the adjunction formula for induced representations

$$tr\ \pi_v(f_v) = 0, \qquad \text{for all } \pi_v = Ind^{G_v}_{P_v}(\sigma_v).$$

Here π_v runs over all induced representations of tempered representations σ_v of M_v for proper parabolic subgroups $P_v = M_v N_v$. See [5], p. 328.

We say *condition* $(*)_v$ holds for f_v if the stable orbital $SO^{G_v}_{\gamma_v}(f_v)$ of f_v vanishes for all regular semisimple $\gamma_v \in G_v$

$$(*)_v \qquad\qquad SO^{G_v}_{\gamma_v}(f_v) = 0 \qquad \text{(for all } \gamma_v \text{ regular semisimple).}$$

Since unstable tori of the group $G_v = GSp(4, F_v)$ are elliptic, the orbital integrals and stable orbital coincide for regular points outside the elliptic locus. Hence, condition $(*)_v$ implies $O^{G_v}_{\gamma_v}(f_v) = 0$ for all regular nonelliptic γ_v.

Let $\tilde{G}_v \subseteq G_v$ denote the subgroup of elements whose value under the similitude character is a square in $(F_v)^*$. Then $tr\ (\pi_v \otimes \mu_v)(f_v) = tr\ \pi_v(f_v)$ for all quadratic characters μ_v, if $supp(f_v) \subseteq \tilde{G}_v$. Note $\tilde{G}_v = Z_v \cdot Sp(4, F_v)$, and $\tilde{G}_v$ is open in G_v. Every element in G_v, which is stably conjugate to an element of $\tilde{G}_v$, is contained in $\tilde{G}_v$.

Remark 5.5. We later use the following auxiliary result, which is related to Proposition 5.3. For $F_v = \mathbb{R}$ there exists some σ_v in the discrete series and some K-finite infinitely differentiable function f_v with compact support satisfying $(*)_v$ and $tr\ \pi_-(\sigma_v)(f_v) > 0$. This is constructed by smooth truncation of a function $f_{\pi_+(\sigma_v)} - f_{\pi_+(\sigma_v)}$, where $f_{\pi_\pm(\sigma_v)}$ are pseudocoefficients. We leave this as an exercise. See also [91], (4.7.1).

Lemma 5.7 (Stability). *For σ_v in the discrete series of M_v, the character T of $\pi_+(\sigma_v) \oplus \pi_-(\sigma_v)$ is a stable distribution. In other words, for any locally constant function f_v on G_v, condition $(*)_v$ implies $T(f_v) = 0$.*

Proof of Lemma 5.7. In the Archimedean case this follows from [90], Lemma 5.2. In the non-Archimedean case one can argue as in [76], Proposition 12.5.3 and Corollary 12.5, to show that the stability of a distribution T is equivalent to the statement $< T, r(\rho_v) >_e = 0$ for all discrete series representations ρ_v of M_v. Since for $T = tr\ \pi_+(\sigma_v) + tr\ \pi_-(\sigma_v)$ the latter means $< \pi_+(\sigma_v) + \pi_-(\sigma_v), \pi_+(\rho_v) - \pi_-(\rho_v) >_e = 0$, this follows from the scalar product formulas proved in Chap. 4, Lemma 4.22. There are two cases that have to be distinguished. Either $\sigma_v \ncong \sigma_v^*$. Then we can apply the orthogonality relations, since both $\pi_\pm(\sigma_v)$ belong to the discrete series, and since furthermore $\pi_\varepsilon(\sigma_v) \cong \pi_{\varepsilon'}(\rho_v)$ implies $\sigma_v \cong \rho_v$ and $\varepsilon = \varepsilon'$. Or $\sigma_v \cong \sigma_v^*$. Then $< \pi_\varepsilon(\sigma_v), \pi_{\varepsilon'}(\rho_v) >_e = 1$ for $\sigma_v \cong \rho_v$ and is zero otherwise as shown in Chap. 4. So this gives the proof. □

Remark 5.6. The same argument proves stability for the traces $T = tr\ \pi_v$ of discrete series representations π_v of G_v for which π_v is not isomorphic to one of the representations $\pi_\pm(\sigma_v)$, σ_v in the discrete series.

Lemma 5.7 implies the existence of auxiliary functions $f_v \in A(G_v)$ as follows

Proposition 5.3 (Instability). *For σ_v in the discrete series of M_v, there exists a locally constant function f_v with condition (ES) and support in $\tilde{G}_v$ such that $(*)_v$ holds and such that $tr\ \pi_-(\sigma_v) > 0$.*

Proof of Proposition 5.3. We can assume v is non-Archimedean. Recall $f_v \in A(G_v)$, and this implies $T(f_v) = 0$ for $T = tr\ \pi_+(\sigma_v) + tr\ \pi_-(\sigma_v)$ by Lemma 5.7. Therefore, $tr\ \pi_-(\sigma_v)(f_v) > 0$ and $tr\ r(\sigma_v)(f_v) < 0$ are equivalent statements. Hence, it is enough to show $tr\ \sigma_v(f_v^{M_v}) < 0$ for a matching function $f_v^{M_v}$. The character of σ_v does not vanish in any neighborhood of the identity element for at least one elliptic torus T_v in M_v. Such a T_v defines two conjugacy classes of tori in G_v, which are stable conjugate, together with admissible isomorphisms between T_v and these tori. Since regular points are smooth points of the conjugation map, one can easily construct matching functions f_v and $f_v^{M_v}$ with support in these tori sufficiently near to the identity such that $(*)_v$ holds for f_v, but such that $tr\ \sigma_v(f_v^{M_v}) < 0$. This is done by the implicit function theorem. Simply consider bump functions f_v with small support near $\gamma_v \in T_v$ such that $SO_{\gamma_v}^{G_v}(f_v) = 0$ and such that $O_{\gamma_v}^\kappa(f_v) = SO_{\gamma_v}^{M_v}(f_v^{M_v}) \neq 0$ for a corresponding bump function with sufficiently small support near γ_v. Then the support of f_v is in $\tilde{G}_v$, and f_v satisfies (ES). Furthermore, $tr\ r(f_v) < 0$ and $f_v \in A(G_v)$. This proves the claim. □

We now prove the following results.

