Chapter II
Measures and Integrals. The General Theory

In this chapter we shall develop a general integration theory for cone-valued functions with respect to operator-valued measures. The structure of locally convex cones will allow the use of many of the main concepts of classical measure theory for (extended) real-valued functions. Section 1 introduces measurability for cone-valued functions on a set X with respect to a (weak) σ-ring of subsets of X. This notion does not involve any reference to a particular measure. Bounded operator-valued measures will be defined in Section 3. The introduction of its modulus allows the extension of any given measure to a full locally convex cone containing the given cone and its neighborhood system, thus greatly facilitating the expansion of our concepts. This yields a new understanding of the variation of a measure, not as a separate positive real-valued measure associated with the given one, but as a component of its extension. The development of an integration theory for cone-valued functions with respect to an operator-valued measure follows in Section 4. Section 5 contains the general convergence theorems for sequences of functions and measures, that is variations and adaptations of the dominated convergence theorem. Chapter II concludes with a long list of special cases and examples in Section 6, demonstrating the generality of the approach. These examples include classical real-valued measure theory as well as settings with vector-, cone-, functional- and operator-valued measures and functions.

1. Measurable Cone-Valued Functions

Throughout the following let X be a set, $(\mathcal{P},\mathcal{V})$ a locally convex cone with dual $\mathcal{P}^*$. Endowed with the pointwise algebraic operations and order, the $\mathcal{P}$-valued functions on X form again a cone, denoted by $\mathcal{F}(X,\mathcal{P})$. As usual, we say that a function $f \in \mathcal{F}(X,\mathcal{P})$ is *supported by a set* $E \subset X$ if $f(x) = 0$ for all $x \in X \setminus E$. For a positive real-valued function φ on X and $f \in \mathcal{F}(X,\mathcal{P})$ we denote by $\varphi_{\otimes} f \in \mathcal{F}(X,\mathcal{P})$ the mapping

W. Roth, *Operator-Valued Measures and Integrals for Cone-Valued Functions*, Lecture Notes in Mathematics 1964,

$$x \mapsto \varphi(x)f(x) : X \to \mathcal{P}.$$

For an element a of $\mathcal{P}$ or of $\mathcal{V}$ we shall also use its symbol to denote the constant function $x \mapsto a$, hence $\varphi \otimes a$ for $x \mapsto \varphi(x)a$.

1.1 Weak σ-Rings. We shall develop our measure and integration theory with respect to a family $\mathfrak{R}$ of subsets of X with the following properties:

(R1) $\emptyset \in \mathfrak{R}$.
(R2) *If* $E_1, E_2 \in \mathfrak{R}$, *then* $E_1 \bigcup E_2 \in \mathfrak{R}$ *and* $E_1 \setminus E_2 \in \mathfrak{R}$.
(R3) *If* $E_n \in \mathfrak{R}$ *for* $n \in \mathbb{N}$ *and* $E_n \subset E$ *for some* $E \in \mathfrak{R}$, *then* $\bigcup_{n \in \mathbb{N}} E_n \in \mathfrak{R}$.

We shall call a family $\mathfrak{R}$ with these properties a *(weak) σ-ring*. (Condition (R3) is weaker then the usual one for σ-rings.) As $E_1 \cap E_2 = E_1 \setminus (E_1 \setminus E_2)$, Condition (R2) implies that $E_1 \cap E_2 \in \mathfrak{R}$ whenever $E_1, E_2 \in \mathfrak{R}$. Of course, any σ-algebra is a σ-ring in this sense, and a σ-ring $\mathfrak{R}$ is a σ-algebra if and only if $X \in \mathfrak{R}$. However, because we shall require boundedness for measures defined on $\mathfrak{R}$, using σ-algebras from the beginning would impose undue limitations. We may, however, associate with $\mathfrak{R}$ in a canonical way the σ-algebra

$$\mathfrak{A}_{\mathfrak{R}} = \{A \subset X \mid A \cap E \in \mathfrak{R} \quad \text{for all} \quad E \in \mathfrak{R}\}$$

of *measurable* subsets of X. As usual, χ_E stands for the characteristic (or indicator) function on X of a subset $E \subset X$, and $\mathcal{S}_{\mathfrak{R}}(X, \mathcal{P})$ is the subcone of $\mathcal{F}(X, \mathcal{P})$ of all $\mathcal{P}$-valued step functions supported by $\mathfrak{R}$, that is functions $h = \sum_{i=1}^{n} \chi_{E_i} \otimes a_i$ with $E_i \in \mathfrak{R}$ and $a_i \in \mathcal{P}$. If the sets E_i are pairwise disjoint, then we shall call the above the *standard representation* for the step function h. Measurability for vector-valued functions has been introduced in various places (see for example Dunford & Schwartz [55], III.2.10). A suitable adaptation for cone-valued functions needs to consider the presence of unbounded elements in $\mathcal{P}$ and the absence of negatives. We shall therefore employ the relative topologies.

1.2 Measurable Functions. We shall say that a function $f \in \mathcal{F}(X, \mathcal{P})$ is *measurable with respect to the σ-ring* $\mathfrak{R}$ if for every $v \in \mathcal{V}$, with respect to the symmetric relative v-topology of $\mathcal{P}$

(M1) $f^{-1}(O) \cap E \in \mathfrak{R}$ *for every open subset* O *of* $\mathcal{P}$ *and every* $E \in \mathfrak{R}$.
(M2) $f(E)$ *is separable in* $\mathcal{P}$ *for every* $E \in \mathfrak{R}$.

Note that Condition (M1) means that $f^{-1}(O) \in \mathfrak{A}_{\mathfrak{R}}$ for all open subsets O of $\mathcal{P}$. Obviously the functions in $\mathcal{S}_{\mathfrak{R}}(X, \mathcal{P})$ are measurable.

Proposition 1.3. *A function $f \in \mathcal{F}(X, \overline{\mathbb{R}})$ is measurable if and only if it is measurable in the usual sense with respect to the σ-algebra $\mathfrak{A}_{\mathfrak{R}}$.*

Proof. Let $f \in \mathcal{F}(X, \overline{\mathbb{R}})$. The neighborhood system for $\overline{\mathbb{R}}$ consists of the positive reals $\varepsilon > 0$. The symmetric relative topology therefore coincides with the usual topology on the elements of $\mathbb{R}$, while $+\infty$ is an isolated point. The range of f, hence $f(E)$ for every $E \in \mathfrak{R}$, is separable in any case. Thus for measurability we require that $f^{-1}(O) \in \mathfrak{A}_{\mathfrak{R}}$ for every open subset of $\mathbb{R}$ and also that $f^{-1}(+\infty) \in \mathfrak{A}_{\mathfrak{R}}$. This coincides with the usual definition of measurability. $\square$

Theorem 1.4. *A function $f \in \mathcal{F}(X, \mathcal{P})$ is measurable if and only if for every $E \in \mathfrak{R}$, $v \in \mathcal{V}$ and $\varepsilon > 0$ there are sets $E_n \in \mathfrak{R}$, $n \in \mathbb{N}$, such that $\bigcup_{n \in \mathbb{N}} E_n = E$ and $f(x) \in v_\varepsilon\big(f(y)\big)$ whenever $x, y \in E_n$ for some $n \in \mathbb{N}$.*

Proof. First assume that the function $f \in \mathcal{F}(X, \mathcal{P})$ is measurable. For $E \in \mathfrak{R}$ and $v \in \mathcal{V}$ let $A = \{a_n \mid n \in \mathbb{N}\}$ be a dense subset (with respect to the symmetric relative v-topology) of $f(E)$. For $a \in \mathcal{P}$ and $\varepsilon > 0$ the sets

$$\overset{\circ}{v}_\varepsilon(a) = \bigcup_{0<\varepsilon'<\varepsilon} v_{\varepsilon'}(a) \qquad \text{and} \qquad (a)\overset{\circ}{v}_\varepsilon = \bigcup_{0<\varepsilon'<\varepsilon} (a)v_{\varepsilon'}$$

are open in the upper and lower relative v-topologies, respectively, and their intersection $\overset{\circ}{v}{}^s_\varepsilon(a)$ is an open neighborhood of a in the symmetric relative v-topology. Set

$$E_n = f^{-1}\big(\overset{\circ}{v}{}^s_{(\varepsilon/3)}(a_n)\big) \cap E \in \mathfrak{R}.$$

Then Lemma I.4.1(a) shows that $f(x) \in v_\varepsilon\big(f(y)\big)$ whenever $x, y \in E_n$. Furthermore, for every $x \in E$ there is some $a_n \in \overset{\circ}{v}{}^s_{\varepsilon/2}\big(f(x)\big)$. Thus $f(x) \in \overset{\circ}{v}{}^s_{\varepsilon/2}(a_n) \subset \overset{\circ}{v}{}^s_\varepsilon(a_n)$, hence $x \in E_n$. This shows $\bigcup_{n \in \mathbb{N}} E_n = E$, as required.

For the converse, assume that the above condition holds for the function $f \in \mathcal{F}(X, \mathcal{P})$ and let $E \in \mathfrak{R}$ Then for every $m \in \mathbb{N}$ there are $E_n^m \in \mathfrak{R}$, such that $\bigcup_{n \in \mathbb{N}} E_n^m = E$ and $f(x) \in v_{(1/m)}\big(f(y)\big)$ whenever $x, y \in E_n^m$ for some $n \in \mathbb{N}$. Choose $a_n^m = f(x_n^m)$ for some $x_n^m \in E_n^m$. Then the set $A = \{a_n^m \mid n, m \in \mathbb{N}\}$ is seen to be dense in $f(E)$, which yields Condition (M1). For (M2), let $O \subset \mathcal{P}$ be open in the symmetric relative v-topology. With the sets E_n^m from above set

$$F = \bigcup \big\{ E_n^m \mid E_n^m \subset f^{-1}(O) \big\}.$$

Then $F \in \mathfrak{R}$ by (R2) and $F \subset f^{-1}(O) \cap E$. For $x \in f^{-1}(O) \cap E$ on the other hand, there is $m \in \mathbb{N}$ such that $v^s_{(1/m)}\big(f(x)\big) \subset O$. We find $x \in E_n^m$ for some $n \in \mathbb{N}$. Then for every $y \in E_n^m$ we have $f(y) \in v_{(1/m)}\big(f(x)\big) \subset O$, hence $y \in f^{-1}(O)$. This shows $x \in E_n^m \subset F$, hence $f^{-1}(O) \cap E = F \in \mathfrak{R}$. $\square$

We can indeed assume that the sets $E_n \in \mathfrak{R}$ from Theorem 1.4 are disjoint, since otherwise we may set

$$G_1 = E_1 \qquad \text{and} \qquad G_n = E_n \setminus \bigcup_{i=1}^{n-1} E_i$$

for $n \geq 2$. Then $G_n \in \mathfrak{R}$ and $G_n \subset E_n$. The sets G_n are disjoint and their union equals the union of the sets E_n, that is the given set $E \in \mathfrak{R}$.

Corollary 1.5. *The measurable functions form a subcone of* $\mathcal{F}(X, \mathcal{P})$.

Proof. Clearly, αf is measurable, whenever $f \in \mathcal{F}(X, \mathcal{P})$ is measurable and $\alpha \geq 0$. We proceed to show that for measurable functions $f, g \in \mathcal{F}(X, \mathcal{P})$ their sum $f + g$ is also measurable. In a first step, given $E \in \mathfrak{R}$ and $v \in \mathcal{V}$, using Theorem 1.4 and Lemma I.4.1(c) we can find sets $E_n \in \mathfrak{R}$ and $\lambda_n \geq 0$ such that $\bigcup_{n \in \mathbb{N}} E_n = E$ and $0 \leq f(x) + \lambda_n$ and $0 \leq g(x) + \lambda_n$ whenever $x \in E_n$. Now, given $\varepsilon > 0$ we set $\delta_n = \varepsilon/(2 + 2\lambda_n)$ and again using Theorem 1.4, for every $n \in \mathbb{N}$ we find sets $E_n^m \in \mathfrak{R}$ such that $\bigcup_{m \in \mathbb{N}} E_n^m = E_n$ and $f(x) \in v_{\delta_n}\big(f(y)\big)$ as well as $g(x) \in v_{\delta_n}\big(g(y)\big)$ whenever $x, y \in E_n^m$. Following I.4.1(b) this yields

$$f(x) \leq (1 + \delta) f(y) + \delta(1 + \lambda) v = (1 + \delta) f(z) + (\varepsilon/2) v$$

and, likewise $g(x) \leq (1 + \delta) f(y) + (\varepsilon/2) v$. Thus

$$f(x) + g(x) \leq (1 + \delta)\big(f(y) + g(y)\big) + \varepsilon v$$

and indeed $f(x) + g(x) \in v_\varepsilon\big(f(y) + g(y)\big)$. As $\bigcup_{m,n \in \mathbb{N}} E_n^m = E$, this proves that the function $f + g$ is also measurable. $\square$

Theorem 1.6. *For measurable functions* $f, g \in \mathcal{F}(X, \mathcal{P})$ *and* $v \in \mathcal{V}$ *the set* $\{x \in X \mid f(x) \preccurlyeq_v g(x)\}$ *is measurable, that is in* $\mathfrak{A}_{\mathfrak{R}}$.

Proof. Let $f, g \in \mathcal{F}(X, \mathcal{P})$ be measurable, $v \in \mathcal{V}$ and $E \in \mathfrak{R}$. According to Theorem 1.4, for $0 < \varepsilon \leq 1$ there are disjoint sets $E_i \in \mathfrak{R}$ such that $\bigcup_{i \in \mathbb{N}} = E$ and $f(x) \in v_\varepsilon\big(f(y)\big)$ as well as $g(x) \in v_\varepsilon\big(g(y)\big)$ whenever $x, y \in E_i$ for some $i \in \mathbb{N}$. Set

$$F_\varepsilon = \bigcup \big\{E_i \mid f(x_i) \in v_\varepsilon\big(g(x_i)\big) \quad \text{for some} \quad x_i \in E_i\big\} \in \mathfrak{R}.$$

If $x \in F_\varepsilon$, then $f(x) \in v_\varepsilon\big(f(x_i)\big)$, $f(x_i) \in v_\varepsilon\big(g(x_i)\big)$ and $g(x_i) \in v_\varepsilon\big(g(x)\big)$, hence $f(x) \in v_{(7\varepsilon)}\big(g(x)\big)$ by Lemma 2,1(a). This shows

$$\big\{x \in E \mid f(x) \in v_\varepsilon\big(g(x)\big)\big\} \subset F_\varepsilon \subset \big\{x \in E \mid f(x) \in v_{7\varepsilon}\big(g(x)\big)\big\}.$$

Then

$$F = \bigcap_{n \in \mathbb{N}} F_{(\frac{1}{n})} = \{x \in E \mid f(x) \preccurlyeq_v g(x)\} \in \mathfrak{R}$$

as well. As $F = E \bigcap \{x \in X \mid f(x) \preccurlyeq_v g(x)\}$, our claim follows. $\square$

We shall use different patterns of pointwise convergence for sequences of cone-valued functions. For functions $(f_n)_{n \in \mathbb{N}}$ and f in $\mathcal{F}(X, \mathcal{P})$, a subset $F \subset X$ and a neighborhood $v \in \mathcal{V}$ we shall write

$$f_n \overset{v}{\underset{F}{\searrow}} f, \qquad f_n \overset{v}{\underset{F}{\nearrow}} f, \qquad \text{or} \qquad f_n \xrightarrow[F]{v} f$$

if $(f_n)_{n\in\mathbb{N}}$ converges to f pointwise on F with respect to the upper, lower or symmetric relative v-topology of $\mathcal{P}$, that is if for every $x \in F$ and $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $f_n(x) \in v_\varepsilon\big(f(x)\big)$, $f_n(x) \in \big(f(x)\big)v_\varepsilon$ or $f_n(x) \in v_\varepsilon^s\big(f(x)\big)$ for all $n \geq n_0$, respectively. Convergence in this sense for all $v \in \mathcal{V}$ means convergence in the (global) upper, lower or symmetric relative topology of $\mathcal{P}$. We shall denote this by

$$f_n \underset{F}{\searrow} f, \qquad f_n \underset{F}{\nearrow} f, \qquad \text{or} \qquad f_n \underset{F}{\rightarrow} f.$$

All the above notions of convergence are compatible with the algebraic operations in $\mathcal{P}$ (see Lemma I.4.1(d)).

Theorem 1.7. *If for $f \in \mathcal{F}(X,\mathcal{P})$ and every $E \in \mathfrak{R}$ and $v \in \mathcal{V}$ there is a sequence of measurable functions $f_n \in \mathcal{F}(X,\mathcal{P})$ such that $f_n \overset{v}{\underset{E}{\rightarrow}} (f)$, then f is also measurable.*

Proof. Let $f \in \mathcal{F}(X,\mathcal{P})$, $E \in \mathfrak{R}$ and $v \in \mathcal{V}$. Suppose that there is a sequence of measurable functions $(f_n)_{n\in\mathbb{N}}$ such that $f_n \overset{v}{\underset{E}{\rightarrow}} (f)$. If for all $n \in \mathbb{N}$ the sets $A_n = \{a_n^i \mid i \in \mathbb{N}\}$ are dense (in the symmetric relative v-topology) in $f_n(E)$, then $A = \bigcup_{n\in\mathbb{N}} A_n$ is obviously dense in $f(E)$, which is thus seen to be separable. Now let $O \subset \mathcal{P}$ be open in the symmetric relative v-topology. For $\varepsilon > 0$ let U_ε be the topological interior of the set $O_\varepsilon = \{a \in O \mid v_\varepsilon^s(a) \subset O\}$. For $m, n \in \mathbb{N}$ set

$$E_n^m = E \bigcap_{k\geq n} (f_k)^{-1}\big(U_{(1/m)}\big) \in \mathfrak{R}$$

and $F = \bigcup_{m,n\in\mathbb{N}} E_n^m \in \mathfrak{R}$. If $x \in f^{-1}(O) \cap E$, then there is $\varepsilon > 0$ such that $v_\varepsilon^s\big(f(x)\big) \subset O$. For $m \geq 7/\varepsilon$ there is $n \in \mathbb{N}$ such that $f_k(x) \in v_{(1/m)}^s\big(f(x)\big)$ for all $k \geq n$. For any such k let $a \in v_{(1/m)}^s\big(f_k(x)\big)$. Then

$$v_{(1/m)}^s(a) \subset v_{(3/m)}^s\big(f_k(x)\big) \subset v_{(7/m)}^s\big(f(x)\big) \subset v_\varepsilon^s\big(f(x)\big) \subset O$$

by Lemma I.4.1(a). Thus $a \in O_{(1/m)}$, hence $v_{(1/m)}^s\big(f_k(x)\big) \subset O_{(1/m)}$, therefore $f_k(x) \in U_{(1/m)}$ and $x \in E_n^m \subset F$. This shows $f^{-1}(O) \subset F$. For $x \in F$, on the other hand, there are $m, n \in \mathbb{N}$ such that $f_k(x) \in U_{(1/m)} \subset O_{(1/m)}$ for all $k \geq n$. There is such k such that $f(x) \in v_{(1/m)}^s\big(f_k(x)\big)$, hence $f(x) \in O$. Thus $f^{-1}(O) \cap E = F \in \mathfrak{R}$. □

Theorem 1.8. *Let $f \in \mathcal{F}(X,\mathcal{P})$ be measurable.*

(a) Let $\varphi : X \to \mathbb{R}$. If φ is positive and measurable with respect to $\mathfrak{A}_\mathfrak{R}$, then $\varphi_\otimes f \in \mathcal{F}(X,\mathcal{P})$ is also measurable.

(b) Let $\Phi : X \to X$. If $\Phi^{-1}(A) \in \mathfrak{A}_\mathfrak{R}$ for all $A \in \mathfrak{A}_\mathfrak{R}$, and $\Phi(E) \subset F$ for every $E \in \mathfrak{R}$ with some $F \in \mathfrak{R}$, then the function then $f\circ\Phi \in \mathcal{F}(X,\mathcal{P})$ is also measurable.

(c) Let $(\mathcal{N},\mathcal{U})$ be a locally convex cone and let $\Psi:\mathcal{P}\to\mathcal{N}$. If for every $u\in\mathcal{U}$ there is $v\in\mathcal{V}$ such that the mapping Ψ is continuous with respect to the symmetric relative v- and u-topologies of $\mathcal{P}$ and $\mathcal{N}$, then the function $\Psi\circ f\in\mathcal{F}(X,\mathcal{N})$ is also measurable.
The assumption on Ψ holds in particular if $\Psi:\mathcal{P}\to\mathcal{N}$ is a continuous linear operator.

Proof. (a) Our claim is obvious if φ is a real-valued step function (supported by $\mathfrak{A}_{\mathfrak{R}}$,) since the validity of the criterion from Theorem 1.4 for $\varphi_{\otimes}f$ follows straight from its validity for f. Generally, there is a sequence $(\psi_n)_{n\in\mathbb{N}}$ of positive real-valued step functions that converges pointwise from below to φ. All the functions $f_n=\psi_{n\otimes}f$ are measurable by the above. If $\varphi(x)=0$ for $x\in X$, then $\psi_n(x)=0$ for all $n\in\mathbb{N}$. Otherwise, for $v\in\mathcal{V}$ and $\varepsilon>0$ choose $\lambda>0$ such that $0\le f(x)+\lambda v$ and set $\gamma=\min\bigl\{1+\varepsilon,\ 1+\frac{\varepsilon}{2\lambda\varphi(x)}\bigr\}$. There is $n_0\in\mathbb{N}$ such that $\psi_n(x)\le\varphi(x)\le\gamma\psi_n(x)$, hence

$$\psi_n(x)(f(x)+\lambda v)\le\varphi(x)(f(x)+\lambda v)\le\gamma\psi_n(x)(f(x)+\lambda v)$$

for all $n\ge n_0$. This shows

$$f_n(x)+\lambda\psi_n(x)v\le\varphi(x)f(x)+\lambda\varphi(x)v\le\varphi(x)f(x)+\gamma\lambda\psi_n(x)v,$$

hence $f_n(x)\le\varphi(x)f(x)+\varepsilon v$ by the cancellation law for positive elements (see Lemma I.4.2 in [100]), as $\gamma\lambda\psi_n<\lambda\psi_n(x)+\varepsilon$. Likewise, the above implies

$$\varphi(x)f(x)+\lambda\varphi(x)v\le\gamma f_n(x)+\gamma\lambda\psi_n(x)v\le\gamma f_n(x)+\gamma\lambda\varphi(x)v,$$

and $\varphi(x)f(x)\le\gamma f_n(x)+\varepsilon v$ as well. Thus $f_n(x)\in v_\varepsilon^s\bigl(\varphi(x)f(x)\bigr)$ for all $n\ge n_0$. This shows $f_n\underset{X}{\to}\varphi_{\otimes}f$, and by Theorem 1.7 the function $\varphi_{\otimes}f$ is seen to be measurable.

For (b), let f and $\Phi:X\to X$ be as stated, let $g=f\circ\Phi$ and $v\in\mathcal{V}$. For $E\in\mathfrak{R}$ we have $\Phi(E)\subset F$ for some $F\in\mathfrak{R}$, hence $g(E)\subset f(F)$ which is separable in the symmetric relative v-topology. Secondly, for an open subset O of $\mathcal{P}$ we have $f^{-1}(O)\in\mathfrak{A}_{\mathfrak{R}}$, hence $g^{-1}(O)=\Phi^{-1}\bigl(f^{-1}(O)\bigr)\in\mathfrak{A}_{\mathfrak{R}}$, and the function g is seen to be measurable.

For Part (c), let f and $\Psi:\mathcal{P}\to\mathcal{N}$ be as stated, let $g=f\circ\Psi$ and $u\in\mathcal{U}$. Let $v\in\mathcal{V}$ be such that Ψ is continuous with respect to the symmetric relative v- and u-topologies of $\mathcal{P}$ and $\mathcal{N}$. For every $E\in\mathfrak{R}$, the set $f(E)$ is separable with respect to the symmetric relative v-topology of $\mathcal{P}$, hence its continuous image $g(E)=\Psi\bigl(f(E)\bigr)$ is separable with respect to the symmetric relative u-topology of $\mathcal{N}$. Secondly, for an open subset O of $\mathcal{N}$ its inverse image $\Psi^{-1}(O)$ is open in $\mathcal{P}$, hence $g^{-1}(O)=f^{-1}\bigl(\Psi^{-1}(O)\bigr)\in\mathfrak{A}_{\mathfrak{R}}$, and the function g is seen to be measurable. For the additional statement in (c), suppose that $\Psi:\mathcal{P}\to\mathcal{N}$ is a continuous linear operator. Given $u\in\mathcal{U}$ there is $v\in\mathcal{V}$ such that $\Psi(a)\le\Psi(b)+u$ holds whenever $a\le b+v$ for

$a, b \in \mathcal{P}$. Thus $a \in v_\varepsilon(b)$, that is $a \le \gamma b + \varepsilon v$ with some $1 \le \gamma \le 1+\varepsilon$ implies $\Psi(a) \le \gamma \Psi(b) + \varepsilon$, hence $\Psi(a) \in u_\varepsilon\bigl(\Psi(b)\bigr)$. Likewise, $a \in v_\varepsilon^s(b)$ implies that $\Psi(a) \in u_\varepsilon^s\bigl(\Psi(b)\bigr)$. The function Ψ is therefore continuous with respect to the symmetric relative v- and u-topologies of $\mathcal{P}$ and $\mathcal{N}$. □

In the literature the terms *weak measurability* or *scalar measurability* are often used for a vector-valued function f if all the scalar-valued functions $\mu \circ f$ for linear functionals μ in the dual of the range of f are measurable in the usual sense. The following theorem states that measurability in our sense implies scalar measurability. The converse holds true for functions with bounded values and separable ranges.

Theorem 1.9. *Let $f \in \mathcal{F}(X, \mathcal{P})$.*

(a) If f is measurable, then the $\overline{\mathbb{R}}$-valued functions $\mu \circ f$ are measurable for all $\mu \in \mathcal{P}^*$.
(b) If the values of f are bounded, $f(E)$ is separable in the symmetric relative v-topology for all $E \in \mathfrak{R}$ and $v \in \mathcal{V}$, and the $\overline{\mathbb{R}}$-valued functions $\mu \circ f$ are measurable for all $\mu \in \mathcal{P}^*$, then f is measurable.

Proof. For (a), suppose that the function $f \in \mathcal{F}(X, \mathcal{P})$ is measurable, and let $\mu \in \mathcal{P}^*$, that is $\mu \in v^\circ$ for some $v \in \mathcal{V}$. Recall from Proposition I.4.5 and Example I.4.37(a) that a continuous linear functional $\mu : \mathcal{P} \to \overline{\mathbb{R}}$ is also continuous, if we endow $\mathcal{P}$ with the symmetric relative v-topology and $\overline{\mathbb{R}}$ with its given symmetric topology (which of course coincides with its symmetric relative topology). Following Theorem 1.8(c), this shows that the function $\mu \circ f : X \to \overline{\mathbb{R}}$ is measurable whenever the function f is measurable.

Now suppose that the assumptions of Part (b) hold for the function $f \in \mathcal{F}(X, \mathcal{P})$. We shall verify the criterion of Theorem 1.4 for measurability. Recall from Proposition I.4.2(iv) that on the subcone $\mathcal{B}$ of bounded elements of $\mathcal{P}$ the corresponding given and relative topologies coincide. As the values of f are supposed to be bounded, this will greatly facilitate our arguments.

In a first step, let us consider an element $a \in \mathcal{B}$ and neighborhood $v \in \mathcal{V}$. Then $b \notin v(a)$, that is $b \not\le a + v$, for an element $b \in \mathcal{B}$ implies that $b \not\le a + v/2$, as indeed otherwise, for $\lambda > 0$ such that $a \le \lambda v$ and $\varepsilon = \min\bigl\{1/9,\, 1/(3\lambda)\bigr\}$ there would be $1 \le \gamma \le 1+\varepsilon$ such that

$$b \le \gamma\,(a + v/2) + \varepsilon v = \gamma a + \left(\frac{\gamma}{2} + \varepsilon\right) v \le \gamma a + \frac{2}{3} v.$$

Because

$$\gamma a = a + (\gamma - 1)a \le a + (\gamma - 1)\lambda v \le a + \varepsilon \lambda v \le a + \frac{1}{3} v,$$

this yields $a \le b + v$, contradicting our assumption. Consequently, following Theorem 3.2 in [175] (see also Corollary 4.34 in Chapter I), there is a linear functional $\mu \in v^\circ$ such that

$$\mu(b) > \mu(a) + \frac{1}{2}.$$

Now, in a second step of our argument, consider an element $a \in \mathcal{B}$ and for $v \in \mathcal{V}$ the symmetric neighborhood

$$v^s(a) = v(a) \cap (a)v = \{c \in \mathcal{P} \mid c \leq a + v \quad \text{and} \quad a \leq c + v\}.$$

Given a set $E \in \mathfrak{R}$, let $\{b_i\}_{i\in\mathbb{N}}$ be a countable subset of $f(E) \setminus v^s(a) \subset \mathcal{B}$ that is dense with respect to the (given) symmetric topology. Such a subset exists because on $\mathcal{B}$ the given and the relative topologies of $\mathcal{P}$ coincide. For each $i \in \mathbb{N}$ we have either $b_i \notin v(a)$ or $b_i \notin (a)v$. Accordingly, we may choose linear functionals $\mu_i \in v^\circ$ corresponding to the elements b_i such that either

$$\mu_i(b) > \mu_i(a) + \frac{1}{2} \qquad \text{or} \qquad \mu_i(a) > \mu_i(b) + \frac{1}{2}$$

if $b_i \notin v(a)$ or $b_i \notin (a)v$, respectively. We denote

$$O_i = \left(-\infty\,,\ \mu_i(a) + \frac{1}{4}\right] \qquad \text{or} \qquad O_i = \left[\mu_i(a) - \frac{1}{4}\,,\ +\infty\right]$$

in these respective cases and set $A_i = \mu_i^{-1}(O_i) \subset \mathcal{P}$ and $A = \bigcap_{i\in\mathbb{N}} A_i$. For every $c \in (v/4)^s(a)$, that is $c \leq a + v/4$ and $a \leq c + v/4$ we have $\mu(c) \leq \mu(a) + 1/4$ and $\mu(a) \leq \mu(c) + 1/4$ for all $\mu \in v^\circ$, hence $c \in A_i$ for all $i \in \mathbb{N}$. This shows $v^s(a) \subset A$. We shall proceed to verify that $A \cap f(E) \subset (2v)^s(a)$. For this, consider any element $c \in f(E) \setminus (2v)^s(a)$. First we observe that $v^s(c) \cap v^s(a) = \emptyset$, because the existence of an element $d \in v^s(c) \cap v^s(a)$ would lead to $c \in (2v)^s(a)$, contradicting our choice of c. Thus $f(E) \cap (v/4)^s(c) \subset f(E) \setminus v^s(a)$ holds as well, and there is some $b_i \in (v/4)^s(c)$. We have

$$\mu_i(b_i) \leq \mu_i(c) + \frac{1}{4} \qquad \text{and} \qquad \mu_i(c) \leq \mu_i(b) + \frac{1}{4}$$

since $\mu_i \in v^\circ$. Recall that either $b_i \notin v(a)$ or $b_i \notin (a)v$. In the first case, this implies $\mu_i(b_i) > \mu_i(a) + 1/2$, hence $\mu_i(c) > \mu_i(a) + 1/4$, and $c \notin A_i$. In the second case, we have $\mu_i(b_i) < \mu_i(a) - 1/2$, hence $\mu_i(c) < \mu_i(a) - 1/4$ and, likewise $c \notin A_i$. Thus indeed $c \notin A$. Summarizing, we verified that

$$v^s(a) \subset A \qquad \text{and} \qquad A \cap f(E) \subset (2v)^s(a).$$

Let us apply this to an element $a = f(x)$ for some $x \in E$. By our assumption, all the $\overline{\mathbb{R}}$-valued functions $\varphi_i = \mu_i \circ f$ are measurable, hence the sets $F_i = \varphi_i^{-1}(O_i)$ are contained in $\mathfrak{A}_{\mathfrak{R}}$. Likewise,

$$F = \bigcap_{i\in\mathbb{N}} F_i = \bigcap_{i\in\mathbb{N}} f^{-1}\big(\mu_i^{-1}(O_i)\big) = \bigcap_{i\in\mathbb{N}} f^{-1}(A_i) = f^{-1}(A) \in \mathfrak{A}_{\mathfrak{R}}.$$

Thus

$$f^{-1}\big(v^s(a)\big) \subset F \qquad \text{and} \qquad F \cap E \subset f^{-1}\big((2v)^s(a)\big).$$

Now in the third and final step of our argument we shall verify the criterion of Theorem 1.4: For $E \in \mathfrak{R}$ and $v \in \mathcal{V}$ let $\{a_n\}_{n\in\mathbb{N}}$ be a subset of $f(E)$ that is dense with respect to the symmetric relative v-topology, hence with respect to the given symmetric v-topology. For each element a_n and the neighborhood $u = (\varepsilon/4)v \in \mathcal{V}$ in place of v choose the set $F = F_n \in \mathfrak{A}_{\mathfrak{R}}$ as in the last part of the preceding step and set $E_n = F_n \cap E$. Then

$$f^{-1}\big(u^s(a_n)\big) \cap E \subset E_n \qquad \text{and} \qquad E_n \subset f^{-1}\big((2u)^s(a)\big)$$

holds for all $n \in \mathbb{N}$ by the above. Thus, firstly, for $x, y \in E_n$ we have $f(x), f(y) \in (2u)^s(a_n)$. But this obviously implies that $f(x) \in (4u)\big(f(y)\big) \subset v_\varepsilon\big(f(y)\big)$. Secondly, for any $x \in E$ there is some a_n such that $f(x) \in u^s(a_n)$, that is $x \in f^{-1}\big(u^s(a_n)\big) \cap E \subset E_n$. This demonstrates $\bigcup_{n\in\mathbb{N}} E_n = E$ and completes our argument. $\square$

2. Inductive Limit Neighborhoods for Cone-Valued Functions

Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone. In preparation of our integration theory for cone-valued functions with respect to an operator-valued measure, we shall introduce appropriate neighborhoods for the cone $\mathcal{F}(X, \mathcal{P})$ and corresponding subcones of measurable functions. Our integrals will constitute continuous linear operators on these cones. First, in order to allow greater generally, we shall extend the given neighborhood system of $\mathcal{V}$.

2.1 Infinity as a Neighborhood. We shall adjoin the maximal element ∞ to the neighborhood system $\mathcal{V}$ such that $a \le b + \infty$ holds for all $a, b \in \mathcal{P}$. The addition and multiplication by scalars involving this element is defined in a canonical way: We set $v + \infty = \infty$, $0 \cdot \infty = 0$ and $\alpha \cdot \infty = \infty$ for all $v \in \mathcal{V}$ and $\alpha > 0$. The augmented neighborhood system which includes this infinite element and $0 \in \mathcal{P}$ will be denoted by $\overline{\mathcal{V}}$, that is $\overline{\mathcal{V}} = \mathcal{V} \cup \{0, \infty\}$. Obviously, $(\overline{\mathcal{V}}, \mathcal{V})$ is a full locally convex cone.

2.2 Inductive Limit Neighborhoods. Let X and $\mathfrak{R}$ be as before, and let $\mathcal{F}\big(X, \overline{\mathcal{V}}\big)$ be the family of $\overline{\mathcal{V}}$-valued functions on X, endowed with the pointwise operations and order. For functions $f, g \in \mathcal{F}(X, \mathcal{P})$ and $s \in \mathcal{F}\big(X, \overline{\mathcal{V}}\big)$ we write $f \le g + s$ if $f(x) \le g(x) + s(x)$ for all $x \in X$. The addition and multiplication by scalars for functions $s, t \in \mathcal{F}\big(X, \overline{\mathcal{V}}\big)$ is defined pointwise, and $s \le t$ means that $f \le g + s$ implies $f \le g + t$ for $f, g \in \mathcal{F}(X, \mathcal{P})$.

An ($\mathfrak{R}$-*compatible) inductive limit neighborhood* for $\mathcal{F}(X,\mathcal{P})$ is a convex subset $\mathfrak{v}$ of measurable functions in $\mathcal{F}\big(X,\overline{\mathcal{V}}\big)$ such that for every $E \in \mathfrak{R}$ there is $v_E \in \mathcal{V}$ and $s \in \mathfrak{v}$ such that $\chi_{E}{}_{\otimes} v_E \le s$. Measurability is meant with respect to $\mathfrak{R}$ and the locally convex cone $(\overline{\mathcal{V}},\mathcal{V})$. For functions $f,g \in \mathcal{F}(X,\mathcal{P})$ and an inductive limit neighborhood $\mathfrak{v}$ we denote

$$f \le g + \mathfrak{v} \qquad \text{if} \qquad f \le g + s, \quad \text{for some} \quad s \in \mathfrak{v}.$$

We define sums and multiples by positive scalars for inductive limit neighborhoods through the addition and multiplication of their elements. A canonical order relation is given by

$$\mathfrak{v} \le \mathfrak{u} \quad \text{if for every } s \in \mathfrak{v} \text{ there is } t \in \mathfrak{u} \text{ such that } s \le t.$$

Inductive limit neighborhoods include uniform neighborhoods, consisting of a single constant function $x \mapsto v$; and if $X \in \mathfrak{R}$, that is if $\mathfrak{R}$ is a σ-algebra, then the uniform neighborhoods form a base for the family of all inductive limit neighborhoods.

2.3 The Cone $\boldsymbol{\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})}$. We shall in the sequel deal with measurable functions in $\mathcal{F}(X,\mathcal{P})$ that can be reached from below by step functions; more precisely: We denote by $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ the subcone of all measurable functions $f \in \mathcal{F}(X,\mathcal{P})$ such that for every inductive limit neighborhood $\mathfrak{v}$ there is $h \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P})$ satisfying $h \le f + \mathfrak{v}$.

Lemma 2.4. *Let* $f \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$.

(a) For every inductive limit neighborhood $\mathfrak{v}$ *there is* $\lambda \ge 0$ *such thats* $0 \le f + \lambda\mathfrak{v}$.
(b) There is $E \in \mathfrak{R}$ *such that* $f(x) \ge 0$ *for all* $x \in X \setminus E$, *and for every* $v \in \mathcal{V}$ *there is* $\lambda \ge 0$ *such that* $0 \le f + \lambda\, \chi_{E^{\otimes}} v$.

Proof. Let $f \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$. For (a), given an inductive limit neighborhood $\mathfrak{v}$, there is a step function $h = \sum_{i=1}^{n} \chi_{E_i^{\otimes}} a_i \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P})$ such that $h \le f + \mathfrak{v}$, that is $h \le f + s$ for some $s \in \mathfrak{v}$. We may assume that the sets $E_i \in \mathfrak{R}$ are disjoint and $E = \bigcup_{i=1}^{n} E_i \in \mathfrak{R}$. There is $v \in \mathcal{V}$ such that $\chi_{E^{\otimes}} v \le \mathfrak{v}$, and in turn there is $\lambda \ge 0$ such that $0 \le a_i + \lambda v$ for all $i = 1,\ldots,n$. This shows $0 \le f(x) + s(x) + \lambda v$ for all $x \in E$ and $0 \le f(x) + s(x)$ for all $x \in X \setminus E$. Thus $0 \le f + (s + \lambda\, \chi_{E^{\otimes}} v)$, hence indeed $0 \le f + (1+\lambda)\mathfrak{v}$.

For (b), let the inductive limit neighborhood $\mathfrak{v}$ consist of all $\mathcal{V}$-valued functions that are supported by some set in $\mathfrak{R}$. By (a) there is $\lambda \ge 0$ and a function $s \in \mathfrak{v}$ such that $0 \le f + \lambda s$. Because s is supported by some set $E \in \mathfrak{R}$, that is $s(x) = 0$ for all $x \in X \setminus E$, we have indeed $f(x) \ge 0$ for all $x \in X \setminus E$. Now let $v \in \mathcal{V}$ and let $\mathfrak{v}$ consist of the single function $x \to v$. Then $0 \le f + \lambda\mathfrak{v}$ with some $\lambda \ge 0$ by (a). Thus $0 \le f + \lambda\, \chi_{E^{\otimes}} v$, as claimed. $\square$

The following lemma provides a more straightforward characterization of the functions in $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$, avoiding the use of inductive limit neighborhoods.

Lemma 2.5. *A measurable $\mathcal{P}$-valued function f is in $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ if and only if*

(i) there is $E \in \mathfrak{R}$ such that $f(x) \geq 0$ for all $x \in X \setminus E$, and
(ii) for every $v \in \mathcal{V}$ there is $h \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P})$ such that $h \leq f + \chi_{X \otimes} v$.

Proof. If $f \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$, then (i) follows from Lemma 2.4(b). Statement (ii) follows from the definition of the cone $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ if we consider the singleton inductive limit neighborhood $\mathfrak{v} = \{\chi_{X \otimes} v\}$. For the converse, suppose that (i) and (ii) hold for the measurable function $f \in \mathcal{F}(X,\mathcal{P})$, and let $\mathfrak{v}$ be an inductive limit neighborhood. For the set $E \in \mathfrak{R}$ from (i) there is $v \in \mathcal{V}$ such that $\chi_{E \otimes} v \leq s$ for some $s \in \mathfrak{v}$. According to (ii), let $h \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P})$ such that $h \leq f + \chi_{X \otimes} v$ and set $h' = \chi_{E \otimes} h \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P})$. Then $h' \leq f + s$, hence $h' \leq f + \mathfrak{v}$. This shows $f \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$. $\square$

Note that a function $\chi_{F \otimes} a$ for $F \in \mathfrak{A}_{\mathfrak{R}}$ and $a \in \mathcal{P}$ is contained in $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ if and only if either $a \geq 0$ or $F \in \mathfrak{R}$.

Lemma 2.6. *Let $f \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ and let φ be a positive real-valued function on X, measurable with respect to $\mathfrak{A}_{\mathfrak{R}}$. If either f is positive or if φ is bounded, then $\varphi_{\otimes} f \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$.*

Proof. Following Theorem 1.8(a), the function $\varphi_{\otimes} f$ is measurable. If f is positive, then $\varphi_{\otimes} f$ is also positive, hence in $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$. Otherwise, there is $\rho > 0$ such that $0 \leq \varphi(x) \leq \rho$ for all $x \in X$. Given an inductive limit neighborhood $\mathfrak{v}$ there is $h \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P})$ such that $h \leq f + (1/2\rho)\mathfrak{v}$. Also, there is $\lambda > 0$ such that $0 \leq f + \lambda\mathfrak{v}$. As φ is bounded and measurable, there is a real-valued positive step function ψ on X such that

$$\psi(x) \leq \varphi(x) \leq \psi(x) + \frac{1}{2\lambda}$$

for all $x \in X$. Then $l = \psi_{\otimes} h \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P})$, and indeed

$$l \leq \psi_{\otimes} f + \frac{1}{2\rho}\psi_{\otimes} v \leq \psi_{\otimes} f + \frac{1}{2}v + (\varphi - \psi)_{\otimes}(f + \lambda\mathfrak{v}) \leq \varphi_{\otimes} f + \mathfrak{v}.$$

$\square$

Lemma 2.6 implies in particular that $\chi_{F \otimes} f \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ whenever $f \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ and $F \in \mathfrak{A}_R$. Also, if φ is a positive real-valued measurable function and $a \in \mathcal{P}$, then $\varphi_{\otimes} a \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ if $a \geq 0$, and $(\chi_E \varphi)_{\otimes} a = \varphi_{\otimes}(\chi_{E \otimes} a) \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ for every $E \in \mathfrak{R}$ in general if the function φ is bounded.

A sequence $(f_n)_{n \in \mathbb{N}}$ in $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ is said to be *bounded below* if for every inductive limit neighborhood $\mathfrak{v}$ there is $\lambda \geq 0$ such that $0 \leq f_n + \lambda\mathfrak{v}$ for all $n \in \mathbb{N}$.

Theorem 2.7. *Let $f \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ and $E \in \mathfrak{R}$. For every inductive limit neighborhood $\mathfrak{v}$, every $v \in \mathcal{V}$ and $\varepsilon > 0$ there is $1 \le \gamma \le 1+\varepsilon$ and a bounded below sequence $(h_n)_{n\in\mathbb{N}}$ of step functions in $\mathcal{S}_{\mathfrak{R}}(X,\mathcal{P})$ such that*

(i) $h_n \le \gamma f + \mathfrak{v}$ for all $n \in \mathbb{N}$.
(ii) For every $x \in E$ there is $n_0 \in \mathbb{N}$ such that $f(x) \le h_n(x) + v$ for all $n \ge n_0$.

Proof. Let $f \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$, $E \in \mathfrak{R}$, let $\mathfrak{v}$ be an inductive limit neighborhood, let $v \in \mathcal{V}$ and $\varepsilon > 0$. Following Lemma 2.4(b) we may assume that $f(x) \ge 0$ for all $x \in X \setminus E$. There is $u \in \mathcal{V}$ such that both $u \le v$ and $\chi_{E^{\otimes}} u \le \mathfrak{v}$. Again using 2.4(b) we find $\lambda \ge 0$ such that $0 \le f + \lambda \chi_{E^{\otimes}} u$ We set $\delta = \min\{1, \frac{\varepsilon}{3}, \frac{1}{4(1+\lambda)}\}$ and $\gamma = (1+\delta)^2 \le 1+\varepsilon$. By Theorem 1.4 there is a partition of E into disjoint subsets $E_i \in \mathfrak{R}$, $i \in \mathbb{N}$, such that $f(x) \in u_\delta\big(f(y)\big)$ holds for all $x, y \in E_i$. Thus

$$f(x) \le (1+\delta) f(y) + \delta(1+\lambda) u$$

by Lemma I.4.1(b). We set $a_i = (1+\delta) f(x_i) \in \mathcal{P}$ for some $x_i \in E_i$. Thus for any $x \in E_i$ we have

$$f(x) \le a_i + \delta(1+\lambda) u \le a_i + v$$

and

$$a_i \le (1+\delta)^2 f(x) + \delta(1+\delta)(1+\lambda) u \le \gamma f(x) + \frac{1}{2} u.$$

Thus

$$\sum_{i=1}^{n} \chi_{E_i^{\otimes}} a_i \le \chi_{E^{\otimes}}(\gamma f) + \frac{1}{2}\chi_{E^{\otimes}} u \le \chi_{E^{\otimes}}(\gamma f) + \frac{1}{2}\mathfrak{v}.$$

Furthermore, there is $h_0 \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P})$ such that $h_0 \le \gamma f + \frac{1}{2}\mathfrak{v}$, and therefore

$$\chi_{(X\setminus E)^{\otimes}} h_o \le \chi_{(X\setminus E)^{\otimes}}(\gamma f) + \frac{1}{2}\mathfrak{v}$$

holds as well. Now we choose the step functions

$$h_n = \sum_{i=1}^{n} \chi_{E_i^{\otimes}} a_i \; + \; \chi_{(X\setminus E)^{\otimes}} h_0.$$

Adding the above yields indeed

$$h_n \le \gamma f + \mathfrak{v}$$

for all $n \in \mathbb{N}$, hence (i). Part (ii) of our claim follows directly from the above, as

$$f(x) \le h_n(x) + v \qquad \text{for all} \qquad x \in \bigcup_{i=1}^{n} E_i.$$

Finally, given an inductive limit neighborhood $\mathfrak{u}$, there is $\lambda \geq 0$ such that $0 \leq h_0 + \lambda\mathfrak{u}$, that is $0 \leq h_0 + \lambda s$ for some $s \leq \mathfrak{u}$. Also there is $u \in \mathcal{V}$ such that $\chi_{E^{\otimes}}u \in \mathfrak{u}$ and $\rho \geq 0$ such that $0 \leq f + \rho\chi_{E^{\otimes}}u$. The latter implies $0 \leq a_i + \rho u$ for all $i \in \mathbb{N}$, hence

$$0 \leq h_n + \lambda s + \rho\chi_{E^{\otimes}}u \leq h_n + (\lambda + \rho)\mathfrak{u}$$

for all $n \in \mathbb{N}$. The sequence $(h_n)_{n\in\mathbb{N}}$ is therefore bounded below. Finally, if $f \geq 0$, then we may choose $h_0 = 0$, and as all the elements $a_i = (1+\delta)f(x_i)$ are also positive, we realize that $h_n \geq 0$ for all $n \in \mathbb{N}$. $\square$

If $(\mathcal{P}, \mathcal{V})$ is indeed a full locally convex cone, as will frequently occur in the subsequent sections, then the preceding result can obviously be simplified. We shall formulate this in a corollary.

Corollary 2.8. *Let $(\mathcal{P}, \mathcal{V})$ be a full locally convex cone. Let $f \in \mathcal{F}_{\mathfrak{R}}(X, \mathcal{P})$ and $E \in \mathfrak{R}$. For every inductive limit neighborhood $\mathfrak{v}$ and $\varepsilon > 0$ there is $1 \leq \gamma \leq 1 + \varepsilon$ and a bounded below sequence $(h_n)_{n\in\mathbb{N}}$ of step functions in $\mathcal{S}_{\mathfrak{R}}(X, \mathcal{P})$ such that*

(i) $h_n \leq \gamma f + \mathfrak{v}$ for all $n \in \mathbb{N}$.
(ii) For every $x \in E$ there is $n_0 \in \mathbb{N}$ such that $f(x) \leq h_n(x)$ for all $n \geq n_0$.

Proof. We choose $v \in \mathcal{V}$ such that $\chi_{E^{\otimes}}v \leq (1/2)\mathfrak{v}_w$ and apply Theorem 2.7 with this v, the inductive limit neighborhood $(1/2)\mathfrak{v}_w$ and the given $\varepsilon > 0$. There is a sequence $(h_n)_{n\in\mathbb{N}}$ as in 2.7. The functions $h'_n = h_n + \chi_{E^{\otimes}}v$ then satisfy our claim. $\square$

3. Operator-Valued Measures

Let $(\mathcal{P}, \mathcal{V})$ be a quasi-full locally convex cone and let $(\mathcal{Q}, \mathcal{W})$ be a locally convex complete lattice cone (see Sections 5 and 6 in Chapter I). Let $\mathfrak{L}(\mathcal{P}, \mathcal{Q})$ denote the cone of all (uniformly) continuous linear operators from $\mathcal{P}$ to $\mathcal{Q}$. Recall from Section 3 in Chapter I that a continuous linear operator between locally convex cones is monotone with respect to the respective weak preorders. Because $\mathcal{Q}$ carries its weak preorder, this implies monotonicity with respect to the given orders of $\mathcal{P}$ and $\mathcal{Q}$ as well. Let X be a set, $\mathfrak{R}$ a (weak) σ-ring of subsets of X. An $\mathfrak{L}(\mathcal{P}, \mathcal{Q})$-*valued measure* θ on $\mathfrak{R}$ is a set function

$$E \mapsto \theta_E : \mathfrak{R} \to \mathfrak{L}(\mathcal{P}, \mathcal{Q})$$

such that $\theta(\emptyset) = 0$ and

$$\theta_{(\bigcup_{i\in\mathbb{N}} E_i)} = \sum_{i=1}^{\infty} \theta_{E_i}$$

holds whenever the sets $E_i \in \mathfrak{R}$ are disjoint and $\bigcup_{i=1}^{\infty} E_i \in \mathfrak{R}$. Convergence for the series on the right-hand side is meant in the following way: For every $a \in \mathcal{P}$ the series $\sum_{i=1}^{\infty} \theta_{E_i}(a)$ is order convergent in $\mathcal{Q}$ in the sense of I.5.26. (Recall from Proposition I.5.42 that order convergence is implied by convergence in the symmetric relative topology.)

Lemma 3.1. *Let $E \in \mathfrak{R}$ and $a \in \mathcal{P}$.*

(a) If $E_i \in \mathfrak{R}$ are such that $E_i \subset E_{i+1}$ for all $i \in \mathbb{N}$ and $E = \cup_{i=1}^{\infty} E_i$, then $\theta_E(a) = \lim_{i\to\infty} \theta_{E_i}(a)$.

(b) If $E_i \in \mathfrak{R}$ are such that $E \supset E_i \supset E_{i+1}$ for all $i \in \mathbb{N}$, and $\cap_{i=1}^{\infty} E_i = \emptyset$, then $0 \leq \varliminf_{i\to\infty} \theta_{E_i}(a) + \mathfrak{O}\big(\theta_E(a)\big)$ and $\varlimsup_{i\to\infty} \theta_{E_i}(a) \leq \mathfrak{O}\big(\theta_E(a)\big)$.

Proof. For Part (a), let $F_1 = E_1$ and $F_i = E_i \setminus F_{i-1}$ for $i > 1$. The sets F_i are disjoint, $E_n = \cup_{i=1}^{n} F_i$ and $E = \cup_{i=1}^{\infty} F_i$. From the countable additivity of the measure θ we infer that $\theta_{E_n}(a) = \sum_{i=1}^{n} \theta_{F_i}(a)$ and $\theta_E(a) = \sum_{i=1}^{\infty} \theta_{F_i}(a)$, hence our claim. For Part (b), let $F_i = E \setminus E_i$ for $i \in \mathbb{N}$. Thus $F_i \subset F_{i+1}$ and $\cup_{i=1}^{\infty} F_i = E$. This shows $\theta_E(a) = \lim_{i\to\infty} \theta_{E_i}(a)$ by Part (a). Furthermore, $\theta_E(a) = \theta_{E_i}(a) + \theta_{F_i}(a)$ holds for all $i \in \mathbb{N}$ by Part (a). Using the limit rules in Lemma I.5.19 we infer that

$$\theta_E(a) \leq \varliminf_{i\to\infty} \theta_{E_i}(a) + \theta_E(a) \leq \varlimsup_{i\to\infty} \theta_{E_i}(a) + \theta_E(a) \leq \theta_E(a),$$

hence equality for these terms as $\mathcal{Q}$ carries the weak preorder which is supposed to be antisymmetric. Now the cancellation rule in Proposition I.5.10(a) yields our claim. □

For our upcoming integration theory for $\mathcal{P}$-valued functions with respect to an $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measure θ (see Section 4 below) we shall also have to assign values of θ to the neighborhoods in $\mathcal{P}$. This will be done by the introduction of its modulus $|\theta|$. Recall that we require the locally convex cone $(\mathcal{P},\mathcal{V})$ to be quasi-full.

3.2 The Modulus of a Measure. Throughout the following, let θ be a fixed $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measure on $\mathfrak{R}$. For a neighborhood $v \in \mathcal{V}$ and a set $E \in \mathfrak{R}$, *modulus (or semivariation) of θ* is defined as

$$\begin{aligned} &|\theta|(E,v) \\ &\quad = \sup\left\{ \sum_{i=1}^{n} \theta_{E_i}(s_i) \;\middle|\; s_i \in \mathcal{P},\ s_i \leq v,\ E_i \in \mathfrak{R} \text{ disjoint subsets of } E \right\}. \end{aligned}$$

The following is obvious from this definition.

Lemma 3.3. *Let $v \in \mathcal{V}$ and $E \in \mathfrak{R}$. If $v \in \mathcal{P}$, then $|\theta|(E,v) = \theta_E(v)$.*

Proof. Let $E \in \mathfrak{R}$ and $v \in \mathcal{V} \cap \mathcal{P}$. If $s_i \in \mathcal{P}$ such that $s_i \leq v$ and $E_i \in \mathfrak{R}$ are disjoint subsets of E for $i = 1, \ldots, n$, then

$$\sum_{i=1}^{n} \theta_{E_i}(s_i) \le \sum_{i=1}^{n} \theta_{E_i}(v) = \theta_{(\cup_{i=1}^{n} E_i)}(v) \le \theta_E(v).$$

Thus $|\theta|(E,v) \le \theta_E(v)$. The reverse inequality is obvious, as we may choose $E_1 = E$ and $s_1 = v$ in 3.2. □

Lemma 3.4. *Let $v \in \mathcal{V}$ and $E \in \mathfrak{R}$. Then*

(a) $0 \le |\theta|(E,v)$ *and* $|\theta|(\emptyset, v) = 0$.
(b) $\theta_E(a) \le \theta_E(b) + |\theta|(E,v)$ *whenever* $a \le b + v$ *for* $a, b \in \mathcal{P}$.
(c) If $E_i \in \mathfrak{R}$ *are disjoint sets such that* $E = \bigcup_{i=1}^{\infty} E_i$,
then $|\theta|(E,v) = \sum_{i=1}^{\infty} |\theta|(E_i, v)$.

Proof. Part (a) is obvious. Part (b) follows as the locally convex cone is supposed to be quasi-full. Indeed, for $a \le b + v$ there is $s \le v$ such that $a \le b + s$. This implies $\theta_E(a) \le \theta_E(b) + \theta_E(s)$, and as $\theta_E(s) \le |\theta|(E,v)$, our claim follows immediately from $\bigwedge 1$. For Part (c), let $E = \cup_{i=1}^{\infty} E_i$ for disjoint sets $E_i \in \mathfrak{R}$. Let $F_1, \ldots, F_n \in \mathfrak{R}$ be disjoint subsets of E and $s_k \in \mathcal{P}$ such that $s_k \le v$ for $k = 1, \ldots, n$. Then

$$\theta_{F_k}(s_k) = \sum_{i=1}^{\infty} \theta_{(F_k \cap E_i)}(s_k)$$

for every $k = 1, \ldots, n$ by the countable additivity of θ, hence

$$\begin{aligned} \sum_{k=1}^{n} \theta_{F_k}(s_k) &= \sum_{k=1}^{n} \left(\sum_{i=1}^{\infty} \theta_{(F_k \cap E_i)}(s_k) \right) \\ &= \sum_{i=1}^{\infty} \left(\sum_{k=1}^{n} \theta_{(F_k \cap E_i)}(s_k) \right) \\ &\le \sum_{i=1}^{\infty} |\theta|(E_i, v) \end{aligned}$$

by the limit rules established in Section 5 of Chapter I. For the converse inequality, let $n \in \mathbb{N}$ and for each $i = 1, \ldots, n$, let $F_1^i, \ldots, F_{n_i}^i \in \mathfrak{R}$ be disjoint subsets of E_i and $s_1^i, \ldots, s_{n_i}^i \le v$. Then

$$\sum_{i=1}^{n} \left(\sum_{k=1}^{n_i} \theta_{F_k^i}(s_k^i) \right) \le |\theta|(E,v),$$

as the sets $F_k^i \subset E$ are pairwise disjoint. Now taking the supremum over all such choices of sets F_i^k yields with $(\bigvee 1)$

$$\sum_{i=1}^{n} |\theta|(E_i, v) \le |\theta|(E,v), \qquad \text{hence} \qquad \sum_{i=1}^{\infty} |\theta|(E_i, v) \le |\theta|(E,v),$$

as $n \in \mathbb{N}$ was arbitrary. □

Lemma 3.5. *Let* $E \in \mathfrak{R}$, $\alpha > 0$ *and* $u, v \in \mathcal{V}$. *Then*

(a) $|\theta|(E, \alpha v) = \alpha |\theta|(E, v)$.
(b) $|\theta|(E, u + v) = |\theta|(E, u) + |\theta|(E, v)$.

Proof. Part (a) is obvious. For Part (b), let $E_1, \ldots, E_n \in \mathfrak{R}$ be disjoint subsets of E and let $r_i \in \mathcal{P}$ such that $r_i \leq u+v$ for $i = 1, \ldots, n$. According to (QF2) in I.6.1 there are elements $s_i, t_i \in \mathcal{P}$ such that $s_i \leq u$, $t_i \leq v$ and $s_i \leq r_i + t_i$. This shows

$$\sum_{i=1}^{n} \theta_{E_i}(r_i) \leq \sum_{i=1}^{n} \theta_{E_i}(s_i) + \sum_{i=1}^{n} \theta_{E_i}(t_i) \leq |\theta|(E, u) + |\theta|(E, v).$$

As the sets $E_i \in \mathfrak{R}$ and the elements $r_i \leq u+v$ were chosen arbitrarily, this shows $|\theta|(E, u+v) \leq |\theta|(E, u) + |\theta|(E, v)$. For the converse inequality, let $E_1, \ldots, E_n \in \mathfrak{R}$ and $F_1, \ldots, F_m \in \mathfrak{R}$ be two collections of disjoint subsets of E. We may assume that $\bigcup_{i=1}^{n} E_i = \bigcup_{k=1}^{m} F_k = E$. Let $s_i \leq u$ and $t_k \leq v$ for $s_i, t_k \in \mathcal{P}$. Then

$$\sum_{i=1}^{n} \theta_{E_i}(s_i) + \sum_{k=1}^{m} \theta_{F_k}(t_k) = \sum_{i=1}^{n} \sum_{k=1}^{m} \theta_{(E_i \cap F_k)}(s_i + t_k) \leq |\theta|(E, u + v)$$

by the above. Taking first the supremum over all choices for the sets $E_i \in \mathfrak{R}$ and the elements $s_i \leq u$ on the left-hand side of this inequality and using $(\bigvee 1)$ yields

$$|\theta|(E, u) + \sum_{k=1}^{m} \theta_{F_k}(t_k) \leq |\theta|(E, v + u).$$

In a second step, we obtain $|\theta|(E, u) + |\theta|(E, v) \leq |\theta|(E, u + v)$ if we repeat this argument for the sets $F_k \in \mathfrak{R}$ and the elements $t_k \leq v$. $\square$

3.6 Bounded Measures. Let $(\mathcal{P}, \mathcal{V})$ be a quasi-full locally convex cone and let $(\mathcal{Q}, \mathcal{W})$ be a locally convex complete lattice cone. We shall say that an $\mathfrak{L}(\mathcal{P}, \mathcal{Q})$-valued measure θ on $\mathfrak{R}$ is $\mathfrak{R}$-*bounded* or *of bounded semivariation on* $\mathfrak{R}$ if

(BV) *For every* $w \in \mathcal{W}$ *and* $E \in \mathfrak{R}$ *there is* $v \in \mathcal{V}$ *such that* $|\theta|(E, v) \leq w$.

In the sequel we shall always assume boundedness in this sense.

