Chapter 12
The Homotopy of Algebras over Operads

Introduction

In this chapter we apply the adjoint construction of model structures to the category of operads and to categories algebras over operads. For this purpose, we use the adjunction $\mathtt{F} : \mathcal{M} \rightleftarrows \mathcal{O} : U$ between operads and Σ_*-objects, respectively the adjunction $\mathtt{P}(-) : \mathcal{E} \rightleftarrows {}_{\mathtt{P}}\mathcal{E} : U$ between $\mathtt{P}$-algebras and their underlying category $\mathcal{E}$.

The construction of these adjoint model structures is studied in [26] in the dg-context and in [4, 58] in a more general setting. The difficulty is to check condition (2) in proposition 11.1.14. Indeed, in many usual cases, this condition is not satisfied unless we restrict ourself to cellular objects. For this reason, we have to use *semi-model categories*, structures introduced in [29] to enlarge the applications of theorem 11.1.13. The rough idea is to restrict the lifting and factorization axioms of model categories to morphisms with a cofibrant domain. By [58], the category of operads inherits such a semi-model structure, and so do the categories of algebras over a Σ_*-cofibrant operads.

The main purpose of this chapter is to review the definition of these semi-model categories. First of all, in §12.1, we recall the definition of a semi-model category, borrowed from [29], and we review the construction of adjoint model structures in this setting. In §12.2, we survey briefly the definition of semi-model structures on categories of operads. In §12.3, we address the definition of semi-model structures on categories of algebras over operads.

In this book, the semi-model category of operads only occurs in examples of applications of the main results. For this reason, we only sketch the proof of the axioms for this semi-model category. But we give comprehensive proofs of the axioms of semi-model categories for categories of algebras over operads, because we use this structure in the next part. The main verification is a particular case of statements about modules over operads used in the next part of the book and deferred to an appendix.

B. Fresse, *Modules over Operads and Functors*, Lecture Notes in Mathematics 1967, 185
DOI: 10.1007/978-3-540-89056-0_12,

In §12.4, we study the semi-model categories of algebras over a cofibrant operad, for which the lifting and factorization axioms of semi-model categories hold in wider situations. In §12.5, we survey results about the homotopy of extension and restriction functors $\phi_! : {}_{\mathtt{P}}\mathcal{E} \rightleftarrows {}_{\mathtt{Q}}\mathcal{E} : \phi^*$ associated to an operad morphism $\phi : \mathtt{P} \to \mathtt{Q}$.

The statements of §§12.4-12.5 are proved in the next part of the book as applications of our results on the homotopy of modules over operads.

12.1 Semi-Model Categories

The rough idea of semi-model categories is to assume all axioms of model categories, including the lifting axiom M4 and the factorization axiom M5, but only for morphisms $f : X \to Y$ whose domain X is a cofibrant object. This restriction allows us to relax condition (2) of proposition 11.1.14 in the definition of model categories by adjunction and to enlarge the applications of this construction.

12.1.1 The Axioms of Semi-Model Categories. Explicitly, the structure of a *semi-model category* consists of a category $\mathcal{A}$ equipped with classes of weak-equivalences, cofibrations and fibrations so that axioms M1, M2, M3 of model categories hold, but where the lifting axiom M4 and the factorization axiom M5 are replaced by the weaker requirements:

M4'. i. The fibrations have the right lifting property with respect to the acyclic cofibrations $i : A \to B$ whose domain A is cofibrant.
ii. The acyclic fibrations have the right lifting property with respect to the cofibrations $i : A \to B$ whose domain A is cofibrant.

M5'. i. Any morphism $f : A \to B$ such that A is cofibrant has a factorization $f = pi$, where i is a cofibration and p is an acyclic fibration.
ii. Any morphism $f : A \to B$ such that A is cofibrant has a factorization $f = qj$, where j is an acyclic cofibration and q is a fibration.

Besides, a semi-model category is assumed to satisfy:

M0' (*initial object axiom*): The initial object of $\mathcal{A}$ is cofibrant.

In the context of semi-model categories, the lifting axiom M4' and the factorization axiom M5' are not sufficient to imply that the initial object is cofibrant. Therefore we add this assertion as an axiom.

In a semi-model category, the class of (acyclic) cofibrations is not fully characterized by the left lifting axioms M4', and similarly as regards the class of (acyclic) fibrations. As a byproduct, the class of (acyclic) cofibrations is not stable under the composition of morphisms, and similarly as regards the class of acyclic fibrations. The axioms imply only that a (possibly transfinite) composite of (acyclic) cofibrations with a cofibrant domain forms still an (acyclic) cofibration.

Similarly, not all (acyclic) cofibrations are stable under pushouts, not all (acyclic) fibrations are stable under pullbacks. The axioms imply only that (acyclic) cofibrations are stable under pushouts over cofibrant domains.

On the other hand, since usual semi-model categories are defined by adjunction from a cofibrantly generated model category (see next), the class of (acyclic) fibrations is stable under composites and pullbacks in applications. Besides, these properties are used to generalize the construction of the homotopy category of model categories. For these reasons, the next assertions are taken as additional axioms of semi-model categories:

M6' (*fibration axioms*):

i. The class of (acyclic) fibrations is stable under (possibly transfinite) composites.
ii. The class of (acyclic) fibrations is stable under pullbacks.

But we do not use these properties in this book.

The result of proposition 11.1.4 can be generalized in the context of semi-model categories:

12.1.2 Proposition. *The following assertion holds in every semi-model category $\mathcal{A}$:*

P1'. The pushout of a weak-equivalence along a cofibration

$$\begin{array}{ccc} A & \rightarrowtail & C \\ {\scriptstyle\sim}\downarrow & & \vdots \\ B & \cdots\rightarrow & D \end{array}$$

gives a weak-equivalence $C \xrightarrow{\sim} D$ provided that A and B are cofibrant in $\mathcal{A}$.

Proof. Careful inspection of the proof of proposition 11.1.4 in [27, Proposition 13.1.2]. □

The properness axiom P1 does not make sense in the context of semi-model categories because only cofibrations with a cofibrant domain are characterized by the axioms.

12.1.3 Cofibrantly Generated Semi-Model Categories. The notion of a cofibrantly generated model category has a natural generalization in the context of semi-model categories. Again, a cofibrantly generated semi-model category consists of a semi-model category $\mathcal{A}$ equipped with a set of generating cofibrations $\mathcal{I}$, respectively a set of generating acyclic cofibrations $\mathcal{J}$, so that:

G1. The fibrations are characterized by the right lifting property with respect to acyclic generating cofibrations $j \in \mathcal{J}$.