Proposition 5.4. *For irreducible σ_v in the discrete series $\mathcal{E}\Theta_-(\sigma_v) = \{\pi_-(\sigma_v)\}$.*

Proposition 5.5. *Suppose σ_v is irreducible and generic. Then $\pi_v \in \mathcal{E}\Theta_+^{glr}(\sigma_v)$ implies $\pi_v \cong \theta_+(\sigma_v)$.*

In some cases these statements are known already, for instance, in the Archimedean case. For p-adic fields (completions of $\mathbb{Q}$), Proposition 5.3 follows from the density lemma (Chap. 4, Sect. 5.1) and Lemma 5.6 in Sect. 5.3. Still another case: If σ_v is cuspidal non-Archimedean such that $\sigma_v \cong \sigma_v^*$, then $\mathcal{E}\Theta_+(\sigma_v) = \{\pi_+(\sigma_v)\}$ (see Chap. 4 and [106], p. 64, lines 10-15, and p. 55), which is stronger than Proposition 5.5. The distribution T of Lemma 5.7 is stable in this case for a trivial reason. Namely $\pi_+(\sigma_v)$ and $\pi_-(\sigma_v)$ are the two constituents of an induced representation called $1 \times \sigma_v$ (Lemma 4.19). Since the orbital integral of any $f_v \in A(G_v)$ has elliptic support, this implies $T(f_v) = 0$ since the trace of the induced representation $1 \times \sigma_v$ vanishes on the regular elliptic locus. In particular, we obtain

Auxiliary functions. For non-Archimedean p-adic local fields F_v and given an irreducible cuspidal representation σ_v with $\sigma_v \cong \sigma_v^*$, there exist functions $f_v \in A(G_v)$ with support in the regular elliptic point of $\tilde{G}_v$ such that $tr\ \pi_v(f_v) > 0$ holds for all $\pi_v \in \mathcal{E}\Theta_-(\sigma_v \otimes \mu_v)$, μ_v quadratic.

Proof of Proposition 5.4. The Archimedean case is well known, so assume v is non-Archimedean.

Suppose for some σ_v in the discrete series of M_v the assertion of Proposition 5.4 is false. Then there is a $\pi_v \in \mathcal{E}\Theta_-(\sigma_v)$ not isomorphic to $\pi_-(\sigma_v)$. Choose a global field F for the given local field F_v such that the central character of σ_v is induced from a grössen character ω of F. For reasons to become clear soon, we write v'' for the place v from now on.

We choose additional auxiliary non-Archimedean places v, v', for which $F/\mathbb{Q}$ is split. Choose $\sigma_v, \sigma_{v'}$ at these auxiliary places in the discrete series such that $\sigma_v \cong \sigma_v^*$ and $\sigma_{v'} \cong \sigma_{v'}^*$ with central characters $\omega_v\ \omega_{v'}$, respectively. Then we fix auxiliary functions $f_v \in A(G_v)$ and $f_{v'} \in A(G_{v'})$ with condition (ES) and support in $\tilde{G}_v$ and $\tilde{G}_{v'}$, respectively, as constructed above (Proposition 5.3).

By the density lemma formulated in Sect. 5.1 and the appropriate choice of central characters, we find a global cuspidal automorphic representation σ of $M(\mathbb{A}_F)$ such that $\sigma \ncong \sigma^*$ realizes the given discrete series representations σ_v, $\sigma_{v'}$, and $\sigma_{v''}$ at the places v, v', and v''. Consider the set C of classes of irreducible cuspidal automorphic representation π' in the global L-packet of σ, which specialize to the given representations $\pi_-(\sigma_v), \pi_-(\sigma_{v'})$, and $\pi_{v''}$ at the places v, v',and v''. Such a π' is never CAP, and by the density lemma we can assume C is nonempty.

We now apply Lemma 5.8 from Sect. 5.5 for the auxiliary functions $f_v, f_{v'}$ chosen above. Hence, conditions (ii), (iv), (v), and (vi) of Lemma 5.8 hold for $w = v$. For our choice of π (i) and (iii) are also satisfied. Furthermore, $f_{v''}$ may be arbitrary. Then all assumptions for Lemma 5.8 are satisfied. We get – since $f_{v''}$ is now arbitrary –

$$\sum_{\pi', (\pi')^{v,v'} = \pi^{v,v'}} m(\pi') \cdot tr\ \pi'_v(f_v) \cdot tr\ \pi'_{v'}(f_{v'}) \ = \ 0.$$

The right side vanishes, since $\pi_{v''} \ncong \pi_\pm(\sigma_{v''})$ at the local place v'', where we originally started from. Furthermore, as we know, the summation is over theta lifts.

Hence, every π'_v is in $\mathcal{E}\Theta_{\pm}(\sigma_v)$ up to a quadratic character twist (Lemma 5.2, part 1), and similarly for $\pi'_{v'}$.

Since the places v, v' are split in F, they are p-adic. Hence $\mathcal{E}\Theta_{-}(\sigma_v) = \{\pi_{-}(\sigma_v)\}$ from Sect. 5.3, and similarly $\mathcal{E}\Theta_{-}(\sigma_{v'}) = \{\pi_{-}(\sigma_{v'})\}$. From our choice of $f_v, f_{v'}$ (Proposition 5.4), therefore,

$$tr\ \pi'_v(f_v) \cdot tr\ \pi'_{v'}(f_{v'}) > 0$$

holds for all relevant π' in the sum, unless either $\pi_v \in \mathcal{E}\Theta_{+}(\sigma_v)$ or $\pi_{v'} \in \mathcal{E}\Theta_{+}(\sigma_{v'})$ holds up to quadratic character twists. But then the principle of exchange at v or v', respectively (Lemma 5.4), implies that π_v and $\pi_{v'}$ must be both in the plus space. From the assumption $\sigma_v \cong \sigma_v^*$ and $\sigma_{v'} \cong \sigma_{v'}^*$, we know from [106] that $\mathcal{E}\Theta_{+}(\sigma_v) = \{\theta_{+}(\sigma_v)\}$ and $\mathcal{E}\Theta_{+}(\sigma_{v'}) = \{\theta_{+}(\sigma_{v'})\}$. Hence, $tr\ \pi'_v(f_v) = tr\ \pi'_v \otimes \mu'_v(f_v) = tr\ \pi_{+}(\sigma_v)(f_v) = -tr\ \pi_{-}(\sigma_v)(f_v)$, and similarly at the place v'. Since the signs of v and v' cancel, again

$$tr\ \pi'_v(f_v) \cdot tr\ \pi'_{v'}(f_{v'}) > 0.$$

In other words, this term is positive for all relevant π' appearing in the trace formula stated above. Furthermore, $m(\pi') \geq 0$ and $m(\pi) \geq 1$. Since the total sum of the terms $tr\ \pi'_v(f_v) \cdot tr\ \pi'_{v'}(f_{v'})$ is zero, this gives a contradiction and completes the proof of Proposition 5.4. $\square$

Proof of Proposition 5.5. Fix a global cuspidal representation $\sigma \ncong \sigma^*$ of $M(\mathbb{A}_F)$ for some number field F. An irreducible cuspidal automorphic representation π in the global L-packet of σ will be called *strange* if for some place it is strange locally, i.e., if $\pi_v \in \mathcal{E}\Theta_{+}^{glr}(\sigma_v)$ for $\pi_v \ncong \pi_{+}(\sigma_v)$. Notice such a v is never Archimedean, since in this case the theta lift is understood well enough. We therefore have to show that strange representations π cannot arise at non-Archimedean places.

