Chapter 2
The Boundedness of Calderón-Zygmund Operators on Wavelet Spaces

We first define test functions and wavelet spaces on spaces of homogeneous type. Then we prove the main result of this chapter, namely that Calderón-Zygmund operators whose kernels satisfy an additional smoothness condition are bounded on *wavelet spaces*. This result will be a crucial tool to provide wavelet expansions of functions and distributions on spaces of homogeneous type in the next chapter.

We first introduce test functions on spaces of homogeneous type.

Definition 2.1. Fix $0 < \gamma, \beta < \theta$. A function f defined on X is said to be a test function of type (x_0, r, β, γ), $x_0 \in X$, and $r > 0$, if f satisfies the following conditions:

(i) $|f(x)| \le C \frac{r^\gamma}{(r+\rho(x,x_0))^{1+\gamma}}$;

(ii) $|f(x) - f(y)| \le C\left(\frac{\rho(x,y)}{r+\rho(x,x_0)}\right)^\beta \frac{r^\gamma}{(r+\rho(x,x_0))^{1+\gamma}}$ for all $x, y \in X$ with $\rho(x,y) \le \frac{1}{2A}(r+\rho(x,x_0))$.

Such functions exist and the reader will find a recipe two lines after Definition 1.2. If f is a test function of type (x_0, r, β, γ), we write $f \in \mathcal{M}(x_0, r, \beta, \gamma)$, and the norm of f in $\mathcal{M}(x_0, r, \beta, \gamma)$ is defined by

$$\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)} = \inf\{C : (i) \text{ and } (ii) \text{ hold}\}.$$

One should observe that if $f \in \mathcal{M}(x_0, r, \beta, \gamma)$, then

$$\|f\|_1 \approx \|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)}.$$

We say that a function f is a *scaling function* if $f \in \mathcal{M}(x_0, r, \beta, \gamma)$ and $\int f(x)d\mu(x) = 1$.

Now fix $x_0 \in X$ and denote $\mathcal{M}(\beta,\gamma) = \mathcal{M}(x_0, 1, \beta, \gamma)$. It is easy to see that $\mathcal{M}(x_1, r, \beta, \gamma) = \mathcal{M}(\beta, \gamma)$ with equivalent norms for all $x_1 \in X$ and $r > 0$. Furthermore, it is also easy to check that $\mathcal{M}(\beta,\gamma)$ is a Banach space with respect to the norm in $\mathcal{M}(\beta,\gamma)$.

D. Deng and Y. Han, *Harmonic Analysis on Spaces of Homogeneous Type*, Lecture Notes in Mathematics 1966,

Definition 2.2. A function f defined on X is said to be a wavelet of type (x_0, r, β, γ) if $f \in \mathcal{M}(x_0, r, \beta, \gamma)$ and $\int f(x)d\mu(x) = 0$. We denote this by $f \in \mathcal{M}_0(x_0, r, \beta, \gamma)$.

These wavelets are named *molecules* by Guido Weiss. A compactly supported molecule is an *atom*. Atomic decompositions preluded wavelet analysis, as indicated in the Introduction. Moreover Caderón-Zygmund operators T satisfying $T(1) = T^*(1) = 0$ have the remarkable property map a molecule into a molecule. We use the notation $\mathcal{M}_0(\beta, \gamma)$, when the dependence in x_0 and r can be forgotten, as a space of wavelets with regularity (β, γ).

To study the boundedness of Calderón-Zygmund singular integral operators on a *wavelet space*, we define the following "strong" weak boundedness property.

Definition 2.3. An operator T defined by a distributional kernel K, is said to have the "strong weak boundedness property" if there exist $\eta > 0$ and $C < \infty$ such that

$$|\langle K, f\rangle| \leq Cr \tag{2.1}$$

for all $f \in C_0^\eta(X \times X)$ with $\text{supp}(f) \subseteq B(x_1, r) \times B(y_1, r), x_1$ and $y_1 \in X, \|f\|_\infty \leq 1, \|f(\cdot, y)\|_\eta \leq r^{-\eta}$, and $\|f(x, \cdot)\|_\eta \leq r^{-\eta}$ for all x and $y \in X$.

If T has the "strong weak boundedness property", we write $T \in SWBP$.