G2. The acyclic fibrations are characterized by the right lifting property with respect to generating cofibrations $i \in \mathcal{I}$.

The small object argument is also supposed to hold for the set of generating cofibrations $\mathcal{I}$ (respectively, generating acyclic cofibrations $\mathcal{J}$) but we can relax the smallness assumption. Namely, we may assume:

S1'. The domain A of every generating cofibration (respectively, generating acyclic cofibration) is small with respect to relative $\mathcal{I}$-cell (respectively, $\mathcal{J}$-cell) complexes

$$K = L_0 \to \cdots \to L_{\lambda-1} \xrightarrow{j_\lambda} L_\lambda \to \cdots \to \operatorname*{colim}_{\lambda<\mu} L_\lambda = L$$

such that K is a cofibrant object.

In a cofibrantly generated semi-model category, the axioms imply only that:

K1'. The relative $\mathcal{I}$-cell (respectively, $\mathcal{J}$-cell) complexes with a cofibrant domain are cofibrations (respectively, acyclic cofibrations).
K2'. The cofibrations (respectively, acyclic cofibrations) with a cofibrant domain are retracts of relative $\mathcal{I}$-cell (respectively, $\mathcal{J}$-cell) complexes.

Our motivation to use semi-model categories comes from the following proposition which weaken the conditions of proposition 11.1.14 to define semi-model structures by adjunction:

12.1.4 Theorem. *Suppose we have an adjunction $F : \mathcal{X} \rightleftarrows \mathcal{A} : U$, where $\mathcal{A}$ is any category with limits and colimits and $\mathcal{X}$ is a cofibrantly generated model category. Let $\mathcal{I}$, respectively $\mathcal{J}$, be the set of generating (acyclic) cofibrations of $\mathcal{X}$ and set $F\mathcal{I} = \{F(i),\ i \in \mathcal{I}\}$, respectively $F\mathcal{J} = \{F(j),\ j \in \mathcal{J}\}$. Consider also the set $F\mathcal{X}_c = \{F(i),\ i$ cofibration in $\mathcal{X}\}$.*

Under assumptions (1-3) below, the category $\mathcal{A}$ inherits a cofibrantly generated semi-model structure with $F\mathcal{I}$ (respectively, $F\mathcal{J}$) as generating (acyclic) cofibrations and so that the functor $U : A \to \mathcal{X}$ creates weak-equivalences.

(1) The functor $U : \mathcal{A} \to \mathcal{X}$ preserves colimits over non-empty ordinals.
(2) For any pushout

$$\begin{array}{ccc} F(K) & \longrightarrow & A \\ {\scriptstyle F(i)}\downarrow & & \vdots\, {\scriptstyle f} \\ F(L) & \cdots\cdots\to & B \end{array}$$

such that A is an $\mathcal{X}$-cofibrant $F\mathcal{X}_c$-cell complex, the morphism $U(f)$ forms a cofibration (respectively an acyclic cofibration) in $\mathcal{X}$ whenever i is a cofibration (respectively an acyclic cofibration) with a cofibrant domain.
(3) The object $UF(0)$ is cofibrant.

The functor $U : \mathcal{A} \to \mathcal{X}$ creates the class of fibrations too and preserves cofibrations with a cofibrant domain.

In accordance with the conventions of §11.1.17, we say that an object $A \in \mathcal{A}$ is $\mathcal{X}$-cofibrant if the functor $U : \mathcal{A} \to \mathcal{X}$ maps the initial morphism $F(0) \to A$ to a cofibration.

Proof. This theorem follows from a careful inspection of the arguments of theorem 11.1.13 and proposition 11.1.14. Use the next lemma to apply condition (2) in the case where the domain of generating (acyclic) cofibrations is not cofibrant. □

12.1.5 Lemma. *Suppose we have a pushout*

$$\begin{array}{ccc} F(K) & \longrightarrow & A \\ {\scriptstyle F(i)}\downarrow & & \vdots\,{\scriptstyle f} \\ F(L) & \cdots\to & B \end{array}$$

such that $U(A)$ is cofibrant and $i : K \to L$ is a cofibration (respectively, an acyclic cofibration), but where K is not necessarily cofibrant. Then we can form a new pushout

$$\begin{array}{ccc} F(M) & \longrightarrow & A \\ {\scriptstyle F(j)}\downarrow & & \vdots\,{\scriptstyle f} \\ F(N) & \cdots\to & B \end{array}$$

such that $j : M \to N$ is still a cofibration (respectively, an acyclic cofibration), but where M is now cofibrant.

Proof. Set $M = U(A)$ and consider the pushout

$$\begin{array}{ccc} K & \longrightarrow & U(A) =: M \\ {\scriptstyle i}\downarrow & & \vdots\,{\scriptstyle j} \\ L & \cdots\to & L \oplus_K U(A) =: N \end{array}$$

where $K \to U(A)$ is the adjoint morphism of $F(K) \to A$. By straightforward categorical constructions, we can form a new pushout

$$\begin{array}{ccc} F(U(A)) & \longrightarrow & A \\ {\scriptstyle F(j)}\downarrow & & \downarrow{\scriptstyle f} \\ F(L \oplus_K U(A)) & \longrightarrow & B \end{array}$$

in which j is substituted to i.

The object $M = U(A)$ is cofibrant by assumption. The morphism j forms a cofibration (respectively, an acyclic cofibration) if i is so. Hence all our requirements are satisfied. □

For our needs, we record that Brown's lemma is also valid in the context of semi-model categories:

12.1.6 Proposition (Brown's lemma)**.** *Let $F : \mathcal{A} \to \mathcal{X}$ be a functor, where $\mathcal{A}$ is a semi-model category and $\mathcal{X}$ is a category equipped with a class of weak-equivalences that satisfies the two-out-of-three axiom. If F maps acyclic cofibrations between cofibrant objects to weak-equivalences, then F maps all weak-equivalences between cofibrant objects to weak-equivalences.*

Proof. The proposition follows from a straightforward generalization of the proof of the standard Brown's lemma. □

¶ The dual version of this statement, in which cofibrations are replaced by fibrations, holds only under a weaker form:

12.1.7 ¶ Proposition (Brown's lemma)**.** *Let $U : \mathcal{A} \to \mathcal{X}$ be a functor, where $\mathcal{A}$ is a semi-model category and $\mathcal{X}$ is a category equipped with a class of weak-equivalences that satisfies the two-out-of-three axiom. If F maps acyclic fibrations between fibrant objects to weak-equivalences, then F maps to weak-equivalences the weak-equivalences $f : X \to Y$ so that X is both cofibrant and fibrant and Y is fibrant.* □

The proof of this statement uses axiom M6'.