For any π' in the global L-packet attached to σ let $S = S(\pi')$ be the set of places v where $\pi'_v \in \mathcal{E}\Theta_{-}(\sigma_v)$ holds up to some quadratic character twist. If strange π exist in this L-packet, we choose π to be minimal with respect to the cardinality of $S(\pi)$ among all the strange π in this L-packet. For this π there exists a non-Archimedean place v'' where $\pi_{v''}$ is strange locally.

We claim the cardinality of $S(\pi)$ is 2 or more. Otherwise the underlying quaternion algebra $D = D(\pi)$ would be split and $S = \emptyset$ (second assertion of Lemma 5.4). In this situation the corresponding theta lift is locally and also globally generic according to results obtained by Howe and Piatetski-Shapiro [41] and also by Soudry. But this implies $\pi_v \cong \theta_{+}(\sigma_v)$ for all v, which contradicts the strangeness assumption. For this recall from [96] and Sect. 5.1 that there is at most one class of irreducible generic representation in $\Theta_{+}(\sigma_v)$. It follows that $\#S(\pi) \geq 2$.

Fix two different places v, v' in $S = S(\pi)$. Choose $f_v, f_{v'}$ to be cuspidal with $supp(f_v) \subseteq \tilde{G}_v$ and $supp(f_{v'}) \subseteq \tilde{G}_{v'}$ such that $f_v, f_{v'}$ satisfy $(*)_v$ ad $(*)_{v'}$, respectively. Furthermore, suppose $tr\ \pi_v(f_v) > 0$ and $tr\ \pi_{v'}(f_{v'}) > 0$ (Proposition 5.3).

Furthermore, choose an auxiliary non-Archimedean place $w \neq v, v', v''$ of residue characteristic different from 2, where σ_w and $\pi_w = \pi_{+}(\sigma_w)$ are unramified.

In particular, $w \notin S(\pi)$. Choose f_w with regular support contained in $\tilde{G}_w$ such that $tr\ \pi_+(\sigma_w)(f_w) > 0$. For example, take a bump function with support in the maximal split torus concentrated at a regular point near the origin, where the character of the unramified representation $\pi_+(\sigma_w)$ is nontrivial. Then, in particular, condition (RS) holds.

With these data fixed we apply Lemma 5.8 from Sect. 5.5. This gives

$$\sum_{\pi', (\pi')^{v,v',w} = \pi^{v,v',w}} m(\pi') \cdot tr\ \pi'_v(f_v) \cdot tr\ \pi'_{v'}(f_{v'}) \cdot tr\ \pi'_w(f_w) \ = \ 0.$$

The right side vanishes since $\pi_{v''} \ncong \pi_\pm(\sigma_{v''})$ at the place v'' where $\pi_{v''}$ is locally strange.

The minimality of π and Propositions 5.3 and 5.4 imply $\pi'_v \in \mathcal{E}\Theta_-(\sigma_v)$ (up to quadratic character twists) for all π' which contribute to the trace formula above, i.e., those π' which are isomorphic to π outside v, v'. The same assertion is true at the place v'. Hence, $tr\ \pi'_v(f_v) = tr\ \pi_-(f_v) > 0$ and also for the place v', by our choice of test functions. It remains for us to consider the auxiliary place w. Here $\pi'_w \otimes \mu_w \in \mathcal{E}\Theta^+(\sigma_w)$ holds up to some quadratic character μ_w. The unramified Sp–O Howe correspondence matches unramified representations with unramified representations. Hence, the restriction of π'_w to $Sp(4, F_w)$ contains an unramified representation since σ_w was unramified by assumption. Hence, up to a quadratic character twist both π_w and π'_w are constituents of $Ind_{Z_w \cdot Sp(4,F_w)}^{G_w}(\tilde{\pi}'_w)$ for the same irreducible unramified representation $\tilde{\pi}'_w$ of $\tilde{G}_w$. This implies $\pi'_w = \chi_w \otimes \pi_w$ for some quadratic character χ_w [104], p. 480, line 5. Therefore, $tr\ \pi'_w(f_w) = tr\ \pi_w(\sigma_w) > 0$ holds independently of π', because the support f_w is contained in $\tilde{G}_w$. This gives a contradiction, since in the trace formula above all these nonnegative terms should sum to zero with at least one of them being positive (for π itself). Hence, there are no strange representations π in the global L-packet of σ. Proposition 5.5 is proven. □

To complete the proof of Theorem 5.2 and also to obtain all the results stated in Sects. 5.2 and 5.3 in full generality, it remains to prove the key formula and the local–global principle (Lemma 5.6) stated in Sect. 5.3. The essential part of the local–global principle is stated in Propositions 5.4 and 5.5. What remains to obtain Lemma 5.6 will be shown below together with the key formula (as stated in Sect. 5.3). So once more we exploit the trace formula.

Proof. Let $\sigma \ncong \sigma^*$ be a cuspidal irreducible automorphic representation of $M(\mathbb{A}_F)$. Then any π in the global L-packet attached to σ is not CAP. Fix π and suppose $e(\pi) \geq 2$. Choose two places $\{v, v'\}$ in $S(\pi)$ and consider Lemma 5.8 in Sect. 5.5 in the special case where the assumptions $(*)_v$ and $(*)_{v'}$ hold simultaneously. In particular, the functions $f_v, f_{v'}$ are cuspidal. We choose f_v and $f_{v'}$ so that $a_+ \neq 0$ and $a'_+ \neq 0$, respectively, and condition (ES) holds in addition with supports in $\tilde{G}_v$ and $\tilde{G}_{v'}$, respectively. This is possible by Proposition 5.3. Lemma 5.7 implies $a_- = -a_+ \neq 0$ and $a'_- = a'_+ \neq 0$, where we used the abbreviations $a_\pm := tr\ \pi'_{\pm\varepsilon}(\sigma_v)(f_v)$ and $a'_\pm := tr\ \pi'_{\pm\varepsilon'}(\sigma_{v'})(f_{v'})$. To satisfy assumption (vi) in Sect. 5.5, if both v and v' are Archimedean, we also choose some additional non-Archimedean place w and some function f_w as in the proof of Proposition 5.5.