Note that if ψ and ϕ are functions satisfying the conditions in Definition 1.15, then $f(x, y) = \psi(x) \times \phi(y)$ satisfies the conditions in Definition 2.3, and hence $|\langle T\psi, \phi\rangle| = |\langle K, f\rangle| \leq Cr$ if T has the "strong weak boundedness property". This means that the strong weak boundedness property implies the weak boundedness property. However, in the standard situation of $\mathbb{R}^n$, the weak boundedness property implies the strong one. Indeed any smooth function $f(x, y), x \in B, y \in B$, supported by $B \times B$ can be written, by a double Fourier series expansion, as $\sum \alpha_j f_j(x) g_j(y)$ with $\sum |\alpha_j| < \infty, \|f_j\|_{C_0^\beta} \leq 1, \|g_j\|_{C_0^\beta} \leq 1$.

If $T \in CZK(\epsilon)$, we say that $T^*(1) = 0$ if $\int T(f)(x)dx = 0$ for all $f \in \mathcal{M}_0(\beta, \gamma)$. Similarly, $T(1) = 0$ if $\int T^*(f)(x)dx = 0$ for all $f \in \mathcal{M}_0(\beta, \gamma)$.

The main result in this chapter is the following theorem.

Theorem 2.4. *Suppose that $T \in CZK(\epsilon) \cap SWBP$, and $T(1) = T^*(1) = 0$. Suppose further that $K(x, y)$, the kernel of T, satisfies the following condition:*

$$\begin{aligned} &|K(x, y) - K(x', y) - K(x, y') + K(x', y')| \\ &\leq C\rho(x, x')^\epsilon \rho(y, y')^\epsilon \rho(x, y)^{-(1+2\epsilon)} \end{aligned} \tag{2.2}$$

for $\rho(x, x'), \rho(y, y') \leq \frac{1}{2A}\rho(x, y)$. Then there exists a constant C such that for each wavelet $f \in \mathcal{M}_0(x_0, r, \beta, \gamma)$ with $x_0 \in X, r > 0$ and $0 < \beta, \gamma < \epsilon$, $Tf \in \mathcal{M}_0(x_0, r, \beta, \gamma)$. Moreover

$$\|T(f)\|_{\mathcal{M}(x_0,r,\beta,\gamma)} \leq C\|T\|\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)} \tag{2.3}$$

where $\|T\|$ denote the smallest constant in the "strong weak boundedness property" and in the estimates of the kernel of T.

Before proving Theorem 2.4, we observe that this theorem will provide wavelet expansions which, as in the standard case of $\mathbb{R}^n$, will be the building blocks of most functional spaces.

To prove Theorem 2.4, we first need the following lemma.

Lemma 2.5. *Suppose that T is a continuous linear operator from $\dot{C}_0^\eta$ to $(\dot{C}_0^\eta)'$ satisfying $T \in CZK(\epsilon) \cap SWBP$ with $\eta < \epsilon$, and $T(1) = 0$. Then there exists a constant C such that*

$$\|T\phi\|_\infty \leq C \tag{2.4}$$

whenever there exist $x_0 \in X$ and $r > 0$ such that $\text{supp}\phi \subseteq B(x_0, r)$ *with $\|\phi\|_\infty \leq 1$ and $\|\phi\|_\eta \leq r^{-\eta}$.*

Proof. We follow the idea of the proof in [M1]. Fix a function $\theta \in C^\infty(\mathbb{R})$ with the following properties: $\theta(x) = 1$ for $|x| \leq 1$ and $\theta(x) = 0$ for $|x| > 2$. Let $\chi_0(x) = \theta(\frac{\rho(x,x_0)}{2r})$ and $\chi_1 = 1 - \chi_0$. Then $\phi = \phi\chi_0$ and for all $\psi \in C_0^\eta(X)$,

$$\begin{aligned}\langle T\phi, \psi\rangle &= \langle K(x,y), \phi(y)\psi(x)\rangle = \langle K(x,y), \chi_0(y)\phi(y)\psi(x)\rangle \\ &= \langle K(x,y), \chi_0(y)[\phi(y) - \phi(x)]\psi(x)\rangle + \langle K(x,y), \chi_0(y)\phi(x)\psi(x)\rangle \\ &:= p + q\end{aligned}$$

where $K(x,y)$ is the distribution kernel of T.