12.1.8 Quillen Adjunctions Between Semi-Model Categories. Some care is necessary to generalize the notion of a Quillen adjunction in the context of semi-model categories: as the lifting axiom M4' is not sufficient to characterize the class of (acyclic) cofibrations and the class of (acyclic) fibrations, the usual equivalent conditions of the definition of a Quillen adjunction are no more equivalent.

Therefore, we say that adjoint functors $F : \mathcal{A} \rightleftarrows \mathcal{X} : U$ between semi-model categories $\mathcal{A}$ and $\mathcal{X}$ define a Quillen adjunction if every one of the following conditions hold:

- A1'. The functor F preserves cofibrations and acyclic cofibrations between cofibrant objects.
- A2'. Same as A2: The functor U preserves fibrations and acyclic fibrations.

Observe however that A2' implies (but is not equivalent to) A1'. Thus, in applications, we only check condition A2'. These properties imply that the pair (F, U) yields an adjunction between homotopy categories, as in the context of model categories.

Say that the functors (F, U) define a Quillen equivalence if we have further:

E1'. For every cofibrant object $X \in \mathcal{X}$, the composite

$$X \xrightarrow{\eta X} UF(X) \xrightarrow{U(i)} U(B)$$

forms a weak-equivalence in $\mathcal{X}$, where ηX refers to the adjunction unit and i arises from a factorization $F(X) \xrightarrow{\sim} B \twoheadrightarrow *$ of the terminal morphism $F(X) \to *$.

E2'. Same as E2: For every fibrant object $A \in \mathcal{A}$, the composite

$$F(Y) \to FU(A) \xrightarrow{\epsilon A} A$$

forms a weak-equivalence in $\mathcal{A}$, where ϵA refers to the adjunction augmentation and Y is any cofibrant replacement of $U(A)$.

The derived functors of a Quillen equivalence of semi-model categories define adjoint equivalences of homotopy categories, as in the context of model categories.

12.1.9 Relative Semi-Model Structures. In certain semi-model categories, the lifting and factorization axioms hold under weaker assumptions on the domain of morphisms. These properties are formalized in a relative notion of a semi-model structure. Next we explain that operads and algebras over cofibrant operads inherits such improved lifting and factorization axioms. But we do not really use these improved semi-model structures that we recall for the sake of completeness only.

Suppose we have an adjunction $F : \mathcal{X} \rightleftarrows \mathcal{A} : U$, where $\mathcal{A}$ is a category with limits and colimits and $\mathcal{X}$ is a model category. Suppose $\mathcal{A}$ is equipped with a semi-model structure so that:

– The functor $U : \mathcal{A} \to \mathcal{X}$ creates weak-equivalences, creates fibrations, and maps the cofibrations $i : A \to B$ so that A is $\mathcal{X}$-cofibrant to cofibrations.

Again, we say that an object $A \in \mathcal{A}$ is $\mathcal{X}$-cofibrant if the functor $U : \mathcal{A} \to \mathcal{X}$ maps the initial morphism $F(0) \to A$ to a cofibration.

In this situation, it makes sense to require the following lifting and factorization axioms:

M4". i. The fibrations have the right lifting property with respect to the acyclic cofibrations $i : A \to B$ such that A is $\mathcal{X}$-cofibrant.
ii. The acyclic fibrations have the right lifting property with respect to the cofibrations $i : A \to B$ such that A is $\mathcal{X}$-cofibrant.

M5". i. Any morphism $f : A \to B$ has a factorization $f = pi$, where i is a cofibration and p is an acyclic fibration, provided that A is $\mathcal{X}$-cofibrant.
ii. Any morphism $f : A \to B$ has a factorization $f = qj$, where j is an acyclic cofibration and q is a fibration, provided that A is $\mathcal{X}$-cofibrant.

If these properties are satisfied, then we say that $\mathcal{A}$ forms a *semi-model category over* $\mathcal{X}$.

The initial object of $\mathcal{A}$ is supposed to be cofibrant by axiom M0' of semi-model categories. As a byproduct, the assumption on the functor $U : \mathcal{A} \to \mathcal{X}$ implies that $U(A)$ is cofibrant if A is cofibrant in $\mathcal{A}$. Accordingly, the lifting and factorization axioms M4"-M5" are stronger than the lifting and factorization axioms M4'-M5' of semi-model categories.

Suppose now we have an adjunction $F : \mathcal{X} \rightleftarrows \mathcal{A} : U$, where $\mathcal{A}$ is any category with limits and colimits and $\mathcal{X}$ is a cofibrantly generated model category. Suppose we have:

(1) Same as assumption (1) of theorem 12.1.4: "*The functor* $U : \mathcal{A} \to \mathcal{X}$ *preserves colimits over non-empty ordinals.*"
(2) Drop the condition that A is an $F\mathcal{X}_c$-cell complex in assumption (2) of theorem 12.1.4: "*For any pushout*

$$\begin{array}{ccc} F(K) & \longrightarrow & A \\ {\scriptstyle F(i)}\downarrow & & \downarrow{\scriptstyle f} \\ F(L) & \longrightarrow & B \end{array}$$

such that A is $\mathcal{X}$-cofibrant, the morphism $U(f)$ forms a cofibration (respectively an acyclic cofibration) in $\mathcal{X}$ whenever i is a cofibration (respectively an acyclic cofibration) with a cofibrant domain."
(3) Same as assumption (3) of theorem 12.1.4: "*The object* $UF(0)$ *is cofibrant.*"

Then $\mathcal{A}$ inherits an adjoint semi-model structure from $\mathcal{X}$ since the requirements of theorem 12.1.4 are fulfilled.