The semilocal trace identity of Lemma 5.8 in this case is simplified considerably. This means that in the sum $\sum' a_{disc}(\pi'_v\pi'_{v'}\pi'_w\pi^{vv'w}) \cdot tr\ \pi'_v(f_v) \cdot tr\ \pi'_{v'}(f_{v'}) \cdot tr\pi'_w(f_w)$ only those representations π' contribute whose local components are in the local L-packet of σ at least up to a quadratic character twist. In particular, π_w is unramified up to a quadratic character twist. In fact all this follows from the last two propositions, and the arguments used for their proof. By the support conditions of f_v, $f_{v'}$, and f_w, quadratic character twists do not have an effect on the trace. Therefore, the contribution $tr\ \pi'_w(f_w) = tr\ \pi_+(f_w) \neq 0$ is independent of π' and can be canceled from both sides of the semilocal identity. Similarly, the nonvanishing terms $a_+ = -a_- \neq 0$ and $a'_+ = -a'_- \neq 0$ are independent of π'. Canceling these terms therefore gives $M_{++} + M_{+-} + M_{-+} + M_{--} = 2(-1)^{e(\pi)}$.

Here $M_{\varepsilon,\varepsilon'}$ is the sum over all multiplicities $m(\pi')$ over all π' such that $\pi' \otimes \mu_v\mu_{v'}\mu_w \cong \pi_\varepsilon(\sigma_v)\pi_{\varepsilon'}(\sigma_v)\pi^{vv'}$ for some local quadratic characters μ_v, $\mu_{v'}$, and μ_w. Since the right side $2(-1)^{e(\pi)}$ equals the left side, which is zero or more, we get $2|e(\pi)$. Furthermore, by the principle of exchange, the condition $M_{++} > 0$ (note $m_{++} > 0$ by assumption) implies $M_{-+} = M_{+-} = 0$. Hence, $M_{++} + M_{--} = 2$ and $e(\pi)$ is even. This almost completes the proof of the key formula. In fact we know $M(\pi') = 1$ from Lemma 5.4 part 2, if the cardinality of $S(\pi')$ is $0, 1$. So by induction on this cardinality, the argument surrounding the statement of the key formula in Sect. 5.3 proves $M_{++} = M_{--} = 1$. It only remains for us to show $m(\pi') = 0$ unless π_v and $\pi_{v'}$ are in the local L-packet of σ_v and $\sigma_{v'}$, respectively. For $\pi_v \in \mu_v \otimes \Theta_+(\sigma_v)$ this follows with the same argument used in the proof of Proposition 5.5. In particular, we always get $\pi'_w \cong \pi_+(\sigma_w)$ at the auxiliary place w. (This could also be seen by moving this place around, i.e., changing to some other place where σ_w is unramified and where $\pi' = \pi_w(\sigma_w)$ holds.) So by the symmetry of v, v' it only remains to show $m(\pi') = 0$ whenever $\pi'_v \cong \mu_v \otimes \pi_-(\sigma_v)$ holds for some quadratic character μ_v but $\pi'_v \ncong \pi_-(\sigma_v)$. Assume this were not true. To obtain a contradiction, again apply the trace formula with $f_{v'}, f_w$ as above, but now with f_v satisfying only condition (ES) but being arbitrary otherwise. Still the assumptions of Lemma 5.8 in Sect. 5.5 are satisfied. From this we obtain a contradiction, provided the characters χ_1 and χ_2 of $\mu_v \otimes \pi_-(\sigma_v)$ and $\pi_-(\sigma_v)$ are linear-independent on the regular elliptic locus of G_v (the support of f_v). Notice the first character, χ_1, appears on the left side of the trace identity, whereas the other character, χ_2, appears on the endoscopic left side of the assertion in Lemma 5.8. So it remains for us to show that linear independence holds on the elliptic locus. For $\sigma_v \ncong \sigma_v^*$ both representations are cuspidal and the claimed linear independence follows from the orthogonality relations for cuspidal characters with respect to the elliptic scalar product. If $\sigma_v \cong \sigma_v^*$, both representations do not belong to the discrete series. But in this case the distribution T of Lemma 5.7 vanishes on the regular elliptic locus. Hence, $-2\chi_1 = r(\mu_v \otimes \sigma_v)$ and $-2\chi_2 = r(\sigma_v)$ holds on the regular elliptic locus, so the linear independence of χ_1, χ_2 on the elliptic locus follows from the corresponding linear independence of $\mu_v \otimes \sigma_v$ and σ_v. Since these representations are in the discrete series, and since they are not isomorphic by our assumptions, they are linear-independent on the regular elliptic locus of M_v. This contradiction implies that any π' in the global L-packet with $m(\pi') > 0$ has its local components π'_v in the local L-packet attached to σ_v for all places v of F. In other words, the local–global principle holds. This, together with the multiplicity result obtained above, implies the key formula. □

Remark 5.7. If we relax the assumptions on $\sigma = (\sigma_1, \sigma_2)$ in definition 5.1, we can similarly consider the case where σ_1 is one-dimensional. If σ_2 is in the discrete series, this is the situation considered in [3]. With the results on L-series shown in [69] (instead of the results obtained by Soudry used in the non-CAP case), the coefficients a_{disc} can be worked out directly, and the arguments from above should extend to give a characterization of the CAP representation in terms of the local Arthur packets. This indeed would give a refined description of the Saito–Kurokawa lift in term of local Arthur packets.

5.5 Appendix on Arthur's Trace Formula

In this appendix we apply the Arthur trace formula. We deduce from it certain local character identities, whose coefficients contain global information. We therefore refer to them as semilocal character relations. This semilocal character identities are useful for several reasons. First, they are the starting point for the proof of the local character identities of the endoscopic lift r in Sect. 5.1. Next, we used them in Sect. 5.4 to show local statements like the stability lemma. Finally, they provide global information, once the local concepts are well understood. For this reason, and to be flexible enough for all these applications, we formulate a number of technical conditions. Under these conditions we prove the semilocal character identities in Lemma 5.8.