To estimate p, let $\lambda_\delta(x,y) = \theta(\frac{\rho(x,y)}{\delta})$. Then

$$\begin{aligned}p &= \langle K(x,y), (1 - \lambda_\delta(x,y))\chi_0(y)[\phi(y) - \phi(x)]\psi(x)\rangle \\ &\quad + \langle K(x,y), \lambda_\delta(x,y)\chi_0(y)[\phi(y) - \phi(x)]\psi(x)\rangle \\ &:= p_{1,\delta} + p_{2,\delta}.\end{aligned} \tag{2.5}$$

Since K is locally integrable on $\Omega = \{(x,y) \in X \times X : x \neq y\}$, the first term on the right hand side of (2.5) satisfies

$$\begin{aligned}|p_{1,\delta}| &= \left| \int_\Omega K(x,y)(1 - \lambda_\delta(x,y))\chi_0(y)[\phi(y) - \phi(x)]\psi(x)d\mu(x)d\mu(y) \right| \\ &\leq C \int_X \int_X |K(x,y)\chi_0(y)[\phi(y) - \phi(x)]\psi(x)|d\mu(x)d\mu(y) \\ &\leq C \int_X |\psi(x)|d\mu(x) = C\|\psi\|_1.\end{aligned}$$

Thus it remains to show that $\lim\limits_{\delta \to 0} p_{2,\delta} = 0$, i.e.,

$$\lim_{\delta\to 0}\langle K(x,y),\lambda_\delta(x,y)\chi_0(y)[\phi(y)-\phi(x)]\psi(x)\rangle = 0, \tag{2.6}$$

and it is here that we use the "strong" weak boundedness property of T :

$$|\langle K,f\rangle| \le Cr \tag{2.7}$$

for all $f \in C_0^\eta(X\times X)$ satisfying $\operatorname{supp} f \subseteq B(x_0,r)\times B(y_0,r), \|f\|_\infty \le 1, \|f(\cdot,y)\|_\eta \le r^{-\eta}$ and $\|f(x,\cdot)\|_\eta \le r^{-\eta}$ for all $x,y\in X$.

To show (2.6), let $\{y_j\}_{j\in\mathbb{Z}} \in X$ be a maximal collection of points satisfying

$$\frac{1}{2}\delta < \inf_{j\neq k}\rho(y_j,y_k) \le \delta. \tag{2.8}$$

By the maximality of $\{y_j\}_{j\in\mathbb{Z}}$, we have that for each $x\in X$ there exists a point y_j such that $\rho(x,y_j)\le\delta$. Let $\eta_j(y) = \theta(\frac{\rho(y,y_j)}{\delta})$ and $\bar\eta_j(y) = \big[\sum_i \eta_i(y)\big]^{-1}\eta_j(y)$. To see that $\bar\eta_j$ is well defined, it suffices to show that for any $y\in X$, there are only finitely many η_j with $\eta_j(y)\neq 0$. This follows from the following fact: $\eta_j(y)\neq 0$ if and only if $\rho(y,y_j)\le 2\delta$ and hence this implies that $B(y_j,\delta)\subseteq B(y,4A\delta)$. Inequalities (2.8) show $B(y_j,\frac{\delta}{4A})\cap B(y_k,\frac{\delta}{4A}) = \phi$, and thus there are at most CA points $y_j\in X$ such that $B(y_j,\frac{\delta}{4A})\subseteq B(y,4A\delta)$. Now let $\Gamma = \{j : \bar\eta_j(y)\chi_0(y)\neq 0\}$. Note that $\#\Gamma \le Cr\delta$ since $\mu(\operatorname{supp}\chi_0)\sim r$ and $\mu(\operatorname{supp}\bar\eta_j)\sim\delta$. We write

$$\lambda_\delta(x,y)\chi_0(y)[\phi(y)-\phi(x)]\psi(x) = \sum_{j\in\Gamma}\lambda_\delta(x,y)\bar\eta_j(y)\chi_0(y)[\phi(y)-\phi(x)]\psi(x),$$

and we obtain

$$\begin{aligned}&\langle K(x,y),\lambda_\delta(x,y)\chi_0(y)[\phi(y)-\phi(x)]\psi(x)\rangle\\ &\quad=\sum_{j\in\Gamma}\langle K(x,y),\lambda_\delta(x,y)\bar\eta_j(y)\chi_0(y)[\phi(y)-\phi(x)]\psi(x)\rangle.\end{aligned}$$

It is then easy to check that $\operatorname{supp}\{\lambda_\delta(x,y)\bar\eta_j(y)\chi_0(y)[\phi(y)-\phi(x)]\psi(x)\}\subseteq B(y_j,3A\delta)\times B(y_j,2\delta)$ and

$$\|\lambda_\delta(x,y)\bar\eta_j(y)\chi_0(y)[\phi(y)-\phi(x)]\psi(x)\|_\infty \le C\delta^\eta$$

where C is a constant depending only on θ,ϕ,ψ,x_0, and r but not on δ and j.