Remarks 3.7. (a) If $(\mathcal{P}, \mathcal{V})$ is a full locally convex cone, then every $\mathfrak{L}(\mathcal{P}, \mathcal{Q})$-valued measure on $\mathfrak{R}$ is bounded. Indeed, let $E \in \mathfrak{R}$ and $w \in \mathcal{W}$. Because the operator $\theta_E : \mathcal{P} \to \mathcal{Q}$ is supposed to be continuous, there is $v \in \mathcal{V}$ such that $\theta_E(a) \leq \theta_E(b) + w$ whenever $a \leq b + v$ for $a, b \in \mathcal{P}$. Following Lemma 3.3, this shows $|\theta|(E, v) = \theta_E(v) \leq w$ in particular.

(b) Let $\mathcal{P} = \mathbb{K}$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, endowed with the equality as order and the usual Euclidean topology; that is $\mathcal{V} = \{\varepsilon\mathbb{B} \mid \varepsilon > 0\}$, where $\mathbb{B}$ is the unit ball in $\mathbb{K}$ and $a \le b + \varepsilon\mathbb{B}$ means that $a \in b + \varepsilon\mathbb{B}$. Let $\mathcal{Q} = \overline{\mathbb{R}}$. Then $\mathfrak{L}(\mathcal{P}, \mathcal{Q})$ can be identified with $\mathbb{K}$, since every linear operator (functional) from $\mathbb{K}$ to $\overline{\mathbb{R}}$ is given by an element $z \in \mathbb{K}$ via the evaluation $a \mapsto \mathfrak{Re}(za)$ for $a \in \mathbb{K}$. This is therefore the case of a real- or complex-valued measure θ. According to 3.2, its modulus is computed as

$$\begin{aligned} |\theta|(E, \mathbb{B}) &= \sup\left\{ \sum_{i=1}^{n} \theta_{E_i} \cdot s_i \;\middle|\; s_i \in \mathbb{B},\ E_i \in \mathfrak{R} \text{ disjoint subsets of } E \right\} \\ &= \sup\left\{ \sum_{i=1}^{n} |\theta_{E_i}| \;\middle|\; E_i \in \mathfrak{R} \text{ disjoint subsets of } E \right\}, \end{aligned}$$

that is the usual *total variation* of the real- or complex-valued measure θ (see II.1.4 in [55]).

(c) If $(\mathcal{P}, \mathcal{V})$ is a locally convex topological vector space, and $\mathcal{Q} = \overline{\mathbb{R}}$, that is the case of a functional-valued measure, our requirement of boundedness corresponds to Dieudonné's notion of *p-domination* in [44] and to Prolla's of *finite p-semivariation* in [155] (Ch. 5.5) for measures with values in the dual of a locally convex vector space.

(d) If $(\mathcal{N}, \|\ \|)$ is a normed space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, then every $\mathcal{N}$-valued measure θ may be considered to be an operator-valued measure in our sense. Indeed, the elements of $\mathcal{N}$ are linear operators from $\mathcal{P} = \mathbb{K}$, endowed with the Euclidean topology, into the standard lattice completion $(\widehat{\mathcal{N}}, \widehat{\mathcal{W}})$ of $\mathcal{N}$ as introduced in I.5.57. The notion of the semivariation of a vector-valued measure as given for example in IV.10.3 in [55] slightly differs from our notion of the modulus, as there it is a real-valued expression (in fact, it is in some sense the norm in $\mathcal{N}$ of our modulus; see Section 8 below), which is however not countably additive in general. We shall consider this example in more detail in Section 6 below.

(e) If $(\mathcal{P}, \mathcal{V})$ is a quasi-full locally convex cone and if we endow the subcone $\mathcal{P}_+$ of its positive elements with the neighborhood system $\widetilde{\mathcal{V}} = \{0\}$, $\big($for this, see also Example I.1.4(b)$\big)$, then every $\mathfrak{L}(\mathcal{P}, \mathcal{Q})$-valued measure θ can be canonically extended to an $\mathfrak{L}(\mathcal{P}_+, \mathcal{Q})$-valued measure on the whole σ-algebra $\mathfrak{A}_{\mathfrak{R}}$: For every set $F \in \mathfrak{A}_{\mathfrak{R}}$ we define the operator $\theta_F \in \mathfrak{L}(\mathcal{P}_+, \mathcal{Q})$ by

$$\theta_F(a) = \sup\{\theta_E(a) \mid E \subset F,\ E \in \mathfrak{R}\} \in \mathcal{Q}$$

for $a \in \mathcal{P}_+$. Linearity of this operator follows from I.5.22, and continuity is trivial, since $\mathcal{P}_+$ is endowed with the neighborhood system $\widetilde{\mathcal{V}} = \{0\}$. For countable additivity on $\mathfrak{A}_{\mathfrak{R}}$ let $F_n \in \mathfrak{A}_{\mathfrak{R}}$, for $n \in \mathbb{N}$, be disjoint sets and let $F = \bigcup_{n\in\mathbb{N}} F_n$. Let $a \in \mathcal{P}_+$. Then

$$\theta_F(a) = \sup\{\theta_E(a) \mid E \subset F,\ E \in \mathfrak{R}\}$$
$$= \sup\left\{\sum_{i=1}^{\infty} \theta_{E\cap F_i}(a) \mid E \subset F,\ E \in \mathfrak{R}\right\} \le \sum_{i=1}^{\infty} \theta_{F_i}(a).$$

For every $n \in \mathbb{N}$, on the other hand, and $E_i \in \mathfrak{R}$ such that $E_i \subset F_i$ for $i = 1, \ldots, n$, we set $E = \bigcup_{i=1}^{n} E_i \in \mathfrak{R}$ and have $\sum_{i=1}^{n} \theta_{E_i}(a) = \theta_E(a) \le \theta_F(a)$. Taking the supremum over all such choices of sets $E_i \in \mathfrak{R}$ yields with Lemma I.5.5(a) that $\sum_{i=1}^{n} \theta_{F_i}(a) \le \theta_F(a)$. This holds for all $n \in \mathbb{N}$ and therefore yields the reverse inequality $\sum_{i=1}^{\infty} \theta_{F_i}(a) \le \theta_F(a)$.

3.8 Extension of a Measure. We may use the modulus of an $\mathfrak{R}$-bounded $\mathfrak{L}(\mathcal{P}, \mathcal{Q})$-valued measure θ to define an extension to an $\mathfrak{R}$-bounded $\mathfrak{L}(\mathcal{P}_\mathcal{V}, \mathcal{Q})$-valued measure, where $(\mathcal{P}_\mathcal{V}, \mathcal{V})$ denotes the standard full extension of the quasi-full locally convex cone $(\mathcal{P}, \mathcal{V})$ as constructed in Section 6.2 of Chapter I, that is

$$\mathcal{P}_\mathcal{V} = \big\{a \oplus v \mid a \in \mathcal{P},\ v \in \mathcal{V} \cup \{0\}\big\}.$$

This follows the extension of a continuous linear operator from $\mathcal{P}$ to $\mathcal{Q}$ into a continuous linear operator from $\mathcal{P}_\mathcal{V}$ to $\mathcal{Q}$ as elaborated in Theorem I.6.3. For $E \in \mathfrak{R}$ and $a \oplus v \in \mathcal{P}_\mathcal{V}$ we set

$$\theta_E(a \oplus v) = \theta_E(a) + |\theta|(E, v).$$

The required properties for a measure are readily checked. Indeed, for a fixed set $E \in \mathfrak{R}$, Lemma 3.5 shows that θ_E is a linear operator on $\mathcal{P}_\mathcal{V}$. In order to verify that this operator is monotone, let $a \oplus v \le b \oplus u$ for $a \oplus v, b \oplus u \in \mathcal{P}_\mathcal{V}$. Let $E_1, \ldots, E_n \in \mathfrak{R}$ be disjoint subsets of E and $s_1, \ldots, s_n \le v$. We set $E_0 = E \setminus \bigcup_{i=1}^{n} E_i$ and $s_0 = 0$. Then $\bigcup_{i=0}^{n} E_i = E$ and $a + s_i \le b + u$ for all $i = 0, \ldots, n$ by our definition of the order in $\mathcal{P}_\mathcal{V}$, hence $a + s_i \le b + t_i$ for some $t_i \le u$ by Condition (QF1) from I.6.1. Thus $\theta_{E_i}(a + s_i) \le \theta_{E_i}(b + t_i)$ for all $i = 0, \ldots, n$ by the monotonicity of the operators θ_{E_i}, hence

$$\begin{aligned} \theta_E(a) + \sum_{i=1}^{n} \theta_{E_i}(s_i) &= \sum_{i=0}^{n} \theta_{E_i}(a + s_i) \\ &\le \sum_{i=0}^{n} \theta_{E_i}(b + t_i) \\ &= \theta_E(b) + \sum_{i=1}^{n} \theta_{E_i}(t_i) \\ &\le \theta_E(b) + |\theta|(E, u). \end{aligned}$$

Taking the supremum over all such choices of sets $E_i \in \mathfrak{R}$ and elements $s_i \le v$ on the left-hand side of this inequality yields

$$\theta_E(a \oplus v) = \theta_E(a) + |\theta|(E, v) \le \theta_E(b) + |\theta|(E, u) = \theta_E(b \oplus u).$$

Furthermore, given $w \in \mathcal{W}$, by the $\mathfrak{R}$-boundedness of the given measure θ, there is $v \in \mathcal{V}$ such that $\theta_E(0 \oplus v) = |\theta|(E, v) \le w$. This implies that the linear operator $\theta_E : \mathcal{P}_\mathcal{V} \to \overline{\mathbb{R}}$ is indeed continuous. The countable additivity of the extended measure follows from Lemma 3.4(c). Furthermore, as $(\mathcal{P}_\mathcal{V}, \mathcal{V})$ is a full cone, the extension of θ remains $\mathfrak{R}$-bounded (see 3.7(a)), that is, $|\theta|(E, 0 \oplus v) = \theta_E(0 \oplus v) = |\theta|(E, v)$ holds for all $E \in \mathfrak{R}$ and $v \in \mathcal{V}$. If on the other hand, θ is an $\mathfrak{R}$-bounded $\mathfrak{L}(\mathcal{P}_\mathcal{V}, \mathcal{Q})$-valued measure on $\mathfrak{R}$, and if θ_0 denotes its restriction to an $\mathfrak{L}(\mathcal{P}, \mathcal{Q})$-valued measure, then we have $|\theta_0|(E, v) \le \theta_E(0 \oplus v)$.

This procedure of extending a given $\mathfrak{R}$-bounded measure from a quasi-full to a full cone yields an interesting new understanding of the *(total) variation* of a given measure, not as a separate positive real-valued measure associated with the given one, but as an integral part of its extension. Because this extension, evaluated at the neighborhoods is also $\mathcal{Q}$- and not necessarily positive real-valued, its countable additivity is preserved, thus removing a major inconvenience that arises in the classical approach (see IV.10.3 in [55]). This therefore avoids the need to introduce the separate terms of *variation* and *semivariation* for a measure (see I.2 in [43]).

The extension of a given measure as carried out in 3.8 will turn out to be invaluable in our upcoming integration theory for cone-valued functions with respect to an operator-valued measure. It does in fact justify the use of a full cone for $\mathcal{P}$, that is the range of the concerned functions and the domain of the linear operators resulting from our measures.

3.9 Composition of Measures and Continuous Linear Operators. Let $(\mathcal{P}, \mathcal{V})$ and $(\widetilde{\mathcal{P}}, \widetilde{\mathcal{V}})$ be quasi-full locally convex cones, and let $(\mathcal{Q}, \mathcal{W})$ and $(\widetilde{\mathcal{Q}}, \widetilde{\mathcal{W}})$ be locally convex complete lattice cones. For continuous linear operators $S \in \mathfrak{L}(\widetilde{\mathcal{P}}, \mathcal{P})$, $T \in \mathfrak{L}(\mathcal{P}, \mathcal{Q})$ and $U \in \mathfrak{L}(\mathcal{Q}, \widetilde{\mathcal{Q}})$ let $U \circ T \circ S \in \mathfrak{L}(\widetilde{\mathcal{P}}, \widetilde{\mathcal{Q}})$ denote their composition, that is the continuous linear operator

$$l \mapsto U\big(T\big(S(l)\big)\big) \,:\, \widetilde{\mathcal{P}} \to \widetilde{\mathcal{Q}}.$$

It is straightforward to verify that this operator is indeed linear and continuous. We shall use this in the following way: If θ is an $\mathfrak{L}(\mathcal{P}, \mathcal{Q})$-valued measure on $\mathfrak{R}$, if $S \in \mathfrak{L}(\widetilde{\mathcal{P}}, \mathcal{P})$ and if the operator $U \in \mathfrak{L}(\mathcal{Q}, \widetilde{\mathcal{Q}})$ is order continuous (see I.5.29), then the set function

$$E \mapsto (U \circ \theta_E \circ S) \,:\, \mathfrak{R} \to \mathfrak{L}(\widetilde{\mathcal{P}}, \widetilde{\mathcal{Q}})$$

is an $\mathfrak{L}(\widetilde{\mathcal{P}}, \widetilde{\mathcal{Q}})$-valued measure, called the *composition of θ with U and S* and denoted as $(U \circ \theta \circ S)$. Countable additivity follows from the order continuity of the operator U. Indeed, let $E_i \in \mathfrak{R}$ be disjoint sets such that $E = \bigcup_{i=1}^{\infty} E_i \in \mathfrak{R}$. Then for every $l \in \widetilde{\mathcal{P}}$ we have $\theta_E\big(S(l)\big) = \sum_{i=1}^{\infty} \theta_{E_i}\big(S(l)\big)$ by the countable additivity of θ, hence

$$\begin{aligned}(U\circ\theta\circ S)_E\,(l) &= U\big(\theta_E\big(S(l)\big)\big)\\ &= U\left(\sum_{i=1}^{\infty}\theta_{E_i}\big(S(l)\big)\right)\\ &= \sum_{i=1}^{\infty}U\big(\theta_{E_i}\big(S(l)\big)\big)\\ &= \sum_{i=1}^{\infty}(U\circ\theta\circ S)_{E_i}\,(l)\end{aligned}$$

by the order continuity of U. The modulus of the measure $(U\circ\theta\circ S)$ can be estimated as follows: Let $E\in\mathfrak{R}$, and for $v\in\mathcal{V}$ let $\tilde{v}\in\widetilde{\mathcal{V}}$ such that $S(l)\le S(m)+v$ whenever $l\le m+\tilde{v}$ for $l,m\in\widetilde{\mathcal{P}}$. If $E_1,\ldots,E_n\in\mathfrak{R}$ are disjoint subsets of E and if $l_i\in\widetilde{\mathcal{P}}$ such that $l_i\le\tilde{v}$ for $i=1,\ldots,n$, then

$$\sum_{i=1}^{n}\big(U\circ\theta\circ S\big)_{E_i}(l_i)=U\left(\sum_{i=1}^{n}\theta_{E_i}\big(S(l_i)\big)\right)\le U\big(|\theta|(E,v)\big).$$

Taking the supremum over all such choices for sets $E_i\in\mathfrak{R}$ and elements $l_i\le\tilde{v}$ yields

$$|U\circ\theta\circ S|(E,\tilde{v})\le U\big(|\theta|(E,v)\big).$$

The $\mathfrak{L}(\widetilde{\mathcal{P}},\widetilde{\mathcal{Q}})$-valued measure $(U\circ\theta\circ S)$ is therefore $\mathfrak{R}$-bounded whenever the $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measure θ is $\mathfrak{R}$-bounded. Indeed, for $E\in\mathfrak{R}$ and $\tilde{w}\in\widetilde{W}$ there is $w\in\mathcal{W}$ such that $U(s)\le U(t)+\tilde{w}$ whenever $s\le t+\tilde{w}$ for $s,t\in\mathcal{Q}$. There is $v\in\mathcal{V}$ such that $|\theta|(E,v)\le w$, hence $|U\circ\theta\circ S|(E,\tilde{v})\le\tilde{w}$ if $\tilde{v}\in\widetilde{V}$ is chosen as above.

We shall in particular make use of the combination of an $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measure θ with an order continuous linear functional $\mu\in\mathcal{Q}^*$. (We choose $\widetilde{\mathcal{P}}=\mathcal{P}$ and the identity operator for S.) The resulting measure $(\mu\circ\theta)$ is $\mathfrak{L}(\mathcal{P},\overline{\mathbb{R}})$-, that is $\mathcal{P}^*$-valued in this case.

3.10 Strong Additivity. Countable additivity of an $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measure θ is meant with respect to order convergence in the locally convex complete lattice cone $(\mathcal{Q},\mathcal{W})$. Order convergence does in general not imply convergence in the weak or indeed convergence in the symmetric relative topology of $\mathcal{Q}$ (see I.5.42). However, the following result based on a well-known theorem by Pettis (see Theorem IV.10.1 in Dunford & Schwartz, [55]) will show that in special cases some stronger type of convergence is implied.

Theorem 3.11. *Let θ be an $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measure, let a be a bounded element of $\mathcal{P}$ and let $\mathcal{Q}_0$ be the subcone of $\mathcal{Q}$ spanned by the set $\{\theta_E(a)\mid E\in\mathfrak{R}\}$. If every continuous linear functional on $\mathcal{Q}_0$ can be extended to an order continuous linear functional on $\mathcal{Q}$, then for disjoint sets $E_i\in\mathfrak{R}$ such that $\cup_{i=1}^{\infty}E_i\in\mathfrak{R}$ the series*

$$\theta_{(\bigcup_{i\in\mathbb{N}} E_i)}(a) = \sum_{i=1}^{\infty} \theta_{E_i}(a)$$

converges in the symmetric topology of $\mathcal{Q}$.

Proof. We shall follow the main lines of the arguments in the proof of Pettis' Theorem as presented in [55]. Let $a \in \mathcal{P}$ be a bounded element, and let $\mathcal{Q}_0$ be the subcone of $\mathcal{Q}$ spanned by the set $\{\theta_E(a) \in \mathcal{Q}_0 \mid E \in \mathfrak{R}\}$. As all the operators θ_E are continuous, the elements of $\mathcal{Q}_0$ are bounded in $\mathcal{Q}$. We may therefore consider $\mathcal{Q}_0$ as a locally convex cone endowed with the symmetric topology generated by the neighborhood system $\mathcal{W}$. Let $\mathcal{Q}_0^{s*}$ be the dual of $\mathcal{Q}_0$ under this topology. According to Proposition II.2.21 in [100], the linear functionals $\mu \in \mathcal{Q}_0^{s*}$ can be expressed as the difference of two elements of the given dual cone (with respect to the given topology) $\mathcal{Q}_0^*$ of $\mathcal{Q}_0$, that is $\mathcal{Q}_0^{s*} = \mathcal{Q}_0^* - \mathcal{Q}_0^*$. As the elements of $\mathcal{Q}_0^*$ were supposed to be order continuous on $\mathcal{Q}_0$, so are the elements of $\mathcal{Q}_0^{s*}$.

Now let us consider a sequence of disjoint sets $E_i \in \mathfrak{R}$ such that $E = \cup_{i=1}^{\infty} E_i \in \mathfrak{R}$. Let $\mathfrak{Z}_0 \subset \mathfrak{R}$ be the set algebra in E generated by the sets E_i, and let $\mathfrak{Z} \subset \mathfrak{R}$ be the σ-algebra in E generated by $\mathfrak{Z}_0$. The algebra $\mathfrak{Z}_0$ is known to be countable (see III.8.4 in [55]). Let $\mathcal{Q}_1$ be the closure (with respect to the symmetric topology) in $\mathcal{Q}_0$ of the subcone that is spanned by the countable set $\{\theta_E(a) \mid E \in \mathfrak{Z}_0\}$.

In a first step, an argument using the separation result from Corollary 4.6 in [172] will demonstrate that $\theta_E(a) \in \mathcal{Q}_1$ for all $E \in \mathfrak{Z}$. For this, assume to the contrary that $\theta_E(a) \notin \mathcal{Q}_1$ for some $E \in \mathfrak{Z}$. Then according to the separation result 4.6 in [172] there is a linear functional $\mu \in \mathcal{Q}_0^{s*}$ such that such that $\mu\big(\theta_E(a)\big) \le -1 \le \mu(l)$ for all $l \in \mathcal{Q}_1$. As $\mathcal{Q}_1$ is a cone, this implies indeed that $\mu(l) \ge 0$ holds for all $l \in \mathcal{Q}_1$. As the linear functional $\mu \in \mathcal{Q}_0^{s*}$ was seen to be order continuous, $G \mapsto \mu\big(\theta_G(a)\big) : \mathfrak{Z} \to \mathbb{R}$ defines a countably additive real-valued measure $(\mu \circ \theta \circ a)$ on $\mathfrak{Z}$. This measure, taking non-negative values on $\mathfrak{Z}_0$ and a negative value on $E \in \mathfrak{R}$ contradicts the uniqueness part of Hahn's extension theorem for measures from an algebra $\mathfrak{Z}_0$ to the σ-algebra $\mathfrak{Z}$ generated by $\mathfrak{Z}_0$ (see III.5.9 in [55] or 12.2.8 in [178]). Thus $\theta_E(a) \in \mathcal{Q}_1$ as claimed.

Now set $F_n = \bigcup_{i=1}^{n} E_i$ for $n \in \mathbb{N}$, and let us assume that, contrary to our claim, there exists a neighborhood $w \in \mathcal{W}$, and a subsequence $(F_m)_{m\in\mathbb{N}}$ of $(F_n)_{n\in\mathbb{N}}$ such that either

$$\theta_E(a) \not\le \theta_{F_m}(a) + w \qquad \text{or} \qquad \theta_{F_m}(a) \not\le \theta_E(a) + w$$

for all $m \in \mathbb{N}$. Then according to Theorem 3.11 in [175] (see also Corollary 4.34 in Chapter I) there are linear functionals $\mu_m \in \mathcal{Q}_0^{s*}$, contained in the polar of the symmetric neighborhood w, such that $\mu_m\big(\theta_E(a)\big) > \mu_m\big(\theta_{F_m}(a)\big) + 1$ for all $m \in \mathbb{N}$. Let $\{l_k \mid k \in \mathbb{N}\}$ be a countable dense (with respect to the symmetric topology) subset of $\mathcal{Q}_1$. Since for every $k \in \mathbb{N}$ the sequence $\big(\mu_m(l_k)\big)_{m\in\mathbb{N}}$ is bounded in $\mathbb{R}$, we may use a Cantor diagonal procedure to find a subsequence $(\mu_{m_j})_{j\in\mathbb{N}}$ of $(\mu_m)_{m\in\mathbb{N}}$ such that the limit $\lim_{j\to\infty} \mu_{m_j}(l_k)$

exists in $\mathbb{R}$ for all $k \in \mathbb{N}$: Indeed, there is a subsequence $(\mu_{m_{(1,j)}})_{j\in\mathbb{N}}$ of $(\mu_m)_{m\in\mathbb{N}}$ such that $\lim_{j\to\infty} \mu_{m_{(1,j)}}(l_1)$ exists in $\mathbb{R}$. Then there is a subsequence $(\mu_{m_{(2,j)}})_{m\in\mathbb{N}}$ of $(\mu_{m_{(1,j)}})_{m\in\mathbb{N}}$ such that $\lim_{j\to\infty} \mu_{m_{(2,j)}}(l_2)$ exists, etc. We set $\mu_{m_j} = \mu_{m_{(j,j)}}$ for all $j \in \mathbb{N}$. Then $(\mu_{m_j})_{j\in\mathbb{N}}$ is a subsequence of each of the sequences $(\mu_{m_{(k,j)}})_{m\in\mathbb{N}}$ for $k \in \mathbb{N}$, thus satisfying our requirement. Now a simple argument will show that the limit $\lim_{j\to\infty} \mu_{m_j}(l)$ exists indeed for all $l \in \mathcal{Q}_1$. In fact, given $l \in \mathcal{Q}_1$ and $\varepsilon > 0$ there is some l_k such that both $l \le l_k + \varepsilon w$ and $l_k \le l + \varepsilon w$, hence $|\mu_{m_j}(l) - \mu_{m_j}(l_k)| \le \varepsilon$ for all $j \in \mathbb{N}$. Moreover, there is $j_0 \in \mathbb{N}$ such that $|\mu_{m_{j_1}}(l_k) - \mu_{m_{j_2}}(l_k)| \le \varepsilon$ whenever $j_1, j_2 \ge j_0$. This implies $|\mu_{m_{j_1}}(l_k) - \mu_{m_{j_2}}(l)| \le 3\varepsilon$. Thus the sequence $\big(\mu_{m_j}(l)\big)_{j\in\mathbb{N}}$ is a Cauchy sequence, hence convergent in $\mathbb{R}$. For every $j \in \mathbb{N}$ let $(\mu_{m_j} \circ \theta \circ a)$ denote the real-valued measure $G \mapsto \mu_{m_j}\big(\theta_G(a)\big) : \mathfrak{Z} \to \mathbb{R}$. Then $\lim_{j\to\infty} (\mu_{m_j} \circ \theta \circ a)(G)$ exists for every $G \in \mathfrak{Z}$ by the above, hence following Nikodým's theorem (see Corollary III.7.4 in [55]) the countable additivity of these measures is uniform in j. As $E = \bigcup_{j=1}^{\infty} F_j$, this contradicts our assumption that $(\mu_{m_j} \circ \theta \circ a)(E) > (\mu_{m_j} \circ \theta \circ a)(F_j) + 1$ holds for all $j \in \mathbb{N}$. $\square$

This result applies in particular if $(\mathcal{Q}, \mathcal{W})$ is the standard lattice completion of some subcone $\mathcal{Q}_0$ of $\mathcal{Q}$ and if the measure θ is $\mathfrak{L}(\mathcal{P}, \mathcal{Q}_0)$-valued. In this case, all continuous linear functionals on $\mathcal{Q}_0$ extend to order continuous linear functionals on $\mathcal{Q}$, as required in Theorem 3.11. If all elements of $\mathcal{P}$ are bounded, then countable additivity of an $\mathfrak{L}(\mathcal{P}, \mathcal{Q}_0)$-valued measure implies convergence of the concerned operators with respect to the strong operator topology of $\mathfrak{L}(\mathcal{P}, \mathcal{Q}_0)$ $\big($see I.7.2(ii)$\big)$.

We shall provide a simple example of a measure that is countably additive with respect to order convergence but not with respect the symmetric topology of $\mathcal{Q}$.

Example 3.12. Let $\mathcal{P} = \mathbb{R}$ with its usual (Euclidean) topology, and let $\mathcal{Q}$ be the cone of all $\overline{\mathbb{R}}$-valued bounded below functions on the interval $[0,1]$, endowed with the pointwise algebraic operations and order, and the constant functions $w > 0$ as neighborhoods. Then $(\mathcal{Q}, \mathcal{W})$ is a locally convex complete lattice cone. Let $\mathfrak{R}$ be the σ-algebra of Borel sets on $X = [0,1]$. For every $E \in \mathfrak{R}$ let θ_E be the linear operator in $\mathfrak{L}(\mathcal{P}, \mathcal{Q})$ that maps $\rho \in \mathbb{R}$ into $\rho\chi_E \in \mathcal{Q}$, where χ_E denotes the characteristic function of the set E. Clearly θ is countably additive with respect to order convergence, but not with respect to uniform convergence, that is convergence with respect to the symmetric topology in $\mathcal{Q}$.

If $(\mathcal{Q}_0, \mathcal{W}_0)$ is locally convex topological vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, then the elements of $\mathcal{Q}_0$ may be considered as continuous linear operators from $\mathcal{P} = \mathbb{K}$, endowed with the Euclidean topology, into the standard lattice completion $(\mathcal{Q}, \mathcal{W})$ of $\mathcal{Q}_0$. (This situation will be explored in greater detail in Example 6.23 below.) A $\mathcal{Q}_0$-valued measure is required to be countably additive with respect to order convergence in $\mathcal{Q}$, that is weak convergence in $\mathcal{Q}_0$. According to Theorem 3.11 (use $a = 1 \in \mathcal{P}$), this implies convergence with respect to the symmetric topology of $\mathcal{Q}$, that is the given topology of $\mathcal{Q}_0$. This result is commonly known as Pettis' theorem.

Corollary 3.13. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex topological vector space over $\mathbb{R}$ or $\mathbb{C}$. For a $\mathcal{P}$-valued measure countable additivity with respect to the weak topology implies countable additivity with respect to the given topology of $\mathcal{P}$.*

3.14 Weak Compactness. A well-known result due to Bartle, Dunford and Schwartz (see Corollary I.2.7 in [43] or Theorem VI.7.3 in [55]) about the relative weak compactness of the range of a vector-valued measure implies the following for operator-valued measures:

Theorem 3.15. *Suppose that $(\mathcal{Q}, \mathcal{W})$ is the standard lattice completion of a Banach space $(\mathcal{Q}_0, \| \, \|)$ over $\mathbb{R}$ or $\mathbb{C}$ and that θ is a bounded $\mathfrak{L}(\mathcal{P}, \mathcal{Q}_0)$-valued measure. Then for every $a \in \mathcal{P}$ and every $E \in \mathfrak{R}$ the set*

$$\{\theta_G(a) \mid F \in \mathfrak{R}, \ \ G \subset E\}$$

is relatively compact in $\mathcal{Q}_0$ with respect to the weak topology $\sigma(\mathcal{Q}_0, \mathcal{Q}_0^*)$.

Proof. Let $a \in \mathcal{P}$ and $E \in \mathfrak{R}$. The family $\mathfrak{R}_E = \{G \in \mathfrak{R}, \ \ G \subset E\}$ is a σ-algebra on E, and the set function

$$G \mapsto \theta_G(a) : \mathfrak{R}_E \to \mathcal{Q}_0$$

is a countably additive $\mathcal{Q}_0$-valued, that is a Banach space-valued measure on $\mathfrak{R}_E$. Our claim then follows directly from Corollary I.2.7 in [43]. $\square$

4. Integrals for Cone-Valued Functions

Throughout this section, let $(\mathcal{P}, \mathcal{V})$ be a full locally convex cone and let $(\mathcal{Q}, \mathcal{W})$ be a locally convex complete lattice cone. Let $\mathfrak{R}$ be a (weak) σ-ring of subsets of X and θ an $\mathfrak{L}(\mathcal{P}, \mathcal{Q})$-valued measure on $\mathfrak{R}$. The requirement that the locally convex cone $(\mathcal{P}, \mathcal{V})$ is full does in fact accommodate quasi-full cones as well. Indeed, in this case we may take advantage of the embedding of a quasi-full cone $(\mathcal{P}, \mathcal{V})$ into the full locally convex cone $(\mathcal{P}_\nu, \mathcal{V})$, that is its standard full extension, as elaborated in I.6, and make use of the corresponding extension of an $\mathfrak{R}$-bounded $\mathfrak{L}(\mathcal{P}, \mathcal{Q})$-valued measure θ to an $\mathfrak{L}(\mathcal{P}_\nu, \mathcal{Q})$-valued measure as constructed in Section 3.8; that is, we may set $\theta_E(v) = |\theta|(E, v)$ for every set $E \in \mathfrak{R}$ and every neighborhood $v \in \mathcal{V} \subset \mathcal{P}_\nu$.

We proceed to define integrals for cone-valued functions with respect to θ. The values of these integrals will be elements of $\mathcal{Q}$. We shall use the cone $\mathcal{F}_{\mathfrak{R}}(X, \mathcal{P})$ of all $\mathcal{P}$-valued measurable functions on X that can be reached from below by $\mathcal{P}$-valued step functions in the sense of Section 2.3. Similarly, $\mathcal{F}_{\mathfrak{R}}(X, \mathcal{V})$ denotes the cone of all measurable $(\mathcal{V} \cup \{0\})$-valued functions on X.

In a first step, we shall define integrals for $\mathcal{P}$- and $\mathcal{V}$-valued step functions on X, that is functions $s = \sum_{i=1}^{n} \chi_{E_i} \otimes a_i$ for $E_i \in \mathfrak{R}$ and elements a_i in $\mathcal{P}$ or $\mathcal{V}$, respectively. We shall denote the corresponding subcones of $\mathcal{F}(X, \mathcal{P})$ by $\mathcal{S}_{\mathfrak{R}}(X, \mathcal{P})$ and $\mathcal{S}_{\mathfrak{R}}(X, \mathcal{V})$. Note that the functions in $\mathcal{S}_{\mathfrak{R}}(X, \mathcal{V})$ are $\big(\mathcal{V} \cup \{0\}\big)$-valued. Obviously, any representation $\sum_{i=1}^{n} \chi_{E_i} \otimes a_i$ for a given step function is not unique. To prepare our definition of the integral for functions in $\mathcal{S}_{\mathfrak{R}}(X, \mathcal{P})$ we observe:

Lemma 4.1. *Let $E_i, F_k \in \mathfrak{R}$ and $a_i, b_k \in \mathcal{P}$ for $i = 1, \ldots, n$ and $k = 1, \ldots, m$. If $\sum_{i=1}^{n} \chi_{E_i} \otimes a_i \le \sum_{k=1}^{m} \chi_{F_k} \otimes b_k$, then $\sum_{i=1}^{n} \theta_{E_i}(a_i) \le \sum_{k=1}^{m} \theta_{F_k}(b_k)$.*

Proof. First we shall verify that for any step function $s = \sum_{i=1}^{n} \chi_{E_i} \otimes a_i$ there exists a representation $\sum_{k=1}^{m} \chi_{F_k} \otimes b_k$ such that the sets $F_k \in \mathfrak{R}$ are pairwise disjoint and such that $\sum_{i=1}^{n} \theta_{E_i}(a_i) = \sum_{k=1}^{m} \theta_{F_k}(b_k)$. We shall use induction with respect to n. For $n = 1$ there is nothing to prove. Assume that our claim holds true for some $n \ge 1$ and let $s = \sum_{i=1}^{n+1} \chi_{E_i} \otimes a_i$. There are disjoint sets $F_k \in \mathfrak{R}$ such that $\sum_{i=1}^{n} \chi_{E_i} \otimes a_i = \sum_{k=1}^{m} \chi_{F_k} \otimes b_k$ satisfying the above. By adding a suitable term $\chi_F \otimes 0$ to the right-hand of the last equation, we may assume that $E_{n+1} \subset \bigcup_{k=1}^{m} F_k$. Hence

$$
\begin{aligned}
s &= \sum_{k=1}^{m} \chi_{F_k} \otimes b_k + \chi_{E_{n+1}} \otimes a_{n+1} \\
&= \sum_{k=1}^{m} \chi_{(F_k \cap E_{n+1})} \otimes (b_k + a_{n+1}) + \sum_{k=1}^{m} \chi_{(F_k \setminus E_{n+1})} \otimes b_k .
\end{aligned}
$$

The sets in the above representation for s are disjoint, and we have indeed

$$
\begin{aligned}
\sum_{i=1}^{n+1} \theta_{E_i}(a_i) &= \sum_{k=1}^{m} \theta_{F_k}(b_k) + \theta_{E_{n+1}}(a_{n+1}) \\
&= \sum_{k=1}^{m} \theta_{(F_k \cap E_{n+1})}(b_k + a_{n+1}) + \sum_{k=1}^{m} \theta_{(F_k \setminus E_{n+1})}(b_k)
\end{aligned}
$$

as claimed. Thus, to prove our claim in Lemma 4.1, we may assume that both families of sets E_i and F_k are pairwise disjoint and that $\sum_{i=1}^{n} \chi_{E_i} \otimes a_i \le \sum_{k=1}^{m} \chi_{F_k} \otimes b_k$. By adding suitable terms $\chi_E \otimes 0$ and $\chi_F \otimes 0$ on the left- and right-hand sides of the above inequality, we may assume in addition that $E = \bigcup_{i=1}^{n} E_i = \bigcup_{k=1}^{m} F_k$. Under these assumptions the sets $E_i \cap F_k$ form a disjoint partition of E, and we have either $E_i \cap F_k = \emptyset$ or $a_i \le b_k$. This yields

$$
\sum_{i=1}^{n} \theta_{E_i}(a_i) = \sum_{i=1}^{n} \sum_{k=1}^{m} \theta_{(E_i \cap F_k)}(a_i) \le \sum_{k=1}^{m} \sum_{i=1}^{n} \theta_{(E_i \cap F_k)}(b_k) = \sum_{k=1}^{m} \theta_{F_k}(b_k),
$$

as claimed. $\square$

4.2 Integrals for $\mathcal{P}$-Valued Step Functions. We are now in a position to define the integral for a $\mathcal{P}$-valued step function

$$h = \sum_{i=1}^{n} \chi_{E_i^{\otimes}} a_i \in \mathcal{S}_{\mathfrak{R}}(X, \mathcal{P})$$

over a measurable set $F \in \mathfrak{A}_{\mathfrak{R}}$ with respect to θ by

$$\int_F h \, d\theta = \sum_{i=1}^{n} \theta_{(E_i \cap F)}(a_i).$$

Lemma 4.1 implies that the sum on the right-hand side is independent of the particular representation for h. The integral represents a monotone linear operator from $\mathcal{S}_{\mathfrak{R}}(X, \mathcal{P})$ into $\mathcal{Q}$.

Lemma 4.3. *Let $F \in \mathfrak{A}_{\mathfrak{R}}$, let $h, g \in \mathcal{S}_{\mathfrak{R}}(X, \mathcal{P})$ and $\alpha \geq 0$. Then*

(a) $\int_F (\alpha h) \, d\theta = \alpha \int_F h \, d\theta$.
(b) $\int_F (g + h) \, d\theta = \int_F g \, d\theta + \int_F h \, d\theta$.
(c) $\int_F g \, d\theta \leq \int_F h \, d\theta$ *whenever* $g \leq h$.
(d) $\int_F g \, d\theta = \int_X (\chi_{F^{\otimes}} g) \, d\theta$.

All these properties are obvious from the definition of the integral and from Lemma 4.1.

We shall demonstrate in the following lemma that, if the full cone $(\mathcal{P}, \mathcal{V})$ is in fact the standard full extension $(\mathcal{P}_{0\mathcal{V}}, \mathcal{V})$ of a quasi-full cone $(\mathcal{P}_0, \mathcal{V})$, and if θ is the canonical extension of an $\mathfrak{R}$-bounded $\mathfrak{L}(\mathcal{P}_0, \mathcal{Q})$-valued measure θ_0, as elaborated in I.6 and 3.8, then the way in which this extension was constructed, guarantees that the integral is already determined by its values on the subcone $\mathcal{S}_{\mathfrak{R}}(X, \mathcal{P}_0)$ of $\mathcal{S}_{\mathfrak{R}}(X, \mathcal{P})$, that is by $\mathcal{P}_0$-valued step functions and the given measure θ_0.

Lemma 4.4. *Let $F \in \mathfrak{A}_{\mathfrak{R}}$ and $g \in \mathcal{S}_{\mathfrak{R}}(X, \mathcal{P})$. If $(\mathcal{P}, \mathcal{V})$ is the standard full extension of the quasi-full cone $(\mathcal{P}_0, \mathcal{V})$, and if θ is the canonical extension of an $\mathfrak{R}$-bounded $\mathfrak{L}(\mathcal{P}_0, \mathcal{Q})$-valued measure θ_0, then*

$$\int_F g \, d\theta = \sup \left\{ \int_F h \, d\theta \;\middle|\; h \in \mathcal{S}_{\mathfrak{R}}(X, \mathcal{P}_0), \quad h \leq g \right\}.$$

Proof. Following 4.3(c), we may assume that $F = X$. Let us first recall and reformulate from 3.2 the definition of the modulus of θ_0 for a set $E \in \mathfrak{R}$ and a neighborhood $v \in \mathcal{V}$.

$$\begin{aligned} |\theta_0|(E, v) &= \sup \left\{ \sum_{i=1}^{n} \theta_{E_i}(s_i) \;\middle|\; s_i \in \mathcal{P}_0,\ s_i \leq v,\ E_i \in \mathfrak{R} \text{ disjoint subsets of } E \right\} \\ &= \sup \left\{ \int_X h \, d\theta \;\middle|\; h \in \mathcal{S}_{\mathfrak{R}}(X, \mathcal{P}_0), \quad h \leq \chi_{E^{\otimes}} v \right\}. \end{aligned}$$

Recall that the extension of θ_0 into θ was constructed by setting $\theta_E(v) = |\theta_0|(E,v)$. Now we consider the case that

$$g = \sum_{i=1}^{n} \chi_{E_i^{\otimes}} v_i \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{V})$$

is a $\mathcal{V}$-valued function. We compute using Lemma I.5.5(a)

$$\begin{aligned}\int_X g\,d\theta &= \sum_{i=1}^{n} \theta_{E_i}(v_i)\\ &= \sum_{i=1}^{n} |\theta_0|(E_i,v_i)\\ &= \sup\left\{ \sum_{i=1}^{n} \int_X h_i\,d\theta \;\middle|\; h_i \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P}_0),\quad h_i \le \chi_{E_i \otimes} v \right\}\\ &= \sup\left\{ \int_X h\,d\theta \;\middle|\; h \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P}_0),\quad h \le g \right\},\end{aligned}$$

as claimed. Now for the general case, let

$$g = \sum_{i=1}^{n} \chi_{E_i^{\otimes}} (a_i + v_i) \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P}),$$

for $a_i \in \mathcal{P}_0$ and $v_i \in \mathcal{V}$. Set $g_1 = \sum_{i=1}^{n} \chi_{E_i^{\otimes}} a_i \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P}_0)$ and $g_2 = \sum_{i=1}^{n} \chi_{E_i^{\otimes}} v_i \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{V})$. Then $g = g_1 + g_2$, and the above yields with property $(\bigvee 1)$

$$\begin{aligned}\int_X g\,d\theta &= \int_X g_1\,d\theta + \int_X g_2\,d\theta\\ &= \sup\left\{ \int_F (g_1 + h)\,d\theta \;\middle|\; h \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P}_0),\quad h \le g_2 \right\}\\ &\le \sup\left\{ \int_F h'\,d\theta \;\middle|\; h' \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P}_0),\quad h' \le g \right\}.\end{aligned}$$

The converse inequality is obvious from 4.3(c). $\square$

Subsequently, with every neighborhood $w \in \mathcal{W}$ we associate the inductive limit neighborhood $\mathfrak{v}_w$, defined as

$$\mathfrak{v}_w = \left\{ s \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{V}) \;\middle|\; \int_X s\,d\theta \le w \right\}.$$

(We shall write $\mathfrak{v}_w(\theta)$ if different measures are involved in our considerations.) The boundedness of θ guarantees that for every $E \in \mathfrak{R}$ there is $v \in \mathcal{V}$ such that $\chi_{E\otimes} v \in \mathfrak{v}_w$. Convexity follows from Lemma 4.3. We have

$\mathfrak{v}_{(\lambda w)} = \lambda \mathfrak{v}_w$ and $\mathfrak{v}_w + \mathfrak{v}_{w'} \leq \mathfrak{v}_{(w+w')}$ for $w, w' \in \mathcal{W}$ and $\lambda > 0$. We proceed to develop the integral over a measurable set $F \in \mathfrak{A}_{\mathfrak{R}}$ for a function $f \in \mathcal{F}_{\mathfrak{R}}(X, \mathcal{P})$ in the following manner: First, for a neighborhood $w \in \mathcal{W}$ we set

$$\int_F^{(w)} f \, d\theta = \sup \left\{ \int_F h \, d\theta \;\middle|\; h \in \mathcal{S}_{\mathfrak{R}}(X, \mathcal{P}), \quad h \leq f + \mathfrak{v}_w \right\}.$$

We note that in the situation of Lemma 4.4, the integral of a function in $\mathcal{F}_{\mathfrak{R}}(X, \mathcal{P})$ is already determined by $\mathcal{P}_0$-valued step functions alone:

Lemma 4.5. *Let $F \in \mathfrak{A}_{\mathfrak{R}}$, $f \in \mathcal{F}_{\mathfrak{R}}(X, \mathcal{P})$ and $w \in \mathcal{W}$. If $(\mathcal{P}, \mathcal{V})$ is the standard full extension of a quasi-full cone $(\mathcal{P}_0, \mathcal{V})$, and if θ is the canonical extension of an $\mathfrak{L}(\mathcal{P}_0, \mathcal{Q})$-valued measure θ_0, then*

$$\int_F^{(w)} f \, d\theta = \sup \left\{ \int_F h \, d\theta \;\middle|\; h \in \mathcal{S}_{\mathfrak{R}}(X, \mathcal{P}_0), \quad h \leq f + \mathfrak{v}_w \right\}.$$

We proceed with a simple observation for step functions.

Lemma 4.6. *Let $F \in \mathfrak{A}_{\mathfrak{R}}$, $f \in \mathcal{S}_{\mathfrak{R}}(X, \mathcal{P})$ and $w \in \mathcal{W}$. Then*

$$\int_F f \, d\theta \leq \int_F^{(w)} f \, d\theta \leq \int_F f \, d\theta + w.$$

Proof. The first part of the inequality is trivial. For the second part, let $h \leq f + \mathfrak{v}_w$ for $h \in \mathcal{S}_{\mathfrak{R}}(X, \mathcal{P})$, that is $h \leq f + s$ for some $\mathcal{V}$-valued step function $s \in \mathfrak{v}_w$. Following Lemma 4.3(b) and (c), this implies

$$\int_F h \, d\theta \leq \int_F f \, d\theta + \int_F s \, d\theta \leq \int_F f \, d\theta + w$$

for each such step function $h \in \mathcal{S}_{\mathfrak{R}}(X, \mathcal{P})$, hence $\int_F^{(w)} f \, d\theta \leq \int_F f \, d\theta + w$ as claimed. $\square$

Proposition 4.7. *Let $E \in \mathfrak{R}$ and $f \in \mathcal{F}_{\mathfrak{R}}(X, \mathcal{P})$ and let $(h_n)_{n \in \mathbb{N}}$ be a bounded below sequence of step functions in $\mathcal{S}_{\mathfrak{R}}(X, \mathcal{P})$ such that for every $x \in E$ there is $n_0 \in \mathbb{N}$ such that $f(x) \leq h_n(x)$ for all $n \geq n_0$. Then*

$$\int_E^{(w)} f \, d\theta \leq \varliminf_{n \to \infty} \int_E h_n \, d\theta + w$$

for every $w \in \mathcal{W}$.

Proof. Let $E \in \mathfrak{R}$, $f \in \mathcal{F}_{\mathfrak{R}}(X, \mathcal{P})$, and let $(h_n)_{n \in \mathbb{N}}$ be a sequence of step functions satisfying our assumptions. For $w \in \mathcal{W}$ let $l \in \mathcal{S}_{\mathfrak{R}}(X, \mathcal{P})$ such that $l \leq f + \mathfrak{v}_w$, that is $l \leq f + s$ for some $s \in \mathfrak{v}_w$. Now we set

$$E_n = \{ x \in E \mid l(x) \leq h_m(x) + s(x) \quad \text{for all} \quad m \geq n \}.$$

All the sets E_n are measurable, $E_n \subset E_{n+1}$ and $E = \bigcup_{n\in\mathbb{N}} E_n$ by our assumption. Given any $u \in \mathcal{W}$ there is $v \in \mathcal{V}$ such that $\theta_E(v) \leq u$, and as the sequence $(h_n)_{n\in\mathbb{N}}$ is bounded below, there is $\rho \geq 0$ such that $0 \leq h_n + \rho\chi_{X^\otimes}v$ for all $n \in \mathbb{N}$. Thus

$$\begin{aligned}\chi_{E_n^\otimes} l &\leq \chi_{E_n^\otimes}(h_n + s) + \chi_{(E\setminus E_n)}(h_n + \rho\chi_{X^\otimes}v)\\ &\leq \chi_{E^\otimes}h_n + \chi_{E_n^\otimes}s + \rho\chi_{(E\setminus E_n)^\otimes}v.\end{aligned}$$

Hence by Lemma 4.3, and because

$$\int_X \chi_{E_n} s\, d\theta \leq \int_X s\, d\theta \leq w, \qquad \text{and} \qquad \int_X \chi_{(E\setminus E_n)^\otimes}v = \theta_{(E\setminus E_n)}(v),$$

when taking the integrals over X in the above inequality, we obtain

$$\int_{E_n} l\, d\theta \leq \int_E h_n\, d\theta + \rho\theta_{(E\setminus E_n)}(v) + w.$$

Because $E_n \subset E_{n+1}$ and $\bigcup_{n\in\mathbb{N}} E_n = E$, Lemma 3.1(a) yields

$$\theta_{(F\cap E)}(a) = \lim_{n\to\infty} \theta_{(F\cap E_n)}(a)$$

for all $F \in \mathfrak{R}$ and $a \in \mathcal{P}$. Considering the definition of the integral for a step function in 4.2, this renders

$$\lim_{n\to\infty} \int_{E_n} l\, d\theta = \int_E l\, d\theta,$$

and Lemma 3.1(b) yields

$$\overline{\lim_{n\to\infty}}\, \theta_{(E\setminus E_n)}(v) \leq \mathfrak{O}\big(\theta_E(v)\big) \leq \varepsilon' u$$

for all $\varepsilon' \geq 0$. Thus, using the limit rules from Lemma I.5.19, we obtain

$$\int_E l\, d\theta \leq \underline{\lim_{n\to\infty}} \int_E h_n\, d\theta + w + \varepsilon' u.$$

Because $u \in \mathcal{W}$ and $\varepsilon' > 0$ were arbitrary, and because $\mathcal{Q}$ carries the weak preorder, this shows

$$\int_E l\, d\theta \leq \underline{\lim_{n\to\infty}} \int_E h_n\, d\theta + w.$$

Our claim follows, since the above inequality holds true for all step functions $l \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P})$ such that $l \leq f + \mathfrak{v}_w$. $\square$

Corollary 4.8. *Let* $F \in \mathfrak{A}_{\mathfrak{R}}$, $f \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ *and* $u, w \in \mathcal{W}$. *Then*

$$\int_F^{(w)} f\,d\theta \leq \int_F^{(u)} f\,d\theta + w.$$

Proof. Let $F \in \mathfrak{A}_{\mathfrak{R}}$, $f \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ and $u, w \in \mathcal{W}$. Let $l \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P})$ such that $l \leq f + \mathfrak{v}_w$, and according to Lemma 2.4, we choose $E \in \mathfrak{R}$ such that both h is supported by E and such that $f(x) \geq 0$ for all $x \in X \setminus E$. For the set $E \cap F \in \mathfrak{R}$, the inductive limit neighborhood $\mathfrak{v} = \mathfrak{v}_u$ and $\varepsilon \geq 0$, let $(h_n)_{n\in\mathbb{N}}$ be a sequence of step functions in $\mathcal{S}_{\mathfrak{R}}(X,\mathcal{P})$ approaching f as in Corollary 2.8. We may assume that the functions h_n are supported by the set $E \cap F$, since we may otherwise replace them by their product with the characteristic function of this set. Proposition 4.7 yields

$$\int_{(E\cap F)}^{(w)} f\,d\theta \leq \varliminf_{n\to\infty} \int_{(E\cap F)} h_n\,d\theta + w.$$

On the other hand, we have

$$\int_F l\,d\theta = \int_{(E\cap F)} l\,d\theta \leq \int_{(E\cap F)}^{(w)} f\,d\theta,$$

since the function l is supported by E. Similarly, for the functions h_n we observe that

$$\int_{(E\cap F)} h_n\,d\theta = \int_F h_n\,d\theta \leq \gamma \int_F^{(u)} f\,d\theta,$$

since $h_n \leq \gamma f + \mathfrak{v}_w$ for all $n \in \mathbb{N}$. Combining all of the above then yields

$$\int_F l\,d\theta \leq \gamma \int_F^{(u)} f\,d\theta + w$$

with some $1 \leq \gamma \leq 1 + \varepsilon$, and indeed

$$\int_F l\,d\theta \leq \int_F^{(u)} f\,d\theta + w,$$

since $\varepsilon > 0$ was chosen independently. Finally, because this last inequality holds true for all $l \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P})$ such that $h \leq f + \mathfrak{v}_w$, our claim follows. $\square$

4.9 Integrals for Functions in $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$. We may now define the *integral over a set* $F \in \mathfrak{A}_{\mathfrak{R}}$ *for a function* $f \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ as

$$\int_F f\,d\theta = \inf_{w\in\mathcal{W}} \int_F^{(w)} f\,d\theta.$$

The above infimum is well-defined and yields an element of the locally convex complete lattice cone $\mathcal{Q}$. Indeed, given any neighborhood $u \in \mathcal{W}$ there is $\lambda \geq 0$ such that $0 \leq f + \lambda\mathfrak{v}_u$. Thus $0 \leq \int_F^{(\lambda u)} f\,d\theta$. According to Corollary 4.8, this yields

$$0 \le \int_F^{(\lambda u)} f\, d\theta \le \int_F^{(w)} f\, d\theta + \lambda u.$$

for all $w \in \mathcal{V}$. This demonstrates that the set $\left\{ \int_F^{(w)} f\, d\theta \mid w \in \mathcal{W} \right\}$ is bounded below, and its infimum exists by $(\bigwedge 1)$. Moreover, our earlier observation in Lemma 4.5 justifies that the above definition of the integral is consistent with the preceding one for step functions. Obviously, the integral is monotone, and we shall proceed to verify that it determines a continuous linear operator from $\mathcal{F}_{\mathfrak{R}}(X, \mathcal{P})$ into $\mathcal{Q}$. For Part (a) of the following lemma, recall from Lemma 2.6 that $\chi_{F} \otimes f \in \mathcal{F}_{\mathfrak{R}}(X, \mathcal{P})$ whenever $f \in \mathcal{F}_{\mathfrak{R}}(X, \mathcal{P})$ and $F \in \mathfrak{A}_{\mathfrak{R}}$. In Part (b) we consider $\mathfrak{R}$ as the index set of a net, directed upward by set inclusion.

Lemma 4.10. *Let* $f \in \mathcal{F}_{\mathfrak{R}}(X, \mathcal{P})$ *and* $F \in \mathfrak{A}_{\mathfrak{R}}$. *Then*

(a) $\int_F f\, d\theta = \int_X (\chi_F \otimes f)\, d\theta$.
(b) $\int_F f\, d\theta = \lim_{E \in \mathfrak{R}} \int_{(E \cap F)} f\, d\theta$.

Proof. For Part (a) we first note that $\chi_F \otimes f \in \mathcal{F}_{\mathfrak{R}}(X, \mathcal{P})$ (see Lemma 2.6). Let $w \in \mathcal{W}$ and $h_0 \in \mathcal{S}_{\mathfrak{R}}(X, \mathcal{P})$ such that $h_0 \le f + \mathfrak{v}_w$. We have

$$\int_F^{(w)} f\, d\theta = \sup \left\{ \int_F h\, d\theta \;\middle|\; h \in \mathcal{S}_{\mathfrak{R}}(X, \mathcal{P}), \quad h \le f + \mathfrak{v}_w \right\}$$

and

$$\int_X^{(w)} \chi_F \otimes f\, d\theta = \sup \left\{ \int_X h'\, d\theta \;\middle|\; h' \in \mathcal{S}_{\mathfrak{R}}(X, \mathcal{P}), \quad h' \le \chi_F \otimes f + \mathfrak{v}_w \right\}$$

First, let $h \in \mathcal{S}_{\mathfrak{R}}(X, \mathcal{P})$ such that $h \le f + \mathfrak{v}_w$. Then $h' = \chi_F \otimes h \le \chi_F \otimes f + \mathfrak{v}_w$, and $\int_X h'\, d\theta = \int_F h\, d\theta$ by 4.3(d). This shows

$$\int_F^{(w)} f\, d\theta \le \int_X^{(w)} \chi_F \otimes f\, d\theta.$$

For the converse inequality, let $h' \in \mathcal{S}_{\mathfrak{R}}(X, \mathcal{P})$ such that $h' \le \chi_F \otimes f + \mathfrak{v}_w$. Then $\chi_F \otimes h' \le \chi_F \otimes f + \mathfrak{v}_w$ and $\chi_{(X \setminus F)} \otimes h_0 \le \chi_{(X \setminus F)} \otimes f + \mathfrak{v}_w$, hence $h = \chi_F \otimes h' + \chi_{(X \setminus F)} \otimes h_0 \le f + 2\mathfrak{v}_w$, and $\int_F h\, d\theta = \int_F h'\, d\theta$. As $\chi_{(X \setminus F)} \otimes h' \le \mathfrak{v}_w$, we have $\int_{(X \setminus F)} h'\, d\theta \le w$, hence $\int_X h'\, d\theta \le \int_F h\, d\theta + w$. This shows

$$\int_X^{(w)} \chi_F \otimes f\, d\theta \le \int_F^{(2w)} f\, d\theta + w.$$

Taking the infima over all $w \in \mathcal{W}$ in the above inequality yields Part (a).

For Part (b) it is therefore sufficient to consider the case $F = X$, because the function f may be replaced by its product with the characteristic function χ_F. Let $E_0 \in \mathfrak{R}$ such that $f(x) \geq 0$ for all $x \in X \setminus E_0$. Then $\chi_{E^\otimes} f \leq \chi_{E'^\otimes} f$ whenever $E_0 \subset E \subset E'$ for $E, E' \in \mathfrak{R}$, hence $\int_E f\,d\theta \leq \int_{E'} f\,d\theta$ by Part (a) and the monotony of the integral. This shows

$$\lim_{E \in \mathfrak{R}} \int_E f\,d\theta = \sup_{E_0 \subset E \in \mathfrak{R}} \int_E f\,d\theta \leq \int_X f\,d\theta.$$

For the converse, let $w \in \mathcal{W}$ and $h \leq f + \mathfrak{v}_w$ for $h \in \mathcal{S}_{\mathfrak{R}}(X, \mathcal{P})$. Because h is supported by a set in $\mathfrak{R}$, there is $E_0 \subset E \in \mathfrak{R}$ such that $\int_X h\,d\theta = \int_E h\,d\theta \leq \int_E^{(w)} f\,d\theta$. Moreover, Corollary 4.8 shows that $\int_E^{(w)} f\,d\theta \leq \int_E f\,d\theta + w$. Thus

$$\int_X h\,d\theta \leq \sup_{E_0 \subset E \in \mathfrak{R}} \int_E f\,d\theta + w.$$

This shows

$$\int_X f\,d\theta \leq \int_X^{(w)} f\,d\theta \leq \sup_{E_0 \subset E \in \mathfrak{R}} \int_E f\,d\theta + w,$$

hence our claim, since $w \in \mathcal{W}$ was arbitrary and $\mathcal{Q}$ carries the weak preorder. $\square$

4.11 Sets of Measure Zero and Properties Holding Almost Everywhere. A set $Z \in \mathfrak{A}_{\mathfrak{R}}$ is said to be *of measure zero (with respect to* θ*)* if $\theta_{(E \cap Z)} = 0$ for all $E \in \mathfrak{R}$. The family $\mathfrak{Z}(\theta)$ of all sets of measure zero is obviously closed for set complements and for countable unions. For a subset F of X we shall say that a pointwise defined property of functions on X holds *θ-almost everywhere on* F if it holds on $F \setminus Z$ with some $Z \in \mathfrak{Z}(\theta)$. In particular, we shall use the symbols $\underset{a.e.F}{\leq}$ or $\underset{a.e.F}{=}$ if the relations $\leq$ or $=$ hold θ-almost everywhere on the set F, respectively; that is for example, $f \underset{a.e.F}{\leq} g + \mathfrak{v}$ for functions $f, g \in \mathcal{F}(X, \mathcal{P})$ and an inductive limit neighborhood $\mathfrak{v}$ means that $\chi_{(F \setminus Z)^\otimes} f \leq \chi_{(F \setminus Z)^\otimes} g + \mathfrak{v}$ holds with some $Z \in \mathfrak{Z}(\theta)$. These relations are of course transitive and compatible with the algebraic operations.

As $\theta_{(E \cap Z)} = 0$ holds for all $E \in \mathfrak{R}$ and $Z \in \mathfrak{Z}(\theta)$, we infer that $\theta_E = \theta_{(E \setminus Z)}$. Now Definition 4.2 yields that $\int_F h\,d\theta = \int_{(F \setminus Z)} h\,d\theta$ for all step functions $h \in \mathcal{S}_{\mathfrak{R}}(X, \mathcal{P})$, $F \in \mathfrak{A}_{\mathfrak{R}}$ and $Z \in \mathfrak{Z}(\theta)$. Considering our definition of the integral in 4.9 we observe that this yields $\int_F f\,d\theta = \int_{(F \setminus Z)} f\,d\theta$ for all $f \in \mathcal{F}_{\mathfrak{R}}(X, \mathcal{P})$ as well. Consequently, $f \underset{a.e.F}{\leq} g$ for functions $f, g \in \mathcal{F}_{\mathfrak{R}}(X, \mathcal{P})$ implies that $\chi_{(F \setminus Z)^\otimes} f \leq \chi_{(F \setminus Z)^\otimes} g$ for some $Z \in \mathfrak{Z}(\theta)$, hence

$$\int_F f\,d\theta = \int_{(F \setminus Z)} f\,d\theta \leq \int_{(F \setminus Z)} g\,d\theta = \int_F g\,d\theta.$$

In particular, any two functions in $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ that coincide θ-almost everywhere on a set $F \in \mathfrak{A}_{\mathfrak{R}}$ have the same integrals over F with respect to θ.

4.12 Integrability over a Set $E \in \mathfrak{R}$. We may now define integrability for cone-valued functions over measurable sets with respect to an operator-valued measure. First, for a set $E \in \mathfrak{R}$ we shall say that a function $f \in \mathcal{F}(X,\mathcal{P})$ is *integrable over* E *with respect to* θ if for every $w \in \mathcal{W}$ and $\varepsilon > 0$ there are functions $f_{(w,\varepsilon)} \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ and $s_{(w,\varepsilon)} \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{V})$ such that

$$f \underset{a.e.E}{\leq} f_{(w,\varepsilon)} \underset{a.e.E}{\leq} \gamma f + s_{(w,\varepsilon)} \qquad \text{and} \qquad \int_E s_{(w,\varepsilon)}\, d\theta \leq \varepsilon w$$

for some $1 \leq \gamma \leq 1+\varepsilon$. Recall that the functions in $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{V})$ are actually $\big(\mathcal{V} \cup \{0\}\big)$-valued. However, in the case of Definition 4.12, without loss of generality we may assume that the function $s_{(w,\varepsilon)} \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{V})$ is indeed $\mathcal{V}$-valued, as we can otherwise replace it by a function $\tilde{s}_{(w,\varepsilon)} = s_{(w,(\varepsilon/2))} + \chi_{X} \otimes v$, where $v \in \mathcal{V}$ is such that $\theta_E(v) \leq (\varepsilon/2)w$, hence $\int_E \tilde{s}_{(w,\varepsilon)}\, d\theta \leq \varepsilon w$.