Make the following assumptions (i)–(vi):

(i) Let σ be an irreducible automorphic representation of $M(\mathbb{A}_F)$ with σ_v, $\sigma_{v'}$ in the discrete series for at least two different places v, v'.
(ii) Suppose that σ is *cuspidal* and $\sigma \ncong \sigma^*$. Then the global L-packet of σ contains cuspidal representations. These are not CAP representations, since otherwise there would exist poles for partial degree 4 or 5 L-functions of π, which contradict the cuspidality of σ. See Sect. 5.2.

Choose signs $\varepsilon, \varepsilon' \in \{\pm\}$. Consider the representations $\pi = \pi_\varepsilon(\sigma_v) \otimes \pi_{\varepsilon'}(\sigma_{v'}) \otimes \pi^{v,v'}$ and $\pi' = \pi_{-\varepsilon}(\sigma_v) \otimes \pi_{-\varepsilon'}(\sigma_{v'}) \otimes \pi^{v,v'}$ in the global L-packet of the weak lift of σ. We write $\pi = \pi_{++}$, $\pi' = \pi_{--}$ in the following. Let $m = m_{++} = m(\pi)$ and $m' = m_{--} = m(\pi')$ denote their multiplicity in the discrete spectrum of $G(\mathbb{A}_F)$. Assume:

(iii) $m_{++} + m_{--} > 0$.

From (iii) it follows that either π or π' is an automorphic representation. By switching $\varepsilon, \varepsilon'$ into their negatives, we can assume without loss of generality $m(\pi) > 0$.

Then the restriction of π to $Sp(4,\mathbb{A}_F)$ is a theta lift and is cuspidal (Lemma 5.2). It is not CAP by assumption (i). So π and π' only contribute to the cuspidal spectrum. Every automorphic representation in the discrete spectrum isomorphic to π or π' belongs to the cuspidal spectrum. This follows from Langlands results on spectral decomposition and the fact that π is not CAP. Therefore, put $m = m_{++} = m_{cusp}(\pi)$ and $m' = m_{--} = m_{cusp}(\pi')$.

Now the essential conditions:

(iv) Let $f_v, f_{v'}$ be locally compact functions with compact support on G_v and $G_{v'}$, respectively, which are in the Hecke algebra in the sense of [6]. Assume that both functions have vanishing orbital integrals at regular nonelliptic semisimple elements. In particular, they are cuspidal.
(v) Assume that either condition $(*)_v$ or condition $(*)_{v'}$ holds.

Furthermore, suppose:

(vi) Condition (RS) holds for the test function f_w at least at one auxiliary non-Archimedean place w, and the case $w \in \{v, v'\}$ is not excluded.

Lemma 5.8. *Suppose assumptions (i)–(vi) hold. Then*

$$\sum_{\pi'}{}' a_{disc}(\pi'_v\pi'_{v'}\pi'_w\pi^{vv'}) \cdot tr\ \pi'_v(f_v) \cdot tr\ \pi'_{v'}(f_{v'}) \cdot tr\ \pi'(f_w)$$

is either zero if for a place $v'' \neq v, v', w$ the class of $\pi_{v''}$ satisfies $\pi_{v''} \notin \{\pi_\pm(\sigma_{v''})\}$, or equal to

$$\frac{1}{2}(-1)^{e(\pi)}(a_+ - a_-)(a'_+ - a'_-) \cdot \tilde{T}(f_w)$$

otherwise. Here $\tilde{T} = tr\ \pi_+(\sigma_w) - tr\ \pi_-(\sigma_w)$ if σ_w is in the discrete series and $\tilde{T} = tr\ \pi_+(\sigma_w)$ otherwise.

Concerning the Notation. In these formulas the summation $\sum'$ is over all classes of global representations π' in the global L-packet of σ for which $(\pi')^{v,v',w} \cong \pi^{v,v',w}$ holds. Here $a_\pm$ and $a'_\pm$ are abbreviations for $a_\pm = tr\ \pi'_{\pm\varepsilon}(\sigma_v)(f_v)$ and $a'_\pm = tr\ \pi'_{\pm\varepsilon'}(\sigma_{v'})(f_{v'})$, respectively. The coefficients $a_{\varepsilon\varepsilon'}$ are zero unless the corresponding multiplicities $m_{\varepsilon\varepsilon'}$ of the weak lifts $\pi_\varepsilon(\sigma_v)\pi_{\varepsilon'}(\sigma_{v'})\pi^{v,v'}$ in the discrete spectrum are nonzero. In fact they are equal to the *multiplicities* of these representations. This follows from [6] (see the references given below) and the Langlands theory of spectral decomposition, since by assumption (ii) all these representations are not CAP, as explained above. In fact $a_{++} := a^G_{disc}(\pi) = m_{cusp}(\pi)$ for any π in the weak lift of σ, which is in the discrete spectrum and is similar for all π' in the global L-packet of σ.

Proof. For simplicity we suppose for the proof $m(\pi) > 0$, as above.

We apply the results obtained by Arthur [6] concerning the trace formula for $G(\mathbb{A}_F)$. Choose any test functions f_λ for the places $\lambda \neq v, v', w$ and put $f =$

$\prod_\lambda f_\lambda$. By assumption f_v and f'_v are cuspidal. So the orbital integrals of these two functions vanish outside the regular elliptic locus and these functions are cuspidal in the sense of [6], p. 538. Since f is cuspidal at the two different places v, v', the Arthur trace formula is simplified. The spectral side is a sum over the traces of the discrete spectrum suitably ordered using the Archimedean infinitesimal characters ([6], Theorem 7.1 and Corollary 7.2) using the notation from [6]

$$\sum_{\pi'} a^G_{disc}(\pi') \cdot I_G(f).$$

Notice $I_G(\pi', f) = tr\ \pi'(f)$ in the notation in [5], p. 325. The coefficients $a^G_{disc}(\pi')$ are complex numbers, and by grouping together the linear combinations of weighted characters defined in [6], formula (4.3), we have

$$\sum_{M_0 \subseteq L_0 \subseteq G_0} |W_0^{L_0}||W_0^G|^{-1} \sum_s |det(s-1)|^{-1}_{\mathfrak{a}^G_{L_0}} \cdot tr\left(M_{Q_0|sQ_0}(0) \circ \rho_{Q_0,t}(s,0,f^1)\right).$$

Here $\rho_{Q_0,t}$ is an induced representation, induced from the part $L^2_{disc,t}(L_0(F)A_{L_0,\infty} \backslash L_0(\mathbb{A}_F))$ of the discrete spectrum of the Levi subgroup L_0. If π' is cuspidal but not CAP, only $L_0 = G_0$ contributes to the coefficient $a_{disc}(\pi)$ and the sum becomes the trace of f on $L^2_{disc,t}(L_0(F)A_{L_0,\infty} \setminus L_0(\mathbb{A}_F))$. Hence, $a_{disc}(\pi) = m_{cusp}(\pi)$ holds in this case. In general, of course, $a_{disc}(\pi) = 0$ unless $m_{disc}(\pi') \neq 0$.