We claim that

$$\|\lambda_\delta(.,y)\bar\eta_j(y)\chi_0(y)[\phi(y)-\phi(.)]\psi(.)\|_\eta \le C, \tag{2.9}$$

and

$$\|\lambda_\delta(x,.)\bar\eta_j(.)\chi_0(.)[\phi(.)-\phi(x)]\psi(x)\|_\eta \le C. \tag{2.10}$$

We accept (2.9) and (2.10) for the moment. Then, since T satisfies the "strong" weak boundedness property, we have

$$\begin{aligned}
&|\langle K(x,y),\lambda_\delta(x,y)\chi_0(y)[\phi(y)-\phi(x)]\psi(x)\rangle| \\
&\quad\le \sum_{j\in\Gamma} |\langle K(x,y),\lambda_\delta(x,y)\bar{\eta}_j(y)\chi_0(y)[\phi(y)-\phi(x)]\psi(x)\rangle| \\
&\quad\le \sum_{j\in\Gamma} C\mu(B(y_j,3A\delta))\delta^\eta \le C\frac{r}{\delta}CA\delta\delta^\eta = CAr\delta^\eta
\end{aligned}$$

which yields (2.6).

It remains to show (2.9) and (2.10). We prove only (2.9) since the proof of (2.10) is similar. To show (2.9) it suffices to show that for $x, x_1 \in X$ and $\rho(x,x_1)\le\delta$,

$$\begin{aligned}
&|\bar{\eta}_j(y)\chi_0(y)|\big|\lambda_\delta(x,y)[\phi(y)-\phi(x)]\psi(x)-\lambda_\delta(x_1,y)[\phi(y)-\phi(x_1)]\psi(x_1)\big| \\
&\quad\le C\rho(x,x_1)^\eta,
\end{aligned}$$

since if $\rho(x,x_1)\ge\delta$, then the expansion on the left above is clearly bounded by

$$\begin{aligned}
&|\bar{\eta}_j(y)\chi_0(y)|\big\{|\lambda_\delta(x,y)[\phi(y)-\phi(x)]\psi(x)|+|\lambda_\delta(x_1,y)[\phi(y)-\phi(x_1)]\psi(x_1)|\big\} \\
&\quad\le C\delta^\eta \le C\rho(x,x_1)^\eta.
\end{aligned}$$

By the construction of $\bar{\eta}_j$, it follows that

$$|\bar{\eta}_j(y)\chi_0(y)|\le C$$

for all $y\in X$. Thus

$$\begin{aligned}
&|\bar{\eta}_j(y)\chi_0(y)||\lambda_\delta(x,y)[\phi(y)-\phi(x)]\psi(x)-\lambda_\delta(x_1,y)[\phi(y)-\phi(x_1)]\psi(x_1)| \\
&\quad\le C|\lambda_\delta(x,y)[\phi(y)-\phi(x)]\psi(x)-\lambda_\delta(x_1,y)[\phi(y)-\phi(x_1)]\psi(x_1)| \\
&\quad\le C|[\lambda_\delta(x,y)-\lambda_\delta(x_1,y)][\phi(y)-\phi(x)]\psi(x)| \\
&\qquad+|\lambda_\delta(x_1,y)[\phi(x)-\phi(x_1)]\psi(x)| \\
&\qquad+|\lambda_\delta(x_1,y)[\phi(y)-\phi(x_1)][\psi(x)-\psi(x_1)]| \\
&\quad:= I+II+III.
\end{aligned}$$