Consequently, for an integrable function $f \in \mathcal{F}(X,\mathcal{P})$ and a net $(f_{(w,\varepsilon)})_{w\in\mathcal{W}}^{\varepsilon>0}$ a of functions in $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ satisfying the above, we shall show that the limit

$$\int_E f\, d\theta = \lim_{\substack{\varepsilon > 0 \\ w \in \mathcal{W}}} \int_E f_{(w,\varepsilon)}\, d\theta$$

exists and is independent of the particular choice for the net $(f_{(w,\varepsilon)})_{w\in\mathcal{W}}^{\varepsilon>0}$. (The index set for this net is $\mathcal{W} \times \{\varepsilon > 0\}$ with the reverse componentwise order.) Indeed, given $w \in \mathcal{W}$ and $\varepsilon > 0$, for all $w_1, w_2 \in \mathcal{W}$ such that $w_1, w_2 \leq w$ and $0 < \varepsilon_1, \varepsilon_2 \leq \varepsilon$ we have

$$f_{(w_1,\varepsilon_1)} \leq \gamma f + s_{(w_1,\varepsilon_1)} \leq \gamma_1 f_{(w_2,\varepsilon)} + s_{(w_1,\varepsilon_1)}$$

for some $1 \leq \gamma \leq 1+\varepsilon$, hence

$$\int_E f_{(w_1,\varepsilon_1)} \leq \gamma_1 \int_E f_{(w_2,\varepsilon_2)} + \varepsilon w.$$

Thus $\big(\int_E f_{(w,\varepsilon)}\big)_{w\in\mathcal{W}}^{\varepsilon>0}$ forms a Cauchy net in the symmetric relative topology of $\mathcal{Q}$, hence is convergent by Proposition I.5.41. The preceding argument together with Lemma I.5.20(c) also shows that this limit is independent of the particular choice for the net $(f_{(w,\varepsilon)})_{w\in\mathcal{W}}^{\varepsilon>0}$.

4.13 Integrability over a Set $F \in \mathfrak{A}_{\mathfrak{R}}$. Obviously, integrability in the sense of 4.12 for a function $f \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ over a set $E \in \mathfrak{R}$ implies integrability over all subsets $G \in \mathfrak{R}$ of E. This observation, together with Lemma 4.10 shows that we may consistently define integrability over sets in the σ-algebra $\mathfrak{A}_{\mathfrak{R}}$ in the following way: We shall say that a function $f \in \mathcal{F}(X,\mathcal{P})$ is *integrable over* $F \in \mathfrak{A}_{\mathfrak{R}}$ *with respect to* θ if f is integrable over the sets $E \cap F$ for all $E \in \mathfrak{R}$ and if the limit

$$\int_F f\,d\theta = \lim_{E\in\mathfrak{R}} \int_{(E\cap F)} f\,d\theta$$

exists in $\mathcal{Q}$. The set of all functions in $\mathcal{F}(X,\mathcal{P})$ that are integrable over F shall be denoted by $\mathcal{F}_{(F,\theta)}(X,\mathcal{P})$. Lemma 4.10(b) implies that $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P}) \subset \mathcal{F}_{(F,\theta)}(X,\mathcal{P})$ for every $F \in \mathfrak{A}_R$ and every $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measure θ on $\mathfrak{R}$.

We may use this definition of integrability also for functions that take the value $\infty \in \overline{\mathcal{V}}$ on a set of measure zero (see Section 2.1).

Theorem 4.14. *Let $F \in \mathfrak{A}_{\mathfrak{R}}$. Then $\mathcal{F}_{(F,\theta)}(X,\mathcal{P})$ is a subcone of $\mathcal{F}(X,\mathcal{P})$ containing $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$. More precisely, for $f,g \in \mathcal{F}_{(F,\theta)}(X,\mathcal{P})$ and $0 \le \alpha \in \mathbb{R}$ we have*

(a) $\int_F (\alpha f)\,d\theta = \alpha \int_F f\,d\theta$
(b) $\int_F (f+g)\,d\theta = \int_F f\,d\theta + \int_F g\,d\theta$
(c) $\int_F f\,d\theta \le \int_F g\,d\theta$ *whenever* $f \underset{a.e.F}{\le} g$.

Proof. In a first case, let us assume that $f,g \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ and that $F = E \in \mathfrak{R}$. Then Part (a) follows trivially from our definition of the integral. For (b), let $w \in \mathcal{W}$, and $h_1 \le f + \mathfrak{v}_w$ and $h_2 \le g + \mathfrak{v}_w$ for $h_1, h_2 \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P})$. Then $h_1 + h_2 \le (f+g) + 2\mathfrak{v}_w$, hence

$$\int_E^{(w)} f\,d\theta + \int_E^{(w)} g\,d\theta \le \int_E^{(2w)} (f+g)\,d\theta,$$

and therefore

$$\int_E f\,d\theta + \int_F g\,d\theta \le \int_E (f+g)\,d\theta.$$

For the converse inequality, let $u \in \mathcal{W}$. For the set $E \in \mathfrak{R}$ the inductive limit neighborhood $\mathfrak{v}_u$ and any $\varepsilon > 0$ choose sequences $(h_n)_{n\in\mathbb{N}}$ and $(l_n)_{n\in\mathbb{N}}$ of step functions in $\mathcal{S}_{\mathfrak{R}}(X,\mathcal{P})$ approaching f and g as in Corollary 2.8, respectively. The sequence $(k_n)_{n\in\mathbb{N}}$, where $k_n = h_n + l_n$ then approaches the function $f+g$ with respect to F, the inductive limit neighborhood $2\mathfrak{v}_u$ and ε. Thus by Proposition 4.7 we have

$$\begin{aligned}
\int_E^{(w)} (f+g)\,d\theta &\le \underline{\lim}_{n\to\infty} \int_E k_n\,d\theta + w \\
&\le \overline{\lim}_{n\to\infty} \int_E h_n\,d\theta + \overline{\lim}_{n\to\infty} \int_E l_n\,d\theta + w \\
&\le \int_E^{(u)} f\,d\theta + \int_E^{(u)} g\,d\theta + w
\end{aligned}$$

for all $w \in \mathcal{W}$. This yields

$$\int_E (f+g)\,d\theta \le \int_E^{(u)} f\,d\theta + \int_E^{(u)} g\,d\theta,$$

since $\mathcal{Q}$ is a locally convex complete lattice cone, and indeed

$$\int_E (f+g)\,d\theta \leq \int_E f\,d\theta + \int_E f\,d\theta$$

after applying the infima over all $u \in \mathcal{W}$ on the right-hand side and using the rules from Section I.5.

Now in a second case, we still suppose that $F = E \in \mathfrak{R}$, and let $f, g \in \mathcal{F}_{(E,\theta)}(X, \mathcal{P})$. Let $(f_{(w,\varepsilon)})_{w\in\mathcal{W}}^{\varepsilon>0}$ and $(g_{(w,\varepsilon)})_{w\in\mathcal{W}}^{\varepsilon>0}$ be nets of functions in $\mathcal{F}_{\mathfrak{R}}(X, \mathcal{P})$ approaching the functions f and g as in 4.12. Then the nets $(\alpha f_{(w,\varepsilon)})_{w\in\mathcal{W}}^{\varepsilon>0}$ and $(f_{(w,\varepsilon)}+g_{(w,\varepsilon)})_{w\in\mathcal{W}}^{\varepsilon>0}$ approach the functions αf and $f+g$, respectively, and the limit rules from Section 4 yield

$$\int_E \alpha f\,d\theta = \lim_{\substack{\varepsilon>0\\ w\in\mathcal{W}}} \int_E \alpha f_{(w,\varepsilon)}\,d\theta = \alpha \int_E f_{(w,\varepsilon)}\,d\theta = \alpha \int_E f\,d\theta$$

and

$$\begin{aligned}
\int_E (f+g)\,d\theta &= \lim_{\substack{\varepsilon>0\\ w\in\mathcal{W}}} \int_E \left(f_{(w,\varepsilon)} + g_{(w,\varepsilon)}\right)d\theta \\
&= \lim_{\substack{\varepsilon>0\\ w\in\mathcal{W}}} \int_E f_{(w,\varepsilon)}\,d\theta + \lim_{\substack{\varepsilon>0\\ w\in\mathcal{W}}} \int_E g_{(w,\varepsilon)}\,d\theta \\
&= \int_E f\,d\theta + \int_E g\,d\theta.
\end{aligned}$$

For Part (c) in this case, suppose that $f \underset{a.e.E}{\leq} g$ and let $(f_{(w,\varepsilon)})_{w\in\mathcal{W}}^{\varepsilon>0}$ and $(g_{(w,\varepsilon)})_{w\in\mathcal{W}}^{\varepsilon>0}$ be nets in $\mathcal{F}_{\mathfrak{R}}(X, \mathcal{P})$ as before. Then

$$f_{(w,\varepsilon)} \underset{a.e.E}{\leq} \gamma f + s_{(w,\varepsilon)} \underset{a.e.E}{\leq} \gamma g + s_{(w,\varepsilon)} \underset{a.e.E}{\leq} \gamma g_{(w,\varepsilon)} + s_{(w,\varepsilon)},$$

hence

$$\int_E f_{(w,\varepsilon)}\,d\theta \leq \gamma \int_E g_{(w,\varepsilon)}\,d\theta + \varepsilon w$$

with some $1 \leq \gamma \leq 1+\varepsilon$ for all $w \in \mathcal{W}$ and $\varepsilon > 0$. According to the limit rules in Section I.5, this yields

$$\int_E f\,d\theta = \lim_{\substack{\varepsilon>0\\ w\in\mathcal{W}}} \int_E f_{(w,\varepsilon)}\,d\theta \leq \lim_{\substack{\varepsilon>0\\ w\in\mathcal{W}}} \int_E g_{(w,\varepsilon)}\,d\theta = \int_E g\,d\theta.$$

For the final and general case, let $F \in \mathfrak{A}_{\mathfrak{R}}$ and $f, g \in \mathcal{F}_{(F,\theta)}(X, \mathcal{P})$. Then the claims of Parts (a),(b) and (c) hold for integrals over all sets $E \cap F$ for $E \in \mathfrak{R}$. The definition of the respective integrals over F together with the limit rules from Lemma I.5.19 yield the validity of these claims for the integrals over F as well. $\square$

Simple examples can show that $F \subset G$ for $F, G \in \mathfrak{A}_{\mathfrak{R}}$ does not necessarily imply that $\mathcal{F}_{(F,\theta)}(X,\mathcal{P}) \subset \mathcal{F}_{(G,\theta)}(X,\mathcal{P})$, but we have the following:

Proposition 4.15. *Let $f \in \mathcal{F}(X,\mathcal{P})$ and $F, G \in \mathfrak{A}_{\mathfrak{R}}$*

(a) If $F, G \in \mathfrak{A}_{\mathfrak{R}}$, then f is integrable over $F \cap G$ if and only if $\chi_{G^{\otimes}} f$ is integrable over F, if and only if $\chi_{F^{\otimes}} f$ is integrable over G. In this case we have $\int_{(F\cap G)} f\,d\theta = \int_F \chi_{G^{\otimes}} f\,d\theta = \int_G \chi_{F^{\otimes}} f\,d\theta$.

(b) If F and G are disjoint and f is integrable over F and G, then f is integrable over $F \cup G$ and $\int_{(F\cup G)} f\,d\theta = \int_F f\,d\theta + \int_G f\,d\theta$.

(c) If $F \subset G$ and f is integrable over F, G and $G \setminus F$, then $\mathfrak{D}\left(\int_F f\,d\theta\right) \leq \mathfrak{D}\left(\int_G f\,d\theta\right)$.

Proof. For Part (a), in a first step let $E \in \mathfrak{R}$. First we observe from Definition 4.12 that a function $f \in \mathcal{F}(X,\mathcal{P})$ is integrable over E if and only if $\chi_{E^{\otimes}} f$ is integrable over E and that $\int_E f\,d\theta = \int_E \chi_{E^{\otimes}} f\,d\theta$. Thus, if for $f \in \mathcal{F}(X,\mathcal{P})$, the function $\chi_{E^{\otimes}} f$ is integrable over X, then by Definition 4.13 the function $\chi_{E^{\otimes}} f$ and therefore f is integrable over E. For the converse, assume that $f \in \mathcal{F}(X,\mathcal{P})$ is integrable over E. Let $E' \in \mathfrak{R}$, $w \in \mathcal{W}$ and $\varepsilon > 0$. According to 4.12 there are $f_{(w,\varepsilon)} \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ and $s_{(w,\varepsilon)} \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{V})$ such that $f \underset{a.e.E}{\leq} f_{(w,\varepsilon)} \underset{a.e.E}{\leq} \gamma f + s_{(w,\varepsilon)}$ with some $1 \leq \gamma \leq 1+\varepsilon$ and $\int_E s_{(w,\varepsilon)}\,d\theta \leq \varepsilon w$. Then we have

$$\chi_{E^{\otimes}} f \underset{a.e.E'}{\leq} \chi_{E^{\otimes}} f_{(w,\varepsilon)} \underset{a.e.E'}{\leq} \gamma f + \chi_{E^{\otimes}} s_{(w,\varepsilon)}$$

and

$$\int_{E'} \chi_{E^{\otimes}} s_{(w,\varepsilon)}\,d\theta \leq \varepsilon w$$

as well. Because the functions $\chi_{E^{\otimes}} f_{(w,\varepsilon)}$ and $\chi_{E^{\otimes}} s_{(w,\varepsilon)}$ are also contained in $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ and $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{V})$, respectively, we conclude that the function $\chi_{E^{\otimes}} f$ is integrable over E' and that

$$\int_{E'} \chi_{E^{\otimes}} f\,d\theta = \lim_{\substack{\varepsilon > 0 \\ w \in \mathcal{W}}} \int_{E'} \chi_{E^{\otimes}} f_{(w,\varepsilon)}\,d\theta = \lim_{\substack{\varepsilon > 0 \\ w \in \mathcal{W}}} \int_X \chi_{(E'\cap E)^{\otimes}} f_{(w,\varepsilon)}\,d\theta.$$

The last equality follows from Lemma 4.10(a). The above holds for all sets $E' \in \mathfrak{R}$, hence using Definition 4.13, we realize that the function $\chi_{E^{\otimes}} f$ is indeed integrable over X and that

$$\begin{aligned}\int_X \chi_{E^{\otimes}} f\,d\theta &= \lim_{E' \in \mathfrak{R}} \int_{E'} \chi_{E^{\otimes}} f\,d\theta = \lim_{\substack{\varepsilon > 0 \\ w \in \mathcal{W}}} \int_X \chi_{E^{\otimes}} f_{(w,\varepsilon)}\,d\theta \\ &= \lim_{\substack{\varepsilon > 0 \\ w \in \mathcal{W}}} \int_E f_{(w,\varepsilon)}\,d\theta = \int_E f\,d\theta\end{aligned}$$

holds. Thus we have verified that a function $f \in \mathcal{F}(X,\mathcal{P})$ is integrable over a set $E \in \mathfrak{R}$ if and only if $\chi_{E^\otimes} f$ is integrable over X and that $\int_E f\,d\theta = \int_X \chi_{E^\otimes} f\,d\theta$ in this case. Now in a second step, let $F \in \mathfrak{A}_{\mathfrak{R}}$ and $f \in \mathcal{F}(X,\mathcal{P})$. By the above, the function f is integrable over all sets $E \cap F$ for $E \in \mathfrak{R}$, if and only if all the functions $\chi_{(E\cap F)^\otimes} f = \chi_{E^\otimes}(\chi_{F^\otimes} f)$ are integrable over X. In this case

$$\int_{(E\cap F)} f\,d\theta = \int_X \chi_{(E\cap F)^\otimes} f\,d\theta = \int_X \chi_{E^\otimes}(\chi_{F^\otimes} f)\,d\theta = \int_E \chi_{F^\otimes} f\,d\theta$$

holds by our first step. According to Definition 4.13 therefore f is integrable over F if and only $\chi_{F^\otimes} f$ is integrable over X and

$$\int_F f\,d\theta = \lim_{E\in\mathfrak{R}} \int_{(E\cap F)} f\,d\theta = \lim_{E\in\mathfrak{R}} \int_E \chi_{F^\otimes} f\,d\theta = \int_X \chi_{F^\otimes} f\,d\theta.$$

In a third and final step for Part (a), let $F, G \in \mathfrak{A}_{\mathfrak{R}}$. From the preceding we conclude that $\chi_{G^\otimes} f$ is integrable over F if and only if $\chi_{F^\otimes}(\chi_{G^\otimes} f) = \chi_{(F\cap G)^\otimes} f$ is integrable over X, that is f is integrable over $F \cap G$, and all the integrals coincide.

For Part (b), suppose that $F \cap G = \emptyset$ and that f is integrable over both F and G. Then both functions $\chi_{F^\otimes} f$ and $\chi_{G^\otimes} f$ are integrable over X by Part (a), hence $\chi_{(F\cup G)^\otimes} f = \chi_{F^\otimes} f + \chi_{G^\otimes} f$ is also integrable over X by Theorem 4.14(b). Thus f is indeed integrable over $F \cup G$ and

$$\int_{(F\cup G)} f\,d\theta = \int_X \chi_{(F\cup G)^\otimes} f = \int_X \chi_{F^\otimes} f + \int_X \chi_{G^\otimes} f = \int_F f\,d\theta + \int_G f\,d\theta$$

by 4.14(b)

For Part (c), suppose that $F \subset G$ and that f is integrable over F, G and $G \setminus F$. Then

$$\int_G f\,d\theta = \int_F f\,d\theta + \int_{(G\setminus F)} f\,d\theta$$

by Part (b), and

$$\mathfrak{O}\left(\int_F f\,d\theta\right) \le \mathfrak{O}\left(\int_F f\,d\theta\right) + \mathfrak{O}\left(\int_{(G\setminus F)} f\,d\omega\right) = \mathfrak{O}\left(\int_G f\,d\theta\right).$$

by Proposition I.5.11(a). $\square$

Proposition 4.16. *Let $f, g \in \mathcal{F}_{(E,\theta)}(X,\mathcal{P})$ for $E \in \mathfrak{R}$ and let $v \in \mathcal{V}$. If $f(x) \preccurlyeq_v g(x)$ holds θ-almost everywhere on E, then $\int_E f\,d\theta \le \int_E g\,d\theta + \mathfrak{O}\big(\theta_E(v)\big)$.*

Proof. Let $E \in \mathfrak{R}$, let $v \in \mathcal{V}$ and $f, g \in \mathcal{F}_{(E,\theta)}(X,\mathcal{P})$ such that $f(x) \preccurlyeq_v g(x)$ θ-almost everywhere on E. In a first case, let us assume in addition that $g \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$. Lemma 2.4(b) implies that there is $\lambda \ge 0$ such that

$0 \leq g(x) + \lambda v$ for all $x \in E$. Recall from Section 2 that $f(x) \preccurlyeq_v g(x)$ means that $f(x) \in v_\varepsilon\big(g(x)\big)$ for all $\varepsilon > 0$. In turn, $f(x) \in v_\varepsilon\big(g(x)\big)$ and $0 \leq g(x) + \lambda v$ implies $f(x) \leq (1+\varepsilon)g(x) + \varepsilon(1+\lambda)v$ by Lemma I.4.1(b). Thus our assumption yields

$$f \underset{a.e.E}{\leq} (1+\varepsilon)g + \varepsilon(1+\lambda)\chi_{E \otimes} v$$

for all $\varepsilon > 0$. By Theorem 4.14(c), this implies

$$\int_E f\, d\theta \leq (1+\varepsilon) \int_E g\, d\theta + \varepsilon(1+\lambda)\theta_E(v).$$

Now we let ε tend to 0 in the right-hand side of this expression. Lemma I.5.21 together with the definition of the zero component in I.5.8 leads to

$$\int_E f\, d\theta \leq \int_E g\, d\theta + \mathfrak{O}\big(\theta_E(v)\big).$$

Now we may argue the general case: Suppose that $f, g \in \mathcal{F}_{(E,\theta)}(X, \mathcal{P})$, let $w \in \mathcal{W}$ and $\varepsilon > 0$, and for g choose the functions $g_{(w,\varepsilon)} \in \mathcal{F}_{\mathfrak{R}}(X, \mathcal{P})$ and $s_{(w,\varepsilon)} \in \mathcal{F}_{\mathfrak{R}}(X, \mathcal{V})$ as in 4.13, that is $g \underset{a.e.E}{\leq} g_{(w,\varepsilon)} \underset{a.e.E}{\leq} \gamma g + s_{(w,\varepsilon)}$ and $\int_E s_{(w,\varepsilon)}\, d\theta \leq \varepsilon w$ for some $1 \leq \gamma \leq 1+\varepsilon$. Then $f(x) \preccurlyeq_v g_{(w,\varepsilon)}(x)$ holds θ-almost everywhere on E, and our first case together with 4.14(c) yields

$$\int_E f\, d\theta \leq \int_E g_{(w,\varepsilon)}\, d\theta + \mathfrak{O}\big(\theta_E(v)\big) \leq \gamma \int_E g\, d\theta + \mathfrak{O}\big(\theta_E(v)\big) + \varepsilon w.$$

Because $w \in \mathcal{W}$ and $\varepsilon > 0$ were arbitrarily chosen, our claim follows. □

The following Proposition 4.17 is an immediate consequence of 4.16 and strengthens Part (c) of Theorem 4.14(c).

Proposition 4.17. *Let $f, g \in \mathcal{F}_{(F,\theta)}(X, \mathcal{P})$ for $F \in \mathfrak{A}_{\mathfrak{R}}$. If $f(x) \preccurlyeq g(x)$ holds θ-almost everywhere on F, then $\int_F f\, d\theta \leq \int_F g\, d\theta$.*

Proof. Let $F \in \mathfrak{A}_{\mathfrak{R}}$ and $f, g \in \mathcal{F}_{(F,\theta)}(X, \mathcal{P})$ such that $f(x) \preccurlyeq g(x)$ holds θ-almost everywhere on F. Let $E \in \mathfrak{R}$, $w \in \mathcal{W}$, and choose $v \in \mathcal{V}$ such that $\theta_E(v) \leq w$. As $f(x) \preccurlyeq g(x)$ implies $f(x) \preccurlyeq_v g(x)$, Proposition 4.16 yields

$$\int_{(E\cap F)} f\, d\theta \leq \int_{(E\cap F)} g\, d\theta + w,$$

hence $\int_{(E\cap F)} f\, d\theta \leq \int_{(E\cap F)} g\, d\theta$, since $w \in \mathcal{W}$ was arbitrarily chosen. Now our definition of the integral over a set $F \in \mathfrak{A}_{\mathfrak{R}}$ in 4.13 together with Lemma I.5.20(c) yields our claim. □

Proposition 4.18. *Let $f \in \mathcal{F}_{(E,\theta)}(X, \mathcal{P})$ for $E \in \mathfrak{R}$.*

(a) If $E_n \in \mathfrak{R}$ such that $E_n \subset E_{n+1}$ for all $n \in \mathbb{N}$, and $E = \bigcup_{n\in\mathbb{N}} E_n$, then $\int_E f\, d\theta = \lim_{n\to\infty} \int_{E_n} f\, d\theta$.

(b) If $E_n \in \mathfrak{R}$ *such that* $E \supset E_n \supset E_{n+1}$ *for all* $n \in \mathbb{N}$, *and* $\bigcap_{n\in\mathbb{N}} E_n = \emptyset$, *then* $0 \le \underline{\lim}_{n\to\infty} \int_{E_n} f\,d\theta \le \overline{\lim}_{n\to\infty} \int_{E_n} f\,d\theta \le \mathfrak{O}\left(\int_E f\,d\theta\right)$.

Proof. For Part (a), let $E_n \in \mathfrak{R}$ such that $E_n \subset E_{n+1}$ for all $n \in \mathbb{N}$, and $E = \bigcup_{n\in\mathbb{N}} E_n \in \mathfrak{R}$. We shall first assume that $f \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$. By Lemma 2.4, for $w \in \mathcal{W}$ there is a neighborhood $v \in \mathcal{V}$ and $\lambda \ge 0$ such that $\theta_E(v) \le w$ and $0 \le f + \lambda\chi_{E\otimes}v$. This implies

$$\chi_{E_n\otimes}f \le \chi_{E_n\otimes}f + \chi_{(E\setminus E_n)\otimes}(f + \lambda\chi_{E\otimes}v) = \chi_{E\otimes}f + \lambda\chi_{(E\setminus E_n)\otimes}v.$$

Thus

$$\int_{E_n} f\,d\theta \le \int_E (f + \lambda\chi_{(E\setminus E_n)\otimes}v)\,d\theta = \int_E f\,d\theta + \lambda\theta_{(E\setminus E_n)}(v).$$

Following Lemma 3.1(b), this yields

$$\overline{\lim_{n\to\infty}} \int_{E_n} f\,d\theta \le \int_E f\,d\theta + \varepsilon w$$

for all $\varepsilon \ge 0$. Now let $h \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P})$ be a step function such that $h \le f + \mathfrak{v}_w$, that is $h \le f + s$ for some $s \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{V})$ such that $\int_X s\,d\theta \le w$. Then $\int_{E_n} h\,d\theta \le \int_{E_n} f\,d\theta + w$ by 4.13(b) and (c), hence

$$\int_E h\,d\theta = \lim_{n\to\infty} \int_{E_n} h\,d\theta \le \underline{\lim}_{n\to\infty} \int_{E_n} f\,d\theta + w.$$

Taking the supremum over all such step functions $h \le f + \mathfrak{v}_w$ yields

$$\int_F^{(w)} f\,d\theta \le \underline{\lim}_{n\to\infty} \int_{E_n} f\,d\theta + w.$$

Combining with the above we infer that

$$\int_E f\,d\theta \le \underline{\lim}_{n\to\infty} \int_{E_n} f\,d\theta + w \le \overline{\lim_{n\to\infty}} \int_{E_n} f\,d\theta + w \le \int_E f\,d\theta + (1+\varepsilon)w.$$

Thus indeed $\int_E f\,d\theta = \lim_{n\to\infty} \int_{E_n} f\,d\theta$, since $w \in \mathcal{W}$ and $\varepsilon > 0$ were arbitrary. Now for the general case, let $f \in \mathcal{F}_{(E,\theta)}(X,\mathcal{P})$. Given $w \in \mathcal{W}$ and $\varepsilon > 0$ choose the functions $f_{(w,\varepsilon)} \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ and $s_{(w,\varepsilon)} \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{V})$ as in Definition 4.12. Then the preceding yields

$$\int_E f\,d\theta \le \int_E f_{(w,\varepsilon)}\,d\theta = \lim_{n\to\infty} \int_{E_n} f_{(w,\varepsilon)}\,d\theta \le \gamma \underline{\lim}_{n\to\infty} \int_{E_n} f\,d\theta + \varepsilon w,$$

and

$$\overline{\lim_{n\to\infty}} \int_{E_n} f\,d\theta \le \overline{\lim_{n\to\infty}} \int_{E_n} f_{(w,\varepsilon)}\,d\theta = \int_E f_{(w,\varepsilon)}\,d\theta \le \gamma \int_E f\,d\theta + \varepsilon w.$$

Our claim from Part (a) follows, since both $w \in \mathcal{W}$ and $\varepsilon > 0$ were arbitrary.

For Part (b), let $E_n \in \mathfrak{R}$ such that $E \supset E_n \supset E_{n+1}$ for all $n \in \mathbb{N}$, and $\bigcap_{n\in\mathbb{N}} E_n = \emptyset$. For the left-hand side of the inequality in (b) we shall again first assume that $f \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$. Let $w \in \mathcal{W}$. Following Lemma 2.4(b), there is $v \in \mathcal{V}$ and $\lambda \geq 0$ such that $\theta_E(v) \leq w$ and $0 \leq f + \lambda\chi_{E\otimes}v$, hence $0 \leq \chi_{E_n\otimes}f + \lambda\chi_{E_n\otimes}v$. Then

$$0 \leq \int_{E_n} (f + \lambda\chi_{E_n\otimes}v)\, d\theta = \int_{E_n} f\, d\theta + \lambda\theta_{E_n}(v).$$

This yields

$$0 \leq \underline{\lim}_{n\to\infty} \int_{E_n} f\, d\theta + \lambda\mathfrak{O}\big(\theta_E(v)\big)$$

by Lemma 3.1(b). Because $\mathfrak{O}\big(\theta_E(v)\big) \leq \varepsilon w$ for all $\varepsilon > 0$, because $w \in \mathcal{W}$ was arbitrary and $\mathcal{Q}$ carries the weak preorder, we infer that $0 \leq \underline{\lim}_{n\to\infty} \int_{E_n} f\, d\theta$. For the general case, that is $f \in \mathcal{F}_{(E,\theta)}(X,\mathcal{P})$, given $w \in \mathcal{W}$ and $\varepsilon >, 0$ we choose functions $f_{(w,\varepsilon)} \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ and $s_{(w,\varepsilon)} \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{V})$ as in Definition 4.12. Then the preceding yields together with the limit rules from Lemma I.5.19

$$\begin{aligned} 0 \leq \underline{\lim}_{n\to\infty} \int_{E_n} f_{(w,\varepsilon)}\, d\theta &\leq \gamma \underline{\lim}_{n\to\infty} \int_{E_n} f\, d\theta + \overline{\lim}_{n\to\infty} \int_{E_n} s_{(w,\varepsilon)}\, d\theta \\ &\leq \gamma \underline{\lim}_{n\to\infty} \int_{E_n} f\, d\theta + \varepsilon w. \end{aligned}$$

Thus indeed $0 \leq \underline{\lim}_{n\to\infty} \int_{E_n} f\, d\theta$, since $w \subset \mathcal{W}$ and $\varepsilon > 0$ were arbitrarily chosen. For the right-hand side of the inequality in (b), let $G_n = E \setminus E_n$. Then $G_n \subset G_{n+1}$, $E = \cup_{n=1}^{\infty} G_n$ and $E = G_n \cup E_n$. Thus

$$\int_{F_n} f\, d\theta + \int_{E_n} f\, d\theta = \int_E f\, d\theta$$

for all $n \in \mathbb{N}$ by 4.15(b). Part (a) of 4.18 yields $\int_E f\, d\theta = \lim_{n\to\infty} \int_{G_n} f\, d\theta$. Again using the limit rules in Lemma I.5.19 we infer that

$$\begin{aligned} \overline{\lim}_{n\to\infty} \int_{E_n} f\, d\theta + \int_E f\, d\theta &= \overline{\lim}_{n\to\infty} \int_{E_n} f\, d\theta + \lim_{n\to\infty} \int_{G_n} f\, d\theta \\ &\leq \overline{\lim}_{n\to\infty} \left(\int_{E_n} f\, d\theta + \int_{G_n} f\, d\theta \right) = \int_E f\, d\theta. \end{aligned}$$

Now the cancellation rule in Proposition I.5.10(a) yields

$$\overline{\lim}_{n\to\infty} \int_{E_n} f\, d\theta \leq \mathfrak{O}\left(\int_E f\, d\theta \right). \quad \square$$

Given a set $F \in \mathfrak{A}_{\mathfrak{A}}$ we shall denote by $\mathcal{F}_{(|F|,\theta)}(X,\mathcal{P})$ the subcone of all functions in $\mathcal{F}(X,\mathcal{P})$ that are integrable over all complements in F of sets in $\mathfrak{R}$, that is

$$\mathcal{F}_{(|F|,\theta)}(X,\mathcal{P}) = \bigcap_{E\in\mathfrak{R}} \mathcal{F}_{(F\setminus E,\theta)}(X,\mathcal{P}).$$

Using this notion, we obtain:

Proposition 4.19. *Let* $f \in \mathcal{F}_{(|F|,\theta)}(X,\mathcal{P})$ *for* $F \in \mathfrak{A}_{\mathfrak{R}}$.
Then $0 \le \underline{\lim}_{E\in\mathfrak{R}} \int_{(F\setminus E)} f\,d\theta \le \overline{\lim}_{E\in\mathfrak{R}} \int_{(F\setminus E)} f\,d\theta \le \mathfrak{O}\left(\int_F f\,d\theta\right)$.

Proof. For every $E \in \mathfrak{R}$ the function f is integrable over $E \cap F \in \mathfrak{R}$ and $F \setminus E \in \mathfrak{A}_{\mathfrak{R}}$. Thus $\int_{(F\setminus E)} f\,d\theta + \int_{(E\cap F)} f\,d\theta = \int_F f\,d\theta$ by 4.15(b). Taking the limit over all $E \in \mathfrak{R}$ and using the definition of the integral in 4.13 and Lemma I.5.19, we obtain

$$\overline{\lim_{E\in\mathfrak{R}}} \int_{(F\setminus E)} f\,d\theta + \int_F f\,d\theta \le \int_F f\,d\theta,$$

hence

$$\overline{\lim_{E\in\mathfrak{R}}} \int_{(F\setminus E)} f\,d\theta \le \mathfrak{O}\left(\int_F f\,d\theta\right)$$

by the cancellation rule Proposition I.5.10(a). For the first part of the inequality in 4.19, we fix $E_0 \in \mathfrak{R}$ and let $E_0 \subset E \in \mathfrak{R}$. Then

$$\int_{(F\setminus E_0)} f\,d\theta = \int_{(F\setminus E)} f\,d\theta + \int_{\big((F\setminus E_0)\cap E\big)} f\,d\theta.$$

Passing to the limits over $E \in \mathfrak{R}$ in this equation and again using I.5.19 and the definition of the integral leads to

$$\int_{(F\setminus E_0)} f\,d\theta \le \underline{\lim_{E\in\mathfrak{R}}} \int_{(F\setminus E)} f\,d\theta + \int_{(F\setminus E_0)} f\,d\theta.$$

Now passing to the limit over $E_0 \in \mathfrak{R}$, we obtain

$$\underline{\lim_{E\in\mathfrak{R}}} \int_{(F\setminus E)} f\,d\theta \le \underline{\lim_{E\in\mathfrak{R}}} \int_{(F\setminus E)} f\,d\theta + \underline{\lim_{E\in\mathfrak{R}}} \int_{(F\setminus E)} f\,d\theta.$$

Following Proposition I.5.10(a) and Proposition I.5.14, the latter implies

$$0 \le \underline{\lim_{E\in\mathfrak{R}}} \int_{(F\setminus E)} f\,d\theta,$$

as claimed. $\square$

5. The General Convergence Theorems

We shall proceed to establish a range of general convergence results for sequences of measures and functions and their respective integrals. These results are modeled after the dominated convergence theorem from classical measure theory. However, the presence of unbounded elements and the general absence of negatives will considerably complicate some technical aspects of the approach. First we shall extend some of the concepts of the preceding section from a single measure to families of measures. Subsequently, we shall set up suitable notions for convergence of sequences of measures and functions. Convergence for sequences of integrals will generally refer to order convergence in $\mathcal{Q}$, though in some special cases we will be able to establish stronger convergence with respect to the symmetric topology.

As in the preceding section, let $(\mathcal{P},\mathcal{V})$ be a full locally convex cone and let $(\mathcal{Q},\mathcal{V})$ be a locally convex complete lattice cone. $\mathfrak{R}$ denotes a (weak) σ-ring of subsets of X. We shall consider $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measures on $\mathfrak{R}$.

5.1 Families of Measures and Properties Holding Almost Everywhere. In the following we shall simultaneously deal with families of measures, and therefore need to extend our notion of properties holding almost everywhere from 4.11 to this situation: Given a (non-empty) family Θ of $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measures, we denote by $\mathfrak{Z}(\Theta)$ the collection of all sets $Z \in \mathfrak{A}_{\mathfrak{R}}$ such that $\theta_{(E\cap Z)} = 0$ for all $E \in \mathfrak{R}$ and $\theta \in \Theta$. This collection is obviously closed for set complements and for countable unions. Correspondingly, for a subset F of X we shall say that a pointwise defined property of functions on X holds Θ-*almost everywhere on* F if it holds on $F \setminus Z$ with some $Z \in \mathfrak{Z}(\Theta)$. If the concerned family Θ of measures is clearly identified, for the sake of simplicity we may use the symbols $\underset{a.e.F}{\leq}$ or $\underset{a.e.F}{=}$ if the relations $\leq$ or $=$ hold Θ-almost everywhere on the set F, respectively.

5.2 Equibounded Families of Measures. A family Θ of measures on $\mathfrak{R}$ is called *equibounded* if for every $E \in \mathfrak{R}$ and $w \in \mathcal{W}$ there is $v \in \mathcal{V}$ such that $|\theta|(E,v) = \theta_E(v) \leq w$ for all $\theta \in \Theta$.

5.3 Integrability with Respect to Equibounded Families of Measures. Likewise, we need to adapt our notation of integrability from Section 4.12 and 4.13. We shall say that a function $f \in \mathcal{F}(X,\mathcal{P})$ is *integrable over a set* $E \in \mathfrak{R}$ *with respect to a family* Θ of $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measures if Θ is equibounded and if for every $w \in \mathcal{W}$ and $\varepsilon > 0$ there are functions $f_{(w,\varepsilon)} \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ and $s_{(w,\varepsilon)} \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{V})$ such that

$$f \underset{a.e.E}{\leq} f_{(w,\varepsilon)} \underset{a.e.E}{\leq} \gamma f + s_{(w,\varepsilon)} \qquad \text{and} \qquad \int_E s_{(w,\varepsilon)}\, d\theta \leq \varepsilon w$$

for some $1 \leq \gamma \leq 1+\varepsilon$ and all $\theta \in \Theta$. The almost-everywhere relation $\underset{a.e.E}{\leq}$ is meant with respect to the family Θ. As in 4.12, we may again assume that the function $s_{(w,\varepsilon)} \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{V})$ is indeed $\mathcal{V}$- rather than $\big(\mathcal{V}\cup\{0\}\big)$-valued. Integrability over a set $F \in \mathfrak{A}_{\mathfrak{R}}$ with respect to Θ then follows as in 4.13: The function $f \in \mathcal{F}(X,\mathcal{P})$ is *integrable over* $F \in \mathfrak{A}_R$ *with respect to* Θ if f is integrable over the sets $E \cap F$ with respect to Θ for all $E \in \mathfrak{R}$ and all $\theta \in \Theta$ the limit

$$\int_F f\,d\theta = \lim_{E\in\mathfrak{R}} \int_{(E\cap F)} f\,d\theta$$

exists. The subcone of all these functions $f \in \mathcal{F}(X,\mathcal{P})$ is denoted by $\mathcal{F}_{(F,\Theta)}(X,\mathcal{P})$.

Likewise, $\mathcal{F}_{(|F|,\Theta)}(X,\mathcal{P})$ denotes the subcone of all functions in $\mathcal{F}(X,\mathcal{P})$ that are integrable with respect to Θ over all complements in F of sets in $\mathfrak{R}$, that is

$$\mathcal{F}_{(|F|,\Theta)}(X,\mathcal{P}) = \bigcap_{E\in\mathfrak{R}} \mathcal{F}_{(F\setminus E,\Theta)}(X,\mathcal{P}).$$

Repeating the argument from Proposition 4.15(a), one can verify that a function $f \in \mathcal{F}(X,\mathcal{P})$ is in $\mathcal{F}_{(F,\Theta)}(X,\mathcal{P})$ or in $\mathcal{F}_{(|F|,\Theta)}(X,\mathcal{P})$ if and only if the function $\chi_{F^\otimes} f$ is contained in $\mathcal{F}_{(X,\Theta)}(X,\mathcal{P})$ or in $\mathcal{F}_{(|X|,\Theta)}(X,\mathcal{P})$, respectively.

While integrability with respect to a family of measures obviously implies integrability with respect to every member of this family, the converse is not always true (see Example 5.15 below).

The following results 5.4 to 5.7 are already of interest for integration with respect to a single measure and might therefore have been placed into the preceding section. We shall, however, also refer to the subsequent more general versions which refer to integration with respect to equibounded families of measures.

Proposition 5.4. *Let* Θ *be an equibounded family of measures on* $\mathfrak{R}$. *Let* $E \in \mathfrak{R}$ *and* $f \in \mathcal{F}_{(E,\Theta)}(X,\mathcal{P})$. *For every* $w \in \mathcal{W}$ *there is* $s \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{V})$ *and* $\lambda \geq 0$ *such that* $0 \underset{a.e.E}{\leq} f + s$ *and* $\int_E s\,d\theta \leq \lambda w$ *for all* $\theta \in \Theta$.

Proof. Let $E \in \mathfrak{R}$, let $f \in \mathcal{F}_{(E,\Theta)}(X,\mathcal{P})$ and $w \in \mathcal{W}$. According to the definition of integrability in 4.12, for $\varepsilon = 1$ there are $g \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ and $s \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{V})$ such that $f \underset{a.e.E}{\leq} g \underset{a.e.E}{\leq} \gamma f + s$ for some $1 \leq \gamma \leq 2$ and $\int_E s\,d\theta \leq w$ for all $\theta \in \Theta$. We choose $v \in \mathcal{V}$ such that $\theta_E(v) \leq w$ for all $\theta \in \Theta$. Following Lemma 2.4(b) there is $G \in \mathfrak{R}$ and $\lambda \geq 0$ such that $0 \leq g + \lambda\chi_{G^\otimes} v$. The latter implies $0 \underset{a.e.E}{\leq} g + \lambda\chi_{E^\otimes} v$, hence

$$0 \underset{a.e.E}{\leq} \frac{1}{\gamma}\,(g + \lambda\chi_{E^\otimes} v) \underset{a.e.E}{\leq} f + \frac{1}{\gamma}\,(s + \lambda\chi_{E^\otimes} v) \underset{a.e.E}{\leq} f + (s + \lambda\chi_{E^\otimes} v).$$

As $s + \lambda\chi_{E^\otimes} v \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{V})$ and $\int_E (s + \lambda\chi_{E^\otimes} v)\,d\theta \leq (1+\lambda)w$ for all $\theta \in \Theta$, our claim follows. □

5.5 The Locally Convex Cone $\big(\mathcal{F}_{(F,\Theta)}(X,\mathcal{P}), \mathfrak{V}(F,\Theta)\big)$. Let Θ be an equibounded family of measures on $\mathfrak{R}$. Endowed with the order $\underset{a.e.F}{\leq}$, that is the given pointwise order Θ-almost everywhere on the set $F \in \mathfrak{A}_{\mathfrak{R}}$, $\mathcal{F}_{(F,\Theta)}(X,\mathcal{P})$ is an ordered cone. We generate a canonical convex quasi-uniform structure (see I.1.3) in the following way: With every $w \in \mathcal{W}$ and $E \in \mathfrak{R}$ we associate the neighborhood $\breve{\mathfrak{v}}_w^E(\Theta)$, defined for functions $f, g \in \mathcal{F}_{(F,\Theta)}(X,\mathcal{P})$ by

$$f \le g + \breve{\mathfrak{v}}_w^E(\Theta) \qquad \text{if} \qquad f \underset{a.e.E}{\le} g + s$$

for some $s \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{V})$ such that $\int_E s\,d\theta \le w$ for all $\theta \in \Theta$. Let $\mathfrak{V}(F,\Theta)$ denote the neighborhood system generated by the neighborhoods $\breve{\mathfrak{v}}_w^E(\Theta)$ for all $w \subset \mathcal{W}$ and $E \in \mathfrak{R}$ such that $E \subset F$. As Θ is equibounded, according to Proposition 5.4, for every function $f \in \mathcal{F}_{(F,\Theta)}(X,\mathcal{P})$, every $w \in \mathcal{W}$ and $E \in \mathfrak{R}$ such that $E \subset F$ there is $s \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{V})$ and $\lambda \ge 0$ such that $0 \underset{a.e.E}{\le} f + s$ and $\int_E s\,d\theta \le \lambda w$ holds for all $\theta \in \Theta$. Thus $0 \le f + \lambda \mathfrak{v}_w^E(\Theta)$. All functions in $\mathcal{F}_{(F,\Theta)}(X,\mathcal{P})$ are therefore bounded below with respect to these neighborhoods. In this way, $\big(\mathcal{F}_{(F,\Theta)}(X,\mathcal{P}), \mathfrak{V}(F,\Theta)\big)$ becomes a locally convex cone as elaborated in I.1.3. Theorem 4.14(c) implies that for every $E \in \mathfrak{R}$ such that $E \subset F$ and every $\theta \in \Theta$ the mapping

$$f \mapsto \int_E f\,d\theta \ : \ \mathcal{F}_{(F,\Theta)}(X,\mathcal{P}) \to \mathcal{Q}$$

is a continuous linear operator. Indeed, for $w \in \mathcal{W}$ we have $\int_E f\,d\theta \le \int_E g\,d\theta + w$ whenever $f \le g + \breve{\mathfrak{v}}_w^E(\Theta)$ for $f, g \in \mathcal{F}_{(F,\Theta)}(X,\mathcal{P})$.

5.6 Subcone-Based Integrability. The following definition of subcone-based integrability is motivated by the fact that in many realizations $(\mathcal{P},\mathcal{V})$ is indeed the standard full extension of some subcone of $\mathcal{P}$, and we might be particularly interested in functions with values in this subcone. Given a subcone $\mathcal{P}_0$ of $\mathcal{P}$ and a neighborhood subsystem $\mathcal{V}_0$ of $\mathcal{V}$, we shall say that a function f in $\mathcal{F}(X,\mathcal{P})$ is $(\mathcal{P}_0,\mathcal{V}_0)$-*based integrable* over a set $E \in \mathfrak{R}$ with respect to an equibounded family Θ of measures if for every $w \in \mathcal{W}$ and $\varepsilon > 0$ there are functions $f_{(w,\varepsilon)} \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P}_0)$ and $s_{(w,\varepsilon)} \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{V}_0)$ such that

$$f \underset{a.e.E}{\le} f_{(w,\varepsilon)} + s_{(w,\varepsilon)}, \quad f_{(w,\varepsilon)} \underset{a.e.E}{\le} \gamma f + s_{(w,\varepsilon)} \quad \text{and} \quad \int_E s_{(w,\varepsilon)}\,d\theta \le \varepsilon w$$

for some $1 \le \gamma \le 1+\varepsilon$ and all $\theta \in \Theta$. The almost-everywhere relation $\underset{a.e.E}{\le}$ is meant with respect to the family Θ. In this context, $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P}_0)$ is the subcone of $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ consisting of all measurable $\mathcal{P}_0$-valued functions such that for every inductive limit neighborhood $\mathfrak{v}$ for $\mathcal{F}(X,\mathcal{P})$ there is a $\mathcal{P}_0$-valued step function $h \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P}_0)$ satisfying $h \le f_{(w,\varepsilon)} + \mathfrak{v}$. Measurability is still defined with respect to the given neighborhood system $\mathcal{V}$ rather than the subsystem $\mathcal{V}_0$. Similarly, $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{V}_0)$ consists of all $\mathcal{V}_0$-valued functions in $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$.

Because $(\mathcal{P}_0, \mathcal{V}_0)$-based integrability over a set $E \in \mathfrak{R}$ implies $(\mathcal{P}_0, \mathcal{V}_0)$-based integrability over all subsets $G \in \mathfrak{R}$ of E, $(\mathcal{P}_0, \mathcal{V}_0)$-based integrability over a set $F \in \mathfrak{A}_{\mathfrak{R}}$ may be defined as in 5.3.

Note that a $(\mathcal{P}_0, \mathcal{V}_0)$-based integrable function is not required to be $\mathcal{P}_0$-valued. Obviously, this notion of subcone-based integrability implies integrability based on the given cone $(\mathcal{P}, \mathcal{V})$ in the sense of 4.13 and 5.3, and the $(\mathcal{P}_0, \mathcal{V}_0)$-based integrable functions form a subcone of $\mathcal{F}_{(F,\Theta)}(X, \mathcal{P})$. For $\mathcal{P}_0 = \mathcal{P}$ and $\mathcal{V}_0 = \mathcal{V}$ the definition of $(\mathcal{P}_0, \mathcal{V}_0)$-based integrability coincides of course with the definition of integrability from 5.3: Clearly, integrability in the sense of 5.3 implies $(\mathcal{P}, \mathcal{V})$-based integrability. For the converse, use $\tilde{f}_{(w,\varepsilon)} = f_{(w,\varepsilon)} + s_{(w,\varepsilon)}$ instead of $f_{(w,\varepsilon)}$ in 5.3.

Other than in the classical scenario (see for example [25], [55], [178] and [179]), our definition of integrability does not generally guarantee that an integrable cone-valued function $f \in \mathcal{F}(X, \mathcal{P})$ can be approximated (even with respect to pointwise convergence) by a sequence of step functions whose integrals then converge towards the integral of f. However, a combination of Theorem 2.7 with Proposition 4.7 yields some corresponding results.

Theorem 5.7. *Let Θ be an equibounded family of measures on $\mathfrak{R}$. Let $E \in \mathfrak{R}$ and let $f \in \mathcal{F}(X, \mathcal{P})$ be $(\mathcal{P}_0, \mathcal{V}_0)$-based integrable over E with respect to Θ for a subcone $\mathcal{P}_0$ of $\mathcal{P}$ and a subsystem $\mathcal{V}_0$ of $\mathcal{V}$. For every $w \in \mathcal{W}$ such that $\theta_E(v) \leq w$ for some $v \in \mathcal{V}_0$ and all $\theta \in \Theta$, and every $\varepsilon > 0$ there is $s \in \mathcal{F}_{\mathfrak{R}}(X, \mathcal{V}_0)$ such that $\int_E s\, d\theta \leq w$ for all $\theta \in \Theta$, $1 \leq \gamma \leq 1 + \varepsilon$ and a bounded below sequence $(h_n)_{n\in\mathbb{N}}$ of $\mathcal{P}_0$-valued step functions such that:*

(i) $h_n \underset{a.e.E}{\leq} \gamma f + s$ *holds for all* $n \in \mathbb{N}$.

(ii) Θ*-almost everywhere on* E, *for* $x \in E$ *there is* $n_0 \in \mathbb{N}$ *such that* $f(x) \leq h_n(x) + s(x)$ *for all* $n \geq n_0$.

(iii) $\int_G f\, d\theta \leq \underline{\lim}_{n\to\infty} \int_G h_n\, d\theta + w$ *and* $\int_G h_n\, d\theta \leq \gamma \int_G f\, d\theta + w$ *for all* $n \in \mathbb{N}$, *all* $G \in \mathfrak{R}$ *such that* $G \subset E$, *and all* $\theta \in \Theta$.

Proof. Let $f \in \mathcal{F}(X, \mathcal{P})$ be $(\mathcal{P}_0, \mathcal{V}_0)$-based integrable over $E \in \mathfrak{R}$. Given $w \in \mathcal{W}$ and $0 < \varepsilon \leq 1$, following our assumption there are $f_{(w,\varepsilon)} \in \mathcal{F}_{\mathfrak{R}}(X, \mathcal{P}_0)$ and $s_{(w,\varepsilon)} \in \mathcal{F}_{\mathfrak{R}}(X, \mathcal{V}_0)$ such that $\int_E s_{(w,\varepsilon)}\, d\theta \leq w/4$ for all $\theta \in \Theta$ and $f \underset{a.e.E}{\leq} f_{(w,\varepsilon)} + s$ and $f_{(w,\varepsilon)} \underset{a.e.E}{\leq} \gamma f + s_{(w,\varepsilon)}$ with some $1 \leq \gamma \leq 1 + \varepsilon/3$. By our assumption there is $v \in \mathcal{V}_0$ such that $\theta_E(v) \leq w/2$ for all $\theta \in \Theta$. We shall apply Theorem 2.7 to the locally convex cone $(\mathcal{P}_0, \mathcal{V}_0)$, the function $f_{(w,\varepsilon)} \in \mathcal{F}_{\mathfrak{R}}(X, \mathcal{P}_0)$, the neighborhood $v \in \mathcal{V}_0$, $\varepsilon/3$ in place of ε, and the inductive limit neighborhood $\mathfrak{v} = \{\chi_X \otimes v\}$. For this we observe that the measurability conditions (M1) and (M2) in Section 1 with respect to the neighborhood system $\mathcal{V}$ imply those with respect to the subsystem $\mathcal{V}_0 \subset \mathcal{V}$. There is $1 \leq \gamma' \leq 1 + \varepsilon/3$ and a bounded below sequence $(h_n)_{n\in\mathbb{N}}$ of $\mathcal{P}_0$-valued step functions such that (i), $h_n(x) \leq \gamma' f_{(w,\varepsilon)}(x) + v$ for all $x \in E$ and $n \in \mathbb{N}$, and (ii), for every $x \in E$ there is $n_0 \in \mathbb{N}$ such that $f_{(w,\varepsilon)}(x) \leq h_n(x) + v$ for all $n \geq n_0$. This yields

$$h_n \underset{a.e.E}{\leq} (\gamma\gamma')f + (\gamma' s_{(w,\varepsilon)} + \chi_{E^{\otimes}} v)$$

for all $n \in \mathbb{N}$. We set

$$s' = \gamma' s_{(w,\varepsilon)} + \chi_{E^{\otimes}} v \in \mathcal{F}_{\mathfrak{R}}(X, \mathcal{V}_0)$$

and observe that $\int_E s'\, d\theta \leq \gamma' w/4 + w/2 \leq w$ for all $\theta \in \Theta$. Because $1 \leq \gamma\gamma' \leq (1+\varepsilon/3)^2 \leq 1+\varepsilon$, this yields Part (i) of our claim with s' in place of s. Part (ii) also follows from the above, since $f_{(w,\varepsilon)}(x) \leq h_n(x) + v$ for $x \in E$ implies that

$$f(x) \leq f_{(w,\varepsilon)}(x) + s_{(w,\varepsilon)}(x) \leq h_n(x) + s_{(w,\varepsilon)}(x) + v \leq h_n(x) + s'(x).$$

For Part (iii) let $G \in \mathfrak{R}$ such that $G \subset E$ and let $\theta \in \Theta$. The second part of (iii) is obvious, since $h_n \underset{a.e.E}{\leq} \gamma f + s'$ implies that $\int_G h_n\, d\theta \leq \gamma \int_G f\, d\theta + w$ for all $n \in \mathbb{N}$. For the first part of (iii), consider the full cone $\mathcal{P}$ and let $h'_n = h_n + \chi_{E^{\otimes}} v \in \mathcal{S}_{\mathfrak{R}}(X, \mathcal{P})$. The sequence $(h'_n)_{n\in\mathbb{N}}$ of step functions approaches $f_{(w,\varepsilon)} \in \mathcal{F}_{\mathfrak{R}}(X, \mathcal{P})$ as required in Proposition 4.7, which therefore yields

$$\int_G f_{(w,\varepsilon)}\, d\theta \leq \int_G^{(w')} f_{(w,\varepsilon)}\, d\theta \leq \varliminf_{n\to\infty} \int_G h'_n\, d\theta + w'$$

for every $\theta \in \Theta$ and all $w' \in \mathcal{W}$, hence

$$\int_G f_{(w,\varepsilon)}\, d\theta \leq \varliminf_{n\to\infty} \int_G h'_n\, d\theta = \varliminf_{n\to\infty} \int_G h_n\, d\theta + \theta_G(v)$$

since $\int_G h'_n\, d\theta = \int_G h_n\, d\theta + \theta_G(v)$. Thus

$$\begin{aligned} \int_G f\, d\theta &\leq \int_G f_{(w,\varepsilon)}\, d\theta + \int_G s_{(w,\varepsilon)}\, d\theta \\ &\leq \varliminf_{n\to\infty} \int_G h_n\, d\theta + \int_G s_{(w,\varepsilon)}\, d\theta + \theta_G(v) \\ &\leq \varliminf_{n\to\infty} \int_G h_n + w \end{aligned}$$

since $\int_G s_{(w,\varepsilon)}\, d\theta \leq w/4$ and $\theta_G(v) \leq w/2$. This yields the first inequality in (iii). $\square$

If the family Θ of measures is equibounded relative to the subsystem $\mathcal{V}_0$ of $\mathcal{V}$, that is if for every $E \in \mathfrak{R}$ and every $w \in \mathcal{W}$ there is $v \in \mathcal{V}_0$ such that $\theta_E(v) \leq w$ for all $\theta \in \Theta$, then the condition on the neighborhood $v \in \mathcal{V}_0$ in Theorem 5.7 is obviously superfluous. Indeed, given $w \in \mathcal{V}$ and any $v \in \mathcal{V}_0$ there is $v' \in \mathcal{V}_0$ as above. Because the neighborhood system $\mathcal{V}_0$ is supposed to be directed downward, there is $v'' \in \mathcal{V}_0$ such that both $v'' \leq v$ and $v'' \leq v'$. Thus $\theta_E(v'') \leq w$ for all $\theta \in \Theta$, and we may apply Theorem 5.7 with v'' in place of v.

For future use, it is worthwhile to formulate as a corollary the simplifications that occur in Theorem 5.7 if the subcone $(\mathcal{P}_0, \mathcal{V}_0)$ of $(\mathcal{P}, \mathcal{V})$ is indeed a full cone, that is if $\mathcal{V}_0 \subset \mathcal{P}_0$.

Corollary 5.8. *Let Θ be an equibounded family of measures on $\mathfrak{R}$. Let $E \in \mathfrak{R}$ and let $f \in \mathcal{F}(X, \mathcal{P})$ be $(\mathcal{P}_0, \mathcal{V}_0)$-based integrable over E with respect to Θ for a subcone $\mathcal{P}_0$ of $\mathcal{P}$ and a subsystem $\mathcal{V}_0 \subset \mathcal{P}_0$ of $\mathcal{V}$. For every $w \in \mathcal{W}$ such that $\theta_E(v) \le w$ for some $v \in \mathcal{V}_0$ and all $\theta \in \Theta$, and every $\varepsilon > 0$, there is $s \in \mathcal{F}_{\mathfrak{R}}(X, \mathcal{V}_0)$ such that $\int_E s\,d\theta \le w$ for all $\theta \in \Theta$, $1 \le \gamma \le 1 + \varepsilon$ and a bounded below sequence $(h_n)_{n \in \mathbb{N}}$ of $\mathcal{P}_0$-valued step functions such that:*

(i) $h_n \underset{a.e.E}{\le} \gamma f + s$ *holds for all* $n \in \mathbb{N}$.
(ii) Θ*-almost everywhere on* E, *for* $x \in E$ *there is* $n_0 \in \mathbb{N}$ *such that* $f(x) \le h_n(x)$ *for all* $n \ge n_0$.
(iii) $\int_G f\,d\theta \le \varliminf\limits_{n \to \infty} \int_G h_n\,d\theta$ *and* $\int_G h_n\,d\theta \le \gamma \int_G f\,d\theta + w$ *for all* $n \in \mathbb{N}$, *all* $G \in \mathfrak{R}$ *such that* $G \subset E$, *and all* $\theta \in \Theta$.

Proof. Given a neighborhood $w \in \mathcal{W}$ satisfying the requirement of the corollary, and $0 < \varepsilon \le 1$ we apply Theorem 5.7 with the neighborhood $w/4 \in \mathcal{W}$ instead of w. As in the proof of 5.7 we choose $v \in \mathcal{V}$ such that $\theta_E(v) \le w/8$. Let $s \in \mathcal{F}_{\mathfrak{R}}(X, \mathcal{V}_0)$ and the sequence $(h_n)_{n \in \mathbb{N}}$ of $\mathcal{P}_0$-valued step functions as in 5.7. We apply Corollary 2.8 to the full cone $(\mathcal{V}_0, \mathcal{V}_0)$ for the function $s \in \mathcal{F}_{\mathfrak{R}}(X, \mathcal{V}_0)$ with the inductive limit neighborhood $\mathfrak{v} = \{\chi_{X \otimes} v\}$: There is a bounded below sequence $(s_n)_{n \in \mathbb{N}}$ of $\mathcal{V}_0$-valued step functions satisfying (i) $s_n \le \gamma' s + \chi_{X \otimes} v$ with some $1 \le \gamma' \le 1 + \varepsilon$ and (ii) for every $x \in E$ there is n_0 such that $s(x) \le s_n(x)$ for all $n \ge n_0$. The latter implies

$$\int_G s(x)\,d\theta \le \varliminf_{n \to \infty} \int_G s_n(x)\,d\theta$$

for all $G \in \mathfrak{R}$ such that $G \subset E$, and all $\theta \in \Theta$, by Proposition 4.7. Now we set

$$h'_n = h_n + s_n + \chi_{E \otimes} v \in \mathcal{S}_{\mathfrak{R}}(X, \mathcal{P}_0) \qquad \text{and} \qquad s' = 3s + 2\chi_{E \otimes} v \in \mathcal{F}_{\mathfrak{R}}(X, \mathcal{V}_0).$$

These are the functions that we use for Corollary 5.8: We have

$$\int_E s'\,d\theta \le 3(w/4) + 2(w/8) = w,$$

and

$$\begin{aligned} h'_n &\underset{aeE}{\le} (\gamma f + s) + s_n + \chi_{E \otimes} v \\ &\le (\gamma f + s) + (\gamma' s + \chi_{X \otimes} v) + \chi_{E \otimes} v \\ &\le \gamma f + s' \end{aligned}$$

since $1+\gamma' \leq 3$. This implies $\int_G h'_n\,d\theta \leq \gamma \int_G f\,d\theta + w$ for all $n \in \mathbb{N}$, all $G \in \mathfrak{R}$ such that $G \subset E$, and all $\theta \in \Theta$. The first part of (iii) follows from the last inequality in the proof of 5.7, that is

$$\begin{aligned}
\int_G f\,d\theta &\leq \underline{\lim}_{n\to\infty} \int_G h_n\,d\theta + \int_G s\,d\theta + \theta_G(v) \\
&\leq \underline{\lim}_{n\to\infty} \int_G h_n\,d\theta + \underline{\lim}_{n\to\infty} \int_G s_n\,d\theta + \theta_G(v) \\
&\leq \underline{\lim}_{n\to\infty} \int_G (h_n + s_n + \chi_{E^{\otimes}} v)\,d\theta \\
&\leq \underline{\lim}_{n\to\infty} \int_G h'_n\,d\theta,
\end{aligned}$$

hence our claim. $\square$

For the following recall the definition of the order topology of a locally complete lattice cone from Section I.5.43. We shall also consider integrals of measurable $\overline{\mathcal{V}}$-, that is $\mathcal{V} \cup \{0,\infty\}$-valued functions (see Section 2.1), if they take the value $\infty \in \overline{\mathcal{V}}$ only on a set of measure zero (see 4.12 and 4.13).