Concerning the geometric side of this trace formula, we obtain from [6], Corollary 7.2,

$$\sum_{\gamma \in (G(F))_{G,S}} a^G(S,\gamma) I_G(\gamma, f).$$

Here $I_G(\gamma, f)$ is the global orbital integral $O^G_\gamma(f) = \prod_v O^{G_v}_{\gamma_v}(f_v)$ of f (see [6], p. 325). Moreover, by our assumption (vi) regarding f_w, the orbital integral at w vanishes unless γ_w is regular semisimple. This implies γ is regular semisimple. Moreover, by our assumption (iv) regarding $f_v, f_{v'}$, the geometric side only involves regular elliptic terms (as in [6], Corollary 7.4). But in this case one can express $a^G(S,\gamma)$ explicitly. As in [6], Corollary 7.4, one obtains the simpler expression for the geometric side

$$\sum_{\gamma \in (G(F)_{ell})} vol(G(F,\gamma)A_{G,\infty} \setminus G(\mathbb{A}_F,\gamma)) \int_{G(\mathbb{A}_F,\gamma)\backslash G(\mathbb{A}_F)} f(x^{-1}\gamma x)dx.$$

Here $G(F,\gamma) = Zent(\gamma, G^0)(F)$. In our case $G = G^0$. Furthermore, in our case G_{der} is simply connected.

Stabilization of the elliptic terms of the geometric side as in [53], using the fundamental lemma proved in Chaps. 6–9, gives for the geometric side of the Arthur trace formula a rearrangement in terms of stable orbital integrals for G and stable orbital integrals for the endoscopic group M

$$\sum_{\pi'} a_{disc}(\pi') \cdot tr\ \pi'(f) \ = \ ST^{G,**}(f) + \frac{1}{4} ST^{M,**}(f^M).$$

The $**$-condition on central terms in [53] can be ignored by the regular support condition (vi) at the places w. Moreover, the global stable orbital $ST^{G,**}(f) = ST^G(f) = 0$ vanishes by the local assumption $(*)_v$ or $(*)_{v'}$ of assumption (v)

$$ST^{G,**}(f) \ + \ \frac{1}{4} ST^{M,**}(f^M) \ = \ \frac{1}{4} ST^{M,**}(f^M).$$

Since M is quasisplit, the terms omitted in $ST^{M,**}(f^M)$ are again the central terms of the stable (semisimple) trace $ST^M(f^M)$ or preferably of a suitable stable trace on the z-extension $\tilde{M} = Gl(2) \times Gl(2)$. One of the functions $f_v, f_{v'}, f_w$ satisfies condition (RS). Without restriction of generality, suppose it is $f_{v'}$. Then there is a matching function $f_{v'}^M$ with regular support by the implicit function theorem using the smoothness of the regular locus of an elliptic torus. Furthermore, we can assume that for the two places v, v' the corresponding functions $f_v^M, f_{v'}^M$ have vanishing orbital integrals for regular nonelliptic elements. Then

$$ST^{M,**}(f^M) \ = \ ST^M(f^M).$$

The z-extension $\tilde{M}$ of M does not have nontrivial endoscopy. The strong cuspidal condition (iv) is inherited by f^M, as well as condition (vi). So again the geometric side of Arthur's trace formula for $f^{\tilde{M}}$ is simple, and in particular only involves elliptic regular terms. Hence, stabilization gives

$$\frac{1}{4} ST^{M,**}(f^M) \ = \ \frac{1}{4} T^{\tilde{M}}(f^{\tilde{M}}).$$

If we compare the geometric terms $T^{\tilde{M}}(f^{\tilde{M}})$ with the spectral side, the simple form of the Arthur trace formula now applied for $\tilde{M}$ yields the character expansion

$$\frac{1}{4} T^{\tilde{M}}(f^{\tilde{M}}) \ = \ \frac{1}{4} \sum_{\sigma} \ a_{disc}(\otimes_v' \sigma_v) \cdot \prod_v tr\ r(\sigma_v)(f_v)$$

for the geometric side. The right side is a sum with σ running over the discrete spectrum of $\tilde{M}(\mathbb{A}_F)$, suitably ordered. To obtain this formula we used the local character identities $tr\ \sigma_v(f_v^{\tilde{M}_v}) \ = \ r(\sigma_v)(f_v)$. By the multiplicity 1 theorem for $Gl(2)$ and $\tilde{M}$ and the spectral theory for $Gl(2)$, we get $a_{disc}(\sigma) = 1$ for all cuspidal representations $\sigma = \otimes_v' \sigma_v$. See [29, 42].

Notice the expansion for $\frac{1}{4} T^{\tilde{M}}(f^{\tilde{M}})$ above is a character expansion in terms of representations of $G(\mathbb{A}_F)$. It involves only representations π which are weak lifts coming from M. All local components π_v that appear are in the local L-packet of an underlying global representation σ in the discrete spectrum of $M(\mathbb{A}_F)$. Comparing this character expansion with the one obtained from the trace formula for $G(\mathbb{A}_F)$ gives the

Formula (CharIdent).

$$\sum_{\pi'} a_{disc}(\pi') \cdot tr\ \pi'(f) \ = \ \frac{1}{4} \cdot \sum_{\sigma} a_{disc}(\sigma) \cdot \prod_{v} tr\ r(\sigma_v)(f_v).$$

A fixed representation $\pi^{v,v',w}$ of $G(\mathbb{A}_F^{v,v',w})$ belongs to the global L-packet of a fixed pair σ, σ^* of representations of $M(\mathbb{A}_F)$. This follows from Proposition 5.2 and the strong multiplicity 1 theorem for $\widetilde{M}$. So the last character identity should imply – so to say from the linear independence of characters of $G(\mathbb{A}_F^{v,v',w})$ and by separating the component $\pi^{v,v',w}$ – the following *semilocal identity*: The term

$$\sum{}' a_{disc}(\pi'_v \pi'_{v'} \pi'_w \pi^{vv'w}) \cdot tr\ \pi'_v(f_v) \cdot tr\ \pi'_{v'}(f_{v'}) \cdot tr\ \pi'_w(f_w)$$

is equal to

$$= \ \frac{1}{2}(-1)^{e(\pi)} a_{disc}(\sigma) \cdot (a_+ - a_-) \cdot (a'_+ - a'_-) \cdot \tilde{T}(f_w)$$

if all local components of $\pi^{v,v',w}$ are in the local L-packets, and it is zero otherwise.