But we have better:

12.1.10 Proposition. *Under these assumptions (1-3) the category $\mathcal{A}$ forms a semi-model category over $\mathcal{X}$.*

Proof. This proposition, like theorem 12.1.4, follows from a careful inspection of the arguments of theorem 11.1.13 and proposition 11.1.14. □

12.1.11 Relative Properness Axioms. In the case of a semi-model category $\mathcal{A}$ over a model category $\mathcal{X}$, it makes sense to improve the properness property of proposition 12.1.2 to cofibrations $i : A \to B$ such that A is $\mathcal{X}$-cofibrant. If the next axiom holds, then we say that $\mathcal{A}$ forms a *(left) proper semi-model category over* $\mathcal{X}$:

P1". The pushout of a weak-equivalence along a cofibration

$$\begin{array}{ccc} A & \rightarrowtail & C \\ \sim\downarrow & & \vdots \\ B & \cdots\rightarrowtail & D \end{array}$$

gives a weak-equivalence $C \xrightarrow{\sim} D$ provided that A and B are $\mathcal{X}$-cofibrant.

12.2 The Semi-Model Category of Operads

In this section, we survey briefly the application of model structures to categories of operads. For this purpose, we assume that the base category $\mathcal{C}$ is equipped with a model structure and forms a cofibrantly generated symmetric monoidal category. Recall that $\mathcal{C}$ is supposed to satisfy the pushout product axiom MM1, as well as the unit axiom MM0.

In this book, the semi-model structure of the category of operads is only used in §17.4, where we study applications of the homotopy theory of modules over operads to categories of algebras over cofibrant dg-operads. Usually, we only deal with the underlying model category of Σ_*-objects and we only use Σ_*-cofibrations of operads and Σ_*-cofibrant operads. Recall that, according to our convention, a Σ_*-cofibration refers to a morphism of operads $\phi : \mathtt{P} \to \mathtt{Q}$ which forms a cofibration in the underlying category of Σ_*-objects and an operad $\mathtt{P}$ is Σ_*-cofibrant if the unit morphism $\eta : I \to \mathtt{P}$ forms a Σ_*-cofibration.

For our needs, we only recall the statement of the result, for which we refer to [4, 26, 58], and we make explicit the structure of cofibrant operads in dg-modules.

Proposition 12.1.10 can be applied to the adjunction

$$\mathtt{F} : \mathcal{M} \rightleftarrows \mathcal{O} : U$$

between operads and Σ_*-objects and returns the following statement:

Theorem 12.2.A (see [26, 58]). *The category of operads $\mathcal{O}$ forms a semi-model category over the category of Σ_*-objects, so that:*

- *The forgetful functor $U : \mathcal{O} \to \mathcal{M}$ creates weak-equivalences, creates fibrations, and maps the cofibrations of operads $i : \mathtt{P} \to \mathtt{Q}$ such that $\mathtt{P}$ is Σ_*-cofibrant to cofibrations.*
- *The morphisms of free operads $\mathtt{F}(i) : \mathtt{F}(M) \to \mathtt{F}(N)$, where $i : M \to N$ ranges over generating (acyclic) cofibrations of Σ_*-objects, form generating (acyclic) cofibrations of the category of operads.*
- *The lifting axiom M4" holds for the (acyclic) cofibrations of operads $i : \mathtt{P} \to \mathtt{Q}$ such that $\mathtt{P}$ is Σ_*-cofibrant.*

– *The factorization axiom M5" holds for the operad morphisms* $\phi : \mathtt{P} \to \mathtt{Q}$ *such that* $\mathtt{P}$ *is* Σ_**-cofibrant.*

□

Thus an operad morphism $\phi : \mathtt{P} \to \mathtt{Q}$ forms a weak-equivalence, respectively a fibration, if its components $\phi : \mathtt{P}(n) \to \mathtt{Q}(n)$ are weak-equivalences, respectively fibrations, in the base model category $\mathcal{C}$.

One has to study the structure of free operads and coproducts to check that assumptions (1-3) of proposition 12.1.10. are fulfilled. This task is achieved in [26] in the context of dg-modules and in [58] in a wider context to return the result of theorem 12.2.A.

By [58], we also have:

Theorem 12.2.B (see [58])**.** *The semi-model category of operads satisfies the axiom of relative properness:*

P1". The pushout of a weak-equivalence along a cofibration

$$\begin{array}{ccc} \mathtt{P} & \rightarrowtail & \mathtt{R} \\ {\scriptstyle\sim}\downarrow & & \vdots \\ \mathtt{Q} & \cdots\to & \mathtt{S} \end{array}$$

gives a weak-equivalence $\mathtt{R} \xrightarrow{\sim} \mathtt{S}$ *provided that* $\mathtt{P}$ *and* $\mathtt{Q}$ *are* Σ_**-cofibrant.*

The authors of [4] observe that the proof of theorem 12.2.A can be simplified in certain usual situations to give a better result:

Theorem 12.2.C (see [4])**.** *Under assumptions (1-3) below, the adjunction*

$$\mathtt{F} : \mathcal{M} \rightleftarrows \mathcal{O} : U$$

creates a full model structure on the category of non-unitary operads $\mathcal{O}_0$*.*

(1) There is a fixed ordinal μ*, such that the domains of generating (acyclic) cofibrations are small with respect to* all *colimits*

$$C = D_0 \to \cdots \to D_{\lambda-1} \xrightarrow{j_\lambda} D_\lambda \to \cdots \to \operatorname*{colim}_{\lambda<\mu} D_\lambda = D.$$

(2) There is a functor of symmetric monoidal model categories $R : \mathcal{C} \to \mathcal{C}$ *which associates a fibrant replacement to any object* $C \in \mathcal{C}$*.*

(3) The category $\mathcal{C}$ *is equipped with a commutative Hopf interval (we refer to* loc. cit. *for the definition of this notion).* □

Recall that an operad $\mathtt{P}$ is non-unitary if we have $\mathtt{P}(0) = 0$.

The assumptions hold for the category of dg-modules, for the category of simplicial sets, but assumption (1) fails for the category of topological spaces (see [28, §2.4]).

The Hopf interval is used to associate a canonical path object to any operad. An argument of Quillen permits to turn round the study of relative cell complexes of operads to prove directly condition (2) of theorem 12.1.4 by using the existence of such path objects (see *loc. cit.* for details).

In the remainder of this section, we take $\mathcal{C} = \mathrm{dg}\,\Bbbk\,\mathrm{Mod}$ and we study the semi-model category of operads in dg-modules. Our purpose is to review the explicit structure of *cofibrant cell dg-operads* obtained in [26]. The result is used in applications of §17.4.

In summary, we check that cofibrant cell dg-operads are equivalent to certain quasi-free objects in operads, just like we prove in §11.2 that cofibrant cell dg-modules are equivalent to quasi-free objects equipped with an appropriate filtration.