Corollary 5.9. *Let Θ be an equibounded family of measures on $\mathfrak{R}$. Let $F \in \mathfrak{A}_{\mathfrak{R}}$ and let $f \in \mathcal{F}(X,\mathcal{P})$ be $(\mathcal{P}_0,\mathcal{V})$-based integrable over F with respect to Θ for a subcone $\mathcal{P}_0$ of $\mathcal{P}$. Then there is a net $(h_i)_{i\in\mathcal{I}}$ of $\mathcal{P}_0$-valued step functions, a net $(s_i)_{i\in\mathcal{I}}$ of measurable $\overline{\mathcal{V}}$-valued functions and a net $(\gamma_i)_{i\in\mathcal{I}}$ in $\mathbb{R}$ such that for every $\theta \in \Theta$:*

(i) For every $x \in F$ there is $i_0 \in \mathcal{I}$ such that $f(x) \leq \gamma_i h_i(x) + s_i(x)$ and $h_i(x) \leq f(x) + s_i(x)$ for all $i \geq i_0$.
(ii) $\lim_{i\in\mathcal{I}} \int_F h_i\,d\theta = \int_F f\,d\theta$ in the order topology of $\mathcal{Q}$.
(iii) $\lim_{i\in\mathcal{I}} \int_F s_i\,d\theta = 0$ in the symmetric topology of $\mathcal{Q}$.
(iv) $\gamma_i \geq 1$ for all $i \in \mathcal{I}$ and $\lim_{i\in\mathcal{I}} \gamma_i = 1$.

Consequently, $\int_F f\,d\theta$ is contained in the closure with respect to the order topology of the subcone of $\mathcal{Q}$ spanned by the set $\{\theta_{(E\cap F)}(a) \mid E \in \mathfrak{R},\ a \in \mathcal{P}_0\}$.

Proof. Suppose that the function $f \in \mathcal{F}(X,\mathcal{P})$ is $(\mathcal{P}_0,\mathcal{V})$-based integrable over the set $F \in \mathfrak{A}_{\mathfrak{R}}$ with respect to Θ and in a first step let $E \in \mathfrak{R}$ be a subset of F. For every $w \in \mathcal{W}$ and $\varepsilon > 0$, let $(s_n^{w,\varepsilon})_{n\in\mathbb{N}}$ be a sequence of $\mathcal{P}_0$-valued step functions, $1 \leq \gamma^{w,\varepsilon} \leq 1+\varepsilon$ and $s^{w,\varepsilon} \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{V})$ as in Theorem 5.7 with εw in place of w. According to 5.7 we have $\int_E s^{w,\varepsilon}\,d\theta \leq \varepsilon w$ for all $\theta \in \Theta$, and

$$\int_E f\,d\theta \leq \underline{\lim}_{n\to\infty} \int_E h_n^{w,\varepsilon}\,d\theta + \varepsilon w \quad \text{and} \quad \overline{\lim}_{n\to\infty} \int_E h_n^{w,\varepsilon}\,d\theta \leq \gamma^{w,\varepsilon} \int_E f\,d\theta + \varepsilon w$$

follows from 5.7(iii). That is, for all $\theta \in \Theta$, both

$$\underline{\lim}_{n\to\infty}\int_E h_n^{w,\varepsilon}\,d\theta \qquad \text{and} \qquad \overline{\lim}_{n\to\infty}\int_E h_n^{w,\varepsilon}\,d\theta$$

are elements of the symmetric relative neighborhood $w_\varepsilon^s\big(\int_E f\,d\theta\big)$. Let the index set $\mathcal{J}$ consist of all triples (w,ε,ϕ), where $w\in\mathcal{W}$, $\varepsilon>0$ and $\phi:\mathcal{W}\times\{\varepsilon>0\}\to\mathbb{N}$. The set $\mathcal{J}$ is ordered and directed upward by $(w_1,\varepsilon_1,\phi_1)\le(w_2,\varepsilon_2,\phi_2)$ if $w_2\le w_1$, $\varepsilon_2\le\varepsilon_1$, and $\phi_1(w,\varepsilon)\le\phi_2(w,\varepsilon)$ for all $w\in\mathcal{W}$ and $\varepsilon>0$. Note that the index set $\mathcal{J}$ does not depend on the subset $E\in\mathfrak{R}$ of F. We set

$$h_j=\chi_{E^\otimes}h_{\phi(w,\varepsilon)}^{w,\varepsilon}$$

for $j=(w,\varepsilon,\phi)\in\mathcal{J}$, as well as

$$s_j=\chi_{E^\otimes}s^{w,\varepsilon}+\chi_{(Z\cup Z_E)^\otimes}\infty \qquad \text{and} \qquad \gamma_j=\gamma^{w,\varepsilon},$$

where ∞ is the infinite element of the augmented neighborhood system $\overline{\mathcal{V}}$ (see 2.1), and $Z=X\setminus\big(\bigcup_{E\in\mathfrak{R}}E\big)$. This is a set of Θ-measure zero. Likewise, $Z_E\in\mathfrak{R}$ is a subset of E of Θ-measure zero and such that the conclusions of 5.7(i) and (ii) hold for all $x\in E\setminus Z_E$. (For this, recall that the union of countably many sets of measure zero is again of measure zero.) Therefore, 5.7(i) and (ii) hold for all $x\in E$, not just Θ-almost everywhere if we replace the function $s^{w,\varepsilon}$ by s_j. Moreover, since the function s_j takes the value $\infty\in\overline{\mathcal{V}}$ only on a zero set, we infer

$$\int_F s_j\,d\theta=\int_E s^{w,\varepsilon}\,d\theta\le\varepsilon w$$

for all $\theta\in\Theta$. We have $1\le\gamma_j\le1+\varepsilon$. Thus

$$\lim_{j\in\mathcal{J}}\int_F s_j\,d\theta=0 \qquad \text{and} \qquad \lim_{j\in\mathcal{J}}\gamma_j=1.$$

The first of these limits is taken in the symmetric topology of $\mathcal{Q}$. Next we shall verify that

$$\int_E f\,d\theta=\lim_{j\in\mathcal{J}}\int_F h_j\,d\theta$$

holds for every $\theta\in\Theta$ in the order topology of $\mathcal{Q}$. Indeed, let $\theta\in\Theta$ and let U be a convex and order convex neighborhood of $\int_E f\,d\theta\in\mathcal{Q}$ in the order topology. As the order topology is coarser than the symmetric relative topology of $\mathcal{Q}$ (see Proposition I.5.44), there are $w_0\in\mathcal{W}$ and $\varepsilon_0>0$ such that U is a neighborhood for every point in the symmetric relative neighborhood ${w_0}_{\varepsilon_0}^s(\int_E f\,d\theta)$. Then for each choice of $w\le w_0$ and $\varepsilon\le\varepsilon_0$ we have both

$$\underline{\lim}_{n\to\infty}\int_E h_n^{w,\varepsilon}\,d\theta\in{w_0}_{\varepsilon_0}^s\left(\int_E f\,d\theta\right) \quad \text{and} \quad \overline{\lim}_{n\to\infty}\int_E h_n^{w,\varepsilon}\,d\theta\in{w_0}_{\varepsilon_0}^s\left(\int_E f\,d\theta\right)$$

by the above. As U is an order topology neighborhood of every element in $v^s_{\varepsilon_0}\left(\int_E f\,d\theta\right)$, there is an integer $\phi_0(w,\varepsilon)\in\mathbb{N}$ such that both

$$\inf_{n\geq\phi_0(w,\varepsilon)}\left\{\int_E h_n^{w,\varepsilon}\,d\theta\right\}\in U \qquad\text{and}\qquad \sup_{n\geq\phi_0(w,\varepsilon)}\left\{\int_E h_n^{w,\varepsilon}\,d\theta\right\}\in U.$$

Now the order convexity of the neighborhood U guarantees that

$$\int_E h_n^{w,\varepsilon}\,d\theta\in U$$

whenever $w\leq w_0$, $\varepsilon\leq\varepsilon_0$ and $n\geq\phi_0(w,\varepsilon)$. We set $j_0=(w_0,\varepsilon_0,\phi_0)\in\mathcal{J}$ and $\phi_0(w,\varepsilon)=1$ if either $w\not\leq w_0$ or $\varepsilon\not\leq\varepsilon_0$. Then the above yields indeed that

$$\int_F h_j\,d\theta=\int_E h^{w,\varepsilon}_{\phi(w,\varepsilon)}\,d\theta\in U$$

for all $j=(w,\varepsilon,\phi)\in\mathcal{J}$ such that $j\geq j_0$, that is $w\leq w_0$, $\varepsilon\leq\varepsilon_0$ and $\phi(w,\varepsilon)\geq\psi_0(w,\varepsilon)$, thus demonstrating our claim. Finally, given $x\in E$, for every $w\in\mathcal{W}$ and $\varepsilon>0$ there is $\phi_0(w,\varepsilon)\in\mathbb{N}$ such that

$$f(x)\leq h_j(x)+s_j(x)\qquad\text{and}\qquad h_j(x)\leq\gamma_j\,f(x)+s_j(x)$$

for all $j=(w,\varepsilon,\phi)\in\mathcal{J}$ such that $\phi(w,\varepsilon)\geq\phi_0(w,\varepsilon)$. If we set $j_0=(w_0,1,\phi_0)$ for any choice of $w_0\in\mathcal{W}$, then the above inequalities hold whenever $j\geq j_0$.

Now in the second step of our construction, for every set $E\in\mathfrak{R}$ we shall construct a net $(h_j^E)_{j\in\mathcal{J}}$ of $(\mathcal{P}_0,\mathcal{V})$-valued step functions as in our first step with respect to the set $E\cap F\in\mathfrak{R}$, that is in particular

$$\int_{(E\cap F)} f\,d\theta=\lim_{j\in\mathcal{J}}\int_F h_j^E\,d\theta$$

for every $\theta\in\Theta$. Similarly, we select the corresponding nets $(s_j^E)_{j\in\mathcal{J}}$ and $(\gamma_j^E)_{j\in\mathcal{J}}$. Now we choose another index set $\mathcal{I}$ consisting of all pairs (E,ψ), where $E\in\mathfrak{R}$ and $\psi:\mathfrak{R}\to\mathcal{J}$, ordered and directed upward by $(E_1,\psi_1)\leq(E_2,\psi_2)$ if $E_1\leq E_2$, and $\psi_1(E)\leq\psi_2(E)$ for all $E\in\mathfrak{R}$. We set $h_i=h^E_{\psi(E)}$ for $i=(E,\psi)\in\mathcal{I}$ and realize that the net $(h_i)_{i\in\mathcal{I}}$ of $(\mathcal{P}_0,\mathcal{V})$-valued step functions satisfies the properties stated in our Corollary. A straightforward diagonal argument similar to the preceding one shows that

$$\int_F f\,d\theta=\lim_{i\in\mathcal{I}}\int_F h_i\,d\theta$$

holds for every $\theta\in\Theta$ in the order topology of $\mathcal{Q}$, as claimed in (ii). Because the integrals of all step functions h_i involved are linear combinations of elements $\theta_{(E\cap F)}(a)$ for $E\in\mathfrak{R}$ and $a\in\mathcal{P}_0$, (ii) does indeed imply that

$\int_F f\,d\theta$ is contained in the closure with respect to the order topology of the subcone of $\mathcal{Q}$ spanned by these elements. Similarly, as claimed in (iii) and in (iv), we verify that

$$\lim_{i\in\mathcal{I}} \int_F s_i\,d\theta = 0 \qquad \text{and} \qquad \lim_{i\in\mathcal{I}} \gamma_i = 1,$$

where the first of these limits is taken in the symmetric topology of $\mathcal{Q}$. For Part (i), let $x \in F$. If $x \notin \bigcup_{E\in\mathfrak{R}} E$, then our claim is trivial, as we have $s_i(x) = \infty \in \overline{\mathcal{V}}$ for all $i \in \mathcal{I}$. Otherwise, there is $E_0 \in \mathfrak{R}$ such that $x \in E_0$. We fix any $w_0 \in \mathcal{W}$ and choose the index $i_0 = (E_0, \psi_0) \in \mathcal{I}$, where $\psi_0 : \mathfrak{R} \to \mathcal{J}$ is the mapping $E \mapsto (w_0, 1, \phi_E) \in \mathcal{J}$. The mapping $\phi_E : \mathcal{W} \times \{\varepsilon > 0\} \to \mathbb{N}$ is chosen as constant $\phi_E(w,\varepsilon) = 1$ if $E_0 \not\subset E$, and otherwise we chose $\phi_E(w,\varepsilon) \in \mathbb{N}$ such that

$$f(x) \le h_j(x) + s_j(x) \qquad \text{and} \qquad h_j(x) \le \gamma_j\, f(x) + s_j(x)$$

for every $j = (w,\varepsilon,\phi) \in \mathcal{J}$ such that $j \ge (w_0, 1, \phi_E)$. This holds for all $E \in \mathfrak{R}$ such that $E_0 \subset E$, hence we infer that

$$f(x) \le h_i(x) + s_i(x) \qquad \text{and} \qquad h_i(x) \le \gamma_i\, f(x) + s_i(x)$$

holds for all $i \ge i_0$. $\square$

It its important to keep in mind that the limit in 5.9(ii) refers to the order topology of $\mathcal{Q}$, not necessarily to order convergence as defined in I.5.18. Because in general the order topology is not known to be Hausdorff, this limit need therefore not be unique.

Corollary 5.9 is of particular interest in case that the locally convex complete lattice cone $(\mathcal{Q},\mathcal{W})$ is indeed the standard completion of some locally convex cone $(\mathcal{Q}_0,\mathcal{W})$ (see I.5.57) and that the measure θ is indeed $\mathfrak{L}(\mathcal{P},\mathcal{Q}_0)$-valued. The closure of $\mathcal{Q}_0$ in $\mathcal{Q}$ with respect to the order topology was seen to be a subcone of the second dual $\mathcal{Q}_0^{**}$ (see Remark I.5.60(a) and Section I.7.3) in this case, and integrals of functions in $\mathcal{F}(X,\mathcal{P})$ are therefore elements of $\mathcal{Q}_0^{**}$. Moreover, if the full locally convex cone $(\mathcal{P},\mathcal{V})$ is indeed the standard full extension of a quasi-full locally convex cone $(\mathcal{P}_0,\mathcal{V})$ (see I.6.2) and if for all $E \in \mathfrak{R}$ the operator θ_E maps the elements of $\mathcal{P}_0$ into $\mathcal{Q}_0$, then a similar statement holds for all functions $f \in \mathcal{F}(X,\mathcal{P})$ that are $(\mathcal{P}_0,\mathcal{V})$-based integrable over F (see Proposition 6.7 below).

Because the values of our measures, that is continuous linear operators from $\mathcal{P}$ into $\mathcal{Q}$, may be restricted to linear operators on a subcone $\mathcal{P}_0$ of $\mathcal{P}$, one may raise the question, if and how such a restriction does affect the integrals of functions with values only in this subcone $\mathcal{P}_0$. Let us be precise: Let $\mathcal{P}_0$ be a subcone of $\mathcal{P}$, and let $\mathcal{V}_0 \subset \mathcal{P}_0$ be a neighborhood subsystem of $\mathcal{V}$. If for a given $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measure θ, for all $E \in \mathfrak{R}$, the restrictions of the linear operators θ_E from the full cone $(\mathcal{P},\mathcal{V})$ to $(\mathcal{Q},\mathcal{W})$ are continuous linear operators from the full cone $(\mathcal{P}_0,\mathcal{V}_0)$ to $(\mathcal{Q},\mathcal{W})$, then, obviously, θ

may also be considered to be an $\mathfrak{L}(\mathcal{P}_0,\mathcal{Q})$-valued measure. This situation requires that for every $E\in\mathfrak{R}$ and $w\in\mathcal{W}$ there is $v\in\mathcal{V}_0$ such that $\theta_E(v)\le w$.

To avoid confusion, we shall denote this restriction of the measure θ to $(\mathcal{P}_0,\mathcal{V}_0)$ by θ_0, and by $\mathcal{F}_{(F,\theta_0)}(X,\mathcal{P}_0)$ the cone of all $\mathcal{P}_0$-valued functions that are integrable over a set $F\in\mathfrak{A}_{\mathfrak{R}}$ with respect to θ_0. Similarly, we shall use $\mathcal{F}_{(F,\Theta_0)}(X,\mathcal{P}_0)$ for functions that are integrable with respect to a family of restricted measures. Because our notions of measurability, of being reached from below by step functions and, consequently, of integrability depend on the given neighborhood system as well as on the cone, we shall have to clarify our notions for this situation.

For a $\mathcal{P}_0$-valued function, measurability with respect to $(\mathcal{P},\mathcal{V})$ obviously implies measurability with respect to $(\mathcal{P}_0,\mathcal{V}_0)$, since $\mathcal{V}_0\subset\mathcal{V}$ (see Conditions (M1) and (M2) in Section 1.2). The cone $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P}_0)$ is however not necessarily a subcone of $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ since the condition for the elements of $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P}_0)$ of being reached from below (see Section 2.3) involves only inductive neighborhoods that use the neighborhoods in $\mathcal{V}_0\subset\mathcal{V}$. Positive functions in $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P}_0)$ are however contained in $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$, since they can be trivially reached from below by the step function $h=0$. Conversely, every $\mathcal{P}_0$-valued function in $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ that can be reached from below by $\mathcal{P}_0$-valued step functions is contained in $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P}_0)$. This implies in particular that $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{V}_0)$ consists of the $\mathcal{V}_0$-valued elements of $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{V})$, that is $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{V}_0)=\mathcal{F}_{\mathfrak{R}}(X,\mathcal{V})\cap\mathcal{F}(X,\mathcal{V}_0)$.

Furthermore, we note that every set $Z\in\mathfrak{A}_{\mathfrak{R}}$ of measure zero with respect to a measure θ is also of measure zero with respect to its restriction θ_0. The almost everywhere notion with respect to θ therefore implies the almost everywhere notion with respect to θ_0. The converse does not necessarily hold true.

Proposition 5.10. *Let $\mathcal{P}_0$ be a subcone of $\mathcal{P}$, and let $\mathcal{V}_0\subset\mathcal{P}_0$ be a neighborhood subsystem of $\mathcal{V}$. Let Θ be an equibounded family of $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measures on $\mathfrak{R}$ such that the family Θ_0 of all restrictions of the measures in Θ to $(\mathcal{P}_0,\mathcal{V}_0)$ is an equibounded family of $\mathfrak{L}(\mathcal{P}_0,\mathcal{Q})$-valued measures. Let $F\in\mathfrak{A}_{\mathfrak{R}}$. If a $\mathcal{P}_0$-valued function f is $(\mathcal{P}_0,\mathcal{V}_0)$-based integrable over F with respect to Θ, then $f\in\mathcal{F}_{(F,\Theta_0)}(X,\mathcal{P}_0)$, and*

$$\int_F f\,d\theta=\int_F f\,d\theta_0$$

holds for all $\theta\in\Theta$.

Proof. Let $\mathcal{P}_0$ be a subcone of $\mathcal{P}$, and let $\mathcal{V}_0\subset\mathcal{P}_0$ be a neighborhood subsystem of $\mathcal{V}$. Let Θ be an equibounded family of $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measures on $\mathfrak{R}$. The family Θ_0 of all restrictions of the measures in Θ to $(\mathcal{P}_0,\mathcal{V}_0)$ is an equibounded family of $\mathfrak{L}(\mathcal{P}_0,\mathcal{Q})$-valued measures if and only if for every $E\in\mathfrak{R}$ and $w\in\mathcal{W}$ there is $v\in\mathcal{V}_0$ such that $\theta_E(v)\le w$ for all $\theta\in\Theta$. By our assumption, Θ satisfies this requirement. Given a set $F\in\mathfrak{A}_{\mathfrak{R}}$, we shall consider $\mathcal{P}_0$-valued functions as elements of the cones $\mathcal{F}_{(F,\Theta)}(X,\mathcal{P})$ or $\mathcal{F}_{(F,\Theta_0)}(X,\mathcal{P}_0)$, respectively. We shall proceed in several steps:

First we observe that every $\mathcal{P}_0$-valued step function $h=\sum_{i=1}^n \chi_{E_i}\otimes a_i$, for $E_i\in\mathfrak{R}$ and $a_i\in\mathcal{P}_0$, is contained in both $\mathcal{F}_{(F,\Theta)}(X,\mathcal{P})$ and $\mathcal{F}_{(F,\Theta_0)}(X,\mathcal{P}_0)$, and we have $\int_F h\,d\theta=\int_F h\,d\theta_0$ for all $\theta\in\Theta$.

In a second step we consider a neighborhood-valued function $s\in\mathcal{F}_{\mathfrak{R}}(X,\mathcal{V}_0)$. As we remarked before, positivity implies that s is also contained in $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{V})$. Given a neighborhood $w\in\mathcal{W}$, the inductive limit neighborhood $\mathfrak{v}_w$ formed by the neighborhoods in $\mathcal{V}$ contains the corresponding neighborhood $\mathfrak{v}_w^0$ formed by the neighborhoods in $\mathcal{V}_0$ as a subset (see Section 4). Thus

$$\begin{aligned}\int_F^{(w)} s\,d\theta_0 &= \sup\left\{\int_F s\,d\theta \;\middle|\; h\in\mathcal{S}_{\mathfrak{R}}(X,\mathcal{P}_0),\quad h\le s+\mathfrak{v}_w^0\right\}\\ &\le \sup\left\{\int_F h\,d\theta \;\middle|\; h\in\mathcal{S}_{\mathfrak{R}}(X,\mathcal{P}),\quad h\le s+\mathfrak{v}_w\right\} = \int_F^{(w)} s\,d\theta.\end{aligned}$$

Taking the infima over all $w\in\mathcal{W}$ on both sides yields

$$\int_F s\,d\theta_0 \le \int_F s\,d\theta.$$

Now in a third step, let $E\in\mathfrak{R}$, and let us consider a $\mathcal{P}_0$-valued function that is $(\mathcal{P}_0,\mathcal{V}_0)$-based integrable over E with respect to Θ. We shall first verify that $f\in\mathcal{F}_{(E,\Theta_0)}(X,\mathcal{P}_0)$. Indeed, the former property requires that for $w\in\mathcal{W}$ and $\varepsilon>0$ there is a $\mathcal{V}_0$-valued function $s_{(w,\varepsilon)}\in\mathcal{F}_{\mathfrak{R}}(X,\mathcal{V})$, and a $\mathcal{P}_0$-valued function $f_{(w,\varepsilon)}\in\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ that can be reached from below by $\mathcal{P}_0$-valued step functions, such that

$$f \underset{a.e.E}{\le} f_{(w,\varepsilon)} \underset{a.e.E}{\le} \gamma f + s_{(w,\varepsilon)}$$

for some $1\le\gamma\le1+\varepsilon$ and such that $\int_E s_{(w,\varepsilon)}\,d\theta\le\varepsilon w$ holds for all $\theta\in\Theta$. As $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{V}_0)=\mathcal{F}_{\mathfrak{R}}(X,\mathcal{V})\cap\mathcal{F}(X,\mathcal{V}_0)$, we have

$$\int_E s_{(w,\varepsilon)}\,d\theta_0 \le \int_E s_{(w,\varepsilon)}\,d\theta \le \varepsilon w$$

for all $\theta_0\in\Theta_0$ by our first step. As we mentioned before, the almost everywhere relation $\underset{a.e.E}{\le}$ with respect to θ implies the same relation with respect to θ_0. The function f is therefore indeed integrable over E with respect to the family Θ_0 of the restricted measures, that is $f\in\mathcal{F}_{(E,\Theta_0)}(X,\mathcal{P}_0)$.

Now let $w\in\mathcal{W}$ and $\varepsilon>0$. We shall apply Corollary 5.8 with the family Θ and the given subcone $(\mathcal{P}_0,\mathcal{V}_0)$ to find a sequence $(h_n)_{n\in\mathbb{N}}$ of $\mathcal{P}_0$-valued step functions as in 5.8. Statements (i) and (iii) refer to the measures $\theta\in\Theta$. However, all functions involved are also contained in $\mathcal{F}_{(E,\Theta_0)}(X,\mathcal{P}_0)$, the integrals with respect to the measures in Θ and in Θ_0 coincide for the step functions h_n, and for the function $s\in\mathcal{F}_{\mathfrak{R}}(X,\mathcal{V}_0)$ in (i) we have

$$\int_F s\,d\theta_0 \le \int_F s\,d\theta \le w$$

by our second step. Property (ii) therefore yields together with Proposition 4.7 that

$$\int_E f\,d\theta_0 \le \varliminf_{n\to\infty} \int_E h_n\,d\theta_0$$

holds for all $\theta_0 \in \Theta_0$. Using this together with our first step and the second part of statement (iii) in 5.8, we obtain

$$\int_E f\,d\theta_0 \le \varliminf_{n\to\infty} \int_E h_n\,d\theta_0 = \varliminf_{n\to\infty} \int_E h_n\,d\theta \le \gamma \int_E f\,d\theta_0 + w$$

and, likewise,

$$\begin{aligned}\int_E f\,d\theta \le \varliminf_{n\to\infty} \int_E h_n\,d\theta &= \varliminf_{n\to\infty} \int_E h_n\,d\theta_0 \\ &\le \gamma \int_E f\,d\theta_0 + \int_E s\,d\theta_0 \le \gamma \int_E f\,d\theta_0 + w\end{aligned}$$

with some $1 \le \gamma \le 1+\varepsilon$. Because $w \in \mathcal{W}$ and $\varepsilon > 0$ were arbitrarily chosen, this yields $\int_E f\,d\theta = \int_E f\,d\theta_0$.

Now for the final step of our argument, let $F \in \mathfrak{A}_{\mathfrak{R}}$, and let $f \in \mathcal{F}(X,\mathcal{P})$ be $\mathcal{P}_0$-valued and $(\mathcal{P}_0,\mathcal{V}_0)$-based integrable over F. Then f is $(\mathcal{P}_0,\mathcal{V}_0)$-based integrable over the sets $E\cap F$, for all $E \in \mathfrak{R}$, hence $f \in \mathcal{F}_{(E\cap F,\,\Theta_0)}(X,\mathcal{P}_0)$ by our second step, and $\int_{(E\cap F)} f\,d\theta = \int_{(E\cap F)} f\,d\theta_0$ by our first step. This shows

$$\int_F f\,d\theta = \lim_{E\in\mathfrak{R}} \int_{(E\cap F)} f\,d\theta = \lim_{E\in\mathfrak{R}} \int_{(E\cap F)} f\,d\theta_0 = \int_F f\,d\theta_0.$$

□

5.11 Sums, Multiples and Order for Measures. Let θ and ϑ be two $\mathfrak{R}$-bounded $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measures, and let $\alpha \ge 0$. We define the $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measures $\theta+\vartheta$ and $\alpha\theta$ by

$$(\theta+\vartheta)_E(a) = \theta_E(a) + \vartheta_E(a)$$

and

$$(\alpha\theta)_E(a) = \alpha\big(\theta_E(a)\big)$$

for $E \in \mathfrak{R}$ and $a \in \mathcal{P}$. The properties of a measure are readily checked.

Corresponding to a subcone $\mathcal{P}_0$ of $\mathcal{P}$ we define an order relation for measures θ and ϑ setting

$$\theta \le_{\mathcal{P}_0} \vartheta \qquad \text{if} \qquad \theta_E(a) \le \vartheta_E(a)$$

holds for all $E \in \mathfrak{R}$ and $a \in \mathcal{P}_0$. We write $\theta \le \vartheta$ for the canonical choice of $\mathcal{P}_0 = \mathcal{P}_+ = \{a \in \mathcal{P} \mid a \ge 0\}$. In this case, for any family Θ of measures and every set $F \in \mathfrak{R}$, every $\mathcal{P}_+$-valued function $f \in \mathcal{F}_{(F,\Theta)}(X,\mathcal{P})$ is seen to be $(\mathcal{P}_0,\mathcal{V})$-based integrable over F with respect to Θ. Note that $\theta \leq \vartheta$ and $\vartheta \leq \theta$ in this sense implies that $\theta = \vartheta$, that is equality for the positive elements implies equality for all elements of $\mathcal{P}$. Indeed, given $E \in \mathfrak{R}$, $a \in \mathcal{P}$ and $w \in \mathcal{W}$ there is $v \in \mathcal{V}$ such that $\theta_E(v) = \vartheta_E(v) \leq w$. There is $\lambda \geq 0$ such that $0 \leq a+\lambda v$. Thus $\theta_E(a+\lambda v) = \vartheta_E(a+\lambda v)$ and $\theta_E(a) \leq \vartheta_E(a)+w$ by the cancellation rules. This shows $\theta_E(a) \leq \vartheta_E(a)$ and likewise, $\vartheta_E(a) \leq \theta_E(a)$.

Proposition 5.12. *Let θ and ϑ be $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measures, let $\alpha \geq 0$, $F \in \mathfrak{A}_{\mathfrak{R}}$, and let $\mathcal{P}_0$ be a subcone of $\mathcal{P}$.*

(a) If $f \in \mathcal{F}_{(F,\{\theta,\vartheta\})}(X,\mathcal{P})$, then $f \in \mathcal{F}_{(F,\theta+\vartheta)}(X,\mathcal{P})$ and $\int_F f\,d(\theta+\vartheta) = \int_F f\,d\theta + \int_F f\,d\vartheta$.
(b) If $f \in \mathcal{F}_{(F,\theta)}(X,\mathcal{P})$, then $f \in \mathcal{F}_{(F,\alpha\theta)}(X,\mathcal{P})$ and $\int_F f\,d(\alpha\theta) = \alpha\int_F f\,d\theta$.
(c) If $\theta \leq_{\mathcal{P}_0} \vartheta$, then $\int_F f\,d\theta \leq \int_F f\,d\vartheta$ holds for every $f \in \mathcal{F}(X,\mathcal{P})$ that is $(\mathcal{P}_0,\mathcal{V})$-based integrable over F with respect to $\Theta = \{\theta,\vartheta\}$.

Proof. Without loss of generality, we may assume that $F = X$. For Part (a), it is clear from our definition of the sum of two measures that our claim, namely $\int_X h\,d(\theta+\vartheta) = \int_X h\,d\theta + \int_X h\,d\vartheta$ holds for all step functions $h \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P})$. Let $\Theta = \{\theta,\vartheta\}$. Every zero set for Θ is obviously a zero set for $\theta+\vartheta$. We shall first show that every function $f \in \mathcal{F}_{(X,\Theta)}(X,\mathcal{P})$ is integrable over every set $E \in \mathfrak{R}$ with respect to $\theta+\vartheta$. Indeed, given $w \in \mathcal{W}$ and $\varepsilon > 0$, let $w' = w/2$ and let the functions $f_{(w',\varepsilon)} \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ and $s_{(w',\varepsilon)} \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{V})$ be as in Definition 5.3, that is

$$f \underset{a.e.E}{\leq} f_{(w',\varepsilon)} \underset{a.e.E}{\leq} \gamma f + s_{(w',\varepsilon)}$$

for some $1 \leq \gamma \leq 1+\varepsilon$ and $\int_E s_{(w',\varepsilon)}\,d\theta \leq \varepsilon w'$ and $\int_E s_{(w',\varepsilon)}\,d\vartheta \leq \varepsilon w'$. The function $s_{(w',\varepsilon)} \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{V})$ is integrable with respect to every measure on $\mathfrak{R}$, and for any $u \in \mathcal{W}$ we realize that

$$\begin{aligned}\int_E^{(u)} s_{(w',\varepsilon)}\,d(\theta+\vartheta) &= \sup\left\{\int_E h\,d(\theta+\vartheta)\,\middle|\, h \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P}),\quad h \leq s_{(w',\varepsilon)} + \mathfrak{v}_w\right\}\\ &\leq \int_E^{(u)} s_{(w',\varepsilon)}\,d\theta + \int_E^{(u)} s_{(w',\varepsilon)}\,d\vartheta.\end{aligned}$$

Taking the respective infima over all neighborhoods $u \in \mathcal{W}$ and using Lemma I.5.20(c), we infer that

$$\int_E s_{(w',\varepsilon)}\,d(\theta+\vartheta) \leq \int_E s_{(w',\varepsilon)}\,d\theta + \int_E s_{(w',\varepsilon)}\,d\vartheta \leq \varepsilon w.$$

This shows integrability for f over E with respect to the family $\Theta = \{\theta,\vartheta,\theta+\vartheta\}$ of measures. Next for $\mathcal{P}_0 = \mathcal{P}$ and $\mathcal{V}_0 = \mathcal{V}$, the family Θ from above, a set $E \in \mathfrak{R}$, a neighborhood $w \in \mathcal{W}$ and $\varepsilon \geq 0$ let $(h_n)_{n\in\mathbb{N}}$

be a sequence of step functions in $\mathcal{S}_{\mathfrak{R}}(X,\mathcal{P})$ approaching the function $f \in \mathcal{F}_{(X,\Theta)}(X,\mathcal{P})$ as in Corollary 5.8. Part (iii) of 5.8 then yields

$$\begin{aligned}
\int_E f\,d(\theta+\vartheta) &\le \underline{\lim}_{n\to\infty} \int_E h_n\,d(\theta+\vartheta) \\
&\le \overline{\lim}_{n\to\infty} \int_E h_n\,d\theta + \overline{\lim}_{n\to\infty} \int_E h_n\,d\vartheta \\
&\le \gamma \int_E f\,d\theta + \gamma \int_E f\,d\vartheta + 2w.
\end{aligned}$$

And similarly,

$$\begin{aligned}
\int_E f\,d\theta + \int_E f\,d\vartheta &\le \underline{\lim}_{n\to\infty} \int_E h_n\,d\theta + \underline{\lim}_{n\to\infty} \int_E h_n\,d\vartheta \\
&\le \underline{\lim}_{n\to\infty} \int_E h_n\,d(\theta+\vartheta) \\
&\le \gamma \int_E f\,d(\theta+\vartheta) + w.
\end{aligned}$$

This in turn shows

$$\int_E f\,d(\theta+\vartheta) = \int_E f\,d(\theta+\vartheta),$$

since $w \in \mathcal{W}$ and $\varepsilon > 0$ were arbitrarily chosen. The latter equality holds for all $E \in \mathfrak{R}$, hence or claim follows from the definition of the integral over X. Part (b) may be verified in a similar way. For Part (c), let $\theta \le_{\mathcal{P}_0} \vartheta$ for a subcone $\mathcal{P}_0$ of $\mathcal{P}$, and let $f \in \mathcal{F}(X,\mathcal{P})$ be $(\mathcal{P}_0,\mathcal{V})$-based integrable over X with respect to $\Theta = \{\theta,\vartheta\}$. For $E \in \mathfrak{R}$, $w \in \mathcal{W}$ and $\varepsilon > 0$ we choose $v \in \mathcal{V}$ such that both $\theta_E(v) \le w$ and $\vartheta_E(v) \le w$. Let $(h_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{P}_0$-valued step functions in $\mathcal{S}_{\mathfrak{R}}(X,\mathcal{P})$ approaching the function f as in Theorem 5.7. We have $\int_E h_n\,d\theta \le \int_E h_n\,d\vartheta$ for all $n \in \mathbb{N}$ since $\theta \le_{\mathcal{P}_0} \vartheta$, hence by Part (iii) of 5.7

$$\int_E f\,d\theta \le \underline{\lim}_{n\to\infty} \int_E h_n\,d\theta + w \le \underline{\lim}_{n\to\infty} \int_E h_n\,d\vartheta + w \le \gamma \int_E f\,d\vartheta + 2w.$$

Thus $\int_E f\,d\theta \le \int_E f\,d\vartheta$, since $w \in \mathcal{W}$ and $\varepsilon > 0$ were arbitrary. Our claim now follows from the definition of the integral over a set $F \in \mathfrak{R}$. $\square$

By the *restriction of a measure* θ on $\mathfrak{R}$ to a subset $F \in \mathfrak{A}_{\mathfrak{R}}$ we mean the measure $\theta|_F$ on $\mathfrak{R}$, defined as

$$(\theta|_F)_E = \theta_{E\cap F}$$

for all $E \in \mathfrak{R}$. It is immediate from the definition of the integral in Section 4 that $f \in \mathcal{F}_{(X,\theta|_F)}(X,\mathcal{P})$ if and only if $f \in \mathcal{F}_{(F,\theta)}(X,\mathcal{P})$ for a function $f \in \mathcal{F}(X,\mathcal{P})$, and that $\int_X f\,d\theta|_F = \int_F f\,d\theta$ in this case.

5.13 Convergence of Sequences of Measures. Let θ and $(\theta_n)_{n\in\mathbb{N}}$ be $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measures on $\mathfrak{R}$. We shall define lower and upper setwise convergence for measures and denote $\theta_n \nearrow \theta$ or $\theta_n \searrow \theta$ if

(i) $\theta_E(a) \le \varliminf\limits_{n\to\infty} \theta_{n\,E}(a)$ or $\varlimsup\limits_{n\to\infty} \theta_{n\,E}(a) \le \theta_E(a)$ holds for all $E \in \mathfrak{R}$ and $a \in \mathcal{P}$, respectively.
(ii) There is a set $E_0 \in \mathfrak{R}$ such that $\theta|_{(X\setminus E_0)} \le_{\mathcal{P}} \theta_n|_{(X\setminus E_0)}$ or $\theta_n|_{(X\setminus E_0)} \le_{\mathcal{P}} \theta|_{(X\setminus E_0)}$ holds for all $n \in \mathbb{N}$, respectively.

We shall denote $\theta_n \longrightarrow \theta$ if both $\theta_n \nearrow \theta$ and $\theta_n \searrow \theta$.

Lemma 5.14. *Let $\Theta = \{\theta_n\}_{n\in\mathbb{N}}$ be equibounded $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measures on $\mathfrak{R}$ such that $\theta_n \nearrow \theta$ for a measure θ. Let $E \in \mathfrak{R}$.*

(a) If $f \in \mathcal{F}_{(E,\Theta)}(X,\mathcal{P})$, then $f \in \mathcal{F}_{(E,\Theta\cup\{\theta\})}(X,\mathcal{P})$.
(b) $\int_E f\,d\theta \le \varliminf\limits_{n\to\infty} \int_E f\,d\theta_n$ for every $f \in \mathcal{F}_{(E,\Theta)}(X,\mathcal{P})$.
(c) $\int_E f\,d\theta = \lim\limits_{n\to\infty} \int_E f\,d\theta_n$ for every invertible function f such that both $f \in \mathcal{F}_{(E,\Theta)}(X,\mathcal{P})$ and $-f \in \mathcal{F}_{(E,\Theta)}(X,\mathcal{P})$.

Proof. Let $\Theta = \{\theta_n\}_{n\in\mathbb{N}}$ be equibounded $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measures on such that $\theta_n \nearrow \theta$ for a measure θ. Let $E \in \mathfrak{R}$. We shall defer the proof of Part (a) since it will use elements of the statement of Part (b). For our proof of Part (b) we shall therefore assume that the function f is integrable over E with respect to the family $\bar{\Theta} = \Theta \cup \{\theta\}$. First, let us consider a step function $h = \sum_{i=1}^n \chi_{E_i} \otimes a_i \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P})$. Using Lemma I.5.19 we observe that $\theta_n \nearrow \theta$ implies

$$\int_X h\,d\theta = \sum_{i=1}^m \theta_{E_i}(a_i) \le \sum_{i=1}^m \left(\varliminf_{n\to\infty} \theta_{n\,E_i}(a_i)\right)$$
$$\le \varliminf_{n\to\infty} \left(\sum_{i=1}^m \theta_{n\,E_i}(a_i)\right) = \varliminf_{n\to\infty} \int_X h\,d\theta_n.$$

Now let $f \in \mathcal{F}_{(E,\bar{\Theta})}(X,\mathcal{P})$. We shall use Corollary 5.8 with $\mathcal{P}_0 = \mathcal{P}$ and $\mathcal{V}_0 = \mathcal{V}$ in order to establish our claim. Given $w \in \mathcal{W}$ and $\varepsilon > 0$ there is a bounded below sequence $(h_k)_{k\in\mathbb{N}}$ of $\mathcal{P}$-valued step functions such that:

(i) there is $1 \le \gamma \le 1+\varepsilon$ and $s \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{V})$ such that $\int_E s\,d\vartheta \le w$ for all $\vartheta \in \bar{\Theta}$, and $h_n \underset{a.e.E}{\le} \gamma f + s$ holds for all $n \in \mathbb{N}$;

(ii) $\bar{\Theta}$-almost everywhere on E, for $x \in E$ there is $n_0 \in \mathbb{N}$ such that $f(x) \le h_n(x)$ for all $n \ge n_0$;

(iii) $\int_E f\,d\vartheta \le \varliminf_{n\to\infty} \int_E h_n\,d\vartheta$ and $\int_E h_n\,d\vartheta \le \gamma \int_E f\,d\vartheta + w$ for all $n \in \mathbb{N}$ and $\vartheta \in \bar{\Theta}$.

For every $k \in \mathbb{N}$ then we have by the above

$$\int_E h_k\,d\theta \le \varliminf_{n\to\infty} \int_X h_k\,d\theta_n \le \gamma \varliminf_{n\to\infty} \int_E f\,d\theta_n + w.$$

Note that this argument implies in particular that the sequence $\left(\int_E f\,d\theta_n\right)_{n\in\mathbb{N}}$ is bounded below. Using (iii), we proceed from this and conclude that

$$\int_E f\,d\theta \le \gamma \varliminf_{n\to\infty} \int_E f\,d\theta_n + w.$$

Claim (b) follows, since this last inequality holds for all $w \in \mathcal{W}$ and $\varepsilon > 0$.

In Part (c) we assume in addition that the negative $-f$ of the function f is also contained in $\mathcal{F}_{(E,\bar{\Theta})}(X,\mathcal{P})$. Then Part (b) yields that both sequences $\left(\int_E f\,d\theta_n\right)_{n\in\mathbb{N}}$ and $\left(\int_E (-f)\,d\theta_n\right)_{n\in\mathbb{N}}$ are bounded below, and

$$\int_E f\,d\theta \le \varliminf_{n\to\infty} \int_E f\,d\theta_n \qquad \text{and} \qquad \int_E (-f)\,d\theta \le \varliminf_{n\to\infty} \int_E (-f)\,d\theta_n.$$

Thus

$$\begin{aligned}
\varlimsup_{n\to\infty} \int_E f\,d\theta_n &= \varlimsup_{n\to\infty} \int_E f\,d\theta_n + \int_E (-f)\,d\theta + \int_E f\,d\theta \\
&\le \varlimsup_{n\to\infty} \int_E f\,d\theta_n + \varliminf_{n\to\infty} \int_E (-f)\,d\theta_n + \int_E f\,d\theta \\
&\le \varlimsup_{n\to\infty} \int_E \big((-f) + f\big)\,d\theta_n + \int_E f\,d\theta \\
&\le \int_E f\,d\theta \le \varliminf_{n\to\infty} \int_E f\,d\theta_n,
\end{aligned}$$

and our claim (c) follows. We shall finally prove Part (a) of the lemma: Let $Z \in \mathfrak{A}_R$ be a zero-set for $\Theta = \{\theta_n\}_{n\in\mathbb{N}}$, that is $\theta_{n\,(E\cap Z)} = 0$ for all $n \in \mathbb{N}$ and $E \in \mathfrak{R}$. As $\theta_n \nearrow \theta$, this implies $\theta_{(E\cap Z)}(a) \le 0$ for all $a \in \mathcal{P}$, hence $\theta_{(E\cap Z)}(a) = 0$ for all $0 \le a \in \mathcal{P}$. However, for every $a \in \mathcal{P}$ there is $v \in \mathcal{V}$ such that $0 \le a + v$. Hence

$$\theta_{(E\cap Z)}(a) = \theta_{(E\cap Z)}(a) + \theta_{(E\cap Z)}(v) = \theta_{(E\cap Z)}(a+v) = 0.$$

Thus $\theta_{(E\cap Z)} = 0$. Every zero-set for Θ is therefore a zero-set for $\Theta \cup \{\theta\}$ as well. Now let $f \in \mathcal{F}_{(|X|,\Theta)}(X,\mathcal{P})$ and let $E \in \mathfrak{R}$. According to Definition 5.3, for every $w \in \mathcal{W}$ and $\varepsilon > 0$ there are functions $f_{(w,\varepsilon)} \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ and $s_{(w,\varepsilon)} \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{V})$ such that

$$f \underset{a.e.E}{\le} f_{(w,\varepsilon)} \underset{a.e.E}{\le} \gamma f + s_{(w,\varepsilon)} \qquad \text{and} \qquad \int_E s_{(w,\varepsilon)}\,d\vartheta \le \varepsilon w$$

for some $1 \le \gamma \le 1 + \varepsilon$ and all $\vartheta \in \Theta$. The almost everywhere relations refer to the family Θ and by the above therefore also to $\Theta \cup \{\theta\}$. Because the function $s_{(w,\varepsilon)} \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{V})$ is integrable over E with respect to every family of measures on $\mathfrak{R}$, we may use Part (b) of the lemma for

$$\int_E s_{(w,\varepsilon)}\, d\theta \le \overline{\lim_{n\to\infty}} \int_E s_{(w,\varepsilon)}\, d\theta_n \le \varepsilon w.$$

This shows that the function f is indeed integrable over E with respect to the family $\Theta \cup \{\theta\}$. $\square$

Example 5.15. The following example will demonstrate that a result corresponding to 5.14(b) for upper convergence of measures is not available in general, that is $\theta_n \searrow \theta$ for measures θ_n, θ on $\mathfrak{R}$ does not necessarily imply that $\overline{\lim\limits_{n\to\infty}} \int_E f\, d\theta_n \le \int_E f\, d\theta$ holds for every integrable function $f \in \mathcal{F}(X,\mathcal{P})$. For this, let $X = [0,1]$, let $\mathfrak{R}$ be the σ-algebra of all Borel sets on X, and let θ be the Lebesgue measure. This may be considered as an $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measure if we set $\mathcal{P} = \mathcal{Q} = \overline{\mathbb{R}}$ (see Examples I.2.1). We define the measures θ_n as $\theta_n(E) = \sqrt{n}\ \theta\big(E \cap [0,\frac{1}{n}]\big)$ for $E \in \mathfrak{R}$. This yields $\theta_n(E) \le \frac{1}{\sqrt{n}}$ for all $E \in \mathfrak{R}$ and $n \in \mathbb{N}$, hence $\theta_n \longrightarrow 0$, that is the zero measure on $\mathfrak{R}$. Now consider the function f on X defined as $f(x) = \frac{1}{\sqrt{x}}$ for $x > 0$ and $f(0) = 0$. As f is positive and measurable, it is contained in $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$, hence in $\mathcal{F}_{(E,\Theta)}(X,\mathcal{P})$, where Θ is the equibounded family $\{\theta_n\}_{n\in\mathbb{N}}$. We calculate $\int_X f\, d\theta_n = \sqrt{n}\int_0^{\frac{1}{n}} f\, d\theta = 2$, hence $\int_X f\, d\theta_n \not\to \int_X f\, d\,0 = 0$, indeed. Note that Part (c) of Lemma 5.14 does not apply in this case. The function f is in fact invertible in $\mathcal{F}(X,\mathcal{P})$ and its inverse $-f$ is integrable with respect to each of the measures θ_n. Indeed, given $\varepsilon > 0$ we may choose $f_\varepsilon(x) = -f(x)$ for $x \ge \varepsilon$ and $f_\varepsilon(x) = 0$ else. Then f_ε is bounded below, hence in $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$, and we have $-f \le f_\varepsilon \le -f + s_\varepsilon$, where $s_\varepsilon(x) = 0$ for $x = 0$ or $x \ge \varepsilon$, and $s_\varepsilon(x) = f(x)$ else. (This function s_ε is $\big(\mathcal{V}\cup\{0\}\big)$-valued as required in Definition 4.12, since the neighborhood system $\mathcal{V}$ of $\overline{\mathbb{R}}$ consists of all strictly positive reals.) For $\varepsilon \le \frac{1}{n}$ we calculate $\int_X s_\varepsilon\, d\theta_n = 2\sqrt{n\varepsilon}$. Thus $-f \in \mathcal{F}_{(E,\theta_n)}(X,\mathcal{P})$ for all $n \in \mathbb{N}$, but $-f$ is not contained in $\mathcal{F}_{(E,\Theta)}(X,\mathcal{P})$ as required in 5.14(c).

5.16 Residual Components. Let $(\theta_n)_{n\in\mathbb{N}}$ be an equibounded sequence of measures, let $F \in \mathfrak{A}_{\mathfrak{R}}$ and $f \in \mathcal{F}_{(F,\{\theta_n\})}(X,\mathcal{P})$. We define the *residual component of f on F with respect to $(\theta_n)_{n\in\mathbb{N}}$* as follows: Let $\mathfrak{F}$ be the collection of all sequences $(E_n)_{n\in\mathbb{N}}$ of sets in $\mathfrak{R}$ such that $E_n \subset F$, $E_n \supset E_{n+1}$ and $\bigcap_{n\in\mathbb{N}} E_n = \emptyset$. Recall that integrability for a function $f \in \mathcal{F}(X,\mathcal{P})$ over F requires integrability over all subsets $E \in \mathfrak{R}$ of F. Thus for $f \in \mathcal{F}_{(F,\{\theta_n\})}(X,\mathcal{P})$ we define

$$\mathfrak{Rs}\big(\theta_n, F, f\big) = \sup_{(E_m)\in\mathfrak{F}} \left\{ \underline{\lim_{m\to\infty}} \left(\overline{\lim_{n\to\infty}} \int_{E_m} f\, d\theta_n \right) \right\}.$$

This appears to be a rather unwieldy expression. It will however turn out to be useful for our continuing investigations.

Lemma 5.17. *Let $(\theta_n)_{n\in\mathbb{N}}$ be an equibounded sequence of measures, and let $F \in \mathfrak{A}_{\mathfrak{R}}$. Then*

(a) $\mathfrak{Rs}\big(\theta_n, F, f\big) \geq 0$ *for all* $f \in \mathcal{F}_{(F,\{\theta_n\})}(X,\mathcal{P})$.
(b) If $\theta_n \leq \omega$ *for a measure* ω *and all* $n \in \mathbb{N}$, *then*
$\mathfrak{Rs}\big(\theta_n, F, f\big) \leq \mathfrak{O}\left(\int_F f\, d\omega\right)$ *for all* $f \in \mathcal{F}_{(|F|,\{\theta_n,\omega\})}(X,\mathcal{P})$.

Proof. Part (a) is trivial, as we may choose the stationary sequence $(E_m)_{m\in\mathbb{N}} \in \mathfrak{F}$, where $E_m = \emptyset$ for all $n \in \mathbb{N}$. For Part (b), suppose that $\theta_n \leq \omega$ holds for a measure ω on $\mathfrak{R}$ and all $n \in \mathbb{N}$, let $\Theta = \{\theta_n, \omega\}_{n\in\mathbb{N}}$, let $f \in \mathcal{F}_{(F,\Theta)}(X,\mathcal{P})$ and $(E_m)_{m\in\mathbb{N}} \in \mathfrak{F}$. For every $w \in \mathcal{W}$ there is by Proposition 5.4 a function $s \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{V})$ and $\lambda \geq 0$ such that $0 \underset{a.e.E_1}{\leq} f + s$ and $\int_{E_1} s\, d\vartheta \leq \lambda w$ for all $\vartheta \in \Theta$. Because $s \geq 0$, this yields for all $m \in \mathbb{N}$

$$\overline{\lim_{n\to\infty}} \int_{E_m} f\, d\theta_n \leq \overline{\lim_{n\to\infty}} \int_{E_m} (f+s)\, d\theta_n \leq \int_{E_m} (f+s)\, d\omega.$$

Thus by Proposition 4.18(b) and Proposition I.5.11

$$\begin{aligned}
\underline{\lim_{m\to\infty}} \left(\overline{\lim_{n\to\infty}} \int_{E_m} f\, d\theta_n \right) &\leq \underline{\lim_{m\to\infty}} \int_{E_m} (f+s)\, d\omega \\
&\leq \mathfrak{O}\left(\int_{E_1} (f+s)\, d\omega \right) \\
&= \mathfrak{O}\left(\int_{E_1} f\, d\omega \right) + \mathfrak{O}\left(\int_{E_1} s\, d\omega \right) \\
&\leq \mathfrak{O}\left(\int_{E_1} f\, d\omega \right) + w,
\end{aligned}$$

since $\mathfrak{O}\left(\int_{E_1} s\, d\omega\right) \leq \varepsilon w$ for all $\varepsilon > 0$. Because $w \in \mathcal{W}$ was arbitrarily chosen, and because $\mathcal{Q}$ carries the weak preorder, this yields

$$\underline{\lim_{m\to\infty}} \left(\overline{\lim_{n\to\infty}} \int_{E_m} f\, d\theta_n \right) \leq \mathfrak{O}\left(\int_{E_1} f\, d\omega \right).$$

Furthermore, Proposition 4.15(c) states that $\mathfrak{O}\left(\int_{E_1} f\, d\omega\right) \leq \mathfrak{O}\left(\int_F f\, d\omega\right)$. Now combining all of the above, we have indeed

$$\mathfrak{Rs}\big(\theta_n, F, f\big) = \sup_{(E_m)\in\mathfrak{F}} \left\{ \underline{\lim_{m\to\infty}} \left(\overline{\lim_{n\to\infty}} \int_{E_m} f\, d\theta_n \right) \right\} \leq \mathfrak{O}\left(\int_F f\, d\omega \right),$$

as claimed. □

Lemma 5.17(b) implies in particular that for a stationary sequence $(\theta_n)_{n\in\mathbb{N}}$ of measures, that is $\theta_n = \theta$ for all $n \in \mathbb{N}$, we have $\mathfrak{Rs}\big(\theta_n, F, f\big) \leq \mathfrak{O}\left(\int_F f\, d\theta\right)$ for all $f \in \mathcal{F}_{(|F|,\theta)}(X,\mathcal{P})$. This leads to the following notation:

For a set $F \in \mathfrak{A}_R$, and an equibounded sequence $(\theta_n)_{n\in\mathbb{N}}$ of measures, a measure θ and a family $\mathfrak{F}$ of functions in $\mathcal{F}_{(F,\{\theta_n,\theta\})}(X,\mathcal{P})$ we shall denote

$$(\theta_n) \underset{\mathfrak{F}}{\overset{F}{\precsim}} \theta \qquad \text{if} \qquad \mathfrak{Rs}\big(\theta_n, F, f\big) \le \mathfrak{O}\left(\int_F f\,d\theta\right)$$

holds for all $f \in \mathfrak{F}$. Setwise convergence of the measures θ_n towards θ, that is $\theta_n \longrightarrow \theta$, does however not necessarily imply that $(\theta_n) \underset{\{f\}}{\overset{F}{\precsim}} \theta$ holds for every integrable function $f \in \mathcal{F}_{(|F|,\{\theta_n,\theta\})}(X,\mathcal{P})$, as our preceding Example 5.15 can demonstrate. Indeed, let us calculate the residual component of the function f in 5.15 on the interval $F = [0,1]$ with respect to the given sequence $(\theta_n)_{n\in\mathbb{N}}$ of measures on $[0,1]$. First, let $(E_m)_{m\in\mathbb{N}}$ be a sequence of Borel sets in $[0,1]$ such that $E_m \supset E_{m+1}$ and $\bigcap_{m\in\mathbb{N}} E_m = \emptyset$. Then

$$\int_{E_m} f\,d\theta_n \le \int_{[0,1]} f\,d\theta_n \le 2$$

for all $k,l \in \mathbb{N}$. This shows $\mathfrak{Rs}\big(\theta_n, F, f\big) \le 2$. For $E_m = \big[0, \frac{1}{m}\big]$, on the other hand, we have $E_m \supset E_{m+1}$ and $\bigcap_{m\in\mathbb{N}} E_m = \emptyset$, and $\int_{E_m} f\,d\theta_n = 2$ whenever $n \ge m$. Thus $\overline{\lim}_{n\to\infty} \int_{E_m} f\,d\theta_n \ge 2$ for all $m \in \mathbb{N}$, and therefore $\mathfrak{Rs}\big(\theta_n, F, f\big) \ge 2$. Together with the above, this yields $\mathfrak{Rs}\big(\theta_n, F, f\big) = 2$. But we have $\theta_n \longrightarrow 0$.

A stronger requirement on the integrability of the function f will however avoid such cases.

5.18 Strongly Integrable Functions. Let Θ be a an equibounded family of measures, and let $E \in \mathfrak{R}$. We shall say that a function $f \in \mathcal{F}(X,\mathcal{P})$ is *strongly integrable over E with respect to Θ* if it is integrable over E in the sense of 5.3, and if in addition, for every $w \in \mathcal{W}$ there is a step function $h \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P})$ such that $\int_G f\,d\theta \le \int_G h\,d\theta + w$ and $\int_G h\,d\theta$ is w-bounded relative to $\int_G f\,d\theta$ in $\mathcal{Q}$, for all $\theta \in \Theta$ and every subset $G \in \mathfrak{R}$ of E. Note that this requirement strengthens the corresponding property from Theorem 5.7 which holds for integrable functions in general.

Similarly, for a set $F \in \mathfrak{A}_{\mathfrak{R}}$, a function $f \in \mathcal{F}(X,\mathcal{P})$ is *strongly integrable over F with respect to Θ* if it is integrable over F in the sense of 5.3 and strongly integrable over the sets $E \cap F$ for all $E \in \mathfrak{R}$. Because strong integrability over a set $E \in \mathfrak{R}$ obviously implies strong integrability over every subset $G \in \mathfrak{R}$ of E, this last part of our definition is consistent with the first one.

It is straightforward to verify that the strongly integrable functions form a subcone of $\mathcal{F}_{(F,\Theta)}(X,\mathcal{P})$.