Since σ and σ^* were supposed to be not isomorphic, and since both σ and σ^* contribute (Proposition 5.2), we got the factor $\frac{1}{2}$ instead of the factor $\frac{1}{4}$ from the sum over the σ on the right side. In the sum the representations vary over all π' with $\pi^{v,v',w}$ fixed up to isomorphism.

Since the Arthur trace formula is not known to converge absolutely, an easy argument which implies the linear independence of characters in the sense above is not known at present. However, assumptions (vi) or (vi) put us into a situation where the above semilocal identity can nevertheless be extracted from the global trace formula.

To extract the semilocal identity stated above from the global Arthur trace formula in this case one uses multipliers at the Archimedean places.

Multipliers. Let $f_\infty \in C_c^\infty(G_\infty, K_\infty)$ be a K_∞-finite test function at the Archimedean places. A W-invariant distribution α with compact support on the Lie algebra of the standard Cartan subgroup h^1 is called a multiplier. In our cases h^1 is the Lie algebra of a maximal split torus; for the general case see [6], Sect. 6. Typical examples are elements in the center of the universal enveloping algebra or W-invariant smooth functions with compact support. Multipliers α act on $C_c^\infty(G_\infty, K_\infty)$ in a natural way as shown by Arthur. Let $f_\infty \mapsto (f_\infty)_\alpha$ denote this action. For an irreducible admissible unitary representation π_∞ of G_∞ let ν_π denote its infinitesimal character viewed as a W-orbit in h^1. Let t_π denote the length of its imaginary part with respect to a suitable norm on h^1 [6]. Then $\pi_\infty((f_\infty)_\alpha)) = \hat{\alpha}(\nu_\pi)\pi_\infty(f_\infty)$ for the Fourier transform $\hat{\alpha}$ of the distribution α. Indeed this formula uniquely characterizes the action.

Notice $(f_\infty)_\alpha$ is cuspidal if f_∞ is cuspidal. This is true since cuspidality is characterized by the vanishing of the traces $tr\ \pi_\infty(f_\infty)$ for all representations π_∞ which are properly induced from irreducible tempered representations. Since $tr\ \pi_\infty((f_\infty)_\alpha)) = \hat{\alpha}(\nu_\pi) tr\ \pi_\infty(f_\infty)$, this property is preserved.

Furthermore, the condition $(*)_\infty$ is preserved by the action of multipliers. See [97] and [6], Definition 1.2.1. Without restriction of generality $F_v = \mathbb{R}$. Then $(*)_\infty$ is equivalent to $T(f) = 0$ for all characters $T = tr\ \pi_\infty$ of discrete series representations of G_v not isomorphic to $\pi_\pm(\sigma_v)$, and for all $T = tr\ \pi_+(\sigma_v) + tr\ \pi_-(\sigma_v)$ attached to discrete series representations σ_v of M_v and all characters T of representations properly induced from tempered representations. This follows from [90], Lemma 5.3. Furthermore, for an irreducible unitary representation σ_∞ the infinitesimal character ν_{σ_∞} determines the infinitesimal character $\nu_\pm$ of the representations $\pi_\pm(\sigma_\infty)$. Indeed

$$\nu_+ = \nu_- = \xi(\nu_{\sigma_\infty})$$

for a suitable "linear" map ξ, up to some shift. This follows from the description of the endoscopic lift in terms of the theta correspondence or from [91], Lemma 4.2.1. The precise nature of the map ξ is of no importance here. For $\nu_v = \xi(\nu_{\sigma_v})$ this implies for the Archimedean places v

$$T((f_v)_\alpha) = \hat{\alpha}(\xi(\nu_v)) \cdot T(f_v) = \widehat{\tilde{\alpha}}(\nu_v) \cdot T(f_v)$$

for $\tilde{\alpha} = \alpha \circ \xi$.

Multipliers act on the global K-finite test function $f = \prod_v f_v$ in $C_c^\infty(G_\infty \times G((\mathbb{A}_F)_f))$ via their action on the Archimedean component f_∞. Hence, the non-Archimedean condition (vi) is preserved by the action of multipliers for trivial reasons.

The infinitesimal character $\nu = \nu_{\pi_\infty}$ of a unitary irreducible admissible representation π_∞ defines a W-orbit. Given ν, a smooth multiplier α is constructed on p. 182ff of [8] such that for $\alpha_m = \alpha * \ldots * \alpha$ (m-fold convolution) the following holds:

$$lim_{m\to\infty} \sum_{\pi'} a_{disc}(\pi') \cdot tr\ \pi'(f_{\alpha_m}) \ = \sum_{\pi',\ \nu_{\pi'}=\nu} a_{disc}(\pi') \cdot tr\ \pi'(f).$$

Recall the left side is the spectral side of the Arthur trace formula in a simple form (for our purposes f is supposed to satisfy the assumption of the trace hypothesis; under the second assumption of condition (vi) this assumption is stable under the action of multipliers). This sum is not necessarily absolutely convergent, so summation is with respect to a suitable ordering using the parameter $t_{\pi'}$. The sum on the right is absolutely convergent owing to the admissibility statement [6], Lemma 4.1. Hence, linear independence of the characters involved holds in the sense of [42], Lemma 16.1.1. We recall that for the above limit formula it suffices to know that $\hat{\alpha}(\nu) = 1$ and $|\hat{\alpha}(\nu_{\pi'})| < 1$ unless the W-orbits of $\nu_{\pi'}$ and ν coincide.

A similar separation of the infinitesimal character can be obtained for the spectral side of the simple trace formula for the endoscopic group. In fact, one can do better.