Recall that $\rho(x,x_1)\le\delta$. If $\rho(x,y)>C\delta$, where C is a constant depending on A but not on δ, then $\lambda_\delta(x,y)=\lambda_\delta(x_1,y)=0$, so $I=0$. Thus we may assume that $\rho(x,y)\le C\delta$ and with θ in (1.7),

$$\begin{aligned}
I &\le C\Big|\frac{\rho(x,y)}{\delta}-\frac{\rho(x_1,y)}{\delta}\Big|\rho(x,y)^\eta \le C\delta^{\eta-1}\rho(x,x_1)^\theta[\rho(x,y)+\rho(x_1,y)]^{1-\theta} \\
&\le C\delta^{\eta-\theta}\rho(x,x_1)^\theta \le C\rho(x,x_1)^\eta
\end{aligned}$$

since we may assume $\eta \leq \theta$. Terms II and III are easy to estimate:

$$II \leq C\rho(x, x_1)^\eta,$$

$$III \leq C\rho(x, x_1)^\eta,$$

since we can assume that $\delta < 1$. This completes the proof of (2.9) and implies

$$|p| \leq C\|\psi\|_1.$$

To finish the proof of Lemma 2.5, we now estimate q. It suffices to show that for $x \in B(x_0, r)$,

$$|T\chi_0(x)| \leq C. \tag{2.11}$$

To see this, it is easy to check that $q = \langle T\chi_0, \phi\psi\rangle$, and hence (2.10) implies

$$|q| \leq \|T\chi_0\|_{L^\infty(B(x_0,r))}\|\phi\psi\|_{L^1(B(x_0,r))} \leq C\|\psi\|_1.$$

To show (2.11), we use Meyer's idea again ([M1]). Let $\psi \in C^\eta(X)$ with supp$\psi \subseteq B(x_0, r)$ and $\int \psi(x)d\mu(x) = 0$. By the facts that $T(1) = 0, \int \psi(x)d\mu(x) = 0$, and the conditions on K, we obtain

$$\begin{aligned}|\langle T\chi_0, \psi\rangle| = |-\langle T\chi_1, \psi\rangle| &= \left|\iint [K(x,y) - K(x_0,y)]\chi_1(y)\psi(x)d\mu(x)d\mu(y)\right| \\ &\leq C\|\psi\|_1.\end{aligned}$$

Thus, $T\chi_0(x) = \omega + \gamma(x)$ for $x \in B(x_0, r)$, where ω is a constant and $\|\gamma\|_\infty \leq C$. To estimate ω, choose $\phi_1 \in C_0^\eta(X)$ with supp $\phi_1 \subseteq B(x_0, r), \|\phi_1\|_\infty \leq 1, \|\phi_1\|_\eta \leq r^{-\eta}$ and $\int \phi_1(x)d\mu(x) = Cr$. We then have, by the "strong" weak boundedness property of T,

$$\left|Cr\omega + \int \phi_1(x)\gamma(x)d\mu(x)\right| = |\langle T\chi_0, \phi_1\rangle| \leq Cr$$

which implies $|\omega| \leq C$ and hence Lemma 2.5. ∎

We remark that the calculation above, together with the dominated convergence theorem and $T1 = 0$, yields the following integral representation:

$$\begin{aligned}&\langle T\phi, \psi\rangle \\ &\quad = \int_\Omega K(x,y)\{\chi_0(y)[\phi(y) - \phi(x)] - \chi_1(y)\phi(x)\}\psi(x)d\mu(y)d\mu(x)\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}&\langle K(x,y), [\phi(y) - \phi(x)]\chi_0(y)\rangle \\ &\quad = \lim_{\delta\to 0}\int_{\rho(x,y)\geq\delta} K(x,y)\chi_0(y)[\phi(y) - \phi(x)]d\mu(y)\end{aligned} \tag{2.13}$$

where χ_0, ϕ and ψ are defined as above.

We return to prove the Theorem 2.4. Fix a function $\theta \in C^1(\mathbb{R})$ with supp $\theta \subseteq \{x \in \mathbb{R} : |x| \leq 2\}$ and $\theta = 1$ on $\{x \in \mathbb{R} : |x| \leq 1\}$. Suppose that $f \in \mathcal{M}_0(x_0, r, \beta, \gamma)$ with $x_0 \in X, r > 0$ and $0 < \beta, \gamma < \epsilon$. We first prove that $T(f)(x)$ satisfies the size condition (i) of Definition 2.1. To do this, we first consider the case where $\rho(x, x_0) \leq 5r$. Set $1 = \xi(y) + \eta(y)$ where $\xi(y) = \theta(\frac{\rho(y,x_0)}{10Ar})$. Then we have

$$T(f)(x) = \int K(x,y)\xi(y)[f(y) - f(x)]d\mu(y) + \int K(x,y)\eta(y)f(y)d\mu(y)$$
$$+f(x)\int K(x,y)\xi(y)d\mu(y) := I + II + III.$$