12.2.1 Quasi-Free Operads in dg-Modules. To begin with, we recall the definition of a twisting cochain and of a quasi-free object in the context of operads. For more background, we refer to [14, 17, 26].

First, a *twisting cochain of operads* consists of a collection of twisting cochains of dg-modules $\partial \in Hom_{\mathcal{C}}(\mathtt{P}(n), \mathtt{P}(n))$ that commute with the action of symmetric groups and satisfy the derivation relation

$$\partial(p \circ_e q) = \partial(p) \circ_e q + \pm p \circ_e \partial(q)$$

with respect to operadic composites. These assumptions ensure that the collection of twisted dg-modules $(\mathtt{P}(n), \partial)$ forms still a dg-operad with respect to the operad structure of $\mathtt{P}$.

An operad $\mathtt{P}$ is *quasi-free (as an operad)* if we have $\mathtt{P} = (\mathtt{F}(M), \partial)$, for a certain twisting cochain of operads $\partial : \mathtt{F}(M) \to \mathtt{F}(M)$, where $\mathtt{F}(M)$ is a free operad. Note that $\partial : \mathtt{F}(M) \to \mathtt{P}(M)$ is determined by its restriction to the generating Σ_*-object $M \subset \mathtt{F}(M)$ since we have the derivation relation

$$\begin{aligned}\partial((\cdots((\xi_1 \circ_{e_2} \xi_2) \circ_{e_3} \cdots) \circ_{e_r} \xi_r) = (&\cdots((\partial\xi_1 \circ_{e_2} \xi_2) \circ_{e_3} \cdots) \circ_{e_r} \xi_r \\ &+ (\cdots((\xi_1 \circ_{e_2} \partial\xi_2) \circ_{e_3} \cdots) \circ_{e_r} \xi_r \\ &+ \cdots + (\cdots((\xi_1 \circ_{e_2} \xi_2) \circ_{e_3} \cdots) \circ_{e_r} \partial\xi_r\end{aligned}$$

for any formal composite $(\cdots((\xi_1 \circ_{e_2} \xi_2) \circ_{e_3} \cdots) \circ_{e_r} \xi_r \in \mathtt{F}(M)$, where $\xi_1, \dots, \xi_r \in M$.

By construction, the generating cofibrations of dg-operads are the morphism of free operads $\mathtt{F}(i \otimes F_r) : \mathtt{F}(C \otimes F_r) \to \mathtt{F}(D \otimes F_r)$, where $i : C \to D$ ranges over the generating cofibrations of dg-modules and F_r is a free Σ_*-object. Recall that the generating cofibrations of dg-modules are inclusions $i : B^{d-1} \to E^d$ where E^d is spanned by an element e_d of degree d, by an element b_{d-1} of degree $d-1$, together with the differential $\delta(e_d) = b_{d-1}$, and B^{d-1} is the submodule of E^d spanned by b_{d-1}.

We use the convention of §11.1.8 to call *cofibrant cell operads* the cell complexes in operads built from generating cofibrations. We prove that cofibrant cell operads are quasi-free operads equipped with a suitable filtration. To obtain this result, we examine the structure of cell attachments:

12.2.2 Lemma. *For a quasi-free operad* $\mathtt{P} = (\mathtt{F}(M), \partial)$*, a cell attachment of generating cofibrations*

$$\begin{array}{ccc} \mathtt{F}(\bigoplus_\alpha B^{d_\alpha - 1} \otimes F_{r_\alpha}) & \xrightarrow{f} & \mathtt{P} \\ {\scriptstyle \mathtt{F}(i_{d_\alpha} \otimes F_{r_\alpha})} \downarrow & & \downarrow {\scriptstyle j} \\ \mathtt{F}(\bigoplus_\alpha E^{d_\alpha} \otimes F_{r_\alpha}) & \longrightarrow & \mathtt{Q} \end{array}$$

returns a quasi-free operad such that $\mathtt{Q} = (\mathtt{F}(M \oplus E), \partial)$*, where* E *is a free* Σ_**-object in graded* $\Bbbk$*-modules* $E = \bigoplus_\alpha \Bbbk\, e_{d_\alpha} \otimes F_{r_\alpha}$ *(together with a trivial differential).*

The twisting cochain $\partial : \mathtt{F}(M \oplus E) \to \mathtt{F}(M \oplus E)$ *is given by the twisting cochain of* $\mathtt{P}$ *on the summand* $M \subset \mathtt{F}(M \oplus E)$ *and is determined on the summand* $E = \bigoplus_\alpha \Bbbk\, e_{d_\alpha} \otimes F_{r_\alpha} \subset \mathtt{F}(M \oplus E)$ *by the relation* $\partial(e_{d_\alpha}) = f(b_{d_\alpha - 1})$*, where* $f : \bigoplus_\alpha \Bbbk\, b_{d_\alpha - 1} \otimes F_{r_\alpha} \to \mathtt{P}$ *represents the attaching map.*

Proof. Straightforward verification. □

By induction, we obtain immediately:

12.2.3 Proposition. *A cofibrant cell operad is equivalent to a quasi-free operad* $\mathtt{P} = (\mathtt{F}(L), \partial)$ *where* L *is a free* Σ_**-object in graded* $\Bbbk$*-modules* $L = \bigoplus_\alpha \Bbbk\, e_{d_\alpha} \otimes F_{r_\alpha}$ *equipped with a basis filtration* $L_\lambda = \bigoplus_{\alpha < \lambda} \Bbbk\, e_{d_\alpha} \otimes F_{r_\alpha}$ *such that* $\partial(L_\lambda) \subset \mathtt{F}(L_{\lambda - 1})$*.* □

12.3 The Semi-Model Categories of Algebras over Operads

The purpose of this section is to define the semi-model structure of the category of algebras over an operad. For this aim, we apply theorem 12.1.4 to the adjunction

$$\mathtt{P}(-) : \mathcal{E} \rightleftarrows {}_{\mathtt{P}}\mathcal{E} : U$$

between the category of $\mathtt{P}$-algebras and the underlying category $\mathcal{E}$. As usual, we assume that $\mathcal{E}$ is any symmetric monoidal category over the base category $\mathcal{C}$ in which the operad is defined. For the needs of this section, we assume as well that $\mathcal{E}$ is equipped with a model structure and forms a cofibrantly generated symmetric monoidal category over $\mathcal{C}$. Recall that $\mathcal{E}$ is supposed to satisfy the pushout product axiom MM1, as well as the unit axiom MM0, like the base category $\mathcal{C}$.