Lemma 5.19. *Let $\Theta = \{\theta, \theta_n\}_{n\in\mathbb{N}}$ be equibounded measures on $\mathfrak{R}$ such that $\theta_n \searrow \theta$. Let $F \in \mathfrak{A}_{\mathfrak{R}}$, and suppose that the function $f \in \mathcal{F}_{(|F|,\Theta)}(X,\mathcal{P})$ is strongly integrable over F with respect to Θ. Then $(\theta_n) \underset{\{f\}}{\overset{F}{\precsim}} \theta$.*

Proof. Let $\Theta = \{\theta, \theta_n\}_{n\in\mathbb{N}}$ be equibounded measures such that $\theta_n \searrow \theta$. As in the first step of the proof of Lemma 5.14, one easily verifies that

$$\overline{\lim_{n\to\infty}} \int_G h \, d\theta_n \le \int_G h \, d\theta$$

holds for every step function $h \in \mathcal{S}_{\mathfrak{R}}(X, \mathcal{P})$ and all $G \in \mathfrak{A}_{\mathfrak{R}}$. Let $F \in \mathfrak{A}_{\mathfrak{R}}$ and suppose that the function $f \in \mathcal{F}_{(F,\Theta)}(X, \mathcal{P})$ is strongly integrable over F with respect to Θ. Let $E_m \in \mathfrak{R}$ for $m \in \mathbb{N}$ be subsets of F such that $E_m \supset E_{m+1}$ and $\bigcap_{m\in\mathbb{N}} E_m = \emptyset$. Following 5.18, the function f is strongly integrable over the set $E = E_1$. Given $w \in \mathcal{W}$, we choose a step function $h = \sum_{i=1}^n \chi_{G_i} \otimes a_i \in \mathcal{S}_{\mathfrak{R}}(X, \mathcal{P})$ as in the first part of 5.18, that is $\int_G h \, d\theta \in \mathcal{B}_w\big(\int_G f \, d\theta\big)$, and $\int_G f \, d\theta \le \int_G h \, d\theta + w$ holds for all $\theta \in \Theta$ and every subset $G \in \mathfrak{R}$ of E, in particular

$$\int_{E_m} f \, d\theta_n \le \int_{E_m} h \, d\theta_n + w$$

holds for all $m, n \in \mathbb{N}$. Thus

$$\overline{\lim_{n\to\infty}} \int_{E_m} f \, d\theta_n \le \overline{\lim_{n\to\infty}} \int_{E_m} h \, d\theta_n + w \le \int_{E_m} h \, d\theta + w$$

for every $m \in \mathbb{N}$, and consequently

$$\underline{\lim_{m\to\infty}} \left(\overline{\lim_{n\to\infty}} \int_{E_m} f \, d\theta_n \right) \le \underline{\lim_{m\to\infty}} \int_{E_m} h \, d\theta + w \le \mathfrak{O}\Big(\int_E h \, d\theta \Big) + w$$

by Proposition 4.18(b). Because $\int_E h \, d\theta$ is w-bounded relative to $\int_E f \, d\theta$, and because $\int_E f \, d\theta$ is bounded relative to $\int_F f \, d\theta$ by Proposition 4.15(c), we have $\mathfrak{O}\big(\int_E h \, d\theta\big) \preccurlyeq_w \mathfrak{O}\big(\int_F f \, d\theta\big)$ by Proposition I.5.13(a). The latter implies

$$\mathfrak{O}\big(\int_E h \, d\theta \big) \le \mathfrak{O}\big(\int_F f \, d\theta \big) + w.$$

Thus, summarizing,

$$\underline{\lim_{m\to\infty}} \left(\overline{\lim_{n\to\infty}} \int_{E_m} f \, d\theta_n \right) \le \mathfrak{O}\Big(\int_F f \, d\theta \Big) + 2w.$$

This holds for all $w \in \mathcal{W}$ and all sequences of sets $E_k \in \mathfrak{R}$ such that $E_k \subset F$ and $\bigcap_{k\in\mathbb{N}} E_k$, and therefore demonstrates

$$\mathfrak{Rs}\big(\theta_n, F, f\big) \le \mathfrak{O}\Big(\int_F f \, d\theta \Big),$$

our claim. $\square$

Lemma 5.20. *Let* $\Theta = \{\theta, \theta_n\}_{n\in\mathbb{N}}$ *be equibounded* $\mathfrak{L}(\mathcal{P},\mathcal{Q})$*-valued measures on* $\mathfrak{R}$ *such that* $\theta_n \searrow \theta$*. Let* $E \in \mathfrak{R}$*.*

(a) Let $f_n, f, f^* \in \mathcal{F}_{(E,\Theta)}(X,\mathcal{P})$ *and* $v \in \mathcal{V}$*. If both* $f_n(x) \preccurlyeq_v f(x)$ *and* $f_n(x) \preccurlyeq_v f^*(x)$ *holds* Θ*-almost everywhere on* E *for all* $n \in \mathbb{N}$*, then* $\overline{\lim\limits_{n\to\infty}} \int_E f_n\, d\theta_n \le \int_E f\, d\theta + \mathfrak{Rs}\big(\theta_n, E, f^*\big) + \mathfrak{O}\big(\sup\{\theta_E(v) \mid \theta \in \Theta\}\big)$*.*

(b) Let $f \in \mathcal{F}_{(E,\Theta)}(X,\mathcal{P})$*. If* $(\theta_n) \overset{E}{\underset{\{f\}}{\prec}} \theta$*, then* $\overline{\lim\limits_{n\to\infty}} \int_E f\, d\theta_n \le \int_E f\, d\theta$*.*

Proof. Let $\Theta = \{\theta, \theta_n\}_{n\in\mathbb{N}}$ be equibounded measures such that $\theta_n \searrow \theta$. As seen before, this implies

$$\overline{\lim_{n\to\infty}} \int_G h\, d\theta_n \le \int_G h\, d\theta$$

for every step function $h \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P})$ and every $G \in \mathfrak{A}_{\mathfrak{R}}$. Let $E \in \mathfrak{R}$, $v \in \mathcal{V}$, and let $f_n, f, f^* \in \mathcal{F}_{(E,\Theta)}(X,\mathcal{P})$ such that both $f_n \preccurlyeq_v f$ and $f_n \preccurlyeq_v f^*$ holds Θ-almost everywhere on E for all $n \in \mathbb{N}$. Let us abbreviate

$$d = \mathfrak{O}\big(\sup\{\theta_E(v) \mid \theta \in \Theta\}\big) \in \mathcal{Q}.$$

Recall from Proposition I.5.11(b) that $\alpha d = d$ for all $\alpha > 0$. Given $w \in \mathcal{W}$ and $\varepsilon > 0$, we shall use Corollary 5.8 for the function f, with $\mathcal{P}_0 = \mathcal{P}$ and $\mathcal{V}_0 = \mathcal{V}$, in order to obtain a sequence $(h_k)_{k\in\mathbb{N}}$ of step functions satisfying 5.8(i), (ii) and (iii). We set

$$G_m = \{x \in E \mid f(x) \preccurlyeq_v h_k(x) \quad \text{for all} \quad k \ge m\}$$

for $m \in \mathbb{N}$. Following Theorem 1.6, the sets G_m are contained in $\mathfrak{R}$, and we have $G_m \subset G_{m+1}$. If we set $G = \bigcup_{m\in\mathbb{N}} G_m$, then 5.8(ii) implies that $E \setminus G \in \mathfrak{Z}(\Theta)$, that is $\int_E g\, d\vartheta = \int_G g\, d\vartheta$ for the functions $g = f_n, f, f^*$ and all $\vartheta \in \Theta$. Because $f_n(x) \preccurlyeq_v h_m(x)$ holds Θ-almost everywhere on G_m, and $f_n(x) \preccurlyeq_v f^*(x)$ holds Θ-almost everywhere on E for all $n \in \mathbb{N}$, and because $\mathfrak{O}\big(\vartheta_E(v)\big) \le d$ for all $\theta \in \Theta$, Proposition 4.16 yields

$$\int_{G_m} f_n\, d\vartheta \le \int_{G_m} h_m\, d\vartheta + d \qquad \text{and} \qquad \int_{(G\setminus G_m)} f_n\, d\vartheta \le \int_{G_m} f^*\, d\vartheta + d.$$

for all $\vartheta \in \Theta$ and $n \in \mathbb{N}$. Thus

$$\begin{aligned}\int_E f_n\, d\theta_n &= \int_{G_m} f_n\, d\theta_n + \int_{(G\setminus G_m)} f_n\, d\theta_n \\ &\le \int_{G_m} h_m\, d\theta_n + \int_{(G\setminus G_m)} f^*\, d\theta_n + d\end{aligned}$$

holds for all $m, n \in \mathbb{N}$. Let $E_m = G \setminus G_m$. Then $E_m \supset E_{m+1}$ and $\bigcap_{m\in\mathbb{N}} E_m = \emptyset$. Using this, we proceed with our argument. For a fixed $m \in \mathbb{N}$, we let n tend to infinity in the preceding inequality, and obtain

$$\begin{aligned}\overline{\lim_{n\to\infty}}\int_E f\,d\theta_n &\le \overline{\lim_{n\to\infty}}\int_{G_m} h_m\,d\theta_n + \overline{\lim_{n\to\infty}}\int_{E_m} f^*\,d\theta_n + d\\ &\le \int_{G_m} h_m\,d\theta + \overline{\lim_{n\to\infty}}\int_{E_m} f^*\,d\theta_n + d\\ &\le \gamma\int_{G_m} f\,d\theta + \overline{\lim_{n\to\infty}}\int_{E_m} f^*\,d\theta_n + d + w.\end{aligned}$$

with some $1\le\gamma\le 1+\varepsilon$. Finally, we let m tend to infinity as well and use Proposition 4.18(a) for

$$\lim_{m\to\infty}\int_{G_m} f\,d\theta = \int_G f\,d\theta = \int_E f\,d\theta.$$

Thus

$$\begin{aligned}\overline{\lim_{n\to\infty}}\int_E f\,d\theta_n &\le \gamma\int_E f\,d\theta + \underline{\lim_{m\to\infty}}\left(\overline{\lim_{n\to\infty}}\int_{E_m} f^*\,d\theta_n\right) + d + w\\ &\le \gamma\int_E f\,d\theta + \mathfrak{Rs}\big(\theta_n, E, f^*\big) + d + w\\ &\le \gamma\left(\int_E f\,d\theta + \mathfrak{Rs}\big(\theta_n, E, f^*\big) + d\right) + w.\end{aligned}$$

The last inequality holds for all $w\in\mathcal{W}$ and $\varepsilon>0$, hence

$$\overline{\lim_{n\to\infty}}\int_E f\,d\theta_n \le \int_E f\,d\theta + \mathfrak{Rs}\big(\theta_n, E, f^*\big) + d,$$

since $\mathcal{Q}$ is endowed with the weak preorder.

For Part (b), we set $f_n = f = f^*$ in Part (a). If $(\theta_n) \underset{\{f\}}{\overset{E}{\prec}} \theta$, that is $\mathfrak{Rs}\big(\theta_n, E, f\big) \le \mathfrak{O}\big(\int_E f\,d\theta\big)$ holds in addition, then

$$\int_E f\,d\theta + \mathfrak{Rs}\big(\theta_n, E, f\big) \le \int_E f\,d\theta + \mathfrak{O}\left(\int_E f\,d\theta\right) = \int_E f\,d\theta$$

follows from Proposition I.5.14. Given $w\in\mathcal{W}$, we choose $v\in\mathcal{V}$ such that $\theta_E(v)\le w$ for all $\theta\in\Theta$. Then obviously $\mathfrak{O}\big(\sup\{\theta_E(v)\mid\theta\in\Theta\}\big)\le w$ holds as well. Part (a) therefore yields

$$\overline{\lim_{n\to\infty}}\int_E f\,d\theta_n \le \int_E f\,d\theta + \mathfrak{Rs}\big(\theta_n, E, f\big) + w \le \int_E f\,d\theta + w$$

for all $w\in\mathcal{W}$. Thus indeed

$$\overline{\lim_{n\to\infty}}\int_E f\,d\theta_n \le \int_E f\,d\theta,$$

since $\mathcal{Q}$ carries the weak preorder. $\square$

Our upcoming convergence theorems will imply that the statements of Lemmas 5.14 and 5.20 do indeed extend to integrals over sets $F \in \mathfrak{A}_{\mathfrak{R}}$, if the concerned functions are contained in $\mathcal{F}_{(|F|,\{\theta_n,\theta\})}(X,\mathcal{P})$.

Lemma 5.21. *Let* $\Theta = \{\theta_n\}_{n\in\mathbb{N}}$ *be equibounded* $\mathfrak{L}(\mathcal{P},\mathcal{Q})$*-valued measures on* $\mathfrak{R}$ *such that* $\theta_n \longrightarrow \theta$ *for a measure* θ. *Let* $F \in \mathfrak{A}_{\mathfrak{R}}$. *If* $f \in \mathcal{F}_{(F,\Theta)}(X,\mathcal{P})$, *then* $f \in \mathcal{F}_{(F,\Theta\cup\{\theta\})}(X,\mathcal{P})$.

Proof. We may assume that $F = X$, since for a function $f \in \mathcal{F}(X,\mathcal{P})$ integrability over F means equivalently that the function $\chi_{F^\otimes} f$ is integrable over X. Let $f \in \mathcal{F}_{(X,\Theta)}(X,\mathcal{P})$. Then $f \in \mathcal{F}_{(E,\Theta)}(X,\mathcal{P})$ for every $E \in \mathfrak{R}$, hence $f \in \mathcal{F}_{(E,\Theta\cup\{\theta\})}(X,\mathcal{P})$ by Lemma 5.14(a). Now let $E_0 \in \mathfrak{R}$ and $v \in \mathcal{V}$ be as in Definition 5.13(ii), that is $\theta|_{(X\setminus E_0)} = \theta_n|_{(X\setminus E_0)}$ holds for all $n \in \mathbb{N}$. Let $E \in \mathfrak{R}$ such that $E_0 \subset E$ and fix $n_0 \in \mathbb{N}$. Then

$$\int_E f\,d\theta = \int_{E_0} f\,d\theta + \int_{E\setminus E_0} f\,d\theta = \int_{E_0} f\,d\theta + \int_{E\setminus E_0} f\,d\theta_{n_0}.$$

Hence

$$\lim_{E\in\mathfrak{R}} \int_E f\,d\theta = \int_{E_0} f\,d\theta + \lim_{E\in\mathfrak{R}} \int_{E\setminus E_0} f\,d\theta_{n_0} = \int_{E_0} f\,d\theta + \int_{X\setminus E_0} f\,d\theta_{n_0}.$$

The function f is therefore indeed integrable over X with respect to θ, and we infer that $f \in \mathcal{F}_{(E,\Theta\cup\{\theta\})}(X,\mathcal{P})$. $\square$

5.22 Convergence of Sequences in $\mathcal{F}(X,\mathcal{P})$. In Section 3 we introduced several notions of pointwise convergence for sequences of $\mathcal{P}$-valued functions. They refer to the lower and upper relative topologies of $\mathcal{P}$, that is for a subset F of X, a sequence $(f_n)_{n\in\mathbb{N}}$ and a function f in $\mathcal{F}(X,\mathcal{P})$ we denote $f_n \underset{F}{\nearrow} f$ or $f_n \underset{F}{\searrow} f$ if for every $x \in F$, $v \in \mathcal{V}$ and $\varepsilon > 0$ there is n_0 such that

$$f(x) \in v_\varepsilon\big(f_n(x)\big) \qquad \text{or} \qquad f_n(x) \in v_\varepsilon\big(f(x)\big)$$

for all $n \geq n_0$, respectively. $f_n \underset{F}{\rightarrow} f$ means that both $f_n \underset{F}{\nearrow} f$ and $f_n \underset{F}{\searrow} f$. Correspondingly, if Θ is a family of measures on $\mathfrak{R}$, then we shall denote $f_n \underset{a.e.F}{\nearrow} f$, $f_n \underset{a.e.F}{\searrow} f$ or $f_n \underset{a.e.F}{\longrightarrow} f$ if this convergence holds Θ-almost everywhere on F, that is on a subset $F \setminus Z$ with some $Z \in \mathfrak{Z}(\Theta)$.

The following version of Fatou's lemma is the first of our main convergence theorems. It refers to lower convergence for both functions and measures.

Theorem 5.23. *Let* $\Theta = \{\theta,\theta_n\}_{n\in\mathbb{N}}$ *be equibounded* $\mathfrak{L}(\mathcal{P},\mathcal{Q})$*-valued measures on* $\mathfrak{R}$ *such that* $\theta_n \nearrow \theta$. *Let* $F \in \mathfrak{A}_{\mathfrak{R}}$, *and let* $f_n, f, f_*, f_{**} \in \mathcal{F}_{(|F|,\Theta)}(X,\mathcal{P})$ *such that* $(\theta_n) \overset{F}{\underset{\{f_*\}}{\prec}} \theta$. *Suppose that* $f_{**} \underset{a.e.F}{\leq} f_n + f_*$ *for all* $n \in \mathbb{N}$, *and that* $f_n \underset{a.e.F}{\nearrow} f$. *Then*

$$\int_F f\,d\theta \leq \varliminf_{n\to\infty} \int_F f_n\,d\theta_n + \mathfrak{O}\left(\int_F f_*\,d\theta\right).$$

Proof. Without loss of generality, we may assume that $F = X$. Indeed, the respective integrals over $F \in \mathfrak{A}_{\mathfrak{R}}$ equal the integrals over X for the products of the concerned functions with χ_F, and these products satisfy the conditions of the theorem with X in place of F. Also, we may assume that the required convergence and boundedness properties hold everywhere on X instead of Θ-everywhere. Indeed, let $\mathfrak{Z}(\Theta)$ be the family of zero subsets of X. Then $f_n \nearrow_{a.e.X} f$ means that $f_n \nearrow_{(X\setminus Y)} f$ for some $Y \in \mathfrak{Z}(\theta)$. Using the fact that $\mathfrak{Z}(\Theta)$ contains countable unions of its members, we can find $Y' \in \mathfrak{Z}(\Theta)$ such that $\chi_{(X\setminus Y')^{\otimes}} f_{**} \leq \chi_{(X\setminus Y')^{\otimes}} (f_n + f_*)$ holds for all $n \in \mathbb{N}$ everywhere on X. Let $Z = Y \cup Y' \in \mathfrak{Z}(\Theta)$. The functions $f'_n = \chi_{(X\setminus Z)^{\otimes}} f_n$, $f' = \chi_{(X\setminus Z)^{\otimes}} f$ and $f'_* = \chi_{(X\setminus Z)^{\otimes}} f_*$, then fulfill everywhere all the assumptions of the theorem and their respective integrals coincide with those of the given functions.

We shall proceed using these simplified assumptions of the theorem for the measures θ_n and θ and the functions f_n, f, f_{**} and f_*. In a first step of this proof we shall discuss the respective integrals of the functions involved over a set $E \in \mathfrak{R}$. Let $w \in \mathcal{W}$ be fixed. Following Proposition 5.4, there is $s \in \mathcal{F}_{\mathfrak{R}}(X, \mathcal{V})$ and $\lambda > 0$ such that $\int_E s \, d\vartheta \leq \lambda w$ for all $\vartheta \in \Theta$ and both $0 \underset{a.e.E}{\leq} f + s$. Using a similar argument as above, that is the replacement of the functions f_n and f by suitable functions f'_n and f' which agree with the former ones Θ-almost everywhere, we may also assume that the last relation holds indeed everywhere on E. Next we choose $0 < \varepsilon < \min\{1, \frac{1}{3\lambda}\}$.

According to Definition 5.3 (see also 4.12), we may assume that $s(x) \in \mathcal{V}$ for all $x \in X$. Thus, under the (now simplified) assumptions of the theorem, for every $x \in E$ there is $n_0 \in \mathbb{N}$ such that $f(x) \in \big(s(x)\big)_\varepsilon\big(f_n(x)\big)$ that is $f(x) \leq \gamma f_n(x) + \varepsilon s(x)$ for all $n \geq n_0$ with some $1 \leq \gamma \leq 1$. According to Lemma I.4.1(c), the latter implies that

$$f(x) \leq (1+\varepsilon) f_n(x) + \varepsilon(1 + 1 + \varepsilon) s(x) \leq (1+\varepsilon) f_n(x) + 3\varepsilon s(x)$$

for all $n \geq n_0$. We choose a neighborhood $v \in \mathcal{V}$ such that $\vartheta_E(v) \leq w$ for all $\vartheta \in \Theta$. Following Theorem 1.6, all the sets

$$E_m = \{x \in E \mid f(x) \preccurlyeq_v (1+\varepsilon) f_n(x) + 3\varepsilon s(x) \quad \text{for all} \quad n \geq m\}$$

are in $\mathfrak{R}$, we have $E_m \subset E_{m+1}$, and $\bigcup_{m\in\mathbb{N}} E_m = E$ by the above. Thus

$$f(x) \preccurlyeq_v (1+\varepsilon) f_n(x) + 3\varepsilon s(x)$$

for all $x \in E_m$ and $n \geq m$. Now Proposition 4.16 yields that

$$\begin{aligned} \int_{E_m} f \, d\vartheta &\leq (1+\varepsilon) \int_{E_m} f_n \, d\vartheta + 3\varepsilon \int_{E_m} s \, d\vartheta + \mathfrak{O}\big(\vartheta(E_m, v)\big) \\ &\leq (1+\varepsilon) \int_{E_m} f_n \, d\vartheta + w \end{aligned}$$

holds for all $\vartheta \in \Theta$. The last part of the inequality follows, since $\int_{E_m} s\,d\vartheta \leq \int_E s\,d\vartheta \leq \lambda w$, $3\varepsilon\lambda < 1$ and $\mathfrak{O}\big(\vartheta(E_m, v)\big) \leq \mathfrak{O}\big(\vartheta(E, v)\big) \leq \varepsilon' w$ for all $\varepsilon' > 0$. Next we use $f_{**} \underset{a.e.X}{\leq} f_n + f_*$ for

$$\int_{E \setminus E_m} f_{**}\,d\vartheta \leq \int_{E \setminus E_m} f_n\,d\vartheta + \int_{E \setminus E_m} f_*\,d\vartheta,$$

multiply the latter by $(1+\varepsilon)$ and add it to the preceding inequality for

$$\int_{E_m} f\,d\vartheta + (1+\varepsilon)\int_{E \setminus E_m} f_{**}\,d\vartheta \leq (1+\varepsilon)\left(\int_E f_n\,d\vartheta + \int_{E \setminus E_m} f_*\,d\vartheta\right) + w.$$

The latter holds true for all $m \in \mathbb{N}$, $n \geq m$ and $\vartheta \in \Theta$. For fixed $m \in \mathbb{N}$, Lemma 5.14(b) yields together with I.5.19

$$\begin{aligned}
\int_{E_m} f\,d\theta + (1+\varepsilon)\int_{E \setminus E_m} f_{**}\,d\theta &\leq \varliminf_{n\to\infty}\int_{E_m} f\,d\theta_n + (1+\varepsilon)\varliminf_{n\to\infty}\int_{E \setminus E_m} f_{**}\,d\theta_n \\
&\leq \varliminf_{n\to\infty}\left(\int_{E_m} f\,d\theta_n + (1+\varepsilon)\int_{E \setminus E_m} f_{**}\,d\theta_n\right) \\
&\leq (1+\varepsilon)\left(\varliminf_{n\to\infty}\int_E f_n\,d\theta_n + \varlimsup_{n\to\infty}\int_{E \setminus E_m} f_*\,d\theta_n\right) \\
&\quad + w.
\end{aligned}$$

Now we let m tend to infinity and apply Proposition 4.18(a) for

$$\lim_{m\to\infty}\int_{E_m} f\,d\theta = \int_E f\,d\theta$$

and 4.18(b) for

$$0 \leq \varliminf_{m\to\infty}\int_{(E \setminus E_m)} f_*\,d\theta.$$

Moreover, the definition of the residual component in 5.16 together with our assumption $(\theta_n) \underset{\{f_*\}}{\overset{F}{\prec}} \theta$ yields

$$\varliminf_{n\to\infty}\left(\varlimsup_{n\to\infty}\int_{E \setminus E_m} f_*\,d\theta_n\right) \leq \mathfrak{Rs}\big(\theta_n, X, f_*\big) \leq \mathfrak{O}\left(\int_X f_*\,d\theta\right).$$

The preceding inequality therefore leads to

$$\begin{aligned}
\int_E f\,d\theta &\leq \varliminf_{m\to\infty}\int_{E_m} f\,d\theta + (1+\varepsilon)\varliminf_{m\to\infty}\int_{(E \setminus E_m)} f_*\,d\theta \\
&\leq (1+\varepsilon)\varliminf_{n\to\infty}\int_E f_n\,d\theta_n + \mathfrak{O}\left(\int_X f_*\,d\theta\right) + w.
\end{aligned}$$

Because this last inequality holds for all $w \in \mathcal{W}$ and $\varepsilon > 0$, and as $\mathcal{Q}$ carries the weak preorder, we infer that

$$\int_E f\,d\theta \leq \varliminf_{n\to\infty} \int_E f_n\,d\theta_n + \mathfrak{O}\left(\int_X f_*\,d\theta\right).$$

Now in the second step of our proof, we shall extend the preceding inequality from integrals over sets $E \in \mathfrak{R}$ to the corresponding integrals over X. By our assumption there is $E_0 \in \mathfrak{R}$ such that $\theta|_{(X\setminus E_0)} \leq_{\mathcal{P}} \theta_n|_{(X\setminus E_0)}$ holds for all $n \in \mathbb{N}$. Following Proposition 5.12(c), the latter implies that $\int_F g\,d\theta \leq \int_F g\,d\theta_n$ for every $F \in \mathfrak{A}_{\mathfrak{R}}$ such that $F \subset X \setminus E_0$ and $g \in \mathcal{F}_{(F,\Theta)}(X,\mathcal{P})$. Recall that all functions involved in the theorem are in $\mathcal{F}_{(|X|,\Theta)}(X,\mathcal{P}_0)$, hence are integrable over complements of all sets in $\mathfrak{R}$. Using this, for every $E \in \mathfrak{R}$ such that $E_0 \subset E$ we infer that

$$\int_{(X\setminus E)} f_{**}\,d\theta \leq \int_{(X\setminus E)} f_n\,d\theta + \int_{(X\setminus E)} f_*\,d\theta \leq \int_{(X\setminus E)} f_n\,d\theta_n + \int_{(X\setminus E)} f_*\,d\theta,$$

hence

$$\int_E f_n\,d\theta_n + \int_{(X\setminus E)} f_{**}\,d\theta \leq \int_X f_n\,d\theta_n + \int_{(X\setminus E)} f_*\,d\theta$$

for all $n \in \mathbb{N}$. Thus using the above and the result of our first step we obtain

$$\begin{aligned}\int_E f\,d\theta + \int_{(X\setminus E)} f_{**}\,d\theta &\leq \varliminf_{n\to\infty}\left(\int_E f_n\,d\theta_n + \int_{(X\setminus E)} f_{**}\,d\theta\right) + \mathfrak{O}\left(\int_X f_*\,d\theta\right)\\ &\leq \varliminf_{n\to\infty}\int_X f_n\,d\theta_n + \int_{(X\setminus E)} f_*\,d\theta + \mathfrak{O}\left(\int_X f_*\,d\theta\right).\end{aligned}$$

Now we use the definition of the integral for

$$\int_X f\,d\theta = \lim_{E\in\mathfrak{R}} \int_E f\,d\theta$$

and Proposition 4.19 for

$$0 \leq \varliminf_{E\in\mathfrak{R}} \int_{(X\setminus E)} f_{**}\,d\theta \qquad \text{and} \qquad \varlimsup_{E\in\mathfrak{R}} \int_{(X\setminus E)} f_*\,d\theta \leq \mathfrak{O}\left(\int_X f_*\,d\theta\right).$$

Finally, taking the limit over $E \in \mathfrak{R}$, and combining all of the above yields

$$\begin{aligned}\int_X f\,d\theta &\le \underline{\lim}_{E\in\mathfrak{R}} \int_E f\,d\theta + \underline{\lim}_{E\in\mathfrak{R}} \int_{(X\setminus E)} f_{**}\,d\theta \\ &\le \overline{\lim}_{E\in\mathfrak{R}} \left(\int_E f\,d\theta + \int_{(X\setminus E)} f_{**}\,d\theta \right) \\ &\le \underline{\lim}_{n\to\infty} \int_X f_n\,d\theta_n + \mathfrak{O}\left(\int_X f_*\,d\theta \right),\end{aligned}$$

since $\mathfrak{O}\left(\int_X f_*\,d\theta\right) + \mathfrak{O}\left(\int_X f_*\,d\theta\right) = \mathfrak{O}\left(\int_X f_*\,d\theta\right)$ by Proposition I.5.11. This completes our proof. $\square$

Because cone-valued functions do in general not have additive inverses, we require a result corresponding to Theorem 5.23 with respect to upper convergence for both measures and functions.

Theorem 5.24. *Let* $\Theta = \{\theta, \theta_n\}_{n\in\mathbb{N}}$ *be equibounded* $\mathfrak{L}(\mathcal{P},\mathcal{Q})$*-valued measures on* $\mathfrak{R}$ *such that* $\theta_n \searrow \theta$*. Let* $F \in \mathfrak{A}_{\mathfrak{R}}$*, and let* $f_n, f, f^* \in \mathcal{F}_{(|F|,\Theta)}(X,\mathcal{P})$ *such that* $(\theta_n) \underset{\{f^*\}}{\overset{F}{\prec}} \theta$*. Suppose that* $f_n \underset{a.e.F}{\le} f^*$ *for all* $n \in \mathbb{N}$*, and that* $f_n \underset{a.e.F}{\searrow} f$*. Then*

$$\overline{\lim}_{n\to\infty} \int_F f_n\,d\theta_n \le \int_F f\,d\theta + \mathfrak{O}\left(\int_F f^*\,d\theta \right).$$

Proof. Our argument will follow the lines of the proof of Theorem 5.23, though some substantial adaptations will be required. For the reasons given in 5.23, without loss of generality, we may assume that $F = X$, and that the stated convergence and boundedness properties hold everywhere on X instead of Θ-everywhere.

Suppose that the functions f, f_n, f^* and the measures $\Theta = \{\theta, \theta_n\}$ fulfill these simplified assumptions of the theorem. Again, in a first step we shall discuss the respective integrals of the functions involved over a set $E \in \mathfrak{R}$. For this, let $w \in \mathcal{W}$ be fixed. Following Proposition 5.4, there is $s \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{V})$ and $\lambda > 0$ such that both $0 \underset{a.e.E}{\le} f + s$ and $0 \underset{a.e.E}{\le} f^* + s$ and $\int_E s\,d\vartheta \le \lambda w$ for all $\vartheta \in \Theta$. Moreover, we have $f_n \underset{a.e.E}{\le} f^*$ for all $n \in \mathbb{N}$ by or assumption. Using a similar argument as before, we may assume that all these relations hold indeed everywhere on E. Next we choose $0 < \varepsilon < \min\{1, \frac{1}{2\lambda}\}$.

We may assume that $s(x) \in \mathcal{V}$ for all $x \in X$ (see 5.3 and 4.12). Thus, under the (now simplified) assumptions of the theorem, for every $x \in E$ there is $n_0 \in \mathbb{N}$ such that $f_n(x) \in \big(s(x)\big)_\varepsilon\big(f(x)\big)$ that is $f_n(x) \le \gamma f(x) + \varepsilon s(x)$ for all $n \ge n_0$ with some $1 \le \gamma \le 1$. According to Lemma I.4.1(b), the latter implies that

$$f_n(x) \le (1+\varepsilon) f(x) + 2\varepsilon s(x)$$

for all $n \ge n_0$. We choose a neighborhood $v \in \mathcal{V}$ such that $\vartheta_E(v) \le w$ for all $\vartheta \in \Theta$. Following Theorem 1.6, all the sets

$$E_m = \{x \in E \mid f_n(x) \preccurlyeq_v (1+\varepsilon) f(x) + 2\varepsilon s(x) \quad \text{for all} \quad n \ge m\}$$

are in $\mathfrak{R}$, we have $E_m \subset E_{m+1}$, and $\bigcup_{m\in\mathbb{N}} E_m = E$ by the above. Thus

$$f_n(x) \preccurlyeq_v (1+\varepsilon)f(x) + 2\varepsilon s(x) \qquad \text{and} \qquad f_n(x) \leq f^*(x)$$

holds Θ-almost everywhere on E_m for all $n \geq m$. We fix $m \in \mathbb{N}$, recall that $\delta(1-2\varepsilon\lambda) > 0$, and that $\mathfrak{O}\big(\sup\{\theta_E(v) \mid \theta \in \Theta\}\big) \leq \delta w$ for all $\delta > 0$, and use Lemma 5.20(a) for

$$\begin{aligned}\overline{\lim_{n\to\infty}} \int_{E_m} f_n\,d\theta_n &\leq \int_{E_m} \Big((1+\varepsilon)f + 2\varepsilon s\Big)\,d\theta + \mathfrak{Rs}\big(\theta_n, E_m, f^*\big) + (1-2\varepsilon\lambda)w \\ &\leq (1+\varepsilon)\int_{E_m} f\,d\theta + \mathfrak{O}\left(\int_X f^*\,d\theta\right) + w.\end{aligned}$$

The last part of this inequality follows, since $\int_{E_m} s\,d\theta \leq \int_E s\,d\theta \leq \lambda w$, and since

$$\mathfrak{Rs}\big(\theta_n, E_m, f^*\big) \leq \mathfrak{Rs}\big(\theta_n, X, f^*\big) \leq \mathfrak{O}\left(\int_X f^*\,d\theta\right)$$

by our assumption on the function f^*. Next we use $f_n \underset{a.e.E}{\leq} f^*$ for $\int_{(E\setminus E_m)} f_n\,d\vartheta \leq \int_{(E\setminus E_m)} f^*\,d\vartheta$, and Lemma 5.20(b) for

$$\overline{\lim_{n\to\infty}} \int_{(E\setminus E_m)} f_n\,d\theta_n \leq \overline{\lim_{n\to\infty}} \int_{(E\setminus E_m)} f^*\,d\theta_n \leq \int_{(E\setminus E_m)} f^*\,d\theta.$$

Thus, using the limit rules from Lemma I.5.19, we obtain

$$\begin{aligned}\overline{\lim_{n\to\infty}} \int_E f_n\,d\theta_n &\leq \overline{\lim_{n\to\infty}} \int_{E_m} f_n\,d\theta_n + \overline{\lim_{n\to\infty}} \int_{(E\setminus E_m)} f_n\,d\theta_n \\ &\leq (1+\varepsilon)\int_{E_m} f\,d\theta + \int_{(E\setminus E_m)} f^*\,d\theta + \mathfrak{O}\left(\int_X f^*\,d\theta\right) + w.\end{aligned}$$

This holds true for all $m \in \mathbb{N}$. Now we let m tend to infinity and apply Proposition 4.18(a) for

$$\lim_{m\to\infty} \int_{E_m} f\,d\theta = \int_E f\,d\theta$$

and 4.18(b) and Proposition I.5.11 for

$$\begin{aligned}\overline{\lim_{m\to\infty}} \int_{(E\setminus E_m)} f^*\,d\theta &\leq \mathfrak{O}\left(\int_E f^*\,d\theta\right) \\ &\leq \mathfrak{O}\left(\int_E f^*\,d\theta\right) + \mathfrak{O}\left(\int_{(X\setminus E)} f^*\,d\theta\right) = \mathfrak{O}\left(\int_X f^*\,d\theta\right).\end{aligned}$$

Combining all of the above, we obtain

$$\overline{\lim_{n\to\infty}} \int_E f_n \, d\theta_n \leq (1+\varepsilon) \int_E f \, d\theta + \mathfrak{O}\left(\int_X f^* \, d\theta \right) + w.$$

Because this inequality holds for all $w \in \mathcal{W}$ and $\varepsilon > 0$, and as $\mathcal{Q}$ carries the weak preorder, we infer that

$$\overline{\lim_{n\to\infty}} \int_E f_n \, d\theta_n \leq \int_E f \, d\theta + \mathfrak{O}\left(\int_X f^* \, d\theta \right).$$

Now in a second step, we shall extend the preceding inequality from integrals over sets $E \in \mathfrak{R}$ to the corresponding integrals over X. Following our definition of the convergence of measures in 5.13, there is $E_0 \in \mathfrak{R}$ such that $\theta_n|_{(X\setminus E_0)} \leq_{\mathcal{P}} \theta|_{(X\setminus E_0)}$ holds for all $n \in \mathbb{N}$. Following Proposition 5.12(c), the latter implies that $\int_F g \, d\theta_n \leq \int_F g \, d\theta$ for every $F \in \mathfrak{A}_{\mathfrak{R}}$ such that $F \subset X \setminus E_0$ and $g \in \mathcal{F}_{(F,\Theta)}(X,\mathcal{P})$. Using this, for every $E \in \mathfrak{R}$ such that $E_0 \subset E$ we infer that

$$\int_X f_n \, d\theta_n = \int_E f_n \, d\theta_n + \int_{(X\setminus E)} f_n \, d\theta_n \leq \int_E f_n \, d\theta_n + \int_{(X\setminus E)} f^* \, d\theta$$

for all $n \in \mathbb{N}$. Thus using the above and the result of our first step we obtain

$$\begin{aligned} \overline{\lim_{m\to\infty}} \int_X f_n \, d\theta_n &\leq \overline{\lim_{m\to\infty}} \int_E f_n \, d\theta_n + \int_{(X\setminus E)} f^* \, d\theta \\ &\leq \int_E f \, d\theta + \int_{(X\setminus E)} f^* \, d\theta + \mathfrak{O}\left(\int_X f^* \, d\theta \right). \end{aligned}$$

Now we use the definition of the integral for

$$\lim_{E\in\mathfrak{R}} \int_E f \, d\theta = \int_X f \, d\theta$$

and Proposition 4.19 for

$$\underline{\lim_{E\in\mathfrak{R}}} \int_{(X\setminus E)} f^* \, d\theta \leq \mathfrak{O}\left(\int_X f^* \, d\theta \right).$$

Combining all of these observations and taking the limit over $E \in \mathfrak{R}$ in the above inequality yields

$$\overline{\lim_{m\to\infty}} \int_X f_n \, d\theta_n \leq \int_X f \, d\theta + \mathfrak{O}\left(\int_X f^* \, d\theta \right),$$

since

$$\mathfrak{O}\left(\int_X f^* \, d\theta \right) + \mathfrak{O}\left(\int_X f^* \, d\theta \right) = \mathfrak{O}\left(\int_X f^* \, d\theta \right)$$

by Proposition I.5.1. This completes our proof. $\square$

Note that for measures $\theta_n \searrow \theta$ and a stationary sequence of functions, that is $f_n = f^* = f \in \mathcal{F}_{(|F|,\Theta)}(X,\mathcal{P})$, such that $(\theta_n) \underset{\{f\}}{\overset{F}{\prec}} \theta$, Theorem 5.24 yields

$$\overline{\lim_{m\to\infty}} \int_F f\,d\theta_n \le \int_F f\,d\theta,$$

since $\int_F f\,d\theta + \mathfrak{O}\left(\int_F f\,d\theta\right) = \int_F f\,d\theta$ by Proposition I.5.14.

The combination of Theorems 5.23 and 5.24 leads to a version of Lebesgue's theorem on dominated convergence (see Proposition 18 in Chapter 11 of [178]). It refers to symmetric convergence for both measures and functions.

Theorem 5.25. *Let* $\Theta = \{\theta_n\}_{n\in\mathbb{N}}$ *be equibounded* $\mathfrak{L}(\mathcal{P},\mathcal{Q})$*-valued measures on* $\mathfrak{R}$ *such that* $\theta_n \longrightarrow \theta$ *for a measure* θ*. Let* $F \in \mathfrak{A}_{\mathfrak{R}}$*, and let* $f_n, f, f_{**}, f_*, f^{**}, f^* \in \mathcal{F}_{(|F|,\Theta)}(X,\mathcal{P})$ *such that* $(\theta_n) \underset{\{f_*,f^*\}}{\overset{F}{\prec}} \theta$*. Suppose that* $f_{**} \underset{a.e.F}{\le} f_n + f_*$ *and* $f_n + f^{**} \underset{a.e.F}{\le} f^*$ *for all* $n \in \mathbb{N}$*, and that* $f_n \underset{a.e.F}{\longrightarrow} f$*. Then*

$$\int_F f\,d\theta \le \underline{\lim_{n\to\infty}} \int_F f_n\,d\theta_n + \mathfrak{O}\left(\int_X f_*\,d\theta\right)$$

and

$$\overline{\lim_{n\to\infty}} \int_F f_n\,d\theta_n \le \int_F f\,d\theta + \mathfrak{O}\left(\int_F f^*\,d\theta\right).$$

Proof. Let the functions $f_n, f, f_{**}, f_*, f_{**} f^*$ and the measures $\Theta = \{\theta_n\}_{n\in\mathbb{N}}$ and θ be as in the assumptions of the theorem. Following Lemma 5.21, integrability with respect to Θ implies integrability with respect to $\bar{\Theta} = \Theta \cup \{\theta\}$. Our assumptions therefore imply those of Theorem 5.23, and we conclude that

$$\int_F f\,d\theta \le \underline{\lim_{n\to\infty}} \int_F f_n\,d\theta_n + \mathfrak{O}\left(\int_F f_*\,d\theta\right).$$

In order to apply Theorem 5.24 we set $g_n = f_n + f^{**}$ and $g = f + f^{**}$ Then $g_n, g \in \mathcal{F}_{(|F|,\bar{\Theta})}(X,\mathcal{P})$ and $g_n \underset{a.e.F}{\le} f^*$ for all $n \in \mathbb{N}$. Moreover, $f_n \underset{a.e.X}{\searrow} f$ implies that $g_n \underset{a.e.X}{\searrow} g$, since the relative topologies were seen to be compatible with the algebraic operations in $\mathcal{P}$ (see Section I.4). The functions g_n, g therefore fulfill the assumptions of Theorem 5.24, and we infer that

$$\begin{aligned}\overline{\lim_{n\to\infty}} \int_F f_n\,d\theta_n + \underline{\lim_{n\to\infty}} \int_F f^{**}\,d\theta_n &\le \overline{\lim_{n\to\infty}} \int_F (f_n + f^{**})\,d\theta_n \\ &\le \int_F f\,d\theta + \int_F f^{**} d\theta + \mathfrak{O}\left(\int_F f^*\,d\theta\right).\end{aligned}$$

Lemma 5.14(b) yields

$$\int_F f^{**}\,d\theta \le \underline{\lim_{n\to\infty}} \int_F f^{**}\,d\theta_n.$$

Thus using the cancellation law in I.5.10(a), we obtain

$$\overline{\lim_{n\to\infty}} \int_F f_n \, d\theta_n \leq \int_F f \, d\theta + \mathfrak{O}\left(\int_F f^* \, d\theta\right) + \mathfrak{O}\left(\int_F f^{**} \, d\theta\right).$$

Finally, the relations $f_n + f^{**} \underset{a.e.F}{\leq} f^*$ and $f_n \underset{a.e.F}{\nearrow} f$ imply that $f(x) + f^{**}(x) \preccurlyeq f^*(x)$ holds θ-almost everywhere on F, and therefore

$$\int_F f \, d\theta + \int_F f^{**} \, d\theta \leq \int_F f^* \, d\theta$$

by Proposition 4.17. The element $\int_F f^{**} \, d\theta$ of $\mathcal{Q}$ is therefore bounded relative to the element $\int_F f^* d\theta$ (see Proposition I.4.11(b)), and Proposition I.5.14 yields that

$$\mathfrak{O}\left(\int_F f^{**} \, d\theta\right) + \mathfrak{O}\left(\int_F f^* \, d\theta\right) = \mathfrak{O}\left(\int_F f^* \, d\theta\right),$$

thus completing our argument. □

We may use the notions of boundedness from Chapter I.4.24 to formulate a special case of Theorem 2.25 that allows a stronger conclusion. Corresponding to I.4.24(iv) we shall say that a subset $\mathcal{A}$ of $\mathcal{F}(X, \mathcal{P})$ is *bounded above relative to a function* $f \in \mathcal{F}(X, \mathcal{P})$ if for every inductive limit neighborhood $\mathfrak{v}$ there are $\lambda, \rho \geq 0$ such that $g \leq \rho f + \lambda \mathfrak{v}$ holds for all $g \in \mathcal{A}$. Similarly we define boundedness below and (relative) boundedness almost everywhere on a set $F \in \mathfrak{A}_{\mathfrak{R}}$, as well as boundedness for nets and sequences in $\mathcal{F}(X, \mathcal{P})$. Recall the notations from I.4.24 and I.4.25.

Corollary 5.26. *Let* θ *be a bounded* $\mathfrak{L}(\mathcal{P}, \mathcal{Q})$*-valued measure on* $\mathfrak{R}$*. Let* $F \in \mathfrak{A}_{\mathfrak{R}}$*, and let* $f_n, f^* \in \mathcal{F}_{(|F|,\theta)}(X, \mathcal{P})$ *such that* $\int_F f^* \, d\theta \in \mathcal{B}\big(\int_F f \, d\theta\big)$*. Let* $(f_n)_{n\in\mathbb{N}}$ *be a sequence in* $\mathcal{F}_{(|F|,\theta)}(X, \mathcal{P})$ *that is* θ*-almost everywhere on* F *bounded below and bounded above relative to* f^**. If* $f_n \xrightarrow[a.e.F]{} f$*, then*

$$\lim_{n\to\infty} \int_F f_n \, d\theta = \int_F f \, d\theta.$$

Proof. This is an immediately consequence of Theorem 5.25: We set $\Theta = \{\theta\}$ and $f_{**} = f^{**} = 0$. Given $w \in \mathcal{W}$ there are $\lambda, \rho \geq 0$ and $n_0 \in \mathbb{N}$ such that $0 \underset{a.e.F}{\leq} f_n + \lambda \mathfrak{v}_w$ and $f_n \underset{a.e.F}{\leq} \rho f^* + \lambda \mathfrak{v}_w$ holds for all $n \geq n_0$. This means $0 \underset{a.e.F}{\leq} f_n + \lambda s$ and $f_n \underset{a.e.F}{\leq} \rho f^* + \lambda t$ for functions $s, t \in \mathcal{S}_{\mathfrak{R}}(X, \mathcal{V})$ such that both $\int_X s \, d\theta \leq w$ and $\int_X t \, d\theta \leq w$. Now we apply Theorem 5.25 with λs in place of f_* and $\rho f^* + \lambda t$ in place of f^* from 5.25. Then $\mathfrak{O}\big(\int_F \lambda s \, d\theta\big) \leq w$ and $\mathfrak{O}\big(\int_F (\rho f^* + \lambda t) \, d\theta\big) \leq \mathfrak{O}\big(f^*\big) + w$ by Proposition I.5.11. Because $\int_F f d\theta + \mathfrak{O}\big(f^*\big) = \int_F f \, d\theta$ by Proposition I.5.14 and our assumption on the function f^*, and because the neighborhood $w \in \mathcal{W}$ was arbitrarily chosen, our claim follows. □

An *elementary function* is a function $f = \varphi\otimes a \in \mathcal{F}(X,\mathcal{P})$, where φ is a bounded measurable non-negative real-valued function supported by a set $E \in \mathfrak{R}$, and a is an element of $\mathcal{P}$. Note that elementary functions are contained in $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$. Indeed, $\chi_E\otimes a \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ implies $\varphi\otimes a = \varphi\otimes(\chi_E\otimes a) \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ by Lemma 2.6. We make the following observations:

Lemma 5.27. *Let φ be a bounded measurable non-negative real-valued function supported by a set in $\mathfrak{R}$. There is a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of real-valued step functions converging uniformly on X to φ and such that $0 \le \varphi_n \le \varphi$ for all $n \in \mathbb{N}$.*

Proof. Let the function φ be as stated, supported by the set $E \in \mathfrak{R}$. Without loss of generality, we may assume that $0 \le \varphi \le 1$. For $n \in \mathbb{N}$ and $i = 1, \dots n$ we set

$$E_n^i = \left\{ x \in E \;\middle|\; \frac{i-1}{n} < \varphi(x) \le \frac{i}{n} \right\} \in \mathfrak{R}$$

and $\varphi_n = \sum_{i=1}^n \frac{i-1}{n}\, \chi_{E_n^i}$. Then $0 \le \varphi_n(x) \le \varphi(x) \le \varphi_n(x) + 1/n$ holds for all $x \in X$. □

Corollary 5.28. *Let θ be a bounded $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measure. Let φ be a bounded non-negative real-valued function supported by a set in $\mathfrak{R}$, and let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of measurable real-valued functions such that $0 \le \varphi_n \underset{a.e.X}{\le} \varphi$ for all $n \in \mathbb{N}$, converging θ-almost everywhere to φ. Then for every $a \in \mathcal{P}$ the sequence $(\varphi_n\otimes a)_{n\in\mathbb{N}}$ in $\mathcal{F}(X,\mathcal{P})$ is bounded below and θ-almost everywhere bounded above relative to $\varphi\otimes a$, the sequence $\big(\int_X \varphi_n\otimes a\, d\theta\big)_{n\in\mathbb{N}}$ in $\mathcal{Q}$ is bounded below and bounded above relative to the element $\int_X \varphi\otimes a\, d\theta$, and*

$$\varphi_n\otimes a \underset{a.e.X}{\longrightarrow} \varphi\otimes a \qquad \textit{and} \qquad \lim_{n\to\infty}\int_X \varphi_n\otimes a\, d\theta = \int_X \varphi\otimes a\, d\theta.$$

Proof. Let the function φ be as stated, supported by the set $E \in \mathfrak{R}$. We may assume that $0 \le \varphi_n \underset{a.e.X}{\le} \varphi \le 1$ holds for all $n \in \mathbb{N}$. For $a \in \mathcal{P}$ we have $\varphi_n\otimes a \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ for all $n \in \mathbb{N}$ by Lemma 2.6. There is a set $Z \in \mathfrak{A}_{\mathfrak{R}}$ of measure 0 such that the functions $\tilde{\varphi}_n = \chi_{(X\setminus Z)}\otimes\varphi_n$ converge pointwise everywhere to $\tilde{\varphi} = \chi_{(X\setminus Z)}\otimes\varphi$ and that $\tilde{\varphi}_n \le \tilde{\varphi} \le 1$ holds for all $n \in \mathbb{N}$. (For this, recall that a countable union of zero sets is again a zero set.) Theorem 1.7 guarantees that $\tilde{\varphi}$ is measurable, hence $\tilde{\varphi}\otimes a \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$, and the function $\varphi\otimes a$ is integrable by 4.12. Let $x \in X \setminus Z$. If $\varphi(x) = 0$, then that $\varphi_n(x) = 0$ for all $n \in \mathbb{N}$ as well. If $\varphi(x) > 0$, then, given $v \in \mathcal{V}$ and $\varepsilon > 0$, there is $\lambda \ge 1$ such that $0 \le a + \lambda v$ and $n_0 \in \mathbb{N}$ such that $\varphi_n(x) \le \varphi(x) \le (1+\varepsilon)\varphi_n(x)$ for all $n \ge n_0$. Thus

$$\varphi_n(x)(a+\lambda v) \le \varphi(x)(a+\lambda v) \le \varphi(x)(a) + (1+\varepsilon)\lambda\varphi_n(x)v$$

and

$$\varphi(x)(a+\lambda v) \le (1+\varepsilon)\varphi_n(x)(a+\lambda v) \le (1+\varepsilon)\varphi_n(x)(a) + (1+\varepsilon)\lambda\varphi(x)v.$$

Now the cancellation law for positive elements (Lemma I.4.2 in [100]) yields

$$\varphi_n(x)a \le \varphi(x)a + 2\varepsilon\lambda\varphi_n(x)v \le \varphi(x)a + 2\varepsilon\lambda v$$

and

$$\varphi(x)a \le (1+\varepsilon)\varphi_n(x)a + 2\varepsilon\lambda\varphi(x)v \le \varphi_n(x)a + 2\varepsilon\lambda v.$$

This shows $\varphi_n(x)a \in v^s_{2\varepsilon\lambda}\big(\varphi(a)\big)$ for all $n \ge n_0$ and demonstrates

$$\varphi_n(x)a \longrightarrow \varphi(x)a$$

in the symmetric relative topology of $\mathcal{P}$. Thus $\varphi_{n\otimes}a \xrightarrow[a.e.X]{} \varphi_{\otimes}a$ holds as claimed.

Furthermore, given an inductive limit neighborhood $\mathfrak{v}$ there is $v \in \mathcal{V}$ such that $\chi_{E\otimes}v \le \mathfrak{v}$ and $\lambda \ge 0$ such that $0 \le a + \lambda v$. Then

$$0 \le \varphi_{n\otimes}(a+\lambda v) \le \varphi_{n\otimes}a + \lambda\mathfrak{v}$$

and

$$\varphi_{n\otimes}a \le \varphi_{n\otimes}(a+\lambda v) \underset{a.e.X}{\le} \varphi_{\otimes}(a+\lambda v) \le \varphi_{\otimes}a + \lambda\mathfrak{v}$$

for all $n \in \mathbb{N}$. The sequence $(\varphi_{n\otimes}a)_{n\in\mathbb{N}}$ in $\mathcal{F}(X,\mathcal{P})$ is therefore bounded below and θ-almost everywhere bounded above relative to the function $\varphi_{\otimes}a$. Furthermore, for any $w \in \mathcal{W}$ we may choose the inductive limit neighborhood $\mathfrak{v}_w$. Then the above yields $0 \le \int_X \varphi_{n\otimes}a\,d\theta + \lambda w$ as well as $\int_X \varphi_{n\otimes}a\,d\theta \le \int_X \varphi_{\otimes}a\,d\theta + \lambda w$ for all $n \in \mathbb{N}$. Hence the sequence $\big(\int_X \varphi_{n\otimes}a\,d\theta\big)_{n\in\mathbb{N}}$ in $\mathcal{Q}$ is seen to be bounded below and bounded above relative to the element $\int_X \varphi_{\otimes}a\,d\theta$. The convergence statement for the sequence of integrals follows from Corollary 5.26. □

Corollary 5.28 in combination with Lemma 5.27 yields a strengthening of the result of Corollary 5.9, that is the approximation of integrable functions by a net of step functions, for elementary functions $f = \varphi_{\otimes}a \in \mathcal{F}(X,\mathcal{P})$: There is a sequence $(h_n)_{n\in\mathbb{N}}$ of step functions that is bounded below and bounded above relative to f such that

$$h_n \longrightarrow f \text{ and } \lim_{n\to\infty}\int_X h_n\,d\theta = \int_X f\,d\theta.$$

5.29 Remarks. (a) If $(\mathcal{Q},\mathcal{W})$ is the (simplified) standard lattice completion (see I.57) of some subcone $(\mathcal{Q}_0,\mathcal{W}_0)$, that is if $(\mathcal{Q},\mathcal{W})$ is a cone of $\overline{\mathbb{R}}$-valued functions on $\mathcal{P}^*$, then for elements $l,m,n \in \mathcal{Q}$ the statement $l \le m + \mathfrak{O}(n)$ means that $l(\mu) \le m(\mu)$ holds for all $\mu \in \mathcal{P}^*$ such that $n(\mu) < +\infty$. The convergence statements of Theorems 5.23 to 5.25 can then be read in this light. The conclusion of Theorem 5.25 means for example that

$$\left(\int_F f\,d\theta\right)(\mu) \le \varliminf_{n\to\infty}\left(\int_F f_n\,d\theta_n\right)(\mu)$$

holds for all $\mu \in \mathcal{P}^*$ such that $\left(\int_F f_*\,d\theta\right)(\mu) < +\infty$, and

$$\varlimsup_{n\to\infty}\left(\int_F f_n\,d\theta_n\right)(\mu) \le \left(\int_F f\,d\theta\right)(\mu)$$

for all $\mu \in \mathcal{P}*$ such that $\left(\int_F f^*\,d\theta\right)(\mu) < +\infty$.

(b) The convergence statements in the preceding Theorems 5.23 to 5.25 refer to order convergence in $\mathcal{Q}$ for the concerned sequences of integrals. Stronger claims than those might state convergence in the lower, upper and symmetric topologies of $\mathcal{Q}$, respectively. In the context of our approach, such claims are however not valid in general, even for stationary sequences of measures, as the following simple example can show: Let $\mathfrak{R}$ be the σ-algebra of all Borel sets in $X = [0,1]$, let $\mathcal{P} = \overline{\mathbb{R}}$ with its usual order and locally convex cone topology. Let $\mathcal{Q}$ be the cone of all bounded below $\overline{\mathbb{R}}$-valued functions on X, endowed with the pointwise operations and order and the strictly positive constant functions w as neighborhoods. Clearly $(\mathcal{Q}, \mathcal{W})$ is a locally convex complete lattice cone, and order convergence in $\mathcal{Q}$ means pointwise convergence for the concerned functions. We define an $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measure θ on $\mathfrak{R}$, setting $\theta_E(\alpha) = \alpha\chi_E \in \mathcal{Q}$ for $E \in \mathfrak{R}$ and $\alpha \in \mathcal{P}$. It is then straightforward to check that $\int_X h\,d\theta = h$ holds for every $\mathcal{P}$-valued step function h on X, that is the integral over θ yields the identity operator from $\mathcal{F}_{(|X|,\theta)}(X,\mathcal{P})$ into $\mathcal{Q}$. Now, if we consider the stationary sequences $\vartheta_n = \theta_n = \theta$ in Theorems 5.23 to 5.25, a review of the assumptions there reveals that only pointwise convergence is required for the sequences of functions $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{F}_{(|X|,\theta)}(X,\mathcal{P})$. Thus only pointwise, that is order convergence will result for their integrals in general. Note that in this example the measure θ is countably additive only with respect to order convergence in $\mathcal{Q}$, not with respect to the weak (see Section I.4.6) or indeed the symmetric relative topology of $\mathcal{Q}$. We shall demonstrate below (Theorem 5.36) that countable additivity for a measure with respect to the symmetric relative topology of $\mathcal{Q}$ in this situation would indeed imply the above stronger statement of convergence for the corresponding sequence of integrals. This shows in particular that no such measure can represent the identity operator from $\mathcal{F}_{(|X|,\theta)}(X,\mathcal{P})$ into $\mathcal{Q}$.

We shall in the following discuss some special cases where convergence with respect to the symmetric topology does indeed result from Theorems 5.23 to 5.25. For the sake of simplicity we shall restrict ourselves to stationary sequences of measures $\theta_n = \theta$ in this context. The preceding Remark 5.29(b) suggests that we shall need to impose further conditions for this purpose. One of these conditions will refer to the countable additivity of the measure θ, another one will require the availability of sufficiently many order continuous linear functionals on $\mathcal{Q}$.

5.30 Strong Additivity. Countable additivity of an $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measure θ as introduced in Section 3 is meant with respect to order convergence in the locally convex complete lattice cone $(\mathcal{Q},\mathcal{W})$. In Theorem 3.11 we verified that in special cases this implies convergence in a stronger sense. In this context, we shall say that an $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measure θ is *strongly additive* if for every decreasing sequence $(E_n)_{n\in\mathbb{N}}$ of sets in $\mathfrak{R}$ such that $\bigcap_{n\in\mathbb{N}} E_n=\emptyset$, for $a\in\mathcal{P}$ and $w\in\mathcal{W}$ there is $n_0\in\mathbb{N}$ such that

$$\theta_{E_n}(a)\le\mathfrak{O}\big(\theta_{E_1}(a)\big)+w$$

holds for all $n\ge n_0$. Similarly, we shall say that a family Θ of $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measures is *uniformly strongly additive* if it is equibounded and if the above property holds with the same n_0 for all $\theta\in\Theta$.

Note that for strong additivity we do not require that a measure is countably additive with respect to the symmetric topology of $\mathcal{Q}$, since this would be overly restrictive. For $\mathcal{Q}=\overline{\mathbb{R}}$, for example, the element $+\infty$ is isolated, that is both open and closed in the symmetric topology of $\overline{\mathbb{R}}$. Thus, for a disjoint union $E=\bigcup_{i\in\mathbb{N}} E_i$ of sets in $\mathfrak{R}$ such that $\theta_E(a)=+\infty$ for $a\in\mathcal{P}$, countable additivity with respect to the symmetric topology would require that $\theta_{(\cup_{i=1}^n E_i)}(a)=+\infty$ for all n greater than some $n_0\in\mathbb{N}$. Lemma 5.31(b) will however imply that for a uniformly strongly additive family Θ of $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measures a requirement corresponding to 5.30 holds indeed with respect to the symmetric topology of $\mathcal{Q}$; more precisely: Given a decreasing sequence $(E_n)_{n\in\mathbb{N}}$ of sets in $\mathfrak{R}$ such that $\bigcap_{n\in\mathbb{N}} E_n=\emptyset$, $a\in\mathcal{P}$ and $w\in\mathcal{W}$, there is $n_0\in\mathbb{N}$ such that

$$0\le\theta_{E_n}(a)+w\qquad\text{and}\qquad\theta_{E_n}(a)\le\mathfrak{O}\big(\theta_{E_1}(a)\big)+w$$

holds for all $\theta\in\Theta$ and $n\ge n_0$.

There are several well-known results about strong additivity. Our version of Pettis' theorem, that is Theorem 3.11, (see Theorem IV.10.1 in [55]), states that in case that $(\mathcal{P},\mathcal{V})$ is a locally convex topological vector space and $(\mathcal{Q},\mathcal{W})$ is the standard lattice completion of some subcone $(\mathcal{Q}_0,\mathcal{W})$, every $\mathfrak{L}(\mathcal{P},\mathcal{Q}_0)$-valued measure is also strongly additive. The Vitali-Hahn-Saks theorem see Theorem III.7.2 in [55]) implies a theorem by Nikodým which states that every setwise convergent sequence of real- or Banach space-valued measures is in fact uniformly strongly additive (see Corollary III.7.4 and Theorem IV.10.6 in [55]). We shall investigate a few implications of strong additivity. Lemmas 5.31 and 5.32 will strengthen the corresponding statements from Proposition 4.18.

Lemma 5.31. *Suppose that the family Θ of $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measures is uniformly strongly additive. Let $E\in\mathfrak{R}$ and $f\in\mathcal{F}_{(E,\Theta)}(X,\mathcal{P})$.*

(a) If $E_n\in\mathfrak{R}$ such that $E_n\subset E_{n+1}$ for all $n\in\mathbb{N}$, and $E=\bigcup_{n\in\mathbb{N}} E_n$, then for every $w\in\mathcal{W}$ there is $n_0\in\mathbb{N}$ such that $\int_{E_n} f\,d\theta\le\int_E f\,d\theta+w$ for all $\theta\in\Theta$ and $n\ge n_0$.

(b) *If* $E_n \in \mathfrak{R}$ *such that* $E \supset E_n \supset E_{n+1}$ *for all* $n \in \mathbb{N}$, *and* $\bigcap_{n\in\mathbb{N}} E_n = \emptyset$, *then for every* $w \in \mathcal{W}$ *there is* $n_0 \in \mathbb{N}$ *such that* $0 \le \int_{E_n} f\,d\theta + w$ *for all* $\theta \in \Theta$ *and* $n \ge n_0$.

Proof. We shall first prove Part (b) of the lemma. Let $E_n \in \mathfrak{R}$ for $n \in \mathbb{N}$ be subsets of $E \in \mathfrak{R}$ such that $E_n \supset E_{n+1}$ and $\bigcap_{n\in\mathbb{N}} E_n = \emptyset$. In a first step, we shall consider a function $f \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$. Let $w \in \mathcal{W}$. Because the family Θ is supposed to be equibounded, there is $v \in \mathcal{V}$ such that $\theta_E(v) \le w$ for all $\theta \in \Theta$. This implies $\mathfrak{O}\big(\theta_E(v)\big) \le \varepsilon w$ for all $\theta \in \Theta$ and $\varepsilon > 0$. Following Lemma 2.4(b), there is $\lambda \ge 0$ such that $0 \le \chi_{E^\otimes} f + \lambda\chi_{E^\otimes} v$. By 5.30 there is $n_0 \in \mathbb{N}$ such that

$$\theta_{E_n}(v) \le \mathfrak{O}\big(\theta_E(v)\big) + \frac{1}{2\lambda}w \le \frac{1}{\lambda}w$$

for all $\theta \in \Theta$ and $n \ge n_0$. This yields

$$0 \le \int_{E_n} (\chi_{E^\otimes} f + \lambda\chi_{E^\otimes} v)\,d\theta = \int_{E_n} f\,d\theta + \lambda\theta_{E_n}(v) \le \int_{E_n} f\,d\theta + w$$

for all $\theta \in \Theta$ and $n \ge n_0$. Now in the second and general step, let $f \in \mathcal{F}_{(E,\Theta)}(X,\mathcal{P})$. Given $w \in \mathcal{W}$ and $0 < \varepsilon \le 1/2$, let the functions $f_{(w,\varepsilon)} \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ and $s_{(w,\varepsilon)} \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{V})$ be as in the definition of integrability in 5.3, that is

$$f \underset{a.e.E}{\le} f_{(w,\varepsilon)} \underset{a.e.E}{\le} \gamma f + s_{(w,\varepsilon)} \qquad \text{and} \qquad \int_E s_{(w,\varepsilon)}\,d\theta \le \varepsilon w$$

for some $1 \le \gamma \le 1+\varepsilon$ and all $\theta \in \Theta$. Following our first step, there is $n_0 \in \mathbb{N}$ such that $0 \le \int_{E_n} f_{(w,\varepsilon)}\,d\theta + w/2$, hence $0 \le \gamma\int_{E_n} f\,d\theta + \left(\frac{1}{2}+\varepsilon\right)w$ for all $\theta \in \Theta$ and $n \ge n_0$. Because $\gamma \ge 1$ and $\varepsilon \le 1/2$ this yields

$$0 \le \int_{E_n} f\,d\theta + w$$

for all $\theta \in \Theta$ and $n \ge n_0$, our claim in Part (b). For Part (a), let $E_n \in \mathfrak{R}$ such that $E_n \subset E_{n+1}$ for all $n \in \mathbb{N}$, and $E = \bigcup_{n\in\mathbb{N}} E_n$. We set $F_n = E \setminus E_n \in \mathfrak{R}$ and have $F_n \supset F_{n+1}$ and $\bigcap_{n\in\mathbb{N}} F_n = \emptyset$. For a function $f \in \mathcal{F}_{(E,\Theta)}(X,\mathcal{P})$ we may now use Part (b) of the lemma: Given $w \in \mathcal{W}$, there is $n_0 \in \mathbb{N}$ such that $0 \le \int_{F_n} f\,d\theta + w$ holds for all $\theta \in \Theta$ and $n \ge n_0$. This yields

$$\int_{E_n} f\,d\theta \le \int_{E_n} f\,d\theta + \left(\int_{F_n} f\,d\theta + w\right) = \int_E f\,d\theta + w$$

for all $\theta \in \Theta$ and $n \ge n_0$, our claim in Part (a). □

Lemma 5.32. *Suppose that the family* Θ *of* $\mathfrak{L}(\mathcal{P},\mathcal{Q})$*-valued measures is uniformly strongly additive and that the function* $f \in \mathcal{F}(X,\mathcal{P})$ *is strongly integrable over* $E \in \mathfrak{R}$ *with respect to* Θ.