Using (2.13),

$$\begin{aligned}|I| &\leq C\int_{\rho(x,y)\leq 25A^2r} |K(x,y)||f(y) - f(x)|d\mu(y)\\ &\leq C\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)}\int_{\rho(x,y)\leq 25A^2r} \rho(x,y)^{-1}\frac{\rho(x,y)^\beta}{r^{1+\beta}}d\mu(y)\\ &\leq C\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)}r^{-1}.\end{aligned}$$

By Lemma 2.5,

$$|III| \leq C|f(x)| \leq C\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)}r^{-1}.$$

For term II we have

$$\begin{aligned}|II| &\leq C\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)}\int_{\rho(x,y)\geq 10Ar} \rho(x,y)^{-1}\frac{r^\gamma}{\rho(y,x_0)^{1+\gamma}}d\mu(y)\\ &\leq C\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)}r^{-1}\end{aligned}$$

since $\rho(x, x_0) \leq 5r$.

This implies that $T(f)(x)$ satisfies (i) of Definition 2.1 with $\rho(x, x_0) \leq 5r$. Consider now $\rho(x, x_0) = R > 5r$. Following the proof in [M1], set $1 = I(y) + J(y) + L(y)$, where $I(y) = \theta(\frac{4A\rho(y,x)}{R}), J(y) = \theta(\frac{4A\rho(y,x_0)}{R})$, and $f_1(y) = f(y)I(y), f_2(y) = f(y)J(y)$, and $f_3(y) = f(y)L(y)$. Then it is easy to check the following estimates:

$$|f_1(y)| \leq C\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)}\frac{r^\gamma}{R^{1+\gamma}}; \tag{2.14}$$

$$\begin{aligned}|f_1(y) - f_1(y')| &\leq |I(y)||f(y) - f(y')| + |f(y')||I(y) - I(y')| \quad (2.15)\\ &\leq C\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)}\frac{\rho(y,y')^\beta}{R^\beta}\frac{r^\gamma}{R^{1+\gamma}}\end{aligned}$$

for all y and y';

$$|f_3(y)| \leq C\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)} \frac{r^\gamma}{\rho(y,x_0)^{1+\gamma}} \chi_{\{y\in X:\rho(y,x_0)>\frac{1}{4A}R\}}; \tag{2.16}$$

$$\int |f_3(y)|d\mu(y) \leq C\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)} \frac{r^\gamma}{R^\gamma}; \tag{2.17}$$

$$\begin{aligned} \left|\int f_2(y)d\mu(y)\right| &= \left|-\int f_1(y)d\mu(y) - \int f_3(y)d\mu(y)\right| \\ &\leq C\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)} \frac{r^\gamma}{R^\gamma}. \end{aligned} \tag{2.18}$$

We write

$$\begin{aligned} T(f_1)(x) &= \int K(x,y)u(y)[f_1(y) - f_1(x)]d\mu(y) + f_1(x)\int K(x,y)u(y)d\mu(y) \\ &= \sigma_1(x) + \sigma_2(x) \end{aligned}$$

where $u(y) = \theta(\frac{2A\rho(x,y)}{R})$. Applying the estimate (2.15) and Lemma 2.5, we obtain

$$\begin{aligned} |\sigma_1(x)| &\leq C\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)} \int_{\rho(x,y)\leq\frac{R}{A}} \rho(x,y)^{-1}\frac{\rho(x,y)^\beta}{R^\beta}\frac{r^\gamma}{R^{1+\gamma}}d\mu(y) \\ &\leq C\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)} \frac{r^\gamma}{R^{1+\gamma}}; \end{aligned}$$

and

$$|\sigma_2(x)| \leq C|f_1(x)| \leq C\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)} \frac{r^\gamma}{R^{1+\gamma}}.$$

Notice that x is not in the support of f_2. We can write

$$\begin{aligned} T(f_2)(x) &= \int [K(x,y) - K(x,x_0)]f_2(y)d\mu(y) + K(x,x_0)\int f_2(y)d\mu(y) \\ &= \delta_1(x) + \delta_2(x). \end{aligned}$$