For an irreducible unitary representation σ_∞ the infinitesimal character ν_{σ_∞} determines the infinitesimal character $\nu_\pm$ of the representations $\pi_\pm(\sigma_\infty)$, as explained above: $\nu_+ = \nu_- = \xi(\nu_{\sigma_\infty})$ for a suitable "linear" map ξ. For $\nu_v = \xi(\nu_{\sigma_v})$ this implies for the Archimedean places v

$$tr\, r(\sigma_v((f_v)_\alpha) = \hat{\alpha}(\xi(\nu_v)) \cdot tr\, r(\sigma_v)(f_v) = \hat{\alpha}(\xi(\nu_v)) \cdot tr\, \sigma_v((f_v)^{M_v}) = tr\, \sigma_v((f_v)_{\tilde{\alpha}})$$

for $\tilde{\alpha} = \alpha \circ \xi$. If we identify the domains of the functions α and $\tilde{\alpha}$, as we may do, we can symbolically write

$$\left((f_v)_\alpha\right)^{M_v} = \left(f_v^{M_v}\right)_\alpha .$$

In other words, the action of multipliers commutes with the endoscopic matching condition. Since the Weyl group of M_v is a subgroup of the Weyl group of G_v, the smooth multiplier α for G_v can therefore be considered as a smooth ($*$-invariant) multiplier for M_v. This being said, we can consider the formulas (CharIdent) from above for the various test functions f_{α_m}. From the spectral limit formulas both for G and for M we obtain for $m \to \infty$

$$\sum_{\pi', \nu_{\pi'} = \xi(\nu)} a_{disc}(\pi') \cdot tr\, \pi'(f) = \frac{1}{4} \cdot \sum_{\sigma,\ \nu_\sigma = \nu} a_{disc}(\sigma) \cdot \prod_v tr\, \sigma_v((f_v)^{M_v})$$

$$= \frac{1}{4} \cdot \sum_{\sigma,\ \nu_\sigma = \nu} a_{disc}(\sigma) \cdot \prod_v tr\, r(\sigma_v)(f_v).$$

From this formula [6], Lemma 4.1, and [42], Lemma 16.1.1, the semilocal identity follows. This completes the proof of Lemma 5.8. □

Proof of formula (0). See page 180. We now explain how to obtain formula (0) in the non-p-adic case as a complement of the proof of Lemma 5.8. For this choose v, v' as in Lemma 5.8 to be Archimedean for a suitable chosen auxiliary number field F, and a suitably chosen auxiliary global irreducible cuspidal automorphic representation $\sigma \not\cong \sigma^*$ of $M(\mathbb{A}_F)$, and we choose w to be some auxiliary "harmless" place, where the global representation σ is unramified and for which the norm of w is sufficiently large. Furthermore, we choose F and σ so that for some additional auxiliary non-Archimedean place w', where $F/\mathbb{Q}$ splits with residue characteristic different from 2, $\sigma_{w'}^* \not\cong \sigma_{w'}$ holds and is cuspidal. Then $\pi_\pm(\sigma_{w'})$ are cuspidal and up to character twists the only representations in $\mathcal{E}\Theta_\pm(\sigma_{w''})$ (see Proposition 5.1 for the notation). Indeed, the statement on the theta lift uses Waldspurger's proof of the Howe duality for the dual pair $Sp(4) \times O(4)$, whereas the cuspidality statement uses what we already considered the case of local fields, which are completions of $\mathbb{Q}$, in Theorem 4.5. For the Archimedean places we choose f_v to be some fixed auxiliary cuspidal function in the Hecke algebra satisfying $(*)_v$ such that $a_- = -a_+ = tr\ \pi_-(\sigma_v)(f_v) > 0$ (notation as in Sect. 5.4). At the Archimedean place v' we choose two functions $f_{v'}$ such that $tr\ \pi_+(\sigma_{v'})(f_{v'}) = 0$ and $tr\ \pi_-(\sigma_{v'})(f_{v'}) = 1$ or vice versa. At the place w' we choose a function $f_{w'}$

in the Hecke algebra whose traces separate the finitely many cuspidal representations of G_v which appear in the packets $\mathcal{E}\Theta_{\pm}(\sigma_{w'})$ of the theta lift twisted by quadratic characters. For this we can assume $f_{w'}$ to be a linear combination of matrix coefficients of these finitely many cuspidal representations. Finally f_w is chosen with regular support in a maximal split torus of G_w. With these choices the global trace formula is considerably simplified, as observed by Deligne–Kazhdan, by Arthur and by Henniart. In particular, on the spectral side only the cuspidal automorphic spectrum contributes, owing to the cuspidal matrix coefficients $f_{w'}$ at the non-Archimedean place w'. This makes the use of multipliers, which were necessary in the more complicated situation of Lemma 5.8, superfluous. Secondly, the $f_{w'}$ chosen completely suffice for the detection of all local constituents of global endoscopic lifts at the place w'. This is due to Lemma 5.2. The next observation is that "moving" the unramified auxiliary place w to become "sufficiently large" allows us to get rid of the influence of f_w in the trace formula. This follows from well-known finiteness results (i.e., apply the same trick as in the proof of Lemma 5.6 and of the key formula at the end of Sect. 5.4). Furthermore, similarly as for the place w', Lemmas 5.2–5.5 in Sect. 5.2 control the Archimedean constituents of the global endoscopic lift. So the $f_{v'}$ chosen again completely detect all local constituents of global endoscopic lifts at the place v' (Lemmas 5.2 and 5.5 in Sect. 5.2). With these choices made, we can now follow the arguments in the proof of Lemma 5.8 mutatis mutandis to obtain

$$\sum_{\pi'} m(\pi'_v \otimes (\pi')^v) \cdot tr\ \pi'(f_v) \;=\; \frac{1}{2}(a_+ - a_-) \prod_{w \neq v} n(\sigma_w, \pi'_w),$$

where $\pi' = \pi'_v \otimes (\pi')^v$ runs over all global cuspidal representations, which are weak endoscopic lifts (in the sense of Sect. 5.2) of our fixed auxiliary global automorphic representation σ with fixed $(\pi')^v$ outside v. Since $\pi'_v \in \{\pi_{\pm}(\sigma_\infty)\}$ by Lemma 5.5, we then obtain $tr\ \pi'(f_v) = a_{\pm} = \pm a_+$. Canceling $a_+ = -a_- \neq 0$ from the formula leaves us with a formula which is the precise analogue of formula (0) with the unique Archimedean place ∞ of $F = \mathbb{Q}$ now replaced by the fixed chosen Archimedean place v of the number field F. To make a comparison with the situation above we change the notation, and let v denote ∞ from now on. Then for any local non-Archimedean local field F_v of characteristic zero and any irreducible admissible representation σ_v of M_v in the discrete series we may have chosen F and σ such that they extend F_v and σ_v (maybe up to a local twist by a character). With this additional choice made, the analogue of formula (0) has now been established. □