(a) If $E_n \in \mathfrak{R}$ *are such that* $E_n \subset E_{n+1}$ *for all* $n \in \mathbb{N}$, *and* $E = \bigcup_{n\in\mathbb{N}} E_n$, *then for every* $w \in \mathcal{W}$ *there is* $n_0 \in \mathbb{N}$ *such that* $\int_E f\,d\theta \le \int_{E_n} f\,d\theta + \mathfrak{O}\left(\int_E f\,d\theta\right) + w$ *for all* $\theta \in \Theta$ *and* $n \ge n_0$.

(b) If $E_n \in \mathfrak{R}$ *are such that* $E \supset E_n \supset E_{n+1}$ *for all* $n \in \mathbb{N}$, *and* $\bigcap_{n\in\mathbb{N}} E_n = \emptyset$, *then for every* $w \in \mathcal{W}$ *there is* $n_0 \in \mathbb{N}$ *such that* $\int_{E_n} f\,d\theta \le \mathfrak{O}\left(\int_E f\,d\theta\right) + w$ *for all* $\theta \in \Theta$ *and* $n \ge n_0$.

Proof. Again, we shall first prove Part (b) of the Lemma. Let $f \in \mathcal{F}(X,\mathcal{P})$ be strongly integrable over $E \in \mathfrak{R}$ with respect to Θ. For $w \in \mathcal{W}$, according to 5.18 there is a step function $h = \sum_{i=1}^m \chi_{F_i} \otimes a_i \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P})$ such that

$$\int_G f\,d\theta \le \int_G h\,d\theta + w/3$$

and such that $\int_G h\,d\theta$ is w-bounded relative to $\int_G f\,d\theta$ for all $\theta \in \Theta$ and every subset $G \in \mathfrak{R}$ of E. Let $E_n \in \mathfrak{R}$ for $n \in \mathbb{N}$ be subsets of E such that $E_n \supset E_{n+1}$ and $\bigcap_{n\in\mathbb{N}} E_n = \emptyset$. For every $n \in \mathbb{N}$ and $\theta \in \Theta$, we calculate

$$\int_{E_n} h\,d\theta = \sum_{i=1}^m \theta_{(E_n \cap F_i)}(a_i).$$

The measures in Θ are supposed to be uniformly strongly additive. Thus there is $n_0 \in \mathbb{N}$ such that

$$\theta_{(E_n \cap F_i)}(a_i) \le \mathfrak{O}\left(\theta_{(E\cap F_i)}(a_i)\right) + \frac{1}{3m}\,w$$

for all $n \ge n_0$, $\theta \in \Theta$ and $i = 1,\ldots,m$. Thus, using Proposition I.5.11

$$\int_{E_n} h\,d\theta \le \sum_{i=1}^m \mathfrak{O}\left(\theta_{(E\cap F_i)}(a_i)\right) + \frac{1}{3}w = \mathfrak{O}\left(\int_E h\,d\theta\right) + \frac{1}{3}\,w$$

and

$$\int_{E_n} h\,d\theta \le \mathfrak{O}\left(\int_E f\,d\theta\right) + \frac{2}{3}\,w,$$

since $\int_E h\,d\theta \in \mathcal{B}_w\left(\int_E f\,d\theta\right)$, which by Proposition I.5.13(a) implies that $\mathfrak{O}\left(\int_E h\,d\theta\right) \le \mathfrak{O}\left(\int_E h\,d\theta\right) + \varepsilon w$ for all $\varepsilon > 0$. Thus for all $n \ge n_0$ and $\theta \in \Theta$ we infer that

$$\int_{E_n} f\,d\theta \le \int_{E_n} h\,d\theta + \frac{1}{3}w \le \mathfrak{O}\left(\int_E f\,d\theta\right) + w,$$

that is Part (b) of our claim. For Part (a), let $E_n \in \mathfrak{R}$ such that $E_n \subset E_{n+1}$ for all $n \in \mathbb{N}$, and $E = \bigcup_{n\in\mathbb{N}} E_n$. We set $F_n = E \setminus E_n \in \mathfrak{R}$ and have $F_n \supset F_{n+1}$ and $\bigcap_{n\in\mathbb{N}} F_n = \emptyset$. For a function $f \in \mathcal{F}(X,\mathcal{P})$ that is strongly integrable over $E \in \mathfrak{R}$ with respect to Θ we may now use Part (b) of the lemma: Given $w \in \mathcal{W}$, there is $n_0 \in \mathbb{N}$ such that $\int_{F_n} f\,d\theta \le \mathfrak{O}\left(\int_E f\,d\theta\right) + w$

for all $\theta \in \Theta$ and $n \geq n_0$. This yields

$$\int_E f\,d\theta = \int_{E_n} f\,d\theta + \int_{F_n} f\,d\theta \leq \int_{E_n} f\,d\theta + \mathfrak{O}\left(\int_E f\,d\theta\right) + w$$

for all $\theta \in \Theta$ and $n \geq n_0$, our claim in Part (a). □

5.33 Weakly Sequentially Compact Sets of Measures. A family Θ of $\mathfrak{L}(\mathcal{P}, \mathcal{Q})$-valued measures is said to be *weakly sequentially compact* if every sequence $(\theta_n)_{n\in\mathbb{N}}$ in Θ contains a setwise convergent subsequence $(\theta_{n_k})_{k\in\mathbb{N}}$, that is $\theta_{n_k} \longrightarrow \theta$ for some measure θ on $\mathfrak{R}$ (see Definition II.3.18 in [55]). Note that we do not require that $\theta \in \Theta$. As a consequence, every subset of a sequentially compact set is again sequentially compact.

Theorem IV.9.1 in [55] provides a well-known criterion for weak sequential compactness of a family of finite real-valued measures defined on a σ-algebra $\mathfrak{R}$: Such a family Θ is weakly sequentially compact if and only if (i) Θ is equibounded, that is the total variation of its elements is bounded on X, and (ii) Θ is uniformly (strongly) additive. We shall use this to establish a criterion for sequential compactness of a family of functional-valued measures, that is for the case $\mathcal{Q} = \overline{\mathbb{R}}$.

Lemma 5.34. *Suppose that all elements of $\mathcal{P}$ are bounded and that $\mathcal{P}$ is separable in the symmetric relative v-topology for every $v \in \mathcal{V}$. Suppose that $X \in \mathfrak{R}$. Then every uniformly strongly additive family of $\mathcal{P}^*$-valued measures on $\mathfrak{R}$ is weakly sequentially compact.*

Proof. Let Θ be a uniformly strongly additive family of $\mathcal{P}^*$-valued measures on $\mathfrak{R}$. Because we assume that $X \in \mathfrak{R}$, and because uniform strong additivity includes equiboundedness (see 5.30), there is $v \in \mathcal{V}$ such that $\theta_X(v) \leq 1$ for all $\theta \in \Theta$. In a first step of our argument we fix an element $a \in \mathcal{P}$ and choose $\lambda \geq 0$ such that both $0 \leq a + \lambda v$ and $a \leq \lambda v$. The latter is possible because all elements of $\mathcal{P}$ are supposed to be bounded, which implies in particular that $\vartheta_E(a) < +\infty$ for every $\mathcal{P}^*$-valued measure ϑ and $E \in \mathfrak{R}$. For every $\theta \in \Theta$ we may therefore define a real-valued measure θ_a on $\mathfrak{R}$, setting $\theta_a(E) = \theta_E(a)$ for every $E \in \mathfrak{R}$. The above implies that $\theta_a(E) \leq \lambda\theta_E(v)$ and $0 \leq \theta_a(E) + \lambda\theta_E(v)$, hence $|\theta_a(E)| \leq \lambda\theta_E(v) < \lambda\theta_X(v) \leq \lambda$. Using this, we can estimate the usual (total) variation $\mathfrak{var}\,(\theta_a, X)$ of this measure (see Definition III.1.4 in [55]) as follows: For disjoint sets $E_1, \ldots, E_n \in \mathfrak{R}$ we have

$$\sum_{i=1}^{n} |\theta_{E_i}(a)| \leq \lambda \sum_{i=1}^{n} \theta_{E_i}(v) = \lambda\theta_{(\cup_{i=1}^n E_i)}(v) \leq \lambda,$$

hence

$$\mathfrak{var}\,(\theta_a, X) = \sup\left\{ \sum_{i=1}^{n} |\theta_a(E_i)| \mid E_1, \ldots, E_n \in \mathfrak{R},\ \text{disjoint} \right\} \leq \lambda.$$

Thus for the family $\Theta_a = \{\theta_a \mid \theta \in \Theta\}$ of real-valued measures, firstly the total variation of its elements is bounded by λ, and secondly, the countable additivity on $\mathfrak{R}$ is uniform with respect to all measures in Θ_a. The latter follows from our requirement that the family Θ is uniformly strongly countably additive. Indeed, let $E_n \in \mathfrak{R}$ such that $E_n \supset E_{n+1}$ and $\bigcap_{n\in\mathbb{N}} E_n = \emptyset$. Following 5.30, given $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $\theta_{E_n}(a) \leq \mathfrak{O}\big(\theta_E(a)\big) + \varepsilon$ for all $n \geq n_0$ and $\theta \in \Theta$. Because $\theta_E(a)$ is finite, we have $\mathfrak{O}\big(\theta_E(a)\big) = 0$. Thus

$$\theta_a(E_n) = \theta_{E_n}(a) \leq \varepsilon$$

holds for all $\theta_a \in \Theta_a$ and $n \geq n_0$. Now the criterion from Theorem IV.9.1 in [55] (see the remark following 5.33) for weak sequential compactness of finite real-valued measures yields that the set Θ_a is indeed weakly sequentially compact. Now in the second step of our argument, following our assumption of the separability of $\mathcal{P}$, we choose a countable subset $\{a_n\}_{n\in\mathbb{N}}$ of $\mathcal{P}$ that is dense with respect to the symmetric relative v-topology. Let $(\theta_n)_{n\in\mathbb{N}}$ be a sequence in Θ. We shall apply a diagonal procedure in order to construct a weakly convergent subsequence of $(\theta_n)_{n\in\mathbb{N}}$. For each $n \in \mathbb{N}$, the set Θ_{a_n} of real-valued measures was seen to be weakly sequentially compact. Thus there is a subsequence $(\theta_n^1)_{n\in\mathbb{N}}$ of $(\theta_n)_{n\in\mathbb{N}}$ and a real-valued measure ϑ^1 such that $\theta^1_{n\,E}(a_1) \to \vartheta^1(E)$ for all $E \in \mathfrak{R}$. Likewise, there is a subsequence $(\theta_n^2)_{n\in\mathbb{N}}$ of $(\theta_n^1)_{n\in\mathbb{N}}$ and a real-valued measure ϑ^2 such that $\theta^2_{n\,E}(a_2) \to \vartheta^2(E)$ for all $E \in \mathfrak{R}$. And so on... We choose the subsequence $(\theta_n^n)_{n\in\mathbb{N}}$ of $(\theta_n)_{n\in\mathbb{N}}$ and claim that this subsequence converges setwise towards some $\mathcal{P}^*$-valued measure ϑ. Indeed, for every $i \in \mathbb{N}$ we have by our construction $\theta^n_{n\,E}(a_i) \to \vartheta^i(E)$ for all $E \in \mathfrak{R}$. Let $\mathcal{P}_0$ be the subcone of $\mathcal{P}$ spanned by the elements $\{a_n\}_{n\in\mathbb{N}}$. By our assumption $\mathcal{P}_0$ is dense in $\mathcal{P}$ with respect to the symmetric relative v-topology. For a fixed $E \in \mathfrak{R}$ and $a = \sum_{i=1}^n \lambda_i a_i \in \mathcal{P}_0$ for $\lambda_i \geq 0$, set

$$\vartheta_E(a) = \lim_{n\to\infty} \theta^n_{n\,E}(a) = \sum_{i=1}^n \lambda_i \vartheta^i(E) \in \mathbb{R}.$$

Clearly, ϑ_E is a linear functional on $\mathcal{P}_0$, and $a \leq b + v$ for $a, b \in \mathcal{P}_0$ implies that

$$\theta_E(a) \leq \theta_E(b) + \theta_E(v) \leq \theta_E(b) + 1,$$

for all $\theta \in \Theta$. Using the limit rules, this shows in turn that $\vartheta_E(a) \leq \vartheta_E(b) + 1$ holds as well. The linear functional $\vartheta_E : \mathcal{P}_0 \to \overline{\mathbb{R}}$ is therefore continuous with respect to the locally convex topology on $\mathcal{P}$ generated by the single neighborhood $v \in \mathcal{V}$, that is the neighborhood system $\mathcal{V}_v = \{\alpha v \mid \alpha > 0\}$, and can therefore be uniquely extended to a continuous linear functional on the whole cone $\mathcal{P}$ (see Theorem I.5.56). Moreover, for every $a \in \mathcal{P}$ and $0 < \varepsilon \leq 1$ there is some $b \in \mathcal{P}_0$ such that both $a \in v_\varepsilon(b)$ and $b \in v_\varepsilon(a)$. This implies by the above that $\theta^n_{n\,E}(a) \in v_\varepsilon\big(\theta^n_{n\,E}(b)\big)$ and $\theta^n_{n\,E}(b) \in v_\varepsilon\big(\theta^n_{n\,E}(a)\big)$ for all $n \in \mathbb{N}$. There is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $\theta^n_{n\,E}(b) \in v_\varepsilon\big(\vartheta_E(b)\big)$ and $\vartheta_E(b) \in v_\varepsilon\big(\theta^n_{n\,E}(a)\big)$. Now combining all of the above yields with Lemma I.4.1(a)

$$\theta^n_{n\,E}(a) \in v_\varepsilon\big(\theta^n_{n\,E}(b)\big) \subset v_{3\varepsilon}\big(\vartheta_E(b)\big) \subset v_{7\varepsilon}\big(\vartheta_E(a)\big)$$

and likewise

$$\vartheta_E(a) \in v_\varepsilon\big(\vartheta_E(b)\big) \subset v_{3\varepsilon}\big(\theta^n_{n\,E}(b)\big) \subset v_{7\varepsilon}\big(\theta^n_{n\,E}(a)\big)$$

for all $n \geq n_0$. This demonstrates that $\theta^n_{n\,E}(a) \to \vartheta_E(a)$ for all $a \in \mathcal{P}$. All left to show is that the mapping $E \mapsto \vartheta_E : \mathfrak{R} \to \mathcal{P}^*$ is countably additive, that is ϑ is indeed a $\mathcal{P}^*$-valued measure on $\mathfrak{R}$. For this, let $a \in \mathcal{P}$, and let $E_i \in \mathfrak{R}$, for $i \in \mathbb{N}$, be disjoint sets. Using the additivity of the measures θ^n_n and the limit rules, we have

$$\vartheta_{(\cup_{i=1}^{i_0} E_i)}(a) = \lim_{n\to\infty}\Big(\theta^n_{n\,(\cup_{i=1}^{i_0} E_i)}(a)\Big) = \sum_{i=1}^{i_0}\Big(\lim_{n\to\infty}\theta^n_{n\,E_i}(a)\Big) = \sum_{i=1}^{i_0}\vartheta_{E_i}(a).$$

for every $i_0 \in \mathbb{N}$. This shows finite additivity in particular. Given $\varepsilon > 0$, it follows from the uniform strong additivity of the measures in Θ together with Lemma 5.31(b) that there is $i_0 \in \mathbb{N}$ such that $|\theta_{(\cup_{i=i_0+1}^{\infty} E_i)}(a)| \leq \varepsilon$ holds for all $\theta \in \Theta$, hence also $|\vartheta_{(\cup_{i=i_0+1}^{\infty} E_i)}(a)| \leq \varepsilon$. This yields with the above

$$\begin{aligned}\Big|\vartheta_{(\cup_{i=1}^{\infty} E_i)}(a) - \sum_{i=1}^{i_0}\vartheta_{E_i}(a)\Big| &= \Big|\vartheta_{(\cup_{i=1}^{\infty} E_i)}(a) - \vartheta_{(\cup_{i=1}^{i_0} E_i)}(a)\Big| \\ &= \Big|\vartheta_{(\cup_{i=i_0+1}^{\infty} E_i)}(a)\Big| \leq \varepsilon.\end{aligned}$$

Because $\varepsilon > 0$ was arbitrarily chosen, this yields

$$\vartheta_{(\cup_{i=1}^{\infty} E_i)}(a) = \sum_{i=1}^{\infty}\vartheta_{E_i}(a).$$

Summarizing, we have verified that the subsequence $\big(\theta^n_n\big)_{n\in\mathbb{N}}$ of $\big(\theta_n\big)_{n\in\mathbb{N}}$ converges setwise towards the $\mathcal{P}^*$-valued measure ϑ. $\square$

In Section 3.9 we introduced the composition of an operator-valued measure θ with two linear operators. We shall now investigate integrals with respect to this type of measures. Let us recall our notations: Let $(\mathcal{P},\mathcal{V})$ and $(\widetilde{\mathcal{P}},\widetilde{\mathcal{V}})$ be full locally convex cones, and let $(\mathcal{Q},\mathcal{W})$ and $(\widetilde{\mathcal{Q}},\widetilde{\mathcal{W}})$ be locally convex complete lattice cones. For an $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measure θ, a continuous linear operator $S \in \mathfrak{L}(\widetilde{\mathcal{P}},\mathcal{P})$ and an order continuous linear operator $U \in \mathfrak{L}(\mathcal{Q},\widetilde{\mathcal{Q}})$, the $\mathfrak{L}(\widetilde{\mathcal{P}},\widetilde{\mathcal{Q}})$-valued measure $(U \circ \theta \circ S)$ was defined as the set function

$$E \mapsto (U \circ \theta_E \circ S) : \mathfrak{R} \to \mathfrak{L}(\widetilde{\mathcal{P}},\widetilde{\mathcal{Q}}).$$

For a $\widetilde{\mathcal{P}}$-valued function $f \in \mathcal{F}(X,\widetilde{\mathcal{P}})$ and a linear operator $S \in \mathfrak{L}(\widetilde{\mathcal{P}},\mathcal{P})$ we denote by $S \circ f \in \mathcal{F}(X,\mathcal{P})$ the $\mathcal{P}$-valued function

$$x \mapsto S\big(f(x)\big) : X \to \mathcal{P}.$$

Theorem 5.35. *Let $(\mathcal{P},\mathcal{V})$ and $(\widetilde{\mathcal{P}},\widetilde{\mathcal{V}})$ be full locally convex cones, and let $(\mathcal{Q},\mathcal{W})$ and $(\widetilde{\mathcal{Q}},\widetilde{\mathcal{W}})$ be locally convex complete lattice cones. Let Θ be an equibounded family of $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measures, and let $\Upsilon\subset\mathfrak{L}(\mathcal{Q},\widetilde{\mathcal{Q}})$ be an equicontinuous family of continuous and order continuous linear operators. Let $S\in\mathfrak{L}(\widetilde{\mathcal{P}},\mathcal{P})$ such that S is onto and $S(\widetilde{V})\subset\mathcal{V}$. Let*

$$\widetilde{\Theta}=\{(U\circ\theta)\mid U\in\Upsilon,\ \theta\in\Theta\}\qquad\text{and}\qquad\widehat{\Theta}=\{(\theta\circ S)\mid\theta\in\Theta\}$$

be the corresponding families of $\mathfrak{L}(\mathcal{P},\widetilde{\mathcal{Q}})$- and $\mathfrak{L}(\widetilde{\mathcal{P}},\mathcal{Q})$-valued composition measures on $\mathfrak{R}$. Let $F\in\mathfrak{A}_{\mathfrak{R}}$. If the function $f\in\mathcal{F}(X,\widetilde{\mathcal{P}})$ is integrable over F with respect to $\widehat{\Theta}$, then the function $S\circ f\in\mathcal{F}(X,\mathcal{P})$ is integrable over F with respect to $\widetilde{\Theta}$, and

$$\int_F(S\circ f)\ d(U\circ\theta)=U\left(\int_F f\,d(\theta\circ S)\right)$$

holds for all $\theta\in\Theta$ and $U\in\Upsilon$.

Proof. We may assume that $F=X$. Let Θ,Υ,S and $\widetilde{\Theta},\widehat{\Theta}$ be as stated, and let $\theta\in\Theta$ and $U\in\Upsilon$. First, for a step function $h=\sum_{i=1}^n\chi_{E_i}{\otimes}\tilde{a}_i\in\mathcal{S}_{\mathfrak{R}}(X,\widetilde{\mathcal{P}})$ we have

$$\begin{aligned}\int_X(S\circ h)\ d(U\circ\theta)&=\sum_{i=1}^n\big(U\circ\theta\big)_{E_i}\big(S(\tilde{a}_i)\big)\\&=\sum_{i=1}^n U\big(\theta_{E_i}(S(\tilde{a}_i))\big)\\&=U\left(\sum_{i=1}^n\big(\theta\circ S\big)_{E_i}(\tilde{a}_i)\right)\\&=U\left(\int_X h\,d(\theta\circ S)\right).\end{aligned}$$

Next we consider a function $f\in\mathcal{F}_{\mathfrak{R}}(X,\widetilde{\mathcal{P}})$. According to Theorem 1.8(c) the function $S\circ f\in\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ is also measurable. Let $\mathfrak{v}$ be an inductive limit neighborhood for $\mathcal{F}(X,\mathcal{P})$. Then for every $E\in\mathfrak{R}$ there is $v_E\in\mathcal{V}$ such that $\chi_E{\otimes}v_E\le\mathfrak{v}$. Correspondingly, there is $\tilde{v}_E\in\widetilde{\mathcal{V}}$ such that $S(\tilde{v}_E)\le v_E$ (see 2.2). Hence $S\circ(\chi_E{\otimes}\tilde{v}_E)\le\chi_E{\otimes}v_E\le\mathfrak{v}$. This shows that the convex set $\tilde{\mathfrak{v}}$ of all measurable $\widetilde{\mathcal{V}}$-valued functions $\tilde{s}$ such that $S\circ\tilde{s}\le\mathfrak{v}$ is a corresponding inductive limit neighborhood for $\mathcal{F}_{\mathfrak{R}}(X,\widetilde{\mathcal{P}})$. By 2.3 there is a step function $h\in\mathcal{S}_{\mathfrak{R}}(X,\widetilde{\mathcal{P}})$ such that $h\le f+\tilde{\mathfrak{v}}$. Then $S\circ h\in\mathcal{S}_{\mathfrak{R}}(X,\mathcal{P})$ and $S\circ h\le S\circ f+\mathfrak{v}$. This shows $S\circ f\in\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$. Now let $E\in\mathfrak{R}$ and $(U\circ\theta)\in\widetilde{\Theta}$. Given $\tilde{w}\in\widetilde{\mathcal{W}}$ and $\varepsilon>0$ we choose $w\in\mathcal{W}$ such that $U(s)\le U(t)+\tilde{w}$ whenever $s\le t+w$ for $s,t\in\mathcal{Q}$. Correspondingly, there is $v\in\mathcal{V}$ such that $\theta_E(v)\le w$, and $\tilde{v}\in\widetilde{\mathcal{V}}$ such that $S(\tilde{v})\le v$. According to Corollary 2.8, given the inductive limit neighborhood $\tilde{\mathfrak{v}}=\{\chi_X{\otimes}\tilde{v}\}$ there is $1\le\gamma\le 1+\varepsilon$ and a bounded below sequence $(h_n)_{n\in\mathbb{N}}$ of step functions in $\mathcal{S}_{\mathfrak{R}}(X,\widetilde{\mathcal{P}})$ such that: (i) $h_n \le \gamma f + \chi_{X\otimes}\tilde{v}$ for all $n \in \mathbb{N}$ and (ii) for every $x \in E$ there is $n_0 \in \mathbb{N}$ such that $f(x) \le h_n(x)$ for all $n \ge n_0$. Thus (i˜) $S\circ h_n \le \gamma(S\circ f)+\chi_{X\otimes}v$ for all $n \in \mathbb{N}$ and (ii˜) for every $x \in E$ there is $n_0 \in \mathbb{N}$ such that $(S\circ f)(x) \le (S\circ h_n)(x)$ for all $n \ge n_0$. As $(\theta\circ S)_E(\tilde{v}) = \theta_E\big(S(\tilde{v})\big) \le \theta_E(v) \le w$ and $(U\circ\theta)_E(v) = U\big(\theta_E(v)\big) \le U(w) \le \tilde{w}$, this yields

$$\int_E h_n \, d(\theta\circ S) \le \gamma \int_E f \, d(\theta\circ S) + w$$

and

$$\int_E (S\circ h_n)\; d(U\circ\theta) \le \gamma \int_E (S\circ f)\; d(U\circ\theta) + \tilde{w}$$

for all $n \in \mathbb{N}$, as well as

$$\int_E f \, d(\theta\circ S) \le \varliminf_{n\to\infty} \int_E h_n \, d(\theta\circ S)$$

and

$$\int_E (S\circ f)\; d(U\circ\theta) \le \varliminf_{n\to\infty} \int_E (S\circ h_n)\; d(U\circ\theta)$$

with Theorem 5.23. Using our observation for order continuous linear operators from I.5.29 and the latter we infer that

$$\begin{aligned} U\left(\int_E f \, d(\theta\circ S)\right) &\le U\left(\varliminf_{n\to\infty} \int_E h_n \, d(\theta\circ S)\right) \\ &\le \varliminf_{n\to\infty} U\left(\int_E h_n \, d(\theta\circ S)\right) \\ &= \varliminf_{n\to\infty} \int_E (S\circ h_n)\; d(U\circ\theta) \\ &\le \gamma \int_E (S\circ f)\; d(U\circ\theta) + \tilde{w} \end{aligned}$$

and

$$\begin{aligned} \int_E (S\circ f)\; d(U\circ\theta) &\le \varliminf_{n\to\infty} \int_E (S\circ h_n)\; d(U\circ\theta) \\ &= \varliminf_{n\to\infty} U\left(\int_E h_n \, d(\theta\circ S)\right) \\ &\le \gamma\, U\left(\int_E f \, d(\theta\circ s)\right) + \tilde{w}. \end{aligned}$$

This holds true for all $\tilde{w} \in \widetilde{\mathcal{W}}$ and $\varepsilon > 0$ and therefore demonstrates

$$\int_E (S\circ f)\; d(U\circ\theta) = U\left(\int_E f \, d(\theta\circ S)\right)$$

for all $f \in \mathcal{F}_{\mathfrak{R}}(X,\widetilde{\mathcal{P}})$ and $\theta \in \Theta$ and $U \in \Upsilon$.

Next we observe that any set in $\mathfrak{A}_{\mathfrak{R}}$ of measure zero with respect to $\widehat{\Theta}$ is also of measure zero with respect to $\widetilde{\Theta}$. Indeed, if $(\theta \circ S)_E = 0$ for a set $E \in \mathfrak{R}$ and all $\theta \in \Theta$, then $\theta_E\big(S(\tilde{a})\big) = 0$ for all $\tilde{a} \in \widetilde{\mathcal{P}}$. As the operator S is supposed to be surjective, this yields $\theta_E(a) = 0$ for all $a \in \mathcal{P}$, hence $\theta_E = 0$ and $(U \circ \theta)_E = 0$ for all $U \in \Upsilon$. Now let $f \in \mathcal{F}_{(X,\widehat{\Theta})}(X, \widetilde{\mathcal{P}})$. Let $E \in \mathfrak{R}$, let $\tilde{w} \in \widetilde{\mathcal{W}}$ and $\varepsilon > 0$. Because the family Υ was supposed to be equicontinuous, there is $w \in \mathcal{W}$ such that $U(s) \leq U(t) + \tilde{w}$ holds for all $U \in \Upsilon$ whenever $s \leq t + w$ for $s, t \in \mathcal{Q}$. Our definition in 5.3 of integrability with respect to the family $\widehat{\Theta}$ over the set $E \in \mathfrak{R}$ requires that there are functions $f_{(w,\varepsilon)} \in \mathcal{F}_{\mathfrak{R}}(X, \widetilde{\mathcal{P}})$ and $s_{(w,\varepsilon)} \in \mathcal{F}_R(X, \widetilde{\mathcal{V}})$ such that

$$f \underset{a.e.E}{\leq} f_{(w,\varepsilon)} \underset{a.e.E}{\leq} \gamma f + s_{(w,\varepsilon)} \qquad \text{and} \qquad \int_E s_{(w,\varepsilon)}\, d(\theta \circ S) \leq \varepsilon w$$

for some $1 \leq \gamma \leq 1 + \varepsilon$ and all $\theta \in \Theta$. Then $S \circ f_{(w,\varepsilon)} \in \mathcal{F}_{\mathfrak{R}}(X, \mathcal{P})$ and $S \circ s_{(w,\varepsilon)} \in \mathcal{F}_{\mathfrak{R}}(X, \mathcal{V})$ by our assumption that $S(\widetilde{\mathcal{V}}) \subset \mathcal{V}$. By the above we have

$$S \circ f \underset{a.e.E}{\leq} S \circ f_{(w,\varepsilon)} \underset{a.e.E}{\leq} \gamma(S \circ f) + (S \circ s_{(w,\varepsilon)})$$

and

$$\int_E (S \circ s_{(w,\varepsilon)})\; d(U \circ \theta) = U\left(\int_E s_{(w,\varepsilon)}\, d(\theta \circ S)\right) \leq \varepsilon \tilde{w}$$

for all $f \in \mathcal{F}_{\mathfrak{R}}(X, \widetilde{\mathcal{P}})$ and $\theta \in \Theta$ and $U \in \Upsilon$. By Definition 5.3, the function $S \circ f$ is therefore also integrable over E with respect to the family $\widetilde{\Theta}$, and we have

$$\begin{aligned}
\int_E (S \circ f)\; d(U \circ \theta) &= \lim_{\substack{\varepsilon > 0 \\ w \in \mathcal{W}}} \int_E (S \circ f_{(w,\varepsilon)})\; d(U \circ \theta) \\
&= \lim_{\substack{\varepsilon > 0 \\ w \in \mathcal{W}}} U\left(\int_E f_{(w,\varepsilon)}\, d(\theta \circ S)\right) \\
&= U\left(\lim_{\substack{\varepsilon > 0 \\ w \in \mathcal{W}}} \int_E f_{(w,\varepsilon)}\, d(\theta \circ S)\right) \\
&= U\left(\int_E f\, d(\theta \circ S)\right)
\end{aligned}$$

for every $\theta \in \Theta$ and $U \in \Upsilon$. Finally, we verify the second part of Definition 5.3, that is integrability over $F = X$. We have

$$\begin{aligned}\int_X (S\circ f)\; d(U\circ\theta) &= \lim_{E\in\mathfrak{R}} \int_E (S\circ f)\; d(U\circ\theta)\\ &= \lim_{E\in\mathfrak{R}} U\left(\int_E f\, d(\theta\circ S)\right)\\ &= U\left(\lim_{E\in\mathfrak{R}} \int_E f\, d(\theta\circ S)\right)\\ &= U\left(\int_X f\, d(\theta\circ S)\right)\end{aligned}$$

for all $\theta\in\Theta$ and $U\in\Upsilon$. Thus $S\circ f\in\mathcal{F}_{(X,\widetilde{\Theta})}(X,\mathcal{P})$, hence our claim. □

We shall in the following mainly use this result for the special case $\widetilde{\mathcal{P}}=\mathcal{P}$ and the identity operator for S, for $\widetilde{\mathcal{Q}}=\overline{\mathbb{R}}$ and an equicontinuous set Υ of order continuous linear functionals in $\mathcal{P}^*$.

Recall from Section I.5.32 that the order continuous linear functionals are said to support the separation property for a locally convex complete lattice cone $(\mathcal{Q},\mathcal{W})$ if for every neighborhood $w\in\mathcal{W}$ we have $l\le m+w$ for $l,m\in\mathcal{Q}$ whenever $\mu(l)\le\mu(m)+1$ holds for all order continuous lattice homomorphisms $\mu\in w^\circ$.

We are now prepared to formulate and prove a combined version of the Convergence Theorems 5.23, 5.24 and 5.25, that under additional assumptions yields convergence with respect to the upper, lower and symmetric topologies of $\mathcal{Q}$, respectively, for the concerned sequence of integrals. Because we shall deal only with bounded elements of $\mathcal{Q}$, we do not need to consider the relative topologies, since they coincide locally with the given topologies in this case (see Section I.4). Recall that for a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathcal{Q}$ convergence towards $a\in\mathcal{Q}$ in the upper, or lower topology of $\mathcal{Q}$ means that for every $w\in\mathcal{W}$ there is $n_0\in\mathbb{N}$ such that

$$a_n\le a+w, \qquad \text{or} \qquad a\le a_n+w$$

holds for all $n\ge n_0$, respectively. Because these topologies are generally far from Hausdorff, limits need not be unique. Convergence in the symmetric topology combines convergence in both the upper and lower topologies.

Theorem 5.36. *Suppose that the order continuous linear functionals support the separation property for $\mathcal{Q}$. Let θ be a strongly additive $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measure on $\mathfrak{R}$, and let $E\in\mathfrak{R}$. Let $f_n,f,f_{**},f_*,f^{**},f^*\in\mathcal{F}(X,\mathcal{P})$ be bounded-valued measurable functions, and suppose that for every $w\in\mathcal{W}$ there is $v\in\mathcal{V}$ such that $\theta_E(v)\le w$ and such that these functions are $(\mathcal{P},\mathcal{V}_0)$-based integrable over E with respect to θ for the subsystem $\mathcal{V}_0=\{\rho v\mid \rho>0\}$ of $\mathcal{V}$. Suppose that the functions f_* and f^* are strongly integrable over E with respect to θ and that their respective integrals are bounded in $\mathcal{Q}$.*

(a) If $f_{**} \underset{a.e.E}{\leq} f_n + f_*$ *for all* $n \in \mathbb{N}$, *and* $f_n \underset{a.e.E}{\nearrow} f$, *then*

$$\int_E f\, d\theta = \lim_{n\to\infty} \int_E f_n\, d\theta_n$$

with respect to the lower topology of $\mathcal{Q}$.

(b) If $f_n \underset{a.e.E}{\leq} f^*$ *for all* $n \in \mathbb{N}$, *and* $f_n \underset{a.e.E}{\searrow} f$, *then*

$$\int_E f\, d\theta = \lim_{n\to\infty} \int_E f_n\, d\theta_n$$

with respect to the upper topology of $\mathcal{Q}$.

(c) If $f_{**} \underset{a.e.E}{\leq} f_n + f_*$ *and* $f_n + f^{**} \underset{a.e.E}{\leq} f^*$ *for all* $n \in \mathbb{N}$, *and* $f_n \underset{a.e.E}{\longrightarrow} f$, *then*

$$\int_E f\, d\theta = \lim_{n\to\infty} \int_E f_n\, d\theta_n$$

with respect to the symmetric topology of $\mathcal{Q}$.

Proof. We shall deal with Parts (a), (b) and (c) simultaneously. By restricting the measure θ and all the functions involved to the set E, we may assume that $X = E \in \mathfrak{R}$. Let $\mathcal{G} = \{f_n, f, f_{**}, f_*, f^{**} f^*\}$ be the family of the functions used in our statement. This family is countable.

Suppose that contrary to our claim, the sequence $\left(\int_E f_n\, d\theta\right)_{n\in\mathbb{N}}$ does not converge towards $\int_E f\, d\theta$ in the (a) lower, (b) upper or (c) symmetric topology of $\mathcal{Q}$. Then there is $w \in \mathcal{W}$ and a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ such that either

$$\text{(a)} \int_E f\, d\theta \not\leq \int_E f_{n_k}\, d\theta + w \qquad \text{or} \qquad \text{(b)} \int_E f_{n_k}\, d\theta \not\leq \int_E f\, d\theta + w,$$

respectively, holds for all $k \in \mathbb{N}$. In case (c), we can find a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ either as in (a) or in (b). We have $\mu\left(\int_E f\, d\theta\right) < +\infty$ for all $\mu \in \mathcal{Q}^*$ since the integral of f is supposed to be bounded in $\mathcal{Q}$. Let Υ be the family of all order continuous linear functionals in w° and Ω be the corresponding set $\{\mu \circ \theta \mid \mu \in \Upsilon,\}$ of $\mathcal{P}^*$-valued measures on $\mathfrak{R}$. Theorem 5.35 yields that the functions in $\mathcal{G}$ are integrable over E with respect to the family Ω and that $\mu\left(\int_E g\, d\theta\right) = \int_E g\, d(\mu \circ \theta)$ holds for all $g \in \mathcal{G}$ and $\mu \in \Upsilon$. By our assumption the order continuous linear functionals support the separation property for $\mathcal{Q}$, thus there are functionals $\mu_k \in \Upsilon \subset w^\circ$ such that either

$$\text{(a)} \int_X f\, d(\mu_k\circ\theta) = \mu_k\left(\int_X f\, d\theta\right) > \mu_k\left(\int_X f_{n_k}\, d\theta\right) + 1 = \int_X f_{n_k}\, d(\mu_k\circ\theta) + 1$$

or

$$\text{(b)} \int_X f_{n_k}\, d(\mu_k\circ\theta) = \mu_k\left(\int_X f_{n_k}\, d\theta\right) > \mu_k\left(\int_X f\, d\theta\right) + 1 = \int_X f\, d(\mu_k\circ\theta) + 1$$

holds for all $k \in \mathbb{N}$, respectively. We shall proceed as follows:

There is $v \in \mathcal{V}$ such that $\theta_E(v) \le w$ and such that the functions in $\mathcal{G}$ are $(\mathcal{P}, \mathcal{V}_0)$-based integrable over E with respect to θ for the subsystem $\mathcal{V}_0 = \{\rho v \mid \rho > 0\}$ of $\mathcal{V}$. All functions in $\mathcal{G}$ are supposed to be measurable, thus their ranges are separable with respect to the symmetric relative v-topology by (M2) in Section 1.2. For every $g \in \mathcal{G}$, let $\mathcal{A}(g)$ be a countable dense subset in the range of g. Recall that by our assumption all elements of $\mathcal{A}(g)$ are bounded in $\mathcal{P}$. Following Definition 5.6, that is the $(\mathcal{P}, \mathcal{V}_0)$-based integrability of the functions in $\mathcal{G}$, for every $g \in \mathcal{G}$ and $n \in \mathbb{N}$ there is a function $g_n \in \mathcal{F}_{\mathfrak{R}}(X, \mathcal{P})$ and $s_n \in \mathcal{F}_{\mathfrak{R}}(X, \mathcal{V}_0)$ such that

$$g \underset{a.e.E}{\le} g_n + s_n, \qquad g_n \underset{a.e.E}{\le} \gamma_n g + s_n \quad \text{and} \quad \int_E s_n \, d\theta \le \frac{1}{n} w$$

for some $1 \le \gamma_n \le 1 + 1/n$. The latter implies that $\int_E s_n \, d\omega \le 1/n$ for all $\omega \in \Omega$. Again, measurability guarantees that there are countable dense subsets $\mathcal{A}(g_n)$ in the respective ranges of the functions g_n. Now, recalling the definition of the cone $\mathcal{F}_{\mathfrak{R}}(X, \mathcal{P})$ in Section 2.3, for every $n \in \mathbb{N}$ and $m \in \mathbb{N}$ there is a step function $h^m_{g_n} \in \mathcal{S}_{\mathfrak{R}}(X, \mathcal{P})$ such that $h^m_{g_n}(x) \le g_n(x) + (1/m)v$ for all $x \in E$. Obviously, the range $\mathcal{A}(h^m_{g_n})$ of $h^m_{g_n}$ is finite. We denote by $\mathcal{B}$ the union of all the sets $\mathcal{A}(g)$, $\mathcal{A}(g_n)$ and $\mathcal{A}(h^m_{g_n})$, for $g \in \mathcal{G}$ and $n, m \in \mathbb{N}$, and by

$$\mathcal{C} = \left\{ \sum_{i=1}^{n} \rho_i b_i + \delta v \mid b_i \in \mathcal{B},\ 0 \le \rho_i \in \mathbb{Q},\ 0 < \delta \in \mathbb{Q} \right\}.$$

This set is also countable, and all its elements are v-bounded in $\mathcal{P}$ by our assumption on the functions $g \in \mathcal{G}$. Finally, let $\mathcal{P}_0$ be the closure of $\mathcal{C}$ in $\mathcal{P}$ with respect to the symmetric relative v-topology. Then $\mathcal{P}_0$ is a subcone of $\mathcal{P}$, separable, and all of its elements are v-bounded, that is bounded with respect to the neighborhood subsystem $\mathcal{V}_0$, which itself is contained in $\mathcal{P}_0$. Moreover, the above shows that all functions in $\mathcal{G}$ are indeed $(\mathcal{P}_0, \mathcal{V}_0)$-based integrable (see 5.6) over E, hence over all subsets $G \in \mathfrak{R}$ of E, with respect to the family Ω of $\mathcal{P}^*$-valued measures. Proposition 5.10 now yields that all these functions are contained in $\mathcal{F}_{(E,\Omega_0)}(X, \mathcal{P}_0)$, where Ω_0 denotes the family of the restrictions to $\mathcal{P}_0$ of the measures in Ω, and that

$$\int_G g \, d(\mu \circ \theta)_0 = \int_G g \, d(\mu \circ \theta) = \mu \left(\int_G g \, d\theta \right)$$

holds for all $g \in \mathcal{G}$, $\mu \in \Upsilon$ and subsets $G \in \mathfrak{R}$ of E.

We proceed to apply Lemma 5.34 to the cone $(\mathcal{P}_0, \mathcal{V}_0)$ in order to show that the family Ω_0 of $\mathcal{P}_0^*$-valued measures is weakly sequentially compact. As we mentioned before, the elements of $\mathcal{P}_0$ are bounded, and $\mathcal{P}_0$ is separable in the symmetric relative v-topology. For equiboundedness of the family Ω_0, let $\varepsilon > 0$ be a neighborhood for $\mathbb{R}$. Correspondingly, we choose the neighborhood $\varepsilon v \in \mathcal{V}_0$ and conclude that

$$(\mu \circ \theta)_{0\,E}(v) = \mu\big(\theta_E(v)\big) \leq \mu(\varepsilon w) \leq \varepsilon$$

for all $(\mu \circ \theta)_0 \in \Omega_0$, since all functionals $\mu \in \Upsilon$ involved are contained in w°. This shows that Ω_0 is indeed equibounded. Likewise, Ω_0 is seen to be uniformly strongly additive. Indeed, let $E_n \in \mathfrak{R}$ such that $E_n \supset E_{n+1}$ and $\bigcap_{n\in\mathbb{N}} E_n = \emptyset$. Following 5.30, given $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $\theta_{E_n}(v) \leq \varepsilon w$ for all $n \geq n_0$ and $\theta \in \Theta$. Thus

$$(\mu \circ \theta)_{0\,E_n}(v) = (\mu \circ \theta)_{E_n}(v) = \mu\big(\theta_{E_n}(v)\big) \leq \mu\big(\theta_{E_n}(v)\big) \leq \varepsilon$$

for all $(\mu \circ \theta)_0 \in \Omega_0$ and $n \geq n_0$. Thus, following Lemma 5.34, the family Ω_0 of $\mathcal{P}_0^*$-valued measures is weakly sequentially compact.

We may therefore assume that the sequence $\big((\mu_k \circ \theta)_0\big)_{k\in\mathbb{N}}$ from the first part of this proof converges setwise to some bounded $\mathcal{P}_0^*$-valued measure ω. We abbreviate ω_k for $(\mu_k \circ \theta)_0$ and recall that either

$$\text{(a)} \int_X f\,d\omega_k > \int_X f_{n_k}\,d\omega_k + 1 \qquad \text{or} \qquad \text{(b)} \int_X f_{n_k}\,d\omega_k > \int_X f\,d\omega_k + 1$$

holds for all $k \in \mathbb{N}$, respectively.

Next we shall argue that $(\omega_k) \underset{\{f_*,f^*\}}{\overset{E}{\prec}} \omega$. In fact, we shall demonstrate that $\mathfrak{Rs}\big(\omega_k, E, g\big) = 0$ for every $g \in \{f_*, f^*\}$. For this, let $E_m \in \mathfrak{R}$ for $m \in \mathbb{N}$ be subsets of E such that $E_m \supset E_{m+1}$ and $\bigcap_{n\in\mathbb{N}} E_m = \emptyset$. Let $\varepsilon > 0$. Because the function g is supposed to be strongly integrable over E with respect to θ, Lemma 5.32(b) yields that for $\varepsilon > 0$ there is $m_0 \in \mathbb{N}$ such that $\int_{E_m} g\,d\theta \leq \mathfrak{O}\left(\int_E g\,d\theta\right) + \varepsilon w$ for all $m \geq m_0$. Because the element $\int_E g\,d\theta$ is supposed to be bounded in $\mathcal{Q}$, we infer that $\mathfrak{O}\left(\int_E g\,d\theta\right) = 0$ (see Proposition I.5.10(c)). We have

$$\int_{E_m} g\,d\omega_k = \mu_k\left(\int_{E_m} g\,d\theta\right) \leq \varepsilon$$

for all $m \geq m_0$ and $k \in \mathbb{N}$, since $\mu_k \in w^\circ$. Thus

$$\underline{\lim}_{m\to\infty}\left(\overline{\lim}_{k\to\infty}\int_{E_m} g\,d\omega_k\right) \leq \varepsilon$$

for all $\varepsilon > 0$, hence

$$\underline{\lim}_{m\to\infty}\left(\overline{\lim}_{k\to\infty}\int_{E_m} g\,d\omega_k\right) \leq 0.$$

This shows

$$\mathfrak{Rs}\big(\omega_k, E, g\big) = \sup_{(E_m)\in\mathfrak{F}}\left\{\underline{\lim}_{m\to\infty}\left(\overline{\lim}_{k\to\infty}\int_{E_m} g\,d\omega_k\right)\right\} = 0.$$

Now, finally, our preceding convergence theorems will yield a contradiction. We shall apply them to the cones $\mathcal{P}_0$ and $\overline{\mathbb{R}}$, the sequence of measures $(\omega_k)_{k\in\mathbb{N}}$ and ω, and the given functions $f_n, f, f_{**}, f_*, f^{**} f^*$. First, Lemma 5.14(a) states that all functions involved are in $\mathcal{F}_{(E,\Omega_0\cup\{\omega\})}(X,\mathcal{P}_0)$. Moreover, Lemmas 5.14(b) and 5.20(b) demonstrate that

$$\int_E f\,d\omega = \lim_{k\to\infty}\int_E f\,d\omega_k.$$

In case (a), Theorem 5.23 yields

$$\int_F f\,d\omega \le \underline{\lim}_{k\to\infty}\int_F f_{n_k}\,d\omega_k$$

since $\mathfrak{O}\left(\int_F f_*\,d\theta\right) = 0$, contradicting our assumption at the start of this argument. Similarly, in case (b), Theorem 5.24 leads to

$$\overline{\lim}_{k\to\infty}\int_F f_{n_k}\,d\omega_k \le \int_F f\,d\omega,$$

contradicting the corresponding assumption for this case. In case (c), finally, Theorem 5.25 yields

$$\int_F f\,d\omega = \lim_{k\to\infty}\int_F f_{n_k}\,d\omega_k,$$

contradicting the assumptions of both cases (a) and (b). This completes our argument. $\square$

As we established in I.5.57, every locally convex cone can be canonically embedded into a larger locally convex complete lattice cone whose order continuous lattice homomorphisms support the separation property. The corresponding requirement in Theorem 5.36 can therefore be met if we use this standard lattice completion for $\mathcal{Q}$. In Section 6 below we shall identify several special cases where Theorem 5.36 can be applied.

6. Examples and Special Cases

The generality of our approach to measures and integrals allows a wide range of settings, depending on the choices for the locally convex cones $(\mathcal{P},\mathcal{V})$ and $(\mathcal{Q},\mathcal{W})$. We shall present a selection of these special cases in this section. Throughout the following, we shall assume that $(\mathcal{P},\mathcal{V})$ is a quasi-full locally convex cone and that $(\mathcal{Q},\mathcal{V})$ is a locally convex complete lattice cone. $(\mathcal{P}_\mathcal{V},\mathcal{V})$ shall denote the standard full extension of $(\mathcal{P},\mathcal{V})$ into a full cone, as elaborated in Section 6 of Chapter I. $(\mathcal{Q}_0,\mathcal{W}_0)$, on the other hand, will stand for a locally convex cone whose standard lattice completion in the sense of I.5.57 is $(\mathcal{Q},\mathcal{W})$. We shall generally use the notations of the preceding sections. In particular, $\mathfrak{R}$ stands for a weak σ-ring of subsets of a set X, and θ is a bounded measure on $\mathfrak{R}$. The concepts of the preceding Sections 4 and 5, in particular our notions of integrability, will be applied to the full cone $(\mathcal{P}_{\mathcal{V}},\mathcal{V})$ instead of $(\mathcal{P},\mathcal{V})$.

Some of our general notions are considerably simplified in special cases. In the first set of examples we shall discuss the specific insertions for $\mathcal{P}$ and $\mathcal{Q}$ that lead to classical integration theory.

6.1 The case $\mathcal{Q}=\overline{\mathbb{R}}$. If we choose $\mathcal{Q}=\overline{\mathbb{R}}$ with the canonical order and the neighborhoods $\mathcal{V}=\{\varepsilon\in\mathbb{R}\mid\varepsilon>0\}$, then the values of the measure θ are linear functionals in the dual cone $\mathcal{P}^*$ of $\mathcal{P}$, and for each $a\in\mathcal{P}$ the mapping

$$E\mapsto\theta_E(a):\mathfrak{R}\to\overline{\mathbb{R}}$$

is an extended real-valued measure on $\mathfrak{R}$. The modulus of the measure θ is given by

$$|\theta|(E,v)=\sup\left\{\sum_{i=1}^{n}\theta_{E_i}(s_i)\;\middle|\;s_i\in\mathcal{P},\,s_i\le v,\,E_i\in\mathfrak{R}\text{ disjoint subsets of }E\right\},$$

which is an element of $\overline{\mathbb{R}}$, for $E\in\mathfrak{R}$ and $v\in\mathcal{V}$. Boundedness therefore means that for every $E\in\mathfrak{R}$ there is $v\in\mathcal{V}$ such that $|\theta|(E,v)<+\infty$. This coincides with Prolla's notion of *finite p-semivariation* in [155] (Ch. 5.5). A bounded measure can be extended to the full cone $(\mathcal{P}_{\mathcal{V}},\mathcal{V})$ as elaborated in Section 3.8. Integrals of $\mathcal{P}$-valued functions with respect to an $\mathfrak{L}(\mathcal{P},\overline{\mathbb{R}})$-, that is $\mathcal{P}^*$-valued measure are also in $\overline{\mathbb{R}}$. For a meaningful statement in our Convergence Theorems 5.22, 5.24 and 5.25 we need to enforce that $\int_F f_*\,d\theta<+\infty$ and $\int_F f^*\,d\theta<+\infty$ in this case.

6.2 Extended Positive-Valued Functions and Measures. We obtain classical integration theory for extended positive-valued functions with respect to extended positive-valued measures if we choose $\mathcal{P}=\overline{\mathbb{R}}_+$, endowed with the singleton neighborhood system $\mathcal{V}=\{0\}$ (see Example 1.2(b) in Chapter I), and $\mathcal{Q}=\overline{\mathbb{R}}$. The dual $\overline{\mathbb{R}}_+^*$ of $\overline{\mathbb{R}}_+$ consists of all elements of $\overline{\mathbb{R}}_+$ (via the usual multiplication) and the singular functional $\bar{0}$ such that $\bar{0}(\alpha)=0$ for all $\alpha\in\mathbb{R}_+$ and $\bar{0}(+\infty)=+\infty$. Every $\overline{\mathbb{R}}_+^*$-valued measure θ is therefore $\mathfrak{R}$-bounded and can be expressed as the sum of an $\overline{\mathbb{R}}_+$-valued measure θ^1 in the usual sense and a measure θ^0 that takes only the values 0 and $\bar{0}$.

Because the symmetric relative topology renders the Euclidean topology on the interval $(0,+\infty)$, and the elements 0 and ∞ as isolated points (see Example 4.18(a) in Chapter I), $\overline{\mathbb{R}}_+$ is separable in this topology. Our notion of measurability from Section 1 for $\overline{\mathbb{R}}_+$-valued functions therefore coincides with the usual one in this case. Continuity for an $\overline{\mathbb{R}}_+$-valued function defined on a topological space X does however require that this function takes the values 0 and $+\infty$ only on respective subsets of X that are both open and closed.

Because $v = 0$ is the only neighborhood for $\overline{\mathbb{R}}_+$, according to Section 4, the integral of a measurable function f over a set $F \in \mathfrak{A}_R$ with respect to a measure θ is defined as

$$\int_F f\,d\theta = \sup\left\{ \int_F h\,d\theta \;\middle|\; h \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P}),\ h \le f \right\},$$

that is the classical definition of the integral.

6.3 Extended Real-Valued Functions and Positive-Valued Measures. We obtain classical integration theory for $\overline{\mathbb{R}}$-valued functions with respect to positive-valued measures if we choose $\mathcal{P} = \mathcal{Q} = \overline{\mathbb{R}}$. The dual $\overline{\mathbb{R}}^*$ of $\overline{\mathbb{R}}$ consists of all positive reals (via the usual multiplication) and the singular functional $\bar{0}$ such that $\bar{0}(\alpha) = 0$ for all $\alpha \in \mathbb{R}$ and $\bar{0}(+\infty) = +\infty$. Every $\overline{\mathbb{R}}^*$-valued measure θ is therefore $\mathfrak{R}$-bounded and can be expressed as the sum of a positive real-valued measure θ^1 in the usual sense and a measure θ^0 that takes only the values 0 and $\bar{0}$. The notion of measurability from Section 1 for $\overline{\mathbb{R}}$-valued functions coincides with the usual one.

Let f be a measurable and bounded below $\overline{\mathbb{R}}$-valued function, and let $F \in \mathfrak{A}_{\mathfrak{R}}$. For a neighborhood $w = \varepsilon \in \mathcal{W}$ the step functions $s \in \mathfrak{v}_\varepsilon$ are invertible, and for a step function $h \in \mathcal{S}_{\mathfrak{R}}(X,\overline{\mathbb{R}})$ such that $h \le f + \mathfrak{v}_\varepsilon$ we have $h' \le f$ with $h' = h - s \in \mathcal{S}_{\mathfrak{R}}(X,\overline{\mathbb{R}})$ and $\int_F h\,d\theta \le \int_F h\,d\theta + \varepsilon$. This shows

$$\int_F^{(\varepsilon)} f\,d\theta \le \sup\left\{ \int_F h\,d\theta \;\middle|\; h \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P}),\ h \le f \right\} + \varepsilon$$

and consequently

$$\int_F f\,d\theta = \sup\left\{ \int_F h\,d\theta \;\middle|\; h \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P}),\ h \le f \right\},$$

the usual definition.

6.4 Real- or Complex-Valued Functions and Measures. In the preceding example we integrated $\overline{\mathbb{R}}$-valued functions with respect to positive real-valued measures. Alternatively, we may consider real- or complex-valued functions, that is $\mathcal{P} = \mathbb{K}$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ with the usual Euclidean topology and the equality as order. The vector space dual $\mathcal{P}^*_{\mathbb{K}}$, of $\mathbb{K}$ is of course $\mathbb{K}$ itself, whereas its dual $\mathcal{P}^*$ as a locally convex cone consists of the real parts of these evaluations (see Example I.2.1(c)). For $\mathcal{Q}$ we choose the simplified standard lattice completion $\widehat{\mathbb{K}}$ of $\mathbb{K}$ which consists of all bounded below $\overline{\mathbb{R}}$-valued functions on Γ, the unit circle of $\mathbb{K}$, endowed with the (strictly) positive constants as neighborhoods (see Example I.5.62(f)).

We consider $\mathbb{K}$-valued measures $E \mapsto \theta_{\mathbb{K}\,E} : \mathfrak{R} \to \mathbb{K}$ in this case, yielding continuous linear operators θ_E from $\mathbb{K}$ to $\widehat{\mathbb{K}}$ via the convention

$$\theta_E\big(a\big)(\gamma) = \mathfrak{Re}(\gamma a\, \theta_{\mathbb{K}\,E})$$

for $E \in \mathfrak{R}$, $a \in \mathbb{K}$ and $\gamma \in \Gamma$. According to 3.2 we calculate the modulus of such a measure for every $E \in \mathfrak{R}$ as

$$\begin{aligned} |\theta|\big(E, \mathbb{B}\big)(\gamma) &= \sup\left\{ \sum_{i=1}^{n} \mathfrak{Re}(\gamma a\, \theta_{\mathbb{K}\,E}) \;\middle|\; |a_i| \le 1,\ E_i \in \mathfrak{R} \text{ disjoint subsets of } E \right\} \\ &= \sup\left\{ \sum_{i=1}^{n} |\theta_{E_i}| \;\middle|\; E_i \in \mathfrak{R} \text{ disjoint subsets of } E \right\} \end{aligned}$$

for all $\gamma \in \Gamma$, where $\mathbb{B} \in \mathcal{V}$ stands for the unit ball in $\mathbb{K}$. This is of course the usual notation for the *total variation* $\mathfrak{var}(\theta, E)$ of a real- or complex-valued measure on a set E (see III.1.4 in [55] or Section 6.1 in [179]). A simple argument (see Lemmas III.1.5 and III.4.5 in [55]) shows that

$$|\theta|(E, \mathbb{B}) \le 4 \sup\big\{ |\theta_G| \mid G \in \mathfrak{R},\ G \subset E \big\} < +\infty$$

for every $E \in \mathfrak{R}$ in this case. Hence any $\mathbb{K}$-valued measure is $\mathfrak{R}$-bounded in the sense of Section 3.6 and may therefore be extended to the standard full extension

$$\mathcal{P}_{\mathcal{V}} = \{ a + \alpha\mathbb{B} \mid a \in \mathbb{K},\ \alpha \ge 0 \}$$

of $\mathcal{P} = \mathbb{K}$, setting

$$\theta_E\big(a + \alpha\mathbb{B}\big)(\gamma) = \theta_E\big(a\big)(\gamma) + \alpha|\theta|(E, \mathbb{B}) = \mathfrak{Re}(\gamma a\, \theta_{\mathbb{K}\,E}) + \alpha|\theta|(E, \mathbb{B})$$

for all $\gamma \in \Gamma$. The notion of measurability from Section 1 for $\mathbb{K}$-valued functions coincides with the usual one. A measurable function f is contained in $\mathcal{F}_{\mathfrak{R}}(X, \mathbb{K})$ if on every set $E \in \mathfrak{R}$ it can be uniformly approximated by a sequence of step functions. It follows from our convergence theorems that the integral of f over E is the limit of the integrals of this sequence of step functions. Integrability in the sense of 4.12 and 4.13, however reaches beyond this requirement. Integrals of $\mathbb{K}$-valued functions are evaluated in $\widehat{\mathbb{K}}$, that is as $\overline{\mathbb{R}}$-valued functions on Γ. However, since according to Corollary 5.9 these integrals are elements of the order closure of the embedding of $\mathbb{K}$ into $\widehat{\mathbb{K}}$, hence are $\mathbb{K}$-linear by I.5.60(b). We may therefore identify the integral in the usual way with a number in $\mathbb{K}$, setting

$$\left\langle \int_F f\, d\theta \right\rangle_{\mathbb{R}} = \left(\int_F f\, d\theta \right)(1)$$

in the real, and

$$\left\langle \int_F f\,d\theta \right\rangle_{\mathbb{C}} = \left(\int_F f\,d\theta \right)(1) - i \left(\int_F f\,d\theta \right)(i)$$

in the complex case, respectively. Moreover, given $\gamma \in \Gamma$ we have

$$\left\langle \int_F \gamma h\,d\theta \right\rangle_{\mathbb{K}} = \gamma \left\langle \int_F f\,d\theta \right\rangle_{\mathbb{K}}$$

for every step function $h \in \mathcal{S}_{\mathfrak{R}}(X, \mathbb{K})$. Because $h \leq f + \mathfrak{v}_w$ holds if and only if $\gamma h \leq \gamma f + \mathfrak{v}_w$ for $h \in \mathcal{S}_{\mathfrak{R}}(X, \mathbb{K})$ and $f \in \mathcal{F}_{\mathfrak{R}}(X, \mathbb{K})$, we have

$$\left\langle \int_F^{(w)} \gamma f\,d\theta \right\rangle_{\mathbb{K}} = \gamma \left\langle \int_F^{(w)} f\,d\theta \right\rangle_{\mathbb{K}}.$$

Consequently, $\mathcal{F}_{(F,\theta)}(X, \mathbb{K})$ is a vector space over $\mathbb{K}$, and the mapping

$$f \mapsto \left\langle \int_F f\,d\theta \right\rangle_{\mathbb{K}} \;:\; \mathcal{F}_{(F,\theta)}(X, \mathbb{K}) \to \mathbb{K}$$

is linear over $\mathbb{K}$.