Using the estimates on the kernel of T and on f_2 in (2.18), we then get

$$\begin{aligned} |\delta_1(x)| &\leq C\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)} \int_{\rho(x_0,y)\leq\frac{R}{2A}} \frac{\rho(x_0,y)^\epsilon}{R^{1+\epsilon}}\frac{r^\gamma}{\rho(x_0,y)^{1+\gamma}}d\mu(y) \\ &\leq C\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)} \frac{r^\gamma}{R^{1+\gamma}} \end{aligned}$$

since $\gamma < \epsilon$, and

$$|\delta_2(x)| \leq CR^{-1}\left|\int f_2(y)d\mu(y)\right| \leq C\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)} \frac{r^\gamma}{R^{1+\gamma}}.$$

Finally, since x is not in the support of f_3, (2.16) implies

$$|T(f_3)(x)| \le C\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)} \int_{\rho(x,y)\ge \frac{R}{4A}, \rho(x_0,y)\ge \frac{R}{4A}} \rho(x,y)^{-1} \frac{r^\gamma}{\rho(x_0,y)^{1+\gamma}} d\mu(y)$$
$$\le C\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)} \frac{r^\gamma}{R^{1+\gamma}}.$$

This yields that $T(f)(x)$ satisfies (i) of Definition 2.1 for $\rho(x,x_0) > 5r$ and hence, estimate (i) of Definition 2.1 for all $x \in X$.

Now we prove that $T(f)(x)$ satisfies the smoothness condition (ii) of Definition 2.1. To do this, set $\rho(x,x_0) = R$ and $\rho(x,x') = \delta$. We consider first the case where $R \ge 10r$ and $\delta \le \frac{1}{20A^2}(r+R)$. As in the above, set $1 = I(y) + J(y) + L(y)$, where $I(y) = \theta(\frac{8A\rho(y,x)}{R}), J(y) = \theta(\frac{8A\rho(y,x_0)}{R})$, and $f_1(y) = f(y)I(y), f_2(y) = f(y)J(y)$, and $f_3(y) = f(y)L(y)$. We write

$$T(f_1)(x) = \int K(x,y)u(y)[f_1(y) - f_1(x)]d\mu(y)$$
$$+ \int K(x,y)v(y)f_1(y)d\mu(y) + f_1(x)\int K(x,y)u(y)d\mu(y)$$

where $u(y) = \theta(\frac{\rho(x,y)}{2A\delta})$ and $v(y) = 1-u(y)$. Denote the first term of the above right-hand side by $p(x)$ and the last two terms by $q(x)$. The size condition of K and the smoothness of f_1 in (2.15) yield

$$|p(x)| \le C\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)} \int_{\rho(x,y)\le 4A\delta} \rho(x,y)^{-1} \frac{\rho(x,y)^\beta}{R^\beta} \frac{r^\gamma}{R^{1+\gamma}} d\mu(y)$$
$$\le C\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)} \frac{\delta^\beta}{R^\beta} \frac{r^\gamma}{R^{1+\gamma}}.$$

This estimate still holds with x replaced by x' for $\rho(x,x') = \delta$. Thus

$$|p(x) - p(x')| \le C\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)} \frac{\delta^\beta}{R^\beta} \frac{r^\gamma}{R^{1+\gamma}}.$$

For $q(x)$, using the condition $T1 = 0$, we obtain

$$q(x) - q(x') = \int [K(x,y) - K(x',y)]v(y)[f_1(y) - f_1(x)]d\mu(y)$$
$$+[f_1(y) - f_1(x)]\int K(x,y)u(y)d\mu(y)$$
$$= I + II.$$

Using Lemma 2.5 and the estimate for f_1 in (2.15),

$$|II| \le C|f_1(x) - f_1(x')| \le C\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)} \frac{\delta^\beta}{R^\beta} \frac{r^\gamma}{R^{1+\gamma}}.$$

Observing

$$|f_1(y)-f_1(x)||v(y)| \le C\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)} \frac{\rho(x,y)^\beta}{R^\beta}\frac{r^\gamma}{R^{1+\gamma}}$$

for all $y \in X$, we see that I is dominated by

$$\begin{aligned} C \int\limits_{\rho(x,y)\ge 2A\delta} & |K(x,y)-K(x',y)||v(y)||f_1(y)-f_1(x)|d\mu(y) \\ & \le C\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)} \int_{\rho(x,y)\ge 2A\delta} \frac{\rho(x,x')^\epsilon}{\rho(x,y)^{1+\epsilon}}\frac{\rho(x,y)^\beta}{R^\beta}\frac{r^\gamma}{R^{1+\gamma}} d\mu(y) \\ & \le C\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)} \frac{\delta^\beta}{R^\beta}\frac{r^\gamma}{R^{1+\gamma}} \end{aligned}$$

since $\beta < \epsilon$. This implies

$$|T(f_1)(x)-T(f_1)(x')| \le C\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)} \frac{\delta^\beta}{R^\beta}\frac{r^\gamma}{R^{1+\gamma}}.$$