The main result reads:

Theorem 12.3.A. *If* $\mathtt{P}$ *is a* Σ_**-cofibrant operad, then the category of* $\mathtt{P}$*-algebras inherits a cofibrantly generated semi-model structure so that the forgetful functor* $U : {}_{\mathtt{P}}\mathcal{E} \to \mathcal{E}$ *creates weak-equivalences and fibrations. The generating (acyclic) cofibrations are the morphisms of free* $\mathtt{P}$*-algebras* $\mathtt{P}(i) : \mathtt{P}(K) \to \mathtt{P}(L)$ *such that* $i : K \to L$ *is a generating (acyclic) cofibrations of the underlying category* $\mathcal{E}$*.*

¶ In the context where the category $\mathcal{E}$ has regular tensor powers, we obtain further:

¶ **Theorem.** *If* $\mathcal{E}$ *is a (reduced) symmetric monoidal category with regular tensor powers, then the definition of theorem 12.3.A returns a semi-model structure as long as the operad* $\mathtt{P}$ *is* $\mathcal{C}$*-cofibrant.*

This theorem gives as a corollary:

¶ **Proposition.** *Let* $\mathtt{P}$ *be a reduced operad. The category of* $\mathtt{P}$*-algebras in connected* Σ_**-objects* ${}_{\mathtt{P}}\mathcal{M}^0$ *forms a semi-model category as long as the operad* $\mathtt{P}$ *is* $\mathcal{C}$*-cofibrant.* □

¶ The positive stable model category of symmetric spectra Sp^{Σ} does not satisfy axiom MM0, but this difficulty can be turned round. Moreover, a better result holds: according to [23], the category of $\mathtt{P}$-algebras in the positive stable flat model category of symmetric spectra inherits a full model structure, for every operad $\mathtt{P}$ in Sp^{Σ} (not necessarily cofibrant in any sense). Note however that the forgetful functor $U : {}_{\mathtt{P}}\mathrm{Sp}^{\Sigma} \to \mathrm{Sp}^{\Sigma}$ does not preserves cofibrations in general.

In many usual situations, the operad $\mathtt{P}$ is the image of an operad in simplicial sets under the functor $\Sigma^{\infty}(-)_+ : \mathcal{S} \to \mathrm{Sp}^{\Sigma}$. In our sense, we use the category of simplicial sets $\mathcal{C} = \mathcal{S}$ as a base model category and the category of spectra $\mathcal{E} = \mathrm{Sp}^{\Sigma}$ as a symmetric monoidal category over $\mathcal{S}$. The forgetful functor $U : {}_{\mathtt{P}}\mathrm{Sp}^{\Sigma} \to \mathrm{Sp}^{\Sigma}$ seems to preserve cofibrations for an operad in simplicial sets though $\mathtt{P}$ does no form an Sp^{Σ}-cofibrant object in spectra (see for instance the case of the commutative operad in [55]). For a cofibrant $\mathtt{P}$-algebra in spectra A, this property implies that the initial morphism $\eta : \mathtt{P}(0) \to A$ is a cofibration, but A does not form a cofibrant object in the underlying category of spectra unless we assume $\mathtt{P}(0) = \mathrm{pt}$.

The proof of theorem 12.3.A (and theorem 12.3) is outlined in the next paragraph. The technical verifications are achieved in the appendix, §20.1.

Under the assumption of theorem 12.3.A, the pushout-product property of proposition 11.5.1 implies that the functor $\mathrm{S}(\mathtt{P}) : \mathcal{E} \to \mathcal{E}$ preserves cofibrations, respectively acyclic cofibrations, with a cofibrant domain. In §2.4, we observe that the functor $\mathrm{S}(\mathtt{P}) : \mathcal{E} \to \mathcal{E}$ preserves all filtered colimits as well. Thus condition (1) of theorem 12.1.4 is easily seen to be satisfied (and similarly in the context of theorem 12.3). The difficulty is to check condition (2):

12.3.1 Lemma. *Under the assumption of theorem 12.3.A, for any pushout*

$$\begin{array}{ccc} \mathtt{P}(X) & \longrightarrow & A \\ {\scriptstyle \mathtt{P}(i)}\downarrow & & \vdots\, {\scriptstyle f} \\ \mathtt{P}(Y) & \dashrightarrow & B \end{array}$$

such that A is an $\mathcal{E}$-cofibrant $\mathtt{P}(\mathcal{E}_c)$-cell complex, the morphism f forms a cofibration (respectively, an acyclic cofibration) in the underlying category $\mathcal{E}$ if $i : X \to Y$ is so.

The same result holds in the context of theorem 12.3. The technical verification of this lemma is postponed to §20.1.

As usual, we call $\mathcal{E}$-cofibrations the morphisms of $\mathtt{P}$-algebras $i : A \to B$ which form a cofibration in the underlying category $\mathcal{E}$, we call $\mathcal{E}$-cofibrant objects the $\mathtt{P}$-algebras A such that the initial morphism $\eta : \mathtt{P}(0) \to A$ is an $\mathcal{E}$-cofibration.

The verification of lemma 12.3.1 includes a proof that:

12.3.2 Proposition. *The cofibrations of $\mathtt{P}$-algebras $i : A \to B$ such that A is a cofibrant $\mathtt{P}$-algebra are $\mathcal{E}$-cofibrations. Any cofibrant $\mathtt{P}$-algebra A is $\mathcal{E}$-cofibrant.*

The definition of the semi-model structure in theorem 12.3.A is natural with respect to the underlying category $\mathcal{E}$ in the following sense:

12.3.3 Proposition. *Let $\mathtt{P}$ be any Σ_*-cofibrant operad. Let $\rho_! : \mathcal{D} \rightleftarrows \mathcal{E} : \rho^*$ be a Quillen adjunction of symmetric monoidal model categories over $\mathcal{C}$. The functors*

$$\rho_! : {}_{\mathtt{P}}\mathcal{D} \rightleftarrows {}_{\mathtt{P}}\mathcal{E} : \rho^*$$

induced by $\rho_!$ and ρ^* define a Quillen adjunction of semi-model categories.