6.5 The Case that $\mathcal{Q}$ Is the Standard Lattice Completion of Some Subcone Q_0. Suppose that $(\mathcal{Q}, \mathcal{W})$ is the standard lattice completion of a locally convex cone $(Q_0, \mathcal{W}_0)$ (see I.5.57), and suppose that the measure θ is indeed $\mathfrak{L}(\mathcal{P}, \mathcal{Q}_0)$-valued. The closure of $\mathcal{Q}_0$ in $\mathcal{Q}$ with respect to the order topology was seen to be a subcone of the second dual $\mathcal{Q}_0^{**}$ (see Sections I.5.60 and I.7.3) in this case, and following Corollary 5.9, integrals of $(\mathcal{P}, \mathcal{V})$-based integrable functions in $\mathcal{F}(X, \mathcal{P})$ are therefore elements of $\mathcal{Q}_0^{**}$. Stronger statements can be obtained for certain types of integrable functions. We shall develop these in the following remarks:

Remarks 6.6. Let A be a relatively bounded subset of $\mathcal{P}$, that is, A is bounded below, and bounded above relative to some element $a_0 \in \mathcal{P}$. Let $E \in \mathfrak{R}$. We observe the following:

(a) The convex hull of $A \cup \{0\}$, that is the set

$$\tilde{A} = \left\{ \sum_{i=1}^n \alpha_i a_i \;\middle|\; a_i \in A,\ 0 \leq \alpha_i \in \mathbb{R},\ \sum_{i=1}^n \alpha_i \leq 1 \right\}$$

is also bounded below and bounded above relative to a_0. Indeed, given $v \in \mathcal{V}$ let $\lambda, \rho \geq 0$ such that $0 \leq a_0 + \lambda v$, $0 \leq a + \lambda v$ and $a \leq \rho a_0 + \lambda v$ for all $a \in A$. Then for any choice of $a_i \in A$ and $0 \leq \alpha_i \in \mathbb{R}$ such that $\sum_{i=1}^n \alpha_i \leq 1$ we have

$$0 \leq \sum_{i=1}^n \alpha_i (a_i + \lambda v) \leq \sum_{i=1}^n \alpha_i a_i + \lambda v$$

and

$$\sum_{i=1}^{n} \alpha_i a_i \leq \sum_{i=1}^{n} \alpha_i(\rho a_0 + \lambda v) + \rho\Big(1 - \sum_{i=1}^{n} \alpha_i\Big)(a_0 + \lambda v) \leq \rho a_0 + \lambda(1+\rho)v.$$

This yields our claim.

(b) For every $E \in \mathfrak{R}$ the set

$$\mathcal{Z}(A, E) = \left\{ \sum_{i=1}^{n} \theta_{E_i}(a_i) \mid a_i \in A,\ E_i \in \mathfrak{R} \text{ disjoint subsets of } E \right\}$$

is bounded below, and bounded above relative to the element $\theta_E(a_0)$, hence $\mathcal{Z}(A, E)$ is a relatively bounded subset of $\mathcal{Q}_0$. Indeed, given $w \in \mathcal{W}$ there is $v \in \mathcal{V}$ such that $|\theta|(E, v) = \theta_E(v) \leq w$. In turn, there are $\lambda, \rho \geq 0$ such that $0 \leq a + \lambda v$ and $a \leq \rho a_0 + \lambda v$ for all $a \in A$. We may also assume that $0 \leq \rho a_0 + \lambda v$. Now let $a_1, \ldots, a_n \in A$ and let $E_1, \ldots, E_n \in \mathfrak{R}$ be disjoint subsets of E. Then

$$0 \leq \sum_{i=1}^{n} \theta_{E_i}(a_i + \lambda v) = \sum_{i=1}^{n} \theta_{E_i}(a_i) + \lambda \sum_{i=1}^{n} \theta_{E_i}(v) \leq \sum_{i=1}^{n} \theta_{E_i}(a_i) + \lambda w$$

and

$$\sum_{i=1}^{n} \theta_{E_i}(a_i) \leq \sum_{i=1}^{n} \theta_{E_i}(\rho a_0 + \lambda v) \leq \theta_E(\rho a_0 + \lambda v) \leq \rho\theta_E(a_0) + \lambda w.$$

The set $\mathcal{Z}(A, E)$ is therefore bounded below and bounded above relative to the element $\theta_E(a_0)$, thus relatively bounded in $\mathcal{Q}_0$.

(c) Now recall from Section I.5.57 that the order topology of the standard lattice completion $\mathcal{Q}$ of $\mathcal{Q}_0$ coincides with the topology of pointwise convergence on the elements of $\mathcal{Q}_0^*$. Thus, according to I.7.3 the limit in $\mathcal{Q}$ with respect to order convergence of any net in the relatively bounded set $\mathcal{Z}(A, E) \subset \mathcal{Q}_0 \subset \mathcal{Q}$ from (b) is contained in the relative strong second dual $(\mathcal{Q}_0)^{**}_{sr}$ of $\mathcal{Q}_0$.

(d) Let $E \in \mathfrak{R}$, let $\varphi_1, \ldots, \varphi_n$ be non-negative measurable real-valued functions such that $\sum_{i=1}^{n} \varphi_i \leq \chi_E$ and let $a_1, \ldots, a_n \in A$. For each $i = 1, \ldots, n$ let $(\psi^i_k \otimes a_i)_{k \in \mathbb{N}}$ be a sequence of step functions approximating $\varphi_i \otimes a_i$ as in 5.27 and 5.28, that is

$$0 \leq \psi^i_k \leq \varphi_i \qquad \text{and} \qquad \lim_{k \to \infty} \int_X \psi^i_k \otimes a_i \; d\theta = \int_x \varphi_i \otimes a_i \, d\theta.$$

According to 5.27, these step functions $\psi^i_k \otimes a_i$ are of the type

$$\sum_{j=1}^{k} \chi_{E_j^{(i,k)}} \otimes (\alpha_j a_i),$$

with disjoint sets $E_j^{(i,k)} \in \mathfrak{R}$ whose union is E, with $0 \leq \alpha_j \leq 1$ and such that $\sum_{j=1}^{k} \alpha_j \chi_{E_j^{(i,k)}} \leq \varphi_i$. For every $k \in \mathbb{N}$ let

$$h_k = \sum_{i=1}^{n} \psi_i^k \otimes a_i = \sum_{i=1}^{n} \sum_{j=1}^{k} \chi_{E_j^{(i,k)}} \otimes (\alpha_j a_i).$$

As

$$\sum_{i=1}^{n} \sum_{j=1}^{k} \alpha_j \chi_{E_j^{(i,k)}} \leq \sum_{i=1}^{n} \varphi_i \leq \chi_E,$$

the step function h_k can be expressed as

$$h_k = \sum_{l=1}^{p} \chi_{F_l} \otimes b_l,$$

where $F_1, \ldots, F_p \in \mathfrak{R}$ are disjoint subsets of E and $b_1, \ldots, b_p$ are suitable convex combinations of the elements of the relatively bounded set $\tilde{A} = A \cup \{0\}$ $\big($see 6.6(a)$\big)$; more precisely

$$b_l = \sum_{i=1}^{n} \beta_i a_i,$$

where β_i is the sum of all those α_j, for $j = 1, \ldots, k$, such that $F_l \subset E_j^{(i,k)}$. Thus the integral

$$\int_X h_k \, d\theta = \sum_{l=1}^{p} \theta_{F_l}(b_l)$$

is contained in the relatively bounded subset

$$\mathcal{Z}(\tilde{A}, E) = \left\{ \sum_{i=1}^{n} \theta_{E_i}(a_i) \;\middle|\; a_i \in \tilde{A},\ E_i \in \mathfrak{R} \text{ disjoint subsets of } E \right\}$$

of $\mathcal{Q}_0$. We have

$$\int_X \left(\sum_{i=1}^{n} \varphi_i \otimes a_i \right) d\theta = \lim_{k \to \infty} \int_X h_k \, d\theta,$$

hence according to (c), the integral of the function $\sum_{i=1}^{n} \varphi_i \otimes a_i$ is contained in the relative strong second dual $(\mathcal{Q}_0)^{**}_{sr}$ of $\mathcal{Q}_0$. The same applies to integrals of this function over sets $F \in \mathfrak{A}_R$, since the functions φ_i may be replaced by the functions $\chi_F \varphi_i$ in the preceding argument.

(e) If for a function $f \in \mathcal{F}_{(F,\theta)}(X, \mathcal{P})$ there is a net $(f_j)_{j \in \mathcal{J}}$ consisting of functions $\sum_{i=1}^{n} \varphi_i \otimes a_i$ as in (d) such that $\int_F f\, d\theta = \lim_{j \in \mathcal{J}} \int_F f_j\, d\theta$, then according to (c), $\int_F f\, d\theta$ is also contained in $(\mathcal{Q}_0)^{**}_{sr}$.

We summarize:

Proposition 6.7. *Let $(\mathcal{P}, \mathcal{V})$ and $(\mathcal{Q}_0, \mathcal{W}_0)$ be locally convex cones such that $(\mathcal{P}, \mathcal{V})$ is quasi-full, and let θ be an $\mathfrak{L}(\mathcal{P}, \mathcal{Q}_0)$-valued measure. Let $F \in \mathfrak{A}_{\mathfrak{R}}$.*

(a) For every $(\mathcal{P}, \mathcal{V})$-based integrable function in $f \in \mathcal{F}_{(F,\theta)}(X, \mathcal{P})$ the integral $\int_F f\, d\theta$ is contained in $\mathcal{Q}_0^{**}$, the second dual of $\mathcal{Q}_0$.

(b) Let $E \in \mathfrak{R}$ and let A be a relatively bounded subset of $\mathcal{P}$. If for $f \in \mathcal{F}_{(F,\theta)}(X, \mathcal{P})$ there is a net $(f_j)_{j \in \mathcal{J}}$ consisting of functions $\sum_{i=1}^{n} \varphi_i \otimes a_i$, where φ_i are non-negative measurable real-valued functions such that $\sum_{i=1}^{n} \varphi_i \le \chi_E$ and $a_i \in A$, and such that $\int_F f\, d\theta = \lim_{j \in \mathcal{J}} \int_F f_j\, d\theta$, then $\int_F f\, d\theta$ is contained in $(\mathcal{Q}_0)^{**}_{sr}$.

We shall obtain a further strengthening of these observations in some special cases.

6.8 Compact and Weakly Compact Measures. Let θ be an $\mathfrak{L}(\mathcal{P}, \mathcal{Q}_0)$-valued measure, where $(\mathcal{P}, \mathcal{V})$ is a quasi-full and $(\mathcal{Q}_0, \mathcal{W}_0)$ is a locally convex cone such that $(\mathcal{Q}, \mathcal{W})$ is its standard lattice completion. Such a measure θ is called *compact* (or *weakly compact*) if for every $E \in \mathfrak{R}$ and every relatively bounded subset A of $\mathcal{P}$ the subset

$$\mathcal{Z}(A, E) = \left\{ \sum_{i=1}^{n} \theta_{E_i}(a_i) \;\middle|\; a_i \in A,\ E_i \in \mathfrak{R} \text{ disjoint subsets of } E \right\}$$

of $\mathcal{Q}_0$ is relatively compact in the symmetric relative topology $\big($or in the weak topology $\sigma(\mathcal{Q}_0, \mathcal{Q}_0^*)\big)$ of $\mathcal{Q}_0$ (see I.4.6).

Recall from Lemma I.4.7 that the symmetric relative topology is finer than $\sigma(\mathcal{Q}_0, \mathcal{Q}_0^*)$, and from I.5.57 that $\sigma(\mathcal{Q}_0, \mathcal{Q}_0^*)$ is finer than the induced order topology on $\mathcal{Q}_0$ which is however still Hausdorff. The latter two topologies coincide, if all elements of $\mathcal{Q}_0$ are bounded (see I.5.57). Moreover, $\sigma(\mathcal{Q}_0, \mathcal{Q}_0^*)$ coincides with its own relative topology (see I.4.6). We observe that every subset $\mathcal{Z}$ of $\mathcal{Q}_0$ which is relatively compact in the symmetric relative topology is also relatively weakly compact. Indeed, the closure $\overline{\mathcal{Z}}$ of $\mathcal{Z}$ with respect to the symmetric relative topology is contained in its closure $\overline{\mathcal{Z}}^w$ with respect to the weak topology. $\overline{\mathcal{Z}}$ is compact in the former, hence also in the latter topology, thus weakly closed since $\sigma(\mathcal{Q}_0, \mathcal{Q}_0^*)$ is Hausdorff. We infer that $\overline{\mathcal{Z}} = \overline{\mathcal{Z}}^w$, and our claim follows. Every compact measure θ is therefore also weakly compact.

For a set $E \in \mathfrak{R}$ and a relatively bounded subset $A \in \mathcal{P}$ we denote by

$$\mathcal{I}(A,E)=\left\{ \int_X \left(\sum_{i=1}^{n} \varphi_i \otimes a_i \right) d\theta \;\middle|\; a_i \in A,\ 0 \le \varphi_i \text{ measurable},\ \sum_{i=1}^{n} \varphi_i \le \chi_E \right\}.$$

Clearly $\mathcal{Z}(A,E) \subset \mathcal{I}(A,E) \subset \mathcal{Q}$. Conversely, we observed in Remark 6.6(d) that $\mathcal{I}(A,E)$ is contained in the closure of $\mathcal{Z}(A,E)$ with respect to the order topology of $\mathcal{Q}$. If the measure θ is compact (or a weakly compact), then the (weak) closure $\overline{\mathcal{Z}(A,E)}^{w}$ of $\mathcal{Z}(A,E)$ is weakly compact and therefore also compact in the coarser induced order topology, and indeed closed in $\mathcal{Q}$ as the order topology is Hausdorff in this case. This demonstrates that

$$\mathcal{I}(A,E) \subset \overline{\mathcal{Z}(A,E)}^{w} \subset \mathcal{Q}_0$$

in this case. Consequently the set $\mathcal{I}(A,E)$ is also (weakly) compact in $\mathcal{Q}_0$.

We summarize:

Proposition 6.9. *Let $(\mathcal{P},\mathcal{V})$ and $(\mathcal{Q}_0,\mathcal{W}_0)$ be locally convex cones such that $(\mathcal{P},\mathcal{V})$ is quasi-full. An $\mathfrak{L}(\mathcal{P},\mathcal{Q}_0)$-valued measure θ is compact (or weakly compact), if and only if for every $E \in \mathfrak{R}$ and for every relatively bounded subset A of $\mathcal{P}$,*

$$\left\{ \int_X \left(\sum_{i=1}^{n} \varphi_i \otimes a_i \right) d\theta \;\middle|\; a_i \in A,\ 0 \le \varphi_i \ \textit{measurable},\ \sum_{i=1}^{n} \varphi_i \le \chi_E \right\}$$

is a relatively compact (or relatively weakly compact) subset of $\mathcal{Q}_0$.

Corollary 6.10. *Let $(\mathcal{P},\mathcal{V})$ and $(\mathcal{Q}_0,\mathcal{W}_0)$ be locally convex cones such that $(\mathcal{P},\mathcal{V})$ is quasi-full and let θ be an $\mathfrak{L}(\mathcal{P},\mathcal{Q}_0)$-valued relatively compact measure. Let $E \in \mathfrak{R}$, $F \in \mathfrak{A}_R$ and let A be a relatively bounded subset of $\mathcal{P}$. If for $f \in \mathcal{F}_{(F,\theta)}(X,\mathcal{P})$ there is a net $(f_j)_{j \in \mathcal{J}}$ consisting of functions $\sum_{i=1}^{n} \varphi_i \otimes a_i$, where φ_i are non-negative measurable real-valued functions such that $\sum_{i=1}^{n} \varphi_i \le \chi_E$ and $a_i \in A$, and such that $\int_F f\,d\theta = \lim_{j \in \mathcal{J}} \int_F f_j\,d\theta$, then $\int_F f\,d\theta$ is contained in $\mathcal{Q}_0$.*

The following consequence of Theorem 3.15 yields that in certain special circumstances every bounded measure is weakly compact.

Proposition 6.11. *Suppose that $(\mathcal{P},\|\ \|)$ is a finite dimensional normed space and that $(\mathcal{Q},\mathcal{W})$ is the standard lattice completion of a Banach space $(\mathcal{Q}_0,\|\ \|)$. Then every bounded $\mathfrak{L}(\mathcal{P},\mathcal{Q}_0)$-valued measure is weakly compact.*

Proof. Let $(\mathcal{P},\|\ \|)$ and $(\mathcal{Q}_0,\|\ \|)$ be as stated and let θ be a bounded $\mathfrak{L}(\mathcal{P},\mathcal{Q}_0)$-valued measure on $\mathfrak{R}$. We consider both $\mathcal{P}$ and $\mathcal{Q}_0$ as normed spaces over $\mathbb{R}$. Given a basis $\{b_1,\ldots,b_m\}$ of $\mathcal{P}$, there is a constant $\rho > 0$ such that

$$\left\| \sum_{k=1}^{m} \beta_k b_k \right\| \geq \rho \left(\max_{k=1,\ldots,m} |\beta_k| \right)$$

for every choice of scalars $\beta_1, \ldots, \beta_m \in \mathbb{R}$ $\big($see for example Lemma 2.4.1 in [107]$\big)$. Now let $E \in \mathfrak{R}$ and let A be a bounded subset of $\mathcal{P}$. According to the above then there exists $\lambda > 0$ such that

$$A \subset \left\{ \sum_{k=1}^{m} \beta_k b_k \;\middle|\; \beta_k \in \mathbb{R}, \quad |\beta_k| \leq \lambda \right\}.$$

We fix $1 \leq k \leq m$. Theorem 3.15 yields that the set

$$\mathcal{Z}_k = \{\theta_G(b_k) \mid G \in \mathfrak{R}, \;\; G \subset E\}$$

is relatively compact in $\mathcal{Q}_0$ with respect to the weak topology $\sigma(\mathcal{Q}_0, \mathcal{Q}_0^*)$. Now let $E_i \in \mathfrak{R}$, for $i = 1, \ldots, n$, be disjoint subsets of E and in a first step let $0 \leq \beta_k^1 \leq \beta_k^2 \ldots \leq \beta_k^n \leq 1$. Set $F_1 = \bigcup_{i=1}^n E_i$, $F_2 = \bigcup_{i=2}^n E_i$, and so on, and $F_n = E_n$. Then

$$\sum_{i=1}^{n} \beta_k^i \theta_{E_i}(b_k) = \beta_k^1 \theta_{F_1}(b_k) + \sum_{i=2}^{n} (\beta_k^i - \beta_k^{i-1}) \theta_{F_i}(b_k).$$

The element $\sum_{i=1}^n \beta_k^i \theta_{E_i}(b_k)$ is therefore contained in the convex hull $\widetilde{\mathcal{Z}_k}$ of the set $\mathcal{Z}_k$. Following a well-known theorem due to Krein $\big($see Theorem IV.11.4 in [185]$\big)$ this convex hull is again relatively weakly compact in $\mathcal{Q}_0$. So, obviously is the set $-\widetilde{\mathcal{Z}_k}$. Using this, we infer that indeed for every choice of $\beta_k^i \in \mathbb{R}$ such that $|\beta_k^i| \leq 1$ for all $i = 1, \ldots, n$ the element $\sum_{i=1}^n \beta_k^i \theta_{E_i}(b_k)$ is contained in relatively weakly compact set $\mathcal{Y}_k = \widetilde{\mathcal{Z}_k} + (-\widetilde{\mathcal{Z}_k})$.

Thus for every choice of elements $a_i = \sum_{k=1}^m \beta_k^i b_k \in A$ and disjoint subsets $E_i \in \mathfrak{R}$ of E we have $|\beta_k^i| \leq \lambda$ for all $i = 1, \ldots, n$ and $k = 1, \ldots, m$, hence

$$\sum_{i=1}^{n} \theta_{E_i}(a_i) = \sum_{k=1}^{m} \sum_{i=1}^{n} \beta_k^i \theta_{E_i}(b_k) \in \lambda \left(\sum_{k=1}^{m} \mathcal{Y}_k \right).$$

As a finite sum of relatively weakly compact sets $\big($see I.V.2 in [185]$\big)$, the set on the right-hand side is also relatively weakly compact in $\mathcal{Q}_0$, and our claim follows. $\square$

6.12 The Case that $\mathcal{P}$ Is a Locally Convex Vector Space. Let $(\mathcal{P}, \mathcal{V})$ be a locally convex topological vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, endowed with a basis $\mathcal{V}$ of balanced convex neighborhoods, that is subsets of $\mathcal{P}$. Equality is the order on $\mathcal{P}$, and involving the neighborhoods we have $a \leq b + v$ if $a - b \in v$ for $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$. As a locally convex cone $(\mathcal{P}, \mathcal{V})$ is of course quasi-full (see I.6.1). The modulus of an $\mathfrak{L}(\mathcal{P}, \mathcal{Q})$-valued measure θ is given by

$$|\theta|(E,v) = \sup\left\{ \sum_{i=1}^{n} \theta_{E_i}(s_i) \;\middle|\; s_i \in v,\ E_i \in \mathfrak{R} \text{ disjoint subsets of } E \right\} \in \mathcal{Q}.$$

According to Lemma 2.5, the cone $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ as introduced in 2.3 consists of those $\mathcal{P}$-valued functions that vanish outside some set $E \in \mathfrak{R}$ and may be uniformly approximated on X by step functions; more precisely: for $f \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ there is $E \in \mathfrak{R}$ such that $f(x) = 0$ for all $x \in X \setminus E$ and for every $v \in \mathcal{V}$ there exists a step function $h \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P})$ such that $h(x) - f(x) \in v$ for all $x \in X$. Any such function f is measurable by Theorem 1.7. Consequently, the functions in $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ are uniformly bounded on all sets in $\mathfrak{R}$. We have $\alpha f \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ whenever $f \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ and $\alpha \in \mathbb{K}$. Every measurable neighborhood-valued function $s \in \mathcal{F}(X,\mathcal{P}_{\mathcal{V}})$ is however contained $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P}_{\mathcal{V}})$, since its values are positive. For a positive real-valued measurable function φ and a neighborhood $v \in \mathcal{V}$, for example, the function $\varphi_{\otimes} v$ is measurable, hence in $\mathcal{F}(X,\mathcal{P}_{\mathcal{V}})$. Recall that $\mathcal{V}$-valued measurable functions are integrated using the canonical extension of the measure θ to the full cone $(\mathcal{P}_{\mathcal{V}},\mathcal{V})$ as elaborated in Section 3.8.

According to 4.12, a $\mathcal{P}$-valued function f is integrable over a set $E \in \mathfrak{R}$ if for every $w \in \mathcal{W}$ and $\varepsilon > 0$ there are functions $f_{(w,\varepsilon)} \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P}_{\mathcal{V}})$ and $s_{(w,\varepsilon)} \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{V})$ such that

$$f \underset{a.e.E}{\leq} f_{(w,\varepsilon)} \underset{a.e.E}{\leq} \gamma f + s_{(w,\varepsilon)}$$

and $\int_E s_{(w,\varepsilon)}\,d\theta \leq \varepsilon w$ for some $1 \leq \gamma \leq 1+\varepsilon$. A straightforward argument involving the uniform boundedness of the functions in $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ leads to a slight simplification in this case, avoiding the relative topologies: A function $f \in \mathcal{F}(X,\mathcal{P}_{\mathcal{V}})$ is integrable over a set $E \in \mathfrak{R}$ if for every $w \in \mathcal{W}$ there are functions $f_w \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P}_{\mathcal{V}})$ and $s_w \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{V})$ such that

$$\text{(I)} \qquad f \underset{a.e.E}{\leq} f_w \underset{a.e.E}{\leq} f + s_w \qquad \text{and} \qquad \int_E s_w\,d\theta \leq w.$$

The function αf is integrable over E for any $\alpha \in \mathbb{K}$, whenever f is. A function $f \in \mathcal{F}(X,\mathcal{P})$ is $(\mathcal{P},\mathcal{V})$-based integrable over $E \in \mathfrak{R}$ (see 5.6) if there are $f_w \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ and $s_w \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{V})$ such that

$$f \underset{a.e.E}{\leq} f_w + s_w, \qquad f_w \underset{a.e.E}{\leq} f + s_w \quad \text{and} \quad \int_E s_w\,d\theta \leq w.$$

Considering that the functions in $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ can be approximated by step functions, this is equivalent to the following condition for integrability which is only slightly stronger than (I):

For every $w \in \mathcal{W}$ there is a step function $h_w \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P})$ and $s_w \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{V})$ such that

$$\text{(BI 1)} \qquad f(x) - h_w(x) \in s_w(x) \quad \text{a.e. on } E \quad \text{and} \quad \int_E s_w\,d\theta \leq w.$$

(Set $f_w = h_w + s_w \in \mathcal{F}_{\mathfrak{R}}(X, \mathcal{P}_\nu)$ in order to satisfy (I).) Condition (BI 1) yields indeed strong integrability in the meaning of Section 5.18, since it obviously implies that $f \le h_w + s_w$, hence

$$\int_G f\,d\theta \le \int_G h_w\,d\theta + \int_G s_w\,d\theta \le \int_G h_w\,d\theta + w$$

for all subsets $G \in \mathfrak{R}$ of E. Somewhat stronger than (BI 1) is the following sufficient integrability condition: For $v \in \mathcal{V}$ let $\|\ \|_v$ denote the corresponding seminorm on $\mathcal{P}$, that is $\|a\|_v = \inf\{\lambda \ge 0 \mid a \in \lambda v\}$. We require that for every $v \in \mathcal{V}$ and $w \in \mathcal{W}$ there is a step function $h_{(v,w)} \in \mathcal{S}_{\mathfrak{R}}(X, \mathcal{P})$ such that the positive real-valued function $x \mapsto \|f(x) - h_{(v,w)}(x)\|_v$ is measurable and

(BI 2) $$\int_E \|f - h_{(v,w)}\|_v \otimes v\ d\theta \le w.$$

Condition (BI 2) obviously implies (BI 1) since, given $w \in \mathcal{W}$ we choose any $v \in \mathcal{V}$ and set $s_w(x) = \|f(x) - h_{(v,w)}(x)\|\, v$. Then obviously $f(x) - h_{(v,w)}(x) \in s_w(x)$ holds for all $x \in E$, hence (BI 1). Moreover, a function $f \in \mathcal{F}(X, \mathcal{P})$ satisfying (BI 2) is $(\mathcal{P}, \mathcal{V}_0)$-based integrable over E with respect to θ for every one-dimensional neighborhood subsystem $\mathcal{V}_0 = \{\rho v_0 \mid \rho > 0\}$, for $v_0 \in \mathcal{V}$. This is one of the requirements in Theorem 5.36. In the special case that $(\mathcal{P}, \mathcal{V})$ is a normed space, that is $\mathcal{V} = \{\rho\mathbb{B} \mid \rho > 0\}$, where $\mathbb{B}$ is the unit ball in $\mathcal{P}$, condition (BI 2) leads to the well-known notion of *Bochner (or Dunford and Schwartz) integrability* (see for example III.2.17 in [55] or II.2 in [43]). This will be further elaborated in Section 6.18 below.

In all of the above cases, integrability is then extended to sets $F \in \mathfrak{A}_{\mathfrak{R}}$ as in 4.13. Convergence for sequences of $\mathcal{P}$-valued functions as required in Theorems 5.23 to 5.25 and 5.34 refers to pointwise convergence with respect to the vector space topology of $\mathcal{P}$. If the measure θ is strongly additive and if the order continuous linear functionals on the locally convex complete lattice cone $(\mathcal{Q}, \mathcal{W})$ support the separation property (see I.5.32), then the strong convergence statements of Theorem 5.36 apply to functions satisfying (BI 2).

We already observed that the functions which are integrable over a set $E \in \mathfrak{R}$ with respect to any of the above criteria form also a vector space over $\mathbb{K}$ in this case.

Now suppose in addition to the above that $(\mathcal{Q}, \mathcal{W})$ is the standard lattice completion of some subcone $(\mathcal{Q}_0, \mathcal{W}_0)$ and the measure θ is $\mathfrak{L}(\mathcal{P}, \mathcal{Q}_0)$-valued (see 6.5). Then, according to Theorem 3.11 countable additivity for θ refers to the strong operator topology of $\mathfrak{L}(\mathcal{P}, \mathcal{Q}_0)$. Moreover, following Proposition 6.7(a), integrals of $(\mathcal{P}, \mathcal{V})$-based integrable functions in $\mathcal{F}(X, \mathcal{P})$ are elements of the second dual $\mathcal{Q}_0^{**}$ of $\mathcal{Q}_0$. We shall make a few supplementary observations for the case that $(\mathcal{Q}_0, \mathcal{W}_0)$ is indeed a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$:

(i) If $(\mathcal{Q}_0, \mathcal{W}_0)$ is a locally convex topological vector space over $\mathbb{K}$, then the $(\mathcal{P}, \mathcal{V})$-based integrable functions in $\mathcal{F}(X, \mathcal{P})$ form a vector space

$\mathcal{F}_{(F,\theta,BI1)}(X,\mathcal{P})$ over $\mathbb{K}$. The integrals of functions in $\mathcal{F}_{(F,\theta,BI1)}(X,\mathcal{P})$ for $F \in \mathfrak{A}_{\mathfrak{R}}$ are contained in the order closure of $\mathcal{Q}_0$ in $\mathcal{Q}$, hence are $\mathbb{K}$-linear $\big($see I.5.60(b)$\big)$ and therefore elements of the second vector space dual $\mathcal{Q}_{0\mathbb{K}}^{**}$ of $\mathcal{Q}_0$.

(ii) If the locally convex space $\mathcal{Q}_0$ is indeed topologically complete, and if a function $f \in \mathcal{F}(X,\mathcal{P})$ fulfills the integrability criterion (BI1), then for every $E \in \mathfrak{R}$ its integral $\int_E f\,d\theta$ in $\mathcal{Q}$ may be approximated in the symmetric (modular) topology of $\mathcal{Q}$ by a net $\big(\int_E h_i\,d\theta\big)_{i\in\mathcal{I}}$ of integrals over step functions. Integrals over step functions are however contained in the complete subspace $\mathcal{Q}_0$ of $\mathcal{Q}$. The Cauchy sequence $\big(\int_E h_n\,d\theta\big)_{n\in\mathbb{N}}$ is therefore convergent in $\mathcal{Q}_0$ and its limit, that is $\int_E f\,d\theta$ is also contained in $\mathcal{Q}_0$.

(iii) If the locally convex space $\mathcal{Q}_0$ is reflexive, then every bounded $\mathfrak{L}(\mathcal{P},\mathcal{Q}_0)$-valued measure θ is seen to be weakly compact. Indeed, for every $E \in \mathfrak{R}$ and every bounded subset A of $\mathcal{P}$ the set

$$\mathcal{Z}(A,E) = \left\{ \sum_{i=1}^{n} \theta_{E_i}(a_i) \;\middle|\; a_i \in A,\ E_i \in \mathfrak{R} \text{ disjoint subsets of } E \right\}$$

from 6.8 is bounded in $\mathcal{Q}_0$ $\big($see Remark 6.8(a)$\big)$, hence relatively weakly compact, since this holds for all bounded subsets in reflexive spaces.

(iv) If both $\mathcal{P}$ and $\mathcal{Q}_0$ are locally convex topological vector spaces over $\mathbb{K}$, then we denote by $\mathfrak{L}_{\mathbb{K}}(\mathcal{P},\mathcal{Q}_0)$ the space of all continuous $\mathbb{K}$-linear operators from $\mathcal{P}$ into $\mathcal{Q}_0$. If the measure θ is indeed $\mathfrak{L}_{\mathbb{K}}(\mathcal{P},\mathcal{Q}_0)$-valued, then for every $F \in \mathfrak{A}_{\mathfrak{R}}$ the operator

$$f \mapsto \int_F f\,d\theta \;:\; \mathcal{F}_{(F,\theta,BI1)}(X,\mathcal{P}) \to \mathcal{Q}_{0\mathbb{K}}^{**}$$

is also linear over $\mathbb{K}$. According to I.5.60(d) we need to verify two conditions for this. The first one is obvious, because the additivity of the operator is given. Likewise, the second condition in I.5.60(d) is evident for all $\alpha \geq 0$. Thus all left to verify is that

$$\left(\int_F \gamma f\,d\theta\right)(\mu) = \left(\int_F f\,d\theta\right)(\gamma\mu).$$

holds for all $f \in \mathcal{F}_{(F,\theta,BI1)}(X,\mathcal{P})$, $\mu \in \mathcal{Q}_0^*$ and $\gamma \in \Gamma$, the unit circle in $\mathbb{K}$. Indeed, this obviously holds true for every step function $h \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P})$. Because the neighborhoods in $\mathcal{V}$ and in $\mathcal{W}$ are supposed to be balanced, $h \leq f + \mathfrak{v}_w$ holds for $h \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P})$ and $f \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ if and only if $\gamma h \leq \gamma f + \mathfrak{v}_w$. Therefore and because the lattice operations are taken pointwise in $\mathcal{Q}$, we infer that

$$\left(\int_F^{(w)} \gamma f\,d\theta\right)(\mu) = \left(\int_F^{(w)} f\,d\theta\right)(\gamma\mu).$$

Now Definition 4.13 yields our claim. We shall formulate this special case as a separate Proposition:

Proposition 6.13. *Let $(\mathcal{P},\mathcal{V})$ and $(\mathcal{Q}_0,\mathcal{W}_0)$ be locally convex topological vector spaces over $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$ and let θ be a bounded $\mathfrak{L}_{\mathbb{K}}(\mathcal{P},\mathcal{Q}_0)$-valued measure. Then the functions in $\mathcal{F}(X,\mathcal{P})$ satisfying (BI1) form a vector space $\mathcal{F}_{(F,\theta,BI1)}(X,\mathcal{P})$ over $\mathbb{K}$, their integrals are contained in the second vector space dual $\mathcal{Q}_{0\mathbb{K}}^{**}$ of $\mathcal{Q}_0$, and the operator*

$$f \mapsto \int_F f\,d\theta \,:\, \mathcal{F}_{(F,\theta,BI1)}(X,\mathcal{P}) \to \mathcal{Q}_{0\mathbb{K}}^{**}$$

is linear over $\mathbb{K}$.

6.14 Algebra Homomorphisms. Let us consider a special case of 6.13. Suppose that the locally convex vector spaces $(\mathcal{P},\mathcal{V})$ and $(\mathcal{Q}_0,\mathcal{W})$ are indeed topological algebras over $\mathbb{K}$, and that $(\mathcal{Q},\mathcal{W})$ is the standard lattice completion of $\mathcal{Q}_0$. A *topological algebra* $\mathcal{P}$ is an algebra and a locally convex topological vector space such that for a fixed element $a\in\mathcal{P}$ (or $b\in\mathcal{P}$) the linear operator $c\mapsto ac$ (or $c\mapsto cb$) from $\mathcal{P}$ into $\mathcal{P}$ is continuous (see for example 8.1 in [137]). Recall that for a linear operator continuity implies weak continuity. Thus $\mathcal{P}$ is also a topological algebra in its weak topology. Indeed, for a fixed $a\in\mathcal{P}$ and $\mu\in\mathcal{P}^*$, the mapping $c\mapsto\mu(ac):\mathcal{P}\to\mathbb{R}$ is a continuous linear functional. Thus, if the net $(c_i)_{i\in\mathcal{I}}$ in $\mathcal{P}$ converges weakly to $c\in\mathcal{P}$, then $\mu(ac_i)_{i\in\mathcal{I}}$ converges to $\mu(ac)$ in $\mathbb{R}$. The net $(ac_i)_{i\in\mathcal{I}}$ therefore converges weakly to $ac\in\mathcal{P}$.

Now suppose that θ is an $\mathfrak{R}$-bounded measure such that its values θ_E for all $E\in\mathfrak{R}$ of are continuous $\mathbb{K}$-linear operators from $\mathcal{P}$ to $\mathcal{Q}_0$ satisfying the following condition:

(A) $\theta_E(a)\,\theta_E(b)=\theta_E(ab)$ *and* $\theta_E(a)\,\theta_G(b)=0$ *for all* $a,b\in\mathcal{P}$ *and disjoint sets* $E,G\in\mathfrak{R}$.

Both requirements in Condition (A) may be reformulated and combined as

(A') $\theta_E(a)\,\theta_G(b)=\theta_{(E\cap G)}(ab)$ *for all* $E,G\in\mathfrak{R}$ *and* $a,b\in\mathcal{P}$.

Indeed, (A') implies (A), and if (A) holds, then for $a,b\in\mathcal{P}$ and $E,G\in\mathfrak{R}$ we have

$$\theta_E(a)\,\theta_G(b)=\big(\theta_{(E\setminus G)}(a)+\theta_{(E\cap G)}(a)\big)\big(\theta_{(G\setminus E)}(b)+\theta_{(E\cap G)}(b)\big)=\theta_{(E\cap G)}(ab),$$

hence (A'). Endowed with the canonical, that is pointwise multiplication, the $\mathcal{P}$-valued step functions form an algebra, and we obtain

$$\int_X (hl)\,d\theta=\left(\int_X h\,d\theta\right)\left(\int_X l\,d\theta\right)$$

for all $h,l\in\mathcal{S}_{\mathfrak{R}}(X,\mathcal{P})$ as an immediate consequence of (A). Indeed, the functions h and l can be expressed as $h=\sum_{i=1}^n \chi_{E_i}\otimes a_i$ and $l=\sum_{i=1}^n \chi_{E_i}\otimes b_i$ with disjoint sets $E_i\in\mathfrak{R}$ and elements $a_i,b_i\in\mathcal{P}$. Then $hl=\sum_{i=1}^n \chi_{E_i}\otimes a_ib_i$ and

$$\sum_{i=1}^{n} \theta_{E_i}(a_i b_i) = \left(\sum_{i=1}^{n} \theta_{E_i}(a_i) \right) \left(\sum_{i=1}^{n} \theta_{E_i}(b_i) \right),$$

that is our claim.

Now let us denote by $\mathcal{E}_{\mathfrak{R}}(X,\mathcal{P})$ the vector subspace of $\mathcal{F}(X,\mathcal{P})$ generated by all elementary functions. Recall that elementary functions are of the type $\varphi_{\otimes}a$, where φ is a bounded non-negative measurable real-valued function supported by a set in $\mathfrak{R}$, and a is an element of $\mathcal{P}$. Obviously, $\mathcal{E}_{\mathfrak{R}}(X,\mathcal{P})$ forms also an algebra, as the product of two elementary functions $\varphi_{\otimes}a$ and $\psi_{\otimes}b$ is the elementary function $(\varphi\psi)_{\otimes}(ab)$. We would like to establish that the integral defines a multiplicative operator on $\mathcal{E}_{\mathfrak{R}}(X,\mathcal{P})$ as well. However, because integrals of these functions are generally contained in the strong second dual $\mathcal{Q}_0^{**}$ of $\mathcal{Q}_0$ rather than in $\mathcal{Q}_0$ itself, we shall a introduce a continuation of the multiplication to $\mathcal{Q}_0^{**} \subset \mathcal{Q}$ in the following way: For elements $l, m \in \mathcal{Q}$ we denote by $l \bullet m$ the set of all elements $q \in \mathcal{Q}$ for which we can find nets $(l_i)_{i\in\mathcal{I}}$ and $(m_j)_{j\in\mathcal{J}}$ in $\mathcal{Q}_0 \subset \mathcal{Q}$ such that $\lim_{i\in\mathcal{I}} l_i = l$, $\lim_{j\in\mathcal{J}} m_j = m$ and

$$\underline{\lim}_{i\in\mathcal{I}}\, \underline{\lim}_{j\in\mathcal{J}}\, l_i m_j = \overline{\lim}_{i\in\mathcal{I}}\, \overline{\lim}_{j\in\mathcal{J}}\, l_i m_j = \underline{\lim}_{j\in\mathcal{J}}\, \underline{\lim}_{i\in\mathcal{I}}\, l_i m_j = \overline{\lim}_{j\in\mathcal{J}}\, \overline{\lim}_{i\in\mathcal{I}}\, l_i m_j = q.$$

Our introductory remark shows that for elements $l, m \in \mathcal{Q}_0$ we have $l \bullet m = \{lm\}$, since on $\mathcal{Q}_0 \subset \mathcal{Q}$ weak and order convergence coincide (see I.5.57). In general, the set $l \bullet m$ may be empty or contain more than one element of $\mathcal{Q}$. However, if $q \in l \bullet m$ and if $\mu \in \mathcal{Q}^*$ is a multiplicative linear functional, then

$$q(\mu) = \lim_{i\in\mathcal{I}} \lim_{j\in\mathcal{J}} \big(l_i m_j\big)(\mu) = \lim_{i\in\mathcal{I}} \lim_{j\in\mathcal{J}} l_i(\mu)\, m_j(\mu) = l(\mu)\, m(\mu).$$

Now let $f = \varphi_{\otimes}a$ and $g = \psi_{\otimes}b$ be two elementary functions. Their product fg is the elementary function $(\varphi\psi)_{\otimes}(ab)$. Let $(\varphi_n)_{n\in\mathbb{N}}$ and $(\psi_n)_{n\in\mathbb{N}}$ be the sequences of real-valued step functions converging to φ and ψ as in 5.27 and 5.28. Thus

$$\lim_{n\to\infty} \int_X \varphi_{n\otimes}a\, d\theta = \int_X f\, d\theta \qquad \text{and} \qquad \lim_{n\to\infty} \int_X \psi_{n\otimes}b\, d\theta = \int_X g\, d\theta$$

by 5.28. For every fixed $m \in \mathbb{N}$ the sequence $(\varphi_m\psi_n)_{n\in\mathbb{N}}$ converges pointwise to the function $\varphi_m\psi$, and we have $0 \le \varphi_m\psi_n \le \varphi_m\psi$ for all $n \in \mathbb{N}$. This shows

$$\lim_{n\to\infty}\left\{ \left(\int_X \varphi_{m\otimes}a \right) \left(\int_X \psi_{n\otimes}b\, d\theta \right) \right\} = \lim_{n\to\infty} \int_X (\varphi_m\psi_n)_{\otimes}(ab)\, d\theta = \int_X (\varphi_m\psi)_{\otimes}(ab)\, d\theta$$

by Corollary 5.28 and the above. Furthermore, the sequence $(\varphi_m\psi)_{m\in\mathbb{N}}$ converges pointwise to the function $\varphi\psi$, and we have $0 \le \varphi_m\psi \le \varphi\psi$ for all

$m \in \mathbb{N}$. Again using 5.28, this yields

$$\lim_{m\to\infty} \int_X (\varphi_m \psi)_\otimes (ab)\, d\theta = \int_X (\varphi\psi)_\otimes (ab)\, d\theta = \int_X (fg)\, d\theta,$$

hence

$$\lim_{m\to\infty} \lim_{n\to\infty} \left\{ \left(\int_X \varphi_{m\otimes} a \right) \left(\int_X \psi_{n\otimes} b\, d\theta \right) \right\} = \int_X (fg)\, d\theta.$$

Similarly, one verifies

$$\lim_{n\to\infty} \lim_{m\to\infty} \left\{ \left(\int_X \varphi_{m\otimes} a \right) \left(\int_X \psi_{n\otimes} b\, d\theta \right) \right\} = \int_X (fg)\, d\theta.$$

Thus indeed

$$\int_X (fg)\, d\theta \in \left(\int_X f\, d\theta \right) \cdot \left(\int_X g\, d\theta \right).$$

Finally, let $f, g \in \mathcal{E}_{\mathfrak{R}}(X, \mathcal{P})$, that is $f = \sum_{i=1}^{i_0} f_i$ and $g = \sum_{k=1}^{k_0} f_k$ with elementary functions f_i, g_k. For each of these functions there are approximating sequences $(h_n^i)_{n\in\mathbb{N}}$ and $(e_n^k)_{n\in\mathbb{N}}$ of step functions as in the preceding step of our argument. We set $h_n = \sum_{i=1}^{i_0} f_n^i$ and $e_n = \sum_{k=1}^{k_0} e_n^k$. The sequences

$$\left(\int_X h_n\, d\theta \right)_{n\in\mathbb{N}} \qquad \text{and} \qquad \left(\int_X e_n\, d\theta \right)_{n\in\mathbb{N}}$$

in $\mathcal{Q}_0$ then converge to $\int_X f\, d\theta$ and $\int_X g\, d\theta$, respectively. For all $n, m \in \mathbb{N}$ we have

$$\left(\int_X h_m\, d\theta \right) \left(\int_X e_n\, d\theta \right) = \sum_{i=1}^{i_o} \sum_{k=1}^{k_0} \int_X (h_m^i e_n^k)\, d\theta,$$

and for fixed i and k

$$\lim_{n\to\infty} \lim_{n\to\infty} \int_X h_m^i e_n^k\, d\theta = \int_X (f_i g_k)\, d\theta$$

by the above. This yields

$$\lim_{m\to\infty} \lim_{n\to\infty} \left\{ \left(\int_X h_m\, d\theta \right) \left(\int_X e_n\, d\theta \right) \right\} = \sum_{i=1}^{i_o} \sum_{k=1}^{k_0} \int_X (f_i g_k)\, d\theta = \int_X (fg)\, d\theta.$$

Reversing the parts of n and m leads to the same result. Thus indeed

$$\int_X (fg)\, d\theta \in \left(\int_X f\, d\theta \right) \cdot \left(\int_X g\, d\theta \right)$$

holds for all $f, g \in \mathcal{E}_{\mathfrak{R}}(X, \mathcal{P})$, provided that the measure θ satisfies (A).

If both $(\mathcal{P},\mathcal{V})$ and $(\mathcal{Q}_0,\mathcal{W})$ are topological algebras with an involution, that is a continuous operator $a\mapsto a^*$ such that $(a+b)^*=a^*+b^*$, $(\alpha a)^*=\bar{\alpha}a^*, (a^*)^*=a$ and $(ab)^*=b^*a^*$ for a,b in $\mathcal{P}$ or in $\mathcal{Q}_0$, respectively, and if the $\mathfrak{L}(\mathcal{P},\mathcal{Q}_0)$-valued measure θ satisfies

(A*) $\theta_E(a^*)=\big(\theta_E(a)\big)^*$ *for all* $E\in\mathfrak{R}$ *and* $a\in\mathcal{P}$

in addition to (A), then a similar property can be derived for the integrals of functions in $\mathcal{E}_{\mathfrak{R}}(X,\mathcal{P})$. Analogously to the above extension of the multiplication in $\mathcal{Q}_0$, for an elements $l\in\mathcal{Q}$ we denote by $l^\star$ the set of all elements $q\in\mathcal{Q}$ for which we can find a net $(l_i)_{i\in\mathcal{I}}$ in $\mathcal{Q}_0\subset\mathcal{Q}$ such that $\lim_{i\in\mathcal{I}}l_i=l$ and

$$\lim_{i\in\mathcal{I}}l_i^*=q.$$

The continuity of the involution in $\mathcal{Q}_0$ shows that for $l\in\mathcal{Q}_0$ we have $l^\star=\{l^*\}$. Otherwise, the set $l^\star$ may be empty or contain more than one element of $\mathcal{Q}$. Canonically, for a function $f\in\mathcal{F}(X,\mathcal{P})$ we denote by $f^*\in\mathcal{F}(X,\mathcal{P})$ the function $x\mapsto\big(f(x)\big)^*$. Then an argument similar to that for the multiplication yields

$$\int_X f^*\,d\theta\in\left(\int_X f\,d\theta\right)^\star$$

for all $f\in\mathcal{E}_{\mathfrak{R}}(X,\mathcal{P})$ and every $\mathfrak{L}(\mathcal{P},\mathcal{Q}_0)$-valued measure θ which satisfies (A) and (A*).

Because both $\chi_{F^\otimes}f,\chi_{F^\otimes}g\in\mathcal{E}_{\mathfrak{R}}(X,\mathcal{P})$ whenever $f,g\in\mathcal{E}_{\mathfrak{R}}(X,\mathcal{P})$ and $F\in\mathfrak{A}_{\mathfrak{R}}$, and because $\int_F f\,d\theta=\int_X\chi_{F^\otimes}f\,d\theta$, the above properties apply also to integrals over measurable subsets F of X.

We formulate this as a further proposition:

Proposition 6.15. *Let* $(\mathcal{P},\mathcal{V})$ *and* $(\mathcal{Q}_0,\mathcal{W}_0)$ *be topological algebras over* $\mathbb{K}=\mathbb{R}$ *or* $\mathbb{K}=\mathbb{C}$ *and let* θ *be a bounded* $\mathfrak{L}_{\mathbb{K}}(\mathcal{P},\mathcal{Q}_0)$*-valued measure such that* $\theta_E(a)\,\theta_G(b)=\theta_{(E\cap G)}(ab)$ *holds for all* $E,G\in\mathfrak{R}$ *and* $a,b\in\mathcal{P}$. *Then*

$$\int_X(fg)\,d\theta\in\left(\int_X f\,d\theta\right)\boldsymbol{\cdot}\left(\int_X g\,d\theta\right)$$

holds for all $f,g\in\mathcal{E}_{\mathfrak{R}}(X,\mathcal{P})$. *If both* $(\mathcal{P},\mathcal{V})$ *and* $(\mathcal{Q}_0,\mathcal{W})$ *are topological algebras with an involution* $a\mapsto a^*$ *and if* θ *satisfies* $\theta_E(a^*)=\big(\theta_E(a)\big)^*$ *for all* $E\in\mathfrak{R}$ *and* $a\in\mathcal{P}$, *then*

$$\int_X f^*\,d\theta\in\left(\int_X f\,d\theta\right)^\star$$

holds for all $f\in\mathcal{E}_{\mathfrak{R}}(X,\mathcal{P})$.

The case that $\mathcal{Q}_0 = \mathbb{K}$. If $\mathcal{Q}_0 = \mathbb{K}$, that is if the values θ_E of the measure θ are $\mathbb{K}$-linear functionals on the algebra $\mathcal{P}$, then Condition (A) means that all functionals θ_E are multiplicative and that for disjoint sets $E, G \in \mathfrak{R}$ we have either $\theta_E = 0$ or $\theta_G = 0$. Thus θ takes at most one non-zero value, that is some multiplicative $\mathbb{K}$-linear functional in $\mathcal{P}^*$. In special cases (see also Section 4.7 in Chapter III below) we infer that θ is indeed some point evaluation measure. Condition (A*) for an algebra with involution means that $\theta_E(a^*) = \overline{\theta_E(a)}$ holds for all $E \in \mathfrak{R}$ and $a \in \mathcal{P}$.

6.16 Lattice Homomorphisms. In Section 5.1 of Chapter I we defined a locally convex $\vee$-semilattice cone to be a locally convex cone $(\mathcal{P}, \mathcal{V})$ with the following properties: The order in $\mathcal{P}$ is antisymmetric, for any two elements $a, b \in \mathcal{P}$ their supremum $a \vee b$ exists in $\mathcal{P}$ and

($\vee$1) $(a+c) \vee (b+c) = a \vee b + c$ *holds for all* $a, b, c \in \mathcal{P}$.
($\vee$2) $a \leq c + v$ *and* $b \leq c + w$ *for* $a, b, c \in \mathcal{P}$ *and* $v, w \in \mathcal{V}$ *implies that* $a \vee b \leq c + (v + w)$.

In case that the locally convex cone $(\mathcal{P}, \mathcal{V})$ is quasi-full, ($\vee$2) may be replaced by the somewhat simpler condition

($\vee 2'$) $a \leq v$ *for* $a \in \mathcal{P}$ *and* $v \in \mathcal{V}$ *implies that* $a \vee 0 \leq v$.

Indeed, suppose that ($\vee$1) and ($\vee 2'$) hold in a quasi-full cone $(\mathcal{P}, \mathcal{V})$, and that $a \leq c + v$ and $b \leq c + w$ for $a, b, c \in \mathcal{P}$ and $v, w \in \mathcal{V}$. Then $a \leq c + s$ and $b \leq c + t$ for some elements $s \leq v$ and $t \leq w$ by (QF1) in I.6.1. By ($\vee 2'$) we have $s \vee 0 \leq v$ and $t \vee 0 \leq w$ as well. Now $a \leq c + s \vee 0 + t \vee 0$ and $b \leq c + s \vee 0 + t \vee 0$ implies

$$a \vee b \leq c + s \vee 0 + t \vee 0 \leq c + (v + w)$$

as required in ($\vee$2). Recall from Proposition I.5.2 that in a locally convex $\vee$-semilattice cone the lattice operation, that is the mapping $(a, b) \mapsto a \vee b : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ is continuous with respect to the symmetric relative topology.

Topological vector lattices and locally convex complete lattice cones in the sense of I.5 are locally convex $\vee$-semilattice cones. Further specific examples include $\overline{\mathbb{R}}$ and $\overline{\mathbb{R}}_+$ $\big($Examples I.1.4(a) and (b)$\big)$ and cones of non-empty convex subsets of a topological vector space with the set-inclusion as order $\big($Example I.1.4(c)$\big)$. The supremum of two convex sets is their convex hull in this case while infima do not always exist.

In the following let us suppose that $(\mathcal{P}, \mathcal{V})$ is a quasi-full locally convex $\vee$-semilattice cone and that θ is an $\mathfrak{R}$-bounded $\mathfrak{L}(\mathcal{P}, \mathcal{Q})$-valued measure whose values θ_E for all $E \in \mathfrak{R}$ of are continuous linear operators from $\mathcal{P}$ to $\mathcal{Q}$ satisfying the following condition:

(L) $\theta_E(a) \vee \theta_E(b) = \theta_E(a \vee b)$ *and* $\theta_E(a) \vee \theta_G(b) = \theta_E(a) + \theta_G(b)$ *for all* $a, b \geq 0$ *in* $\mathcal{P}$ *and disjoint sets* $E, G \in \mathfrak{R}$.

We shall verify below that (L) implies that its first requirement, that is to say

$$\theta_E(a) \vee \theta_E(b) = \theta_E(a \vee b),$$

holds indeed for all, not only the positive elements of $\mathcal{P}$. First we observe that Condition (L) implies

$$\text{(i)} \qquad \theta_E(a) \wedge \theta_G(b) \leq \mathfrak{O}\left(\theta_E(a) \vee \theta_G(b)\right)$$

for disjoint sets $E, G \in \mathfrak{R}$ and $0 \leq a, b \in \mathcal{P}$, as well as

$$\text{(ii)} \qquad \sup_{i=1,\dots,n} \theta_{E_i}(a_i) = \sum_{i=1}^{n} \theta_{E_i}(a_i)$$

for disjoint sets $E_i \in \mathfrak{R}$ and $a_i \geq 0$ in $\mathcal{P}$. For (i), let $E, G \in \mathfrak{R}$ be disjoint and $0 \leq a, b \in \mathcal{P}$. Then

$$\theta_E(a) \vee \theta_G(b) = \theta_E(a) + \theta_G(b) = \theta_E(a) \vee \theta_G(b) + \theta_E(a) \wedge \theta_G(b)$$

by (L) and Proposition I.5.3. This yields our claim via the cancellation rule in I.5.10(a). We shall prove (ii) by induction: For $n = 1$ there is nothing to prove. Suppose our claim holds for $n \in \mathbb{N}$, and let $E_1, \dots, E_{n+1} \in \mathfrak{R}$ be disjoint sets, and $0 \leq a_1, \dots, a_{n+1} \in \mathcal{P}$. The inequality

$$\sup_{i=1,\dots,n+1} \theta_{E_i}(a_i) \leq \sum_{i=1}^{n+1} \theta_{E_i}(a_i)$$

is obvious. For the converse, using Proposition I.5.3 we infer

$$\begin{aligned}\sum_{i=1}^{n+1} \theta_{E_i}(a_i) &= \sup_{i=1,\dots,n} \theta_{E_i}(a_i) \;+\; \theta_{E_{n+1}}(a_{n+1}) \\ &= \sup_{i=1,\dots,n+1} \theta_{E_i}(a_i) + \sup_{i=1,\dots,n} \theta_{E_i}(a_i) \;\wedge\; \theta_{E_{n+1}}(a_{n+1}).\end{aligned}$$

We have

$$\begin{aligned}&\sup_{i=1,\dots,n} \theta_{E_i}(a_i) \wedge \theta_{E_{n+1}}(a_{n+1}) \\ &\qquad \leq \sup_{i=1,\dots,n} \big(\theta_{E_i}(a_i) \wedge \theta_{E_{n+1}}(a_{n+1})\big) + \mathfrak{O}\Bigg(\sup_{i=1,\dots,n+1} \theta_{E_i}(a_i)\Bigg)\end{aligned}$$

by Proposition I.5.15(b), and for each $i = 1, \dots, n$

$$\theta_{E_i}(a_i) \wedge \theta_{E_{n+1}}(a_{n+1}) \leq \mathfrak{O}\left(\theta_{E_i}(a_i) \vee \theta_{E_{n+1}}(a_{n+1})\right) \leq \mathfrak{O}\left(\sup_{i=1,\dots,n+1} \theta_{E_i}(a_i)\right)$$

by (i). Thus Propositions I.5.10(c) and I.5.11 yield

$$\sum_{i=1}^{n+1} \theta_{E_i}(a_i) \leq \sup_{i=1,\dots,n+1} \theta_{E_i}(a_i)$$

as claimed. Next we shall verify that

$$\text{(iii)} \qquad \left(\sum_{i=1}^{n}\theta_{E_i}(a_i)\right)\vee\left(\sum_{i=1}^{n}\theta_{E_i}(b_i)\right)=\sum_{i=1}^{n}\theta_{E_i}(a_i\vee b_i)$$

holds for disjoint sets $E_i \in \mathfrak{R}$ and $a_i, b_i \in \mathcal{P}$. Indeed, let $E = \bigcup_{i=1}^{n} E_i$. Given $w \in \mathcal{W}$ there is $v \in \mathcal{V}$ such that $\theta_E(v) \le w$ and $\lambda \ge 0$ such that $0 \le a_i + \lambda v$ and $0 \le b_i + \lambda v$ for all $i = 1, \ldots, n$. Because the locally convex cone $\mathcal{P}$ is supposed to be quasi-full, there are $s_i, t_i \in \mathcal{P}$ such that $s_i, t_i \le \lambda v$ and $0 \le a_i + s_i$ and $0 \le b_i + t_i$. Then $s_i \vee 0$, $t_i \vee 0 \le \lambda v$ by our assumptions for a semi lattice cone. We set $s = \sum_{i=1}^{n}(s_i \vee 0) + (t_i \vee 0)$ and conclude that $0 \le s \le n\lambda v$ as well as $0 \le a_i + s$ and $0 \le b_i + s$ for all $i = 1, \ldots, n$. Using this and (ii) from above, we conclude that

$$\begin{aligned}
&\left(\sum_{i=1}^{n}\theta_{E_i}(a_i)\right)\vee\left(\sum_{i=1}^{n}\theta_{E_i}(b_i)\right)+\theta_E(s)\\
&\qquad\qquad= \left(\sum_{i=1}^{n}\theta_{E_i}(a_i+s)\right)\vee\left(\sum_{i=1}^{n}\theta_{E_i}(b_i+s)\right)\\
&\qquad\qquad= \sup_{i=1,\ldots,n}\theta_{E_i}(a_i+s)\vee\sup_{i=1,\ldots,n}\theta_{E_i}(b_i+s)\\
&\qquad\qquad= \sup_{i=1,\ldots,n}\theta_{E_i}(a_i+s)\vee\theta_{E_i}(b_i+s)\\
&\qquad\qquad= \sup_{i=1,\ldots,n}\theta_{E_i}\big((a_i+s)\vee(b_i+s)\big)\\
&\qquad\qquad= \sum_{i=1}^{n}\theta_{E_i}\big((a_i+s)\vee(b_i+s)\big)\\
&\qquad\qquad= \sum_{i=1}^{n}\theta_{E_i}(a_i\vee b_i)\;+\;\theta_E(s).
\end{aligned}$$

Considering that $\mathfrak{O}\big(\theta_E(s)\big) \le w$ and that $w \in \mathcal{W}$ was arbitrarily chosen, now the cancellation law from Proposition I.5.10(a) yields (iii). Note that (iii) implies a strengthening of the first requirement in (L): $\theta_E(a)\vee\theta_E(b)=\theta_E(a\vee b)$ holds for all $E \in \mathfrak{R}$ and all (not necessarily positive) elements $a, b \in \mathcal{P}$.