Note that for $\rho(x,x')=\delta \le \frac{1}{20A^2}(r+R)$ and $R \ge 10r$, x and x' are not in the supports of f_2 and f_3. Using the condition for K and the estimate for f_2 in (2.18), then

$$\begin{aligned} |T(f_2)(x)-T(f_2)(x')| &= \left| \int [K(x,y)-K(x',y)]f_2(y)d\mu(y) \right| \\ & \le \int |K(x,y)-K(x',y)-K(x,x_0)-K(x',x_0)||f_2(y)|d\mu(y) \\ & \quad +|K(x,x_0)-K(x',x_0)|\left|\int f_2(y)d\mu(y)\right| \\ & \le C\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)} \left\{ \int_{\rho(x_0,y)\le \frac{R}{4A}} \frac{\rho(x,x')^\epsilon \rho(y,x_0)^\epsilon}{R^{2+\epsilon}} \frac{r^\gamma}{\rho(y,x_0)^{1+\gamma}} d\mu(y) \right. \\ & \qquad \left. + \frac{\delta^\epsilon}{R^{1+\epsilon}}\frac{r^\gamma}{R^\gamma} \right\} \\ & \le C\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)} \frac{\delta^\epsilon}{R^\epsilon}\frac{r^\gamma}{R^{1+\gamma}} \end{aligned}$$

since $\gamma < \epsilon$. Finally, we have

$$\begin{aligned} |T(f_3)(x)-T(f_3)(x')| &= \left| \int [K(x,y)-K(x',y)]f_3(y)d\mu(y) \right| \\ & \le C \int_{\rho(x,y)\ge \frac{R}{8A} \ge 2A\delta} \frac{\rho(x,x')^\epsilon}{\rho(x,y)^{1+\epsilon}} |f_3(y)|d\mu(y) \le C\|f\|_{\mathcal{M}(x_0,r,\beta,\gamma)} \frac{\delta^\epsilon}{R^\epsilon}\frac{r^\gamma}{R^{1+\gamma}}. \end{aligned}$$

These estimates imply that $T(f)(x)$ satisfies the condition (ii) of Definition 2.1 for the case where $\rho(x,x_0)=R\ge 10r$ and $\rho(x,x')=\delta \le \frac{1}{20A^2}(r+R)$. We now consider the other cases. Note first that if $\rho(x,x_0)=R$ and $\frac{1}{2A}(r+R) \ge$

$\rho(x, x') = \delta \geq \frac{1}{20A^2}(r+R)$, then the estimate (ii) of Definition 2.1 for $T(f)(x)$ follows from the estimate (i) of Definition 2.1 for $T(f)(x)$. So we only need to consider the case where $R \leq 10r$ and $\delta \leq \frac{1}{20A^2}(r+R)$. This case is similar and easier. In fact, all we need to do is to replace R in the proof above by r. We leave these details to the reader. The proof of Theorem 2.4 is completed.

We remark that the condition in (2.2) is also necessary for the boundedness of Calderón-Zygmund operators on *wavelet spaces*. To be precise, in the next chapter, we will prove all kinds of Calderón's identities and use them to provide all kinds of wavelet expansions of functions and distributions on spaces of homogeneous type. Suppose that T is a Calderón-Zygmund operator and maps the wavelet space $\mathcal{M}_0(x_0, r, \beta, \gamma)$ to itself. By the wavelet expansion given in Theorem 3.25 below, $K(x, y)$, the kernel of T, can be written as $K(x,y) = \sum_{\lambda \in \Lambda} T(\widetilde{\psi}_\lambda)(x)\psi_\lambda(y)$. Since $\widetilde{\psi}_\lambda(x)$ is a wavelet, by the assumption on $T, T(\widetilde{\psi}_\lambda)(x)$ is also a wavelet. Then one can easily check that $K(x, y)$ satisfies the condition (2.2) but the exponent ϵ must be replaced by ϵ' with $0 < \epsilon' < \beta, \gamma$. We leave these details to the reader.