Proof. Fibrations and acyclic fibrations are created by forgetful functors in the semi-model categories of $\mathtt{P}$-algebras. For this reason we obtain immediately that ρ^* preserves fibrations and acyclic fibrations. Since the functor $\rho_!$ maps (acyclic) cofibrations to (acyclic) cofibrations and preserves free objects by proposition 3.2.14, we obtain that $\rho_!$ maps generating (acyclic) cofibrations of ${}_{\mathtt{P}}\mathcal{D}$ to (acyclic) cofibrations in ${}_{\mathtt{P}}\mathcal{E}$. Since the functor $\rho_!$ preserves colimits and retracts, we obtain further that $\rho_!$ maps all (acyclic) cofibrations of ${}_{\mathtt{P}}\mathcal{D}$ to (acyclic) cofibrations in ${}_{\mathtt{P}}\mathcal{E}$. □

We have further:

12.3.4 Proposition. *If $\rho_! : \mathcal{D} \rightleftarrows \mathcal{E} : \rho^*$ is a Quillen equivalence, then $\rho_! : {}_{\mathtt{P}}\mathcal{D} \rightleftarrows {}_{\mathtt{P}}\mathcal{E} : \rho^*$ defines a Quillen equivalence as well.*

Proof. Suppose A is a cofibrant object in ${}_{\mathtt{P}}\mathcal{D}$. By proposition 12.3.2, the morphism $\eta : \mathtt{P}(0) \to A$ forms a cofibration in $\mathcal{D}$. Since $\mathtt{P}$ is supposed to be

Σ_*-cofibrant in $\mathcal{D}$, the $\mathtt{P}$-algebra A forms a cofibrant object in $\mathcal{D}$ as well. Since the forgetful functor $U : {}_{\mathtt{P}}\mathcal{E} \to \mathcal{E}$ creates fibrations, any fibrant replacement of $\rho_! A$ in ${}_{\mathtt{P}}\mathcal{E}$ defines a fibrant replacement of $\rho_! A$ in the underlying category. From these observations, we conclude that the composite

$$A \xrightarrow{\eta A} \rho^* \rho_! A \to \rho^* B,$$

where ηA refers to the adjunction unit and B is any fibrant replacement of $\rho_! A$ in ${}_{\mathtt{P}}\mathcal{E}$, forms a weak-equivalence in $\mathcal{D}$ and hence forms a weak-equivalence of $\mathtt{P}$-algebras in $\mathcal{D}$.

Suppose B is a fibrant object in ${}_{\mathtt{P}}\mathcal{E}$. Pick a cofibrant replacement $\mathtt{P}(0) \rightarrowtail A \xrightarrow{\sim} \rho^* B$ of $\rho^* B$ in ${}_{\mathtt{P}}\mathcal{D}$. Use again that the forgetful functor $U : {}_{\mathtt{P}}\mathcal{E} \to \mathcal{E}$ creates fibrations and that the forgetful functor $U : {}_{\mathtt{P}}\mathcal{D} \to \mathcal{D}$ preserves cofibrations to conclude that the composite

$$\rho_! A \to \rho_! \rho^* B \xrightarrow{\epsilon} B,$$

where ϵB refers to the adjunction augmentation, forms a weak-equivalence in $\mathcal{E}$ and hence forms a weak-equivalence of $\mathtt{P}$-algebras in $\mathcal{E}$. □

12.3.5 ¶ Remark. If $\rho_! : \mathcal{D}^0 \rightleftarrows \mathcal{E}^0 : \rho^*$ is a Quillen adjunction, respectively a Quillen equivalence, between (reduced) categories with regular tensor powers, then propositions 12.3.3-12.3.4 hold as long as $\mathtt{P}$ is a $\mathcal{C}$-cofibrant (non-unitary) operad.

12.3.6 Quasi-Free Algebras over Operads in dg-Modules. In the remainder of this section, we take $\mathcal{C} = \operatorname{dg}\Bbbk\operatorname{Mod}$ and we study the structure of cofibrant algebras over a Σ_*-cofibrant dg-operad $\mathtt{P}$. To simplify, we also take $\mathcal{E} = \operatorname{dg}\Bbbk\operatorname{Mod}$ but we can use the principle of generalized point-tensors to extend our results to $\mathtt{P}$-algebras in Σ_*-objects and to $\mathtt{P}$-algebras in right modules over operads.

As usual, we prove that *cofibrant cell* $\mathtt{P}$*-algebras* in dg-modules are equivalent to quasi-free $\mathtt{P}$-algebras equipped with an appropriate filtration. The plan of our constructions parallels the case of operads, addressed in §§12.2.1-12.2.3.

First, we review the definition of a twisting cochain and of a quasi-free object in the category of $\mathtt{P}$-algebras.

A twisting cochain of dg-modules $\partial \in Hom_{\mathcal{C}}(A, A)$ defines a *twisting cochain of* $\mathtt{P}$*-algebras* if $\partial : A \to A$ satisfies the derivation relation

$$\partial(p(a_1, \dots, a_n)) = \sum_{i=1}^{n} \pm p(a_1, \dots, \partial(a_i), \dots, a_n),$$

for every $p \in \mathtt{P}(n)$, $a_1, \dots, a_n \in A$. This assumption ensures that the twisted dg-module (A, ∂) inherits the structure of a $\mathtt{P}$-algebra in dg-modules.

A $\mathtt{P}$-algebra A is *quasi-free* if we have $A = (\mathtt{P}(C), \partial)$ for a certain twisting cochain of $\mathtt{P}$-algebras $\partial : \mathtt{P}(C) \to \mathtt{P}(C)$. Note that $\partial : \mathtt{P}(C) \to \mathtt{P}(C)$ is determined by its restriction to the generating dg-module $C \subset \mathtt{P}(C)$ of the free $\mathtt{P}$-algebra $\mathtt{P}(C)$ since we have the relation

$$\partial(p(x_1, \ldots, x_n)) = \sum_{i=1}^{n} \pm p(x_1, \ldots, \partial(x_i), \ldots, x_n),$$

for every element $p(x_1, \ldots, x_n) \in \mathtt{P}(C)$.

Recall that the generating cofibrations of $\mathtt{P}$-algebras in dg-modules are the morphism of free $\mathtt{P}$-algebras $\mathtt{P}(i) : \mathtt{P}(C) \to \mathtt{P}(D)$ induced by generating cofibrations of dg-modules. Recall that the generating cofibrations of dg-modules are inclusions $i : B^{d-1} \to E^d$ where E^d is spanned by an element e_d of degree d, by an element b_{d-1} of degree $d-1$, together with the differential $\delta(e_d) = b_{d-1}$, and B^{d-1} is the submodule of E^d spanned by b_{d-1}.

We use the convention of §11.1.8 to call *cofibrant cell* $\mathtt{P}$*-algebras* the cell complexes in $\mathtt{P}$-algebras obtained by successive attachments of generating cofibrations of the category of $\mathtt{P}$-algebras. We prove that cofibrant cell $\mathtt{P}$-algebras are quasi-free $\mathtt{P}$-algebras equipped with a suitable filtration. To obtain this result, we examine the structure of cell attachments on quasi-free $\mathtt{P}$-algebras:

12.3.7 Lemma. *For a quasi-free* $\mathtt{P}$*-algebra* $A = (\mathtt{P}(C), \partial)$*, a cell attachment of generating cofibrations*

$$\begin{array}{ccc} \mathtt{P}(\bigoplus_\alpha B^{d_\alpha - 1}) & \xrightarrow{f} & A \\ {\scriptstyle \mathtt{P}((i_{d_\alpha}))}\downarrow & & \vdots\, j \\ \mathtt{P}(\bigoplus_\alpha E^{d_\alpha}) & \cdots\cdots\to & B \end{array}$$

returns a quasi-free $\mathtt{P}$*-algebra such that* $B = (\mathtt{P}(C \oplus E), \partial)$*, where* E *is a free graded* $\Bbbk$*-module* $E = \bigoplus_\alpha \Bbbk\, e_{d_\alpha}$ *(equipped with a trivial differential).*

The twisting cochain $\partial : \mathtt{P}(C \oplus E) \to \mathtt{P}(C \oplus E)$ *is given by the twisting cochain of* A *on the summand* $C \subset \mathtt{P}(C \oplus E)$ *and is determined by the relation* $\partial(e_{d_\alpha}) = f(b_{d_\alpha - 1})$ *on the summand* $E = \bigoplus_\alpha \Bbbk\, e_{d_\alpha} \subset \mathtt{P}(C \oplus E)$*, where* $f : \bigoplus_\alpha \Bbbk\, b_{d_\alpha - 1} \to A$ *represents the attaching map.*

Proof. Straightforward verification. □

By induction, we obtain immediately:

12.3.8 Proposition. *A cofibrant cell* $\mathtt{P}$*-algebra is equivalent to a quasi-free* $\mathtt{P}$*-algebra* $A = (\mathtt{P}(C), \partial)$ *where* C *is a free graded* $\Bbbk$*-module* $C = \bigoplus_\alpha \Bbbk\, e_{d_\alpha}$ *equipped with a basis filtration* $C_\lambda = \bigoplus_{\alpha < \lambda} \Bbbk\, e_{d_\alpha}$ *such that* $\partial(C_\lambda) \subset \mathtt{P}(C_{\lambda - 1})$*.* □

12.4 Addendum: The Homotopy of Algebras over Cofibrant Operads

The result of theorem 12.3.A can be improved if we assume that $\mathtt{P}$ is a cofibrant operad:

Theorem 12.4.A (see [58]). *If $\mathtt{P}$ is a cofibrant operad, then the category of $\mathtt{P}$-algebras in $\mathcal{E}$ forms a semi-model category over $\mathcal{E}$, so that:*

- *The forgetful functor $U : {}_{\mathtt{P}}\mathcal{E} \to \mathcal{E}$ creates weak-equivalences, creates fibrations, and maps the cofibrations of $\mathtt{P}$-algebras $i : A \to B$ such that A is $\mathcal{E}$-cofibrant to cofibrations.*
- *The lifting axiom M4" holds for the (acyclic) cofibrations of $\mathtt{P}$-algebras $i : A \to B$ such that A is $\mathcal{E}$-cofibrant.*
- *The factorization axiom M5" holds for the morphisms of $\mathtt{P}$-algebras $f : A \to B$ such that A is $\mathcal{E}$-cofibrant.*

Besides:

Theorem 12.4.B (see [58]). *The semi-model category of algebras over a cofibrant operad $\mathtt{P}$ satisfies the axiom of relative properness:*

P1". The pushout of a weak-equivalence along a cofibration

$$\begin{array}{ccc} A & \rightarrowtail & C \\ {\scriptstyle\sim}\downarrow & & \vdots \\ B & \dashrightarrow & D \end{array}$$

gives a weak-equivalence $C \xrightarrow{\sim} D$ provided that A and B are $\mathcal{E}$-cofibrant.

We refer to [58] for the proof of these theorems. We do not use these results, which are only mentioned for the sake of completeness.

12.5 The Homotopy of Extension and Restriction Functors – Objectives for the Next Part

By §3.3.5, any morphism of operads $\phi : \mathtt{P} \to \mathtt{Q}$ gives rise to adjoint extension and restriction functors:

$$\phi_! : {}_{\mathtt{P}}\mathcal{E} \rightleftarrows {}_{\mathtt{Q}}\mathcal{E} : \phi^*.$$

In §16, we use the homotopy of modules over operads to prove:

Theorem 12.5.A. *Suppose $\mathtt{P}$ (respectively, $\mathtt{Q}$) is a Σ_*-cofibrant operad so that the category of $\mathtt{P}$-algebras (respectively, $\mathtt{Q}$-algebras) comes equipped with a semi-model structure.*

The extension and restriction functors

$$\phi_! : {}_{\mathtt{P}}\mathcal{E} \rightleftarrows {}_{\mathtt{Q}}\mathcal{E} : \phi^*.$$

define a Quillen adjunction, a Quillen equivalence if ϕ is a weak-equivalence.

¶ In the context of a (reduced) symmetric monoidal category with regular tensor powers, this statement holds whenever the operads $\mathtt{P}$ and $\mathtt{Q}$ are $\mathcal{C}$-cofibrant.

Property A2' (the right adjoint preserves fibrations and acyclic fibrations) of a Quillen adjunction is immediate to check, because fibrations and acyclic fibrations are created by forgetful functors and the restriction functor $\phi^* : {}_{\mathtt{Q}}\mathcal{E} \to {}_{\mathtt{P}}\mathcal{E}$ reduces to the identity if we forget algebra structures.

In §16, we use the representation of the functors $\phi_! : {}_{\mathtt{P}}\mathcal{E} \rightleftarrows {}_{\mathtt{Q}}\mathcal{E} : \phi^*$ by modules over operads and the results of §15 to prove properties E1'-E2' of a Quillen equivalence when $\phi : \mathtt{P} \to \mathtt{Q}$ is a weak-equivalence of operads.

We refer to this chapter for full details on the proof of theorem 12.5.A.