The supremum $f \vee g \in \mathcal{F}(X,\mathcal{P})$ of two functions $f, g \in \mathcal{F}(X,\mathcal{P})$ is canonically defined as the mapping $x \mapsto f(x) \vee g(x)$. If we take into account the continuity of the lattice operation in $\mathcal{P}$, then Theorem 1.4 yields immediately that the supremum of two measurable functions is again measurable, and consequently, a brief review of 2.3 confirms that the subcone $\mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ of $\mathcal{F}(X,\mathcal{P})$ is closed for suprema. As an immediate consequence of (iii) we infer that

$$\int_X (h \vee l)\, d\theta = \left(\int_X h\, d\theta\right) \vee \left(\int_X l\, d\theta\right)$$

holds for all step functions $h, l \in \mathcal{S}_{\mathfrak{R}}(X,\mathcal{P})$. Now let us denote by $\mathcal{S}_{\mathfrak{R}}^{\sigma}(X,\mathcal{P})$ the subcone of all functions $f \in \mathcal{F}_{\mathfrak{R}}(X,\mathcal{P})$ for which there exists a sequence

$(h_n)_{n\in\mathbb{N}}$ of step functions that is bounded below and bounded above relative to f and such that $h_n \longrightarrow f$. According to Corollary 5.26, this implies $\lim\limits_{n\to\infty} \int_X h_n\,d\theta = \int_X f\,d\theta$. Lemma 5.27 and Corollary 5.28 yield in particular that $\mathcal{E}_{\mathfrak{R}}(X,\mathcal{P})$, the subcone generated by all elementary functions, is contained in $\mathcal{S}^{\sigma}_{\mathfrak{R}}(X,\mathcal{P})$. We proceed to establish that the integral with respect to a measure satisfying (L) defines a $\vee$-semilattice homomorphism (see I.5.30) from $\mathcal{S}^{\sigma}_{\mathfrak{R}}(X,\mathcal{P})$ into $\mathcal{Q}$:

Let $f,g \in \mathcal{S}^{\sigma}_{\mathfrak{R}}(X,\mathcal{P})$, and let $(h_n)_{n\in\mathbb{B}}$ and $(l_n)_{n\in\mathbb{B}}$ be the corresponding sequences of step functions approaching f and g as required above. Because of the continuity (with respect to the symmetric relative topology) of the lattice operation in $\mathcal{P}$, this implies $h_n \vee l_n \longrightarrow f \vee g$, that is the sequence $(h_n \vee l_n)_{n\in\mathbb{N}}$ of step functions converges pointwise to the function $f \vee g \in \mathcal{F}(X,\mathcal{P})$. We shall proceed to verify that this sequence is bounded below and bounded above relative to $f \vee g$, hence the function $f\vee g$ is also contained in $\mathcal{S}^{\sigma}_{\mathfrak{R}}(X,\mathcal{P})$. Indeed, let $\mathfrak{v}$ be an inductive limit neighborhood for $\mathcal{F}(X,\mathcal{P})$. There are $\lambda,\rho,\sigma \geq 0$ such that all of the following hold true: $0 \leq f + \lambda\mathfrak{v}$, $0 \leq g + \lambda\mathfrak{v}$ $\big($see Lemma 2.4(a)$\big)$, as well as $0 \leq h_n + \lambda\mathfrak{v}$, $0 \leq l_n + \lambda\mathfrak{v}$, $h_n \leq \rho f + \lambda\mathfrak{v}$ and $l_n \leq \sigma g + \lambda\mathfrak{v}$ for all $n \in \mathbb{N}$. We may indeed assume that $\sigma = \rho$, since otherwise, for example if $\sigma < \rho$, we can suitably adjust

$$l_n \leq (\sigma g + \lambda\mathfrak{v}) + (\rho-\sigma)(g+\lambda\mathfrak{v}) = \rho g + \lambda' v.$$

Using this, we argue as follows: Firstly, the preceding conditions imply that $0 \leq h_n \vee l_n + \lambda\mathfrak{v}$ holds for all $n\in\mathbb{N}$. Secondly, there are $\overline{\mathcal{V}}$-valued functions $s_n, t_n \in \mathfrak{v}$ such that $h_n \leq \rho f + \lambda s_n$ and $l_n \leq \rho g + \lambda t_n$. Let $x \in X$. Because $\mathcal{P}$ is quasi-full, there are $0 \leq u_n, v_n \in \mathcal{P}$ such that $u_n \leq s_n(x)$, $v_n \leq t_n(x)$ and $h_n(x) \leq \rho f(x) + \lambda u_n$ and $l_n(x) \leq \rho g(x) + \lambda v_n$. Thus both $h_n(x), l_n(x) \leq \rho\big(f\vee g\big)(x) + \lambda(u_n+v_n)$ and therefore

$$\big(h_n \vee l_n\big)(x) \leq \rho\big(f\vee g\big)(x) + \lambda(u_n+v_n) \leq \rho\big(f\vee g\big)(x) + \lambda\big(s_n+t_n\big)(x).$$

This shows $h_n \vee l_n \leq \rho(f\vee g) + 2\lambda\mathfrak{v}$ for all $n\in\mathbb{N}$ and verifies our claim. We therefore have

$$\lim_{n\to\infty}\int_X h_n\,d\theta = \int_X f\,d\theta, \qquad \lim_{n\to\infty}\int_X l_n\,d\theta = \int_X g\,d\theta$$

and

$$\lim_{n\to\infty}\int_X h_n \vee l_n\,d\theta = \int_X f\vee g\,d\theta$$

by Corollary 5.26. As

$$\begin{aligned}\lim_{n\to\infty}\int_X h_n\vee l_n\,d\theta &= \lim_{n\to\infty}\left(\left(\int_X h_n\,d\theta\right)\vee\left(\int_X l_n\,d\theta\right)\right)\\ &= \lim_{n\to\infty}\left(\int_X h_n\,d\theta\right)\vee\lim_{n\to\infty}\left(\int_X l_n\,d\theta\right)\end{aligned}$$

by the above and by Proposition I.5.25(a), we conclude that

$$\int_X (f \vee g)\, d\theta = \left(\int_X f\, d\theta\right) \vee \left(\int_X g\, d\theta\right)$$

holds for all functions $f, g \in \mathcal{S}^{\sigma}_{\mathfrak{R}}(X, \mathcal{P})$, provided that the measure θ satisfies (L). In other words, the integral with respect to θ defines a $\vee$-semilattice homomorphism from $\mathcal{S}^{\sigma}_{\mathfrak{R}}(X, \mathcal{P})$ to $\mathcal{Q}$ in the sense of I.5.30. Because both functions $\chi_{F^{\otimes}} f$ and $\chi_{F^{\otimes}} g$ are elements of $\mathcal{S}^{\sigma}_{\mathfrak{R}}(X, \mathcal{P})$ whenever $f, g \in \mathcal{E}_{\mathfrak{R}}(X, \mathcal{P})$ and $F \in \mathfrak{A}_{\mathfrak{R}}$, and because $\int_F f\, d\theta = \int_X \chi_{F^{\otimes}} f\, d\theta$, this applies also to integrals over measurable subsets F of X.

We summarize:

Proposition 6.17. *Let $(\mathcal{P}, \mathcal{V})$ be a quasi-full locally convex $\vee$-semilattice cone and let θ be a bounded $\mathfrak{L}_{\mathbb{K}}(\mathcal{P}, \mathcal{Q}_0)$-valued measure such that $\theta_E(a) \vee \theta_E(b) = \theta_E(a \vee b)$ and $\theta_E(a) \vee \theta_G(b) = \theta_E(a) + \theta_G(b)$ for all $a, b \geq 0$ in $\mathcal{P}$ and disjoint sets $E, G \in \mathfrak{R}$ Then*

$$\int_X (f \vee g)\, d\theta = \left(\int_X f\, d\theta\right) \vee \left(\int_X g\, d\theta\right)$$

holds for all functions $f, g \in \mathcal{S}^{\sigma}_{\mathfrak{R}}(X, \mathcal{P})$.

The case that $\mathcal{Q} = \overline{\mathbb{R}}$. If $\mathcal{Q} = \overline{\mathbb{R}}$, that is if the values θ_E of the measure θ are elements of $\mathcal{P}^*$, then Condition (L) means that (i) all functionals θ_E are lattice homomorphisms and (ii) for disjoint sets $E, G \in \mathfrak{R}$ we have either $\theta_E = 0$ or $\theta_G = 0$.

Similar concepts and results could obviously developed for locally convex $\wedge$-semilattice cones as defined in Section I.5.1 and $\wedge$-semilattice homomorphisms (see I.5.30).

6.18 Cone-Valued Functions and Positive Real-Valued Measures. If $\mathcal{P}$ is a subcone of $\mathcal{Q}$, and if the topology induced onto $\mathcal{P}$ by the neighborhood system $\mathcal{W}$ of $\mathcal{Q}$ is equivalent to the topology induced by its given neighborhood system $\mathcal{V}$, then for every $\rho \in \mathbb{R}_+$ the mapping

$$a \mapsto \rho a : \mathcal{P} \to \mathcal{Q},$$

defines a continuous linear operator. Thus every $\mathbb{R}_+$-valued measure θ on $\mathfrak{R}$, that is

$$E \mapsto \theta_E : \mathfrak{R} \to \mathbb{R}_+$$

is an operator-valued measure in the sense of Section 3. In particular, σ-additivity in our sense follows from σ-additivity for the $\mathbb{R}_+$-valued measure θ in the usual sense using Proposition I.5.22. Indeed, let $E_i \in \mathfrak{R}$ be disjoint sets, $E = \bigcup_{i=1}^{\infty} E_i$ and set $F_n = \bigcup_{i=1}^{n} E_i$. Then $\theta_E = \lim_{n\to\infty} \theta_{F_n} \in \mathbb{R}_+$. For σ-additivity of θ as an $\mathfrak{L}(\mathcal{P}, \mathcal{Q})$-valued measure, we shall first consider the case that $\theta_E = 0$. Then $\theta_{F_n} = 0$ for all $n \in \mathbb{N}$, as this sequence is increasing. For any $a \in \mathcal{P}$ this means

$$\sum_{i=1}^{\infty} \theta_{E_i}(a) = \lim_{n\to\infty} \theta_{F_n}(a) = \theta_E(a) = 0.$$

Otherwise, Proposition I.5.22 yields

$$\sum_{i=1}^{\infty} \theta_{E_i}(a) = \lim_{n\to\infty} \theta_{F_n}(a) = \lim_{n\to\infty} \big(\theta_{F_n} a\big) = \Big(\lim_{n\to\infty} \theta_{F_n}\Big) a = \theta_E\, a = \theta_E(a)$$

as well. For $E \in \mathfrak{R}$ and $w \in \mathcal{W}$ there is $v \in \mathcal{V}$ such that $a \le b + v$ for $a, b \in \mathcal{P}$ implies $a \le b + w$. Then the modulus of the measure θ is given by

$$\begin{aligned} |\theta|(E, v) &= \sup\left\{ \sum_{i=1}^{n} \theta_{E_i}\, s_i \;\middle|\; s_i \le v,\ E_i \in \mathfrak{R} \text{ disjoint subsets of } E \right\} \\ &\le \sup\left\{ \sum_{i=1}^{n} \theta_{E_i} \;\middle|\; E_i \in \mathfrak{R} \text{ disjoint subsets of } E \right\} w \ \le\ \theta_E\, w. \end{aligned}$$

The $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measure θ is therefore bounded in the sense of Section 3.6 and can be extended to the full cone $(\mathcal{P}_{\mathcal{V}}, \mathcal{V})$ (see Section 3.8). In case that $(\mathcal{Q},\mathcal{W})$ is indeed the standard lattice completion of $(\mathcal{P},\mathcal{V})$ as introduced in I.5.57, then Corollary 5.9 (see also 6.5) yields that the integrals of integrable functions in $\mathcal{F}(X,\mathcal{P})$ are indeed elements of the second dual $\mathcal{P}^{**}$ of $\mathcal{P}$.

6.19 Vector-Valued Functions and Real- or Complex-Valued Measures. Let $(\mathcal{P},\mathcal{V})$ be a locally convex topological vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, endowed with a basis $\mathcal{V}$ of balanced convex neighborhoods, and let $(\mathcal{Q},\mathcal{W})$ be the standard lattice completion of $(\mathcal{P},\mathcal{V})$, as defined in Section 5.57 of Chapter I. Then the topology induced by $\mathcal{W}$ onto the embedding of $\mathcal{P}$ into $\mathcal{Q}$ is equivalent to the topology induced by its given neighborhood system $\mathcal{V}$ (see I.5.57). For each $\rho \in \mathbb{K}$ the mapping

$$a \mapsto \rho a : \mathcal{P} \to \mathcal{Q},$$

is therefore a continuous linear operator. Thus every $\mathbb{K}$-valued measure θ on $\mathfrak{R}$, that is

$$E \mapsto \theta_E : \mathfrak{R} \to \mathbb{K}$$

is an operator-valued measure in the sense of Section 3. For σ-additivity, let $E_i \in \mathfrak{R}$ be disjoint sets, $E = \bigcup_{i=1}^{\infty} E_i$ and set $F_n = \bigcup_{i=1}^{n} E_i$. Then $\theta_E = \lim_{n\to\infty} \theta_{F_n} \in \mathbb{K}$, and $\lim_{n\to\infty} \big(\theta_{F_n} a\big) = \big(\lim_{n\to\infty} \theta_{F_n}\big) a$ holds for all $a \in \mathcal{P}$, since $(\mathcal{P},\mathcal{V})$ is a topological vector space, hence the scalar multiplication is continuous. The $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measure θ is indeed strongly additive in the sense of 5.32 since for every decreasing sequence $(E_n)_{n\in\mathbb{N}}$ of sets in $\mathfrak{R}$

such that $\bigcap_{n\in\mathbb{N}} E_n = \emptyset$ and $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $|\theta_{E_n}| \leq \varepsilon$ for all $n \geq n_0$. Because for $a \in \mathcal{P}$ and $w \in \mathcal{W}$ there is $v \in \mathcal{V}$ and $\lambda \geq 0$ such that $a \leq \lambda v \leq \lambda w$, hence

$$\theta_{E_n}(a) = \theta_{E_n}\, a \leq \lambda w,$$

holds for all $n \geq n_0$. The latter follows since the neighborhoods in $\mathcal{V}$ are balanced and convex for $\mathcal{P}$. Recall from 6.4 that the total variation $\mathfrak{var}(\theta, E)$ of a real- or complex-valued measure θ on is always finite. For $E \in \mathfrak{R}$ and $w \in \mathcal{W}$ there is $v \in \mathcal{V}$ such that $a \leq b + v$, that is $a - b \in v$ for $a, b \in \mathcal{P}$ implies $a \leq b + w$. We have $\gamma s \in |\gamma| v$ for all $\gamma \in \mathbb{K}$ whenever $s \in v$ for $s \in \mathcal{P}$ and $v \in \mathcal{V}$. According to 6.12 the modulus of the $\mathfrak{L}(\mathcal{P}, \mathcal{Q})$-valued measure θ is therefore given by

$$\begin{aligned} |\theta|(E, v) &= \sup\left\{ \sum_{i=1}^{n} \theta_{E_i}\, s_i \;\middle|\; s_i \in v,\ E_i \in \mathfrak{R} \text{ disjoint subsets of } E \right\} \\ &\leq \sup\left\{ \sum_{i=1}^{n} |\theta_{E_i}| \;\middle|\; E_i \in \mathfrak{R} \text{ disjoint subsets of } E \right\} w \\ &= \mathfrak{var}(\theta, E)\ w. \end{aligned}$$

The $\mathfrak{L}(\mathcal{P}, \mathcal{Q})$-valued measure θ is therefore bounded in the sense of Section 3.6.

Integrability for $\mathcal{P}$-valued functions had been characterized in 6.12. Integrals of functions that satisfy Condition (BI 1) are elements of the second vector space dual $\mathcal{P}_{\mathbb{K}}^{**}$ of $\mathcal{P}$ $\big($see 6.12(i)$\big)$. According to 6.12(iv), the operator

$$f \mapsto \int_F f\, d\theta \,:\, \mathcal{F}_{(F,\theta,BI1)}(X, \mathcal{P}) \to \mathcal{P}_{\mathbb{K}}^{**}$$

is linear over $\mathbb{K}$. Integrals of functions that satisfy Condition (BI 2) from 6.12 are indeed elements of the closure with respect to the symmetric topology of $\mathcal{P}$ in $\mathcal{P}_{\mathbb{K}}^{**}$. In case of a topologically complete locally convex vector space $\mathcal{P}$, this closure coincides with $\mathcal{P}$.

Neighborhood-valued measurable functions are integrated using the canonical extension of the measure θ to the full cone $(\mathcal{P}_{\mathcal{V}}, \mathcal{V})$ as elaborated in Section 3.8. For a positive real-valued measurable function φ and a neighborhood $v \in \mathcal{V}$, for example, the function $\varphi_{\otimes} v$ is measurable, hence in $\mathcal{F}(X, \mathcal{P}_{\mathcal{V}})$. According to the above for every $F \in \mathfrak{R}$ its integral may be estimated as $\int_F \varphi_{\otimes} v\, d\theta \leq \big(\int_F \varphi\, d\,\mathfrak{var}(\theta)\big)\, w$, where $\mathfrak{var}(\theta)$ is the positive real-valued measure $E \mapsto \mathfrak{var}(\theta, E) : \mathfrak{R} \to \mathbb{R}$ and $w \in \mathcal{W}$ is a neighborhood such that $a \leq b + v$, that is $a - b \in v$ for $a, b \in \mathcal{P}$ implies $a \leq b + w$.

Because the locally convex complete lattice cone $(\mathcal{Q}, \mathcal{W})$ allows sufficiently many order continuous linear functionals, that is the order continuous lattice homomorphisms on $\mathcal{Q}$ support the separation property (see I.5.32 and I.5.57), the strong convergence statements of Theorem 5.36 apply to functions satisfying Condition (BI 2) from 6.12.

Let us consider the special case that $(\mathcal{P},\mathcal{V})$ is a normed space, that is $\mathcal{V}=\{\rho\mathbb{B}\mid\rho>0\}$, where $\mathbb{B}$ is the unit ball in $\mathcal{P}$. A vector-valued function $f\in\mathcal{F}(X,\mathcal{P})$ is called *Bochner (or Dunford and Schwartz) integrable* over a set $E\in\mathfrak{R}$ with respect to a scalar-valued measure θ (see for example III.2.17 in [55] or II.2 in [43]) if for every $\varepsilon>0$ there is a step function $h_\varepsilon\in\mathcal{S}_{\mathfrak{R}}(X,\mathcal{P})$ such that the mapping $x\mapsto\|f(x)-h_\varepsilon(x)\|$ is measurable and

$$\int_E \|f-h_\varepsilon\|\; d\,\mathfrak{var}(\theta)\le\varepsilon.$$

Indeed, if the $\mathcal{P}$-valued function f is Bochner integrable, then given $w\in\mathcal{W}$ there is $\varepsilon>0$ such that $a\le b+\varepsilon\mathbb{B}$, that is $\|a-b\|\le\varepsilon\mathbb{B}$ for $a,b\in\mathcal{P}$ implies $a\le b+w$. We set $h_{(\mathbb{B},w)}=h_\varepsilon\in\mathcal{S}_{\mathfrak{R}}(X,\mathcal{P})$ and compute

$$\int_E \|f-h_\varepsilon\|\circledast\mathbb{B}\; d\,\theta\le\int_E \|f-h_\varepsilon\|\; d\,\mathfrak{var}(\theta)\le w$$

by our preceding considerations, hence (BI 2) from 6.12 holds for f.

6.20 Operator-Valued Functions and Operator-Valued Measures. Let $\mathcal{N}$ and $\mathcal{H}$ be cones, and let $\mathfrak{Z}$ and $\mathfrak{Y}$ be families of subsets of $\mathcal{N}$ and of $\mathcal{H}$, directed upward by set inclusion. Furthermore, let $(\mathcal{M},\mathcal{U})$ and $(\mathcal{L},\mathcal{R})$ be two locally convex cones, and for the respective cones $L(\mathcal{N},\mathcal{M})$ and $L(\mathcal{H},\mathcal{L})$ of linear operators consider the neighborhoods $V_{(Z,u)}$ for $Z\in\mathfrak{Z}$ and $u\in\mathcal{U}$, and $W_{(Y,r)}$ for $Y\in\mathfrak{Y}$ and $r\in\mathcal{R}$ (see Section I.7); that is $S\le U+V_{(Z,u)}$ or $R\le T+W_{(Y,r)}$ for operators $S,U\in L(\mathcal{N},\mathcal{M})$ or $R,T\in L(\mathcal{H},\mathcal{L})$, respectively, if

$$S(z)\le U(z)+u\quad\text{for all }\ z\in Z,\qquad\text{or}\qquad R(y)\le T(y)+r\quad\text{for all }\ y\in Y.$$

Let $\mathfrak{H}(\mathcal{N},\mathcal{M})$ be a subcone of $L(\mathcal{N},\mathcal{M})$ such that all its elements are bounded below with respect to the neighborhoods $V_{(Z,u)}$ and such that the resulting locally convex cone $\big(\mathfrak{H}(\mathcal{N},\mathcal{M}),\mathfrak{V}\big)$ is quasi-full. Similarly, let $\mathfrak{H}(\mathcal{H},\mathcal{L})$ be a subcone of $L(\mathcal{H},\mathcal{L})$ whose elements are bounded below with respect to the neighborhoods $W_{(Y,r)}$ and denote the resulting locally convex cone by $\big(\mathfrak{H}(\mathcal{H},\mathcal{L}),\mathfrak{W}\big)$. Let $\big(\widehat{\mathfrak{H}}(\mathcal{H},\mathcal{L}),\widehat{\mathfrak{W}}\big)$ be a locally convex complete lattice cone containing the latter, for example its (simplified) standard lattice completion (see Sections I.5.57 and I.7). Now in the context of our general theory we may consider integrals for $\mathfrak{H}(\mathcal{N},\mathcal{M})$-valued functions with respect to bounded $\mathfrak{L}\big(\mathfrak{H}(\mathcal{N},\mathcal{M}),\,\widehat{\mathfrak{H}}(\mathcal{H},\mathcal{L})\big)$-valued measures.

This is indeed a rather unwieldy setting. It does however facilitate a considerably wider choice of applications for our theory, as we shall see in Sections 6.22 to 6.23 below. Moreover, note that this point of view generalizes our original one since the given cones $(\mathcal{P},\mathcal{V})$ and $(\mathcal{Q},\mathcal{W})$ may be considered as cones of linear operators from $\mathbb{R}_+$ to $\mathcal{P}$ or to $\mathcal{Q}$, respectively $\big($see Example I.7.1(c)$\big)$.

We shall study two useful special cases in further detail:

(i) *The case* $\mathcal{N} = \mathcal{H}$ *and* $\mathfrak{Z} = \mathfrak{Y}$. In this case every linear operator $T \in L(\mathcal{M},\mathcal{L})$ may be reinterpreted as a linear operator $\overline{T}$ from $\mathfrak{H}(\mathcal{N},\mathcal{M})$ into $L(\mathcal{N},\mathcal{L})$ mapping the operator $U \in \mathfrak{H}(\mathcal{N},\mathcal{M})$ into the operator $T\circ U \in L(\mathcal{N},\mathcal{L})$; that is

$$\big(T\circ U\big)(z) = T\big(U(z)\big) \in \mathcal{L} \qquad \text{for all} \qquad z \in \mathcal{N}.$$

In order to guarantee that the operator $T\circ U$ is bounded below with respect to the neighborhoods $W_{(Y,r)} \in \mathfrak{W}$, and that the operator

$$\overline{T} : \mathfrak{H}(\mathcal{N},\mathcal{M}) \to L(\mathcal{N},\mathcal{L})$$

is continuous with regard to the respective neighborhood systems for these cones, we shall require that T itself is continuous from $(\mathcal{M},\mathcal{U})$ into $(\mathcal{L},\mathcal{R})$, that is $T \in \mathfrak{L}(\mathcal{M},\mathcal{L})$. Indeed, for $Z \in \mathfrak{Z}$ and $r \in \mathcal{R}$ there is $u \in \mathcal{U}$ such that $T(a) \le T(b)+r$ whenever $a \le b+u$ for $a,b \in \mathcal{M}$. Then for operators $S,U \in \mathfrak{H}(\mathcal{N},\mathcal{M})$ such that $S \le U + V_{(\mathfrak{Z},u)}$ we have $S(z) \le U(z)+r$, hence $\big(T\circ S\big)(z) \le \big(T\circ U\big)(z)+r$ for all $z \in Z$. This shows $\overline{T}(S) \le \overline{T}(U)+W_{(Z,r)}$. Moreover, as for every $S \in \mathfrak{H}(\mathcal{N},\mathcal{M})$ we have $0 \le S + \lambda V_{(Z,u)}$ for some $\lambda \ge 0$, the above implies that $0 \le \overline{T}(S) + \lambda W_{(Z,r)}$.

In this way, an $\mathfrak{L}(\mathcal{M},\mathcal{L})$-valued measure θ on $\mathfrak{R}$ may be reinterpreted as an $\mathfrak{L}\big(\mathfrak{H}(\mathcal{N},\mathcal{M}),\, \widehat{\mathfrak{H}}(\mathcal{N},\mathcal{L})\big)$-valued measure, where $(\widehat{\mathfrak{H}}(\mathcal{N},\mathcal{L}),\mathcal{W})$ is a locally convex complete lattice cone containing all the operators $\theta_E \circ U$ for $E \in \mathfrak{R}$ and $U \in \mathfrak{H}(\mathcal{N},\mathcal{M})$. We are using the above identification of a continuous linear operator $T \in \mathfrak{L}(\mathcal{M},\mathcal{L})$ with a continuous linear operator $\overline{T} \in \mathfrak{L}\big(\mathfrak{H}(\mathcal{N},\mathcal{M}),\, \widehat{\mathfrak{H}}(\mathcal{N},\mathcal{L})\big)$. We proceed to calculate the modulus of such a measure: For $E \in \mathfrak{R}$ and $V_{(Z,u)} \in \mathfrak{V}$ we have

$$|\theta|\big(E, V_{(Z,u)}\big) = \sup\left\{ \sum_{i=1}^{n} \theta_{E_i} \circ S_i \;\middle|\; S_i \le V_{(Z,u)},\ E_i \in \mathfrak{R} \text{ disjoint subsets of } E \right\}.$$

The supremum on the right-hand side is taken in the locally convex complete lattice cone $\widehat{\mathfrak{H}}(\mathcal{N},\mathcal{L})$. For $\mathfrak{R}$-boundedness of this measure we require that for every $r \in \mathcal{R}$ and $Z \in \mathfrak{Z}$ there is $u \in \mathcal{U}$ such that $|\theta|\big(E, V_{(Z,u)}\big) \le W_{(Z,r)}$. Note that for $\mathcal{N} = \mathbb{R}_+$ and $\mathfrak{Z} = \big\{\{1\}\big\}$, that is for $\big(\mathfrak{H}(\mathcal{N},\mathcal{M}),\mathfrak{V}\big)$ and $\big(\mathfrak{H}(\mathcal{N},\mathcal{L}),\mathfrak{W}\big)$ being isomorphic to the given cones $(\mathcal{M},\mathcal{U})$ and $(\mathcal{L},\mathcal{R})$ we have $|\theta|\big(E, V_{(\{1\},u)}\big) = |\theta|(E,u)|$. Countable additivity for the $\mathfrak{L}\big(\mathfrak{H}(\mathcal{N},\mathcal{M}),\, \widehat{\mathfrak{H}}(\mathcal{N},\mathcal{L})\big)$-valued measure θ requires that for disjoint sets $E_i \in \mathfrak{R}$ for every $U \in \mathfrak{H}(\mathcal{N},\mathcal{M})$ the series

$$\theta_{(\bigcup_{i\in\mathbb{N}} E_i)} \circ U = \sum_{i=1}^{\infty} \big(\theta_{E_i} \circ U\big)$$

is order convergent in $\widehat{\mathfrak{H}}(\mathcal{N},\mathcal{L})$. In case that $\widehat{\mathfrak{H}}(\mathcal{N},\mathcal{L})$ is the simplified standard lattice completion of $\mathfrak{H}(\mathcal{N},\mathcal{L})$ as constructed in I.7.1, this means that for disjoint sets $E_i \in \mathfrak{R}$

$$\mu\Big(\theta_{(\bigcup_{i\in\mathbb{N}} E_i)}\big(U(a)\big)\Big) = \sum_{i=1}^{\infty} \mu\Big(\theta_{E_i}\big(U(a)\big)\Big)$$

holds for all $U \in \mathfrak{H}(\mathcal{N},\mathcal{M})$, $a \in \bigcup_{Z\in\mathfrak{Z}} Z$ and $\mu \in \mathcal{L}^*$. Also in this case, Corollary 5.9 yields together with Remark I.7.1 that integrals of $\big(\mathfrak{H}(\mathcal{N},\mathcal{M}),\mathfrak{V}\big)$-based integrable functions are indeed linear operators from $\mathcal{N}$ into $\mathcal{L}^{**}$, the second dual of $\mathcal{L}$.

(ii) *The case* $\mathcal{M}=\mathcal{L}$ *and* $\mathcal{U}=\mathcal{R}$. In this case every linear operator $T\in L(\mathcal{H},\mathcal{N})$ may be reinterpreted as a linear operator $\widetilde{T}$ from $\mathfrak{H}(\mathcal{N},\mathcal{M})$ into $L(\mathcal{H},\mathcal{M})$, mapping the operator $U \in \mathfrak{H}(\mathcal{N},\mathcal{M})$ into the operator $U\circ T \in L(\mathcal{H},\mathcal{M})$; that is

$$\big(U\circ T\big)(z) = U\big(T(z)\big) \in \mathcal{M} \qquad \text{for all} \qquad z\in\mathcal{H}.$$

In order to guarantee that the operator $U\circ T$ is bounded below with respect to the neighborhoods $W_{(Y,r)} \in \mathfrak{W}$, and that the operator

$$\widetilde{T} : \mathfrak{H}(\mathcal{N},\mathcal{M}) \to L(\mathcal{H},\mathcal{M})$$

is continuous with regard to the respective neighborhood systems, we shall require that for every $Y\in\mathfrak{Y}$ there is some $Z\in\mathfrak{Z}$ such that $f(Y)\subset Z$. Indeed, for $Y\in\mathfrak{Y}$ and $u\in\mathcal{U}$ let $Z\in\mathfrak{Z}$ such that $f(Y)\subset Z$. Then for operators $S,U\in\mathfrak{H}(\mathcal{N},\mathcal{M})$ such that $S\le U+V_{(\mathfrak{Z},u)}$ we have $S(z)\le U(z)+u$ for all $z\in Z$, hence $\big(S\circ T\big)(y)\le \big(U\circ T\big)(z)+u$ for all $y\in Y$. This shows $\widetilde{T}(S)\le\widetilde{T}(U)+W_{(Y,u)}$. Moreover, as for every $S\in\mathfrak{H}(\mathcal{N},\mathcal{M})$ we have $0\le S+\lambda V_{(Z,u)}$ for some $\lambda\ge 0$, the above implies that $0\le\overline{T}(S)+\lambda W_{(Y,u)}$.

In this way, an $L(\mathcal{H},\mathcal{N})$-valued measure θ satisfying the above requirement may be reinterpreted as an $\mathfrak{L}\big(\mathfrak{H}(\mathcal{N},\mathcal{M}),\,\widehat{\mathfrak{H}}(\mathcal{H},\mathcal{M})\big)$-valued measure, where $(\widehat{\mathfrak{H}}(\mathcal{H},\mathcal{M}),\widehat{\mathcal{W}})$ is a locally convex complete lattice cone containing all the operators $U\circ\theta_E$ for $E\in\mathfrak{R}$ and $U\in\mathfrak{H}(\mathcal{N},\mathcal{M})$, and using the above identification. The modulus of such a measure is calculated for $E\in\mathfrak{R}$ and $V_{(Z,u)}\in\mathfrak{V}$ as

$$|\theta|\big(E,V_{(Z,u)}\big)=\sup\left\{\sum_{i=1}^{n} S_i\circ\theta_{E_i}\;\middle|\; S_i\le V_{(Z,u)},\ E_i\in\mathfrak{R} \text{ disjoint subsets of } E\right\}.$$

The supremum on the right-hand side is taken in the locally convex complete lattice cone $\widehat{\mathfrak{H}}(\mathcal{H},\mathcal{M})$. For $\mathfrak{R}$-boundedness of this measure we require that for every $u\in\mathcal{U}$ and $Y\in\mathfrak{Y}$ there is $Z\in\mathfrak{Z}$ such that $|\theta|\big(E,V_{(Z,u)}\big)\le W_{(Y,u)}$. Countable additivity for the measure θ requires that for disjoint sets $E_i\in\mathfrak{R}$ for every $U\in\mathfrak{H}(\mathcal{N},\mathcal{M})$ the series

$$U \circ \theta_{(\bigcup_{i\in\mathbb{N}} E_i)} = \sum_{i=1}^{\infty} \big(U \circ \theta_{E_i}\big)$$

is order convergent in $\mathfrak{H}(\mathcal{H},\mathcal{M})$. In case that $\widehat{\mathfrak{H}}(\mathcal{H},\mathcal{M})$ is the simplified standard lattice completion of $\mathfrak{H}(\mathcal{H},\mathcal{M})$ as constructed in I.7.1, Corollary 5.9 and Remark I.7.1 yield that integrals of $\big(\mathfrak{H}(\mathcal{N},\mathcal{M}),\mathfrak{V}\big)$-based integrable functions are linear operators from $\mathcal{H}$ into $\mathcal{M}^{**}$, the second dual of $\mathcal{M}$. If both $\mathcal{H}$ and $\mathcal{M}$ are vector spaces over $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$, then these integrals are indeed $\mathbb{K}$-linear operators from $\mathcal{M}$ into the second vector space dual $\mathcal{M}^{**}_{\mathbb{K}}$ of $\mathcal{M}$ (see I.7.1).

6.21 Positive, Real or Complex-Valued Functions and Operator-Valued Measures. This is a special case for the preceding section. Let $(\mathcal{P},\mathcal{V})$ and $(\mathcal{Q},\mathcal{W})$ be locally convex cones, and let $\mathbb{K}=\mathbb{R}_+$, or $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$ if $\mathcal{P}$ and $\mathcal{Q}$ are indeed locally convex topological vector spaces over $\mathbb{R}$ or $\mathbb{C}$, respectively, endowed with their symmetric topologies. We choose $\mathcal{N}=\mathcal{M}=\mathcal{P}$ and $\mathfrak{H}(\mathcal{N},\mathcal{M})=\mathbb{K}$ in the setting of Section 6.20. Depending on the suitable choice for the family $\mathfrak{Z}$ of bounded below subsets of $\mathcal{P}$, the following upper neighborhoods for an element $\alpha\in\mathbb{K}$ will render $\mathbb{K}$ into a quasi-full locally convex $\big($see Example I.7.2(c)$\big)$: For $\mathbb{K}=\mathbb{R}_+$ the family of all $\mathbb{B}^u_\varepsilon(\alpha)=[0,\alpha+\varepsilon]$ for $\varepsilon>0$, or the single neighborhood and $\mathbb{B}^u_0(\alpha)=[0,\alpha]$, both yielding the natural order; for $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$ the Euclidean neighborhoods $\mathbb{B}_\varepsilon(\alpha)=\{\beta\in\mathbb{K}\mid|\beta-\alpha|\le\varepsilon\}$ with equality as the order on $\mathbb{K}$. In order to deal with these cases simultaneously, let us denote by $\mathbb{B}$ either $\mathbb{B}^u_1(0),\mathbb{B}^u_0(0)$ or $\mathbb{B}_1(\alpha)$, that is the respective unit neighborhoods of $0\in\mathbb{K}$, and let $\Gamma=\{0\},\Gamma=\{0,1\}$ or $\Gamma=\{\gamma\in\mathbb{K}\mid|\gamma|=1\}\}$ be the corresponding units spheres.

We set $\mathcal{L}=\mathcal{Q}$ and use the special case (i) in Section 6.20 in order to integrate $\mathbb{K}$-valued functions with respect to an $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measure. For $\widehat{\mathfrak{H}}(\mathcal{P},\mathcal{Q})$ we choose the simplified standard lattice completion of $\mathfrak{L}(\mathcal{P},\mathcal{Q})$. For $E\in\mathfrak{R}$ and the above neighborhoods we calculate the modulus of an $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measure θ as follows:

$$|\theta|(E,\mathbb{B}) = \sup\left\{\sum_{i=1}^{n}\gamma_i\theta_{E_i}\;\middle|\;\gamma_i\in\Gamma,\quad E_i\in\mathfrak{R}\text{ disjoint subsets of }E\right\}.$$

The supremum on the right-hand side of these expressions is taken in the locally convex complete lattice cone $\widehat{\mathfrak{H}}(\mathcal{P},\mathcal{Q})$, that is a cone of $\overline{\mathbb{R}}$-valued functions with the pointwise algebraic and lattice operations. For $\mathbb{K}=\mathbb{R}_+$ and $\mathbb{B}=\mathbb{B}^u_0$ we have of course $|\theta|(E,\mathbb{B})=0$. For the remaining cases boundedness of the $\mathfrak{L}\big(\mathbb{K},\mathfrak{L}(\mathcal{P},\mathcal{Q})\big)$-valued measure θ requires that for every $E\in\mathfrak{R}$, the modulus $|\theta|(E,\mathbb{B})$ is bounded in $\widehat{\mathfrak{H}}(\mathcal{P},\mathcal{Q})$ with respect to all neighborhoods $W_{(Z,w)}$ for $Z\in\mathfrak{Z}$ and $w\in\mathcal{W}$. Let us recall the construction in I.7 of the standard lattice completion $\widehat{\mathfrak{H}}(\mathcal{P},\mathcal{Q})$ of $\mathfrak{L}(\mathcal{P},\mathcal{Q})$ to understand this further: The elements of $\widehat{\mathfrak{H}}(\mathcal{P},\mathcal{Q})$ are $\overline{\mathbb{R}}$-valued functions on the set

$\Upsilon = \big(\bigcup_{Z\in\mathfrak{Z}} Z\big) \times \mathcal{Q}^*$. An element $\varphi \in \widehat{\mathfrak{H}}(\mathcal{P},\mathcal{Q})$, that is an $\overline{\mathbb{R}}$-valued function on Υ is bounded relative to a neighborhood $W_{(Z,w)}$ if there is $\lambda \geq 0$ such that $\varphi(a,\mu) \leq \lambda$ holds for all $a \in Z$ and $\mu \in w^\circ$. Thus for boundedness of the measure θ we require that for every choice of disjoint subsets $E_i \in \mathfrak{R}$ of E and $\gamma_i \subset \Gamma$ we have

$$\sum_{i=1}^{n} \mu\big(\gamma_i E_i(a)\big) = \sum_{i=1}^{n} \mathfrak{Re}(\gamma_i)\mu\big(E_i(a)\big) \leq \lambda$$

for all $a \in Z$ and $\mu \in w^\circ$; or equivalently, that for every $Z \in \mathfrak{Z}$ the subset

$$\left\{ \sum_{i=1}^{n} \theta_{E_i}(a) \;\middle|\; E_i \in \mathfrak{R} \text{ disjoint subsets of } E,\;\; a \in Z \right\}$$

is bounded above in $\mathcal{Q}$. Indeed, in case that $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, both $(\mathcal{P},\mathcal{V})$ and $(\mathcal{Q},\mathcal{W})$ are locally convex vector spaces, and we have $\gamma\mu \in w^\circ$ for all $\gamma \in \Gamma$ whenever $\mu \in w^\circ$ for $w \in \mathcal{W}$.

Recall that all sets $Z \in \mathfrak{Z}$ are required to be bounded below in $\mathcal{P}$. The choice of all these sets for $\mathfrak{Z}$ results in the uniform operator topology for $\mathfrak{L}(\mathcal{P},\mathcal{Q})$ $\big($see I,7.1(i)$\big)$. If the sets in $\mathfrak{Z}$ are also bounded above (as is indeed implied in the case that $\mathcal{P}$ is a locally convex vector space in its symmetric topology), then boundedness of θ as an $\mathfrak{L}\big(\mathbb{K},\mathfrak{L}(\mathcal{P},\mathcal{Q})\big)$-valued measure is already implied by its boundedness as an $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measure. Indeed, given $E \in \mathfrak{R}$ and $w \in \mathcal{W}$ there is $v \in \mathcal{V}$ such that $|\theta|(E,v) \leq w$ (see Sections 3.2 to 3.6). Then for every $Z \in \mathfrak{Z}$ there is $\lambda \geq 0$ such that $z \leq \lambda v$ for all $z \in \mathfrak{Z}$. This implies the above condition for the boundedness of θ.

If $\mathfrak{Z}$ consists of all finite subsets of $\mathcal{P}$, that is if we consider the strong operator topology for $\mathfrak{L}(\mathcal{P},\mathcal{Q})$ $\big($see I,7.1(ii)$\big)$, then boundedness is a much weaker condition for $\theta:$ For every $a \in \mathcal{P}$ the subset

$$\left\{ \sum_{i=1}^{n} \theta_{E_i}(a) \;\middle|\; E_i \in \mathfrak{R} \text{ disjoint subsets of } E \right\}$$

is required to be bounded above in $\mathcal{Q}$.

Countable additivity for the $\mathfrak{L}\big(\mathbb{K},\mathfrak{L}(\mathcal{P},\mathcal{Q})\big)$-valued measure θ demands that for disjoint sets $E_i \in \mathfrak{R}$

$$\mu\big(\theta_{(\bigcup_{i\in\mathbb{N}} E_i)}(a)\big) = \sum_{i=1}^{\infty} \mu\big(\theta_{E_i}(a)\big)$$

holds for all $a \in \bigcup_{Z\in\mathfrak{Z}} Z$ and $\mu \in \mathcal{Q}^*$.

Our notion of measurability for $\mathbb{K}$-valued functions coincides with the usual one (see also Examples 6.3 and 6.4). For $\mathbb{K} = \mathbb{R}_+$ all measurable $\mathbb{K}$-valued functions are in $\mathcal{F}(X,\mathbb{K})$, hence integrable. For $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ with the Euclidean topology, a measurable $\mathbb{K}$-valued function is in $\mathcal{F}(X,\mathbb{K})$ if on every set $E \in \mathfrak{R}$ it can be uniformly approximated by step functions. This implies of course strong integrability in the sense of 5.18. Because $\widehat{\mathfrak{H}}(\mathcal{P},\mathcal{Q})$ was supposed to be the simplified standard lattice completion of $\mathfrak{L}(\mathcal{P},\mathcal{Q})$, the integral to a function $\varphi \in \mathcal{F}(X,\mathbb{K})$ with respect to an $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measure over a set $E \in \mathfrak{R}$ is a linear operator from $\mathcal{P}$ into Q^{**}, contained in the closure of $\mathfrak{L}(\mathcal{P},\mathcal{Q})$ in $\widehat{\mathfrak{H}}(\mathcal{P},\mathcal{Q})$ with respect to the symmetric relative topology. Thus, if the cone $\mathfrak{L}(\mathcal{P},\mathcal{Q})$ is topologically complete with respect to this topology, then this integral is indeed an element of $\mathfrak{L}(\mathcal{P},\mathcal{Q})$.

Let us proceed to discuss the convergence theorems from Section 5: For the sake of simplicity we shall restrict ourselves to the case of a single measure, that is $\theta_n = \theta$ for all $n \in \mathbb{N}$ in Theorems 5.23 to 5.25: Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of integrable $\mathbb{K}$-valued functions that converges pointwise θ-almost everywhere on a set $F \in \mathfrak{A}_{\mathfrak{R}}$ to a function φ in the symmetric relative topology of $\mathbb{K}$. This is of course the usual (Euclidean) notion of convergence, except for the case of $\mathbb{K} = \mathbb{R}_+$ endowed with the neighborhood $\mathbb{B}_0^u$ which renders $0 \in \mathbb{R}_+$ into an isolated point $\big($see Example I.4.37(b)$\big)$. The boundedness conditions from Theorem 5.25 for the sequence $(\varphi_n)_{n\in\mathbb{N}}$ read somewhat differently for the different choices for $\mathbb{K}$: We set $\varphi_{**} = \varphi_* = 0$ in all cases. For $\mathbb{K} = \mathbb{R}_+$ we require that $\varphi_n \underset{a.e. F}{\leq} \varphi^*$ holds for all $n \in \mathbb{N}$ with some integrable function φ^*. For $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ with the Euclidean topology and the order we use an integrable positive-valued function φ^* and the function $f^* = \varphi^*{}_{\otimes}\mathbb{B}$ whose values are in the full cone $\mathbb{K}_{\mathcal{V}} = \{\alpha + \rho\mathbb{B} \mid \alpha \in \mathbb{K},\ \rho \geq 0\}$ to which Theorem 5.25 applies in this case. We therefore require that $|\varphi_n| \underset{a.e. F}{\leq} \varphi^*$ holds for all $n \in \mathbb{N}$ in this case. The assumptions of Theorem 5.25 are now satisfied. Let $T_n = \int_F \varphi_n \, d\theta$, $T = \int_F \varphi \, d\theta$ and $T^* = \int_F \varphi^* \, d\theta$, or $T^* = \int_F (\varphi^*{}_{\otimes}\mathbb{B}) \, d\theta$ in case $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. These integrals are in general elements of $\widehat{\mathfrak{H}}(\mathcal{P},\mathcal{Q})$. The conclusion of Theorem 5.25 now states that

$$T \leq \underline{\lim}_{n\to\infty} T_n \qquad \text{and} \qquad \overline{\lim}_{n\to\infty} T_n \leq T + \mathfrak{O}\,(T^*)$$

in $\widehat{\mathfrak{H}}(\mathcal{P},\mathcal{Q})$, that is

$$T(a,\mu) \leq \underline{\lim}_{n\to\infty} T_n(a,\mu)$$

for all $a \in \bigcup_{Z\in\mathfrak{Z}} Z$ and $\mu \in \mathcal{P}^*$, and indeed

$$T(a,\mu) = \lim_{n\to\infty} T_n(a,\mu)$$

whenever $T^*(a,\mu) < +\infty$. Note that for linear operators $T \in \mathfrak{L}(\mathcal{P},\mathcal{Q})$ as elements of $\widehat{\mathfrak{H}}(\mathcal{P},\mathcal{Q})$ we have $T(a,\mu) = \mu\big(T(a)\big)$.

Now let us investigate the additional assumptions of Theorem 5.36 which will lead to convergence of $(T_n)_{n\in\mathbb{N}}$ towards T in the symmetric topology of $\widehat{\mathfrak{H}}(\mathcal{P},\mathcal{Q})$: We require that $F = E$ is in $\mathfrak{R}$. Strong additivity of the $\mathfrak{L}\big(\mathbb{K},\mathfrak{L}(\mathcal{P},\mathcal{Q})\big)$-valued measure θ in the sense of 5.30 means that for every decreasing sequence $(E_n)_{n\in\mathbb{N}}$ of sets in $\mathfrak{R}$ such that $\bigcap_{n\in\mathbb{N}} E_n = \emptyset$, for

$Z \in \mathfrak{Z}$ and $w \in \mathcal{W}$ there is $n_0 \in \mathbb{N}$ such that

$$\theta_{E_n}(a) \leq \mathfrak{O}\big(\theta_{E_1}(a)\big) + w$$

holds for all $a \in Z$ and $n \geq n_0$. Recall that in case $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ we assume that both $\mathcal{P}$ and $\mathcal{Q}$ are locally convex vector spaces in their respective symmetric topologies, thus $\mathfrak{O}\big(\theta_{E_1}(a)\big) = 0$, and $\theta_{E_n}(a) \leq w$ implies that $\theta_{E_n}(\gamma a) \leq w$ for all $\gamma \in \Gamma$. The above therefore means that the sequence $(\theta_{E_n})_{n\in\mathbb{N}}$ of linear operators converges to 0 in the symmetric topology of $\big(\mathfrak{L}(\mathcal{P},\mathcal{Q}),\mathfrak{W}\big)$. We also need to require that the functions φ_n, φ and φ^* or $\varphi^*{}_{\otimes}\mathbb{B}$ are $(\mathbb{K},\mathfrak{V})$-based integrable in the sense of 5.6. Measurability in the classical sense and boundedness below almost everywhere on the set E is sufficient for this. This condition also yields strong integrability for the functions φ^* or $\varphi^*{}_{\otimes}\mathbb{B}$. Finally, according to 5.36 we require that the element $T^* = \int_E \varphi^*\, d\theta$ or $T^* = \int_E (\varphi^*{}_{\otimes}\mathbb{K})\, d\theta$ is bounded in $\widehat{\mathfrak{H}}(\mathcal{P},\mathcal{Q})$. Under these additional assumptions then Theorem 5.36 yields

$$T = \lim_{n\to\infty} T_n$$

in the symmetric topology of $\widehat{\mathfrak{H}}(\mathcal{P},\mathcal{Q})$. If as in most cases of interest the integrals T_n and T are actually elements of $\mathfrak{L}(\mathcal{P},\mathcal{Q})$, then we infer convergence in the symmetric operator topology of $(\mathfrak{L}(\mathcal{P},\mathcal{Q}),\mathfrak{W})$.

Operator algebras. If $\mathcal{H} = \mathcal{P} = \mathcal{Q}$ is a locally convex topological vector space, then the space of continuous linear operator $\mathfrak{L}(\mathcal{P},\mathcal{P})$ forms a topological algebra, endowed with the composition of operators as its multiplication (see 6.4). We integrate $\mathbb{K}$-valued functions with respect to an $\mathfrak{L}(\mathcal{P},\mathcal{P})$-valued measure θ in this case. The values of the integrals are contained in the simplified standard completion $\widehat{\mathfrak{H}}(\mathcal{P},\mathcal{P})$ of $\mathfrak{L}(\mathcal{P},\mathcal{P})$. For the integral to determine a multiplicative linear operator from $\mathcal{E}_{\mathfrak{R}}(X,\mathbb{K}) = \mathcal{F}_{\mathfrak{R}}(X,\mathbb{K})$ into $\widehat{\mathfrak{H}}(\mathcal{P},\mathcal{P})$ in the sense of Example 6.14 we need to require that the measure θ satisfies Condition (A), that is $\theta_E(a)\,\theta_E(b) = \theta_E(ab)$ and $\theta_E(a)\,\theta_G(b) = 0$ holds for all $a,b \in \mathbb{K}$ and disjoint sets $E,G \in \mathfrak{R}$. As $\theta_E(a) = a\theta_E$ in this case, Condition (A) reads as follows:

(A) $(\theta_E)^2 = \theta_E$ *and* $\theta_E\,\theta_G = 0$ *for disjoint sets* $E,G \in \mathfrak{R}$,

that is the operators $\theta_E \in \mathfrak{L}(\mathcal{P},\mathcal{P})$ are required to be idempotent and pairwise orthogonal for disjoint sets $E,G \in \mathfrak{R}$.

Spectral Measures. For a concrete example, let $\mathcal{H} = \mathcal{P} = \mathcal{Q}$ be a complex Hilbert space with unit ball $\mathbb{U}$ and the neighborhood system $\mathcal{V}=\{\rho\mathbb{U} \mid \rho>0\}$. Let $\mathfrak{R}$ be a weak σ-ring, and as in spectral theory, let θ be a projection-valued measure on $\mathfrak{R}$. We consider θ as an $\mathfrak{L}\big(\mathbb{C},\mathfrak{L}(\mathcal{H},\mathcal{H})\big)$-valued measure in the above sense. Such a measure is seen to be $\mathfrak{R}$-bounded, even if we choose the uniform operator topology for $\mathfrak{L}(\mathcal{H},\mathcal{H})$ $\big($see I.7.2(i)$\big)$, that is the family of all bounded subsets of $\mathcal{H}$ for $\mathfrak{Z}$. Indeed, let $a \in \mathcal{H}$ such that

$\|a\| \le 1$ and let $E_i \in \mathfrak{R}$, for $i = 1, \ldots, n$ be disjoint sets. For a spectral measure the θ_{E_i} are projections onto mutually orthogonal subspaces of $\mathcal{P}$. Thus the elements $a_i = \theta_{E_i}(a)$ are orthogonal and $\sum_{i=1}^{n} \|a_i\|^2 \le \|a\|^2 = 1$ by the Bessel inequality $\big($see Theorem 1 in I.5 of [82]$\big)$. Thus

$$\Big\| \sum_{i=1}^{n} \theta_{E_i}(a) \Big\|^2 = \Big\| \sum_{i=1}^{n} a_i \Big\|^2 = \sum_{i=1}^{n} \|a_i\|^2 \le 1.$$

The set

$$\left\{ \sum_{i=1}^{n} \theta_{E_i}(a) \;\middle|\; E_i \in \mathfrak{R} \text{ disjoint subsets of } E, \quad a \in \mathcal{H}, \ \|a\| \le 1 \right\}$$

is therefore indeed bounded above in $\mathcal{H}$. Countable additivity for a spectral measure is however required only with respect to the strong operator topology for $\mathfrak{L}(\mathcal{H},\mathcal{H})$, which arises if we choose the family of all finite subsets of $\mathcal{H}$ for $\mathfrak{Z}$. (Because projection operators in $\mathfrak{L}(\mathcal{H},\mathcal{H})$ are of norm 1, countable additivity with respect to the uniform operator topology can of course only apply to finite sums of such operators.) Theorem 5.36 therefore yields convergence in the strong but not in the uniform operator topology of $\mathfrak{L}(\mathcal{H},\mathcal{H})$ for spectral measures.

Spectral measures satisfy Condition (A) from above and also Condition (A*) from 6.14, that is $\theta_E(a^*) = \big(\theta_E(a)\big)^*$ for all $E \in \mathfrak{R}$ and $a \in \mathcal{P}$. As $\theta_E(a^*) = \bar{a}\theta_E$ and $\big(\theta_E(a)\big)^* = \big(a\theta_E\big)^* = \bar{a}\big(\theta_E\big)^*$ in this case, this is equivalent to

(A*) $\theta_E = \big(\theta_E\big)^*$ *for all* $E \in \mathfrak{R}$.

This condition holds because the projection operators $\theta_E \in \mathfrak{L}(\mathcal{H},\mathcal{H})$ are self-adjoint. The linear operator $f \mapsto \int_X f\,d\theta$ is therefore multiplicative on the space $\mathcal{F}_{\mathfrak{R}}(X,\mathbb{C})$ of bounded measurable $\mathbb{K}$-valued functions and preserves the involution, that is $\int_X f^*\,d\theta = \big(\int_X f\,d\theta\big)^*$.

6.22 Operator-Valued Functions and Cone-Valued Measures. This is again a special case of 6.20. Let $\mathcal{P}$ be a cone, $(\mathcal{Q},\mathcal{W})$ a locally convex complete lattice cone. We choose $\mathcal{N} = \mathcal{P}$, $\mathcal{M} = \mathcal{L} = \mathcal{Q}$ and $\mathcal{H} = \mathbb{R}_+$ in the setting of 6.20 and use the special case (ii). For $\mathfrak{Z}$ we choose a family of subsets of $\mathcal{P}$, directed upward by set inclusion such that $\bigcup_{Z\in\mathfrak{Z}} Z = \mathcal{P}$, and suppose that the locally convex cone $(\mathfrak{H}(\mathcal{P},\mathcal{Q}),\mathfrak{V})$ of linear operators from $\mathcal{P}$ into $\mathcal{Q}$ is quasi-full. Let $\mathfrak{Y}$ consist of the singleton subset $\{1\}$ of $\mathbb{R}_+$. Then the locally convex cone $\big(L(\mathbb{R}_+,\mathcal{Q}),\mathfrak{W}\big)$ is isomorphic to $(\mathcal{Q},\mathcal{W})$ $\big($see Example I.7.2(d)$\big)$, hence a locally convex complete lattice cone. Similarly, because the cone $\mathcal{P}$ can be identified with the cone $L(\mathbb{R}_+,\mathcal{P})$, we may consider the elements of $\mathcal{P}$ to be linear operators from some quasi-full cone $\mathfrak{H}(\mathcal{P},\mathcal{Q})$ into $L(\mathbb{R}_+,\mathcal{Q})$, that is into $\mathcal{Q}$. Our choice for the families $\mathfrak{Z}$ and $\mathfrak{Y}$ guarantees that these operators are continuous $\big($see 6.20 (ii)$\big)$. Using these settings, case (ii) from 6.20 therefore permits us to consider $\mathfrak{H}(\mathcal{P},\mathcal{Q})$-valued functions together with $\mathcal{P}$-valued measures. Countable additivity requires for a $\mathcal{P}$-valued measure θ that for disjoint sets $E_i \in \mathfrak{R}$ and for every linear operator $T \in \mathfrak{H}(\mathcal{P},\mathcal{Q})$ the series

$$T\Big(\theta_{(\bigcup_{i\in\mathbb{N}} E_i)}\Big) = \sum_{i=1}^{\infty} T\big(\theta_{E_i}\big)$$

is order convergent in $\mathcal{Q}$. The modulus of θ is calculated for $E \in \mathfrak{R}$ and $V_{(Z,w)} \in \mathfrak{V}$ as

$$\begin{aligned}|\theta|\big(E, V_{(Z,w)}\big)\\ = \sup\left\{\sum_{i=1}^{n} T_i\big(\theta_{E_i}\big) \;\middle|\; T_i \le V_{(Z,w)},\ E_i \in \mathfrak{R} \text{ disjoint subsets of } E\right\}.\end{aligned}$$

$\mathfrak{R}$-boundedness in the sense of Section 3.6 requires that for every $E \in \mathfrak{R}$ and $u \in \mathcal{W}$ there is $V_{(Z,w)} \in \mathfrak{V}$ such that $|\theta|\big(E, V_{(Z,w)}\big) \le u$. A bounded $\mathcal{P}$-valued measure then integrates $\mathfrak{H}(\mathcal{P},\mathcal{Q})$-valued functions, and the values of these integrals are elements of $L(\mathbb{R}_+,\mathcal{Q})$, that is $\mathcal{Q}$ itself. If $(\mathcal{Q},\mathcal{W})$ is indeed the standard lattice completion of some locally convex cone $\mathcal{Q}_0$ and if the concerned function is $\big(\mathfrak{H}(\mathcal{P},\mathcal{Q}_0)\big)$-based integrable, then its integral is an element of the subcone $\mathcal{Q}_0^{**}$ of $\mathcal{Q}$.

Let us further consider the special case that $(\mathcal{P},\mathcal{V})$ is a locally convex vector space, that $(\mathcal{Q},\mathcal{W})$ is the standard lattice completion of a locally convex vector space $(\mathcal{Q}_0,\mathcal{W}_0)$ and that $\mathfrak{H}(\mathcal{P},\mathcal{Q}) \subset \mathfrak{L}(\mathcal{P},\mathcal{Q}_0)$. Then countable additivity of an $\mathcal{P}$-valued measure θ is guaranteed by weak convergence of the concerned series $\sum_{i=1}^{n}\theta_{E_i}$ in $\mathcal{P}$ in this case. Indeed, weak convergence in $\mathcal{P}$ implies weak convergence in $\mathcal{Q}$ for the series $\sum_{i=1}^{n}\theta_{E_i}(T) = \sum_{i=1}^{n} T(\theta_{E_i})$ for every operator $T \in \mathfrak{L}(\mathcal{P},\mathcal{Q}_0)$. (see IV.2.1 in [185]). Weak convergence in $\mathcal{Q}_0$, however, coincides with order convergence in $\mathcal{Q}$ in this case (see I.5.57) as required for countable additivity. Moreover, Theorem 3.11 (or Corollary 3.13), that is our version of Pettis' theorem yields that for a vector-valued measure, countable additivity with respect to weak convergence implies countable additivity with respect to strong convergence, that is convergence in the symmetric topology of $\mathcal{P}$. Every such measure is therefore strongly additive in the sense of 5.30.

6.23 Positive, Real or Complex-Valued Functions and Cone- or Vector-Valued Measures. This is a special case for the preceding section. Let $(\mathcal{P},\mathcal{V})$ be a locally convex cone and let and $(\mathcal{Q},\mathcal{W})$ be its standard lattice completion, Let $\mathbb{K} = \mathbb{R}_+$, or $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ if $\mathcal{P}$ is indeed a locally convex vector space over $\mathbb{R}$ or $\mathbb{C}$, respectively, endowed with its symmetric topology. We choose $\mathfrak{H}(\mathcal{P},\mathcal{Q}) = \mathbb{K}$ endowed with one of the suitable topologies arising from the choice for the family $\mathfrak{Z}$ of bounded below subsets of $\mathcal{P}$ (see Example I.7.2(c) and 6.21 above), that is topologies generated by the neighborhoods $\mathbb{B}$ as discussed for the respective cases in 6.21. We shall also use the notation for the unit sphere Γ from 6.21. Using this, the modulus of a $\mathcal{P}$-valued measure θ is given by

$$|\theta|(E,\mathbb{B}) = \sup\left\{\sum_{i=1}^{n}\gamma_i\theta_{E_i}\;\middle|\;\gamma_i\in\Gamma,\quad E_i\in\mathfrak{R}\text{ disjoint subsets of }E\right\}.$$

The supremum on the right-hand side of this expression is taken in the locally convex complete lattice cone $\mathcal{Q}$, that is a cone of $\overline{\mathbb{R}}$-valued functions with the pointwise algebraic and lattice operations. Boundedness is of course guaranteed in the case of $\mathbb{B}=\mathbb{B}_0^u$. In the remaining cases it requires (see the corresponding detailed argument in 6.21) that the set

$$\left\{\sum_{i=1}^{n}\theta_{E_i}\;\middle|\;E_i\in\mathfrak{R}\text{ disjoint subsets of }E\right\}$$

is bounded in $\mathcal{P}$. We will be able to verify that every $\mathcal{P}$-valued measure θ is $\mathfrak{R}$-bounded in this instance. For this call to mind that the elements $\theta_E\in\mathcal{P}$, for all $E\in\mathfrak{R}$, are considered to be continuous linear operators from $\mathbb{K}$ into $\mathcal{N}$, thus are required to be bounded elements of $\mathcal{P}$. Furthermore, recall from I.5.57 that the neighborhood system $\mathcal{V}$ for $\mathcal{P}$ is a generating subset of the neighborhood system $\mathcal{W}$ for the standard lattice completion $\mathcal{Q}$ of $\mathcal{P}$. For $E\in\mathfrak{R}$ let us consider the subset

$$A=\{\theta_{E'}\mid E'\in\mathfrak{R},\ E'\subset E\}$$

of $\mathcal{P}$. We shall use Proposition I.4.25 (which is derived from the Uniform Boundedness Theorem 3.4 in [172]) in order to verify that A is bounded above in $\mathcal{P}$. For this, let $\mu\in\mathcal{P}^*$. Because the elements of $\mathcal{P}^*$ are also order continuous linear functionals on the standard lattice completion $\mathcal{Q}$ of $\mathcal{P}$, we know from 3.9 that $\mu\circ\theta$ is an $\mathfrak{L}(\mathbb{K},\overline{\mathbb{R}})$-valued, that is an $\overline{\mathbb{R}}$-valued measure on $\mathfrak{R}$. This measure is indeed real-valued, since the elements $\theta_{E'}$, for all $E'\in\mathfrak{R}$, were seen to be bounded elements of $\mathcal{P}$. A countably additive real-valued measure is however known to be bounded, that is

$$\big\{\mu(a)\mid a\in A\big\}=\big\{(\mu\circ\theta)_{E'}\mid E'\in\mathfrak{R},\ E'\subset E\big\}$$

is a bounded subset of $\mathbb{R}$. Because this holds true for all linear functionals $\mu\in\mathcal{P}^*$, Proposition I.4.25 yields that the set A is bounded above relative to $0\in\mathcal{P}$, that is bounded above, as claimed. Thus, given $v\in\mathcal{V}$, there is indeed $\lambda\geq 0$ such that $\theta_{E'}\leq\lambda v$ holds for all subsets $E'\in\mathfrak{R}$ of E. We claim that this implies $|\theta|(E,V)\leq 4\lambda v$. For this let us recall the construction of the standard lattice completion $(\mathcal{Q},\mathcal{W})$ of $(\mathcal{P},\mathcal{V})$. Its elements are $\overline{\mathbb{R}}$-valued functions φ on the dual $\mathcal{P}^*$ of $\mathcal{P}$, and we have $\varphi\leq v$ if $\varphi(\mu)\leq 1$ for all $\mu\in v^\circ$. For any such $\mu\in v^\circ$, $\mu\circ\theta$ was seen to be a real-valued countably additive measure on $\mathfrak{R}$. As $(\mu\circ\theta)_{E'} \le \lambda$ for all subsets $E' \in \mathfrak{R}$ of E, we know that is total variation on E, that is $\mathfrak{var}\big(\mu\circ\theta,\, E\big)$ is bounded by the constant 4λ (see 6.4). Thus

$$\big|\theta\big|\big(E,V\big)(\mu) = \mathfrak{var}\big(\mu\circ\theta,\, E\big) \le 4\lambda$$

for all $\mu \in v^\circ$. This demonstrates $|\theta|(E,V) \le 4\lambda v$, as claimed.

Integrals of $\mathbb{K}$-valued functions with respect to a $\mathcal{P}$-valued measure θ were seen to be elements of $\mathcal{P}^{**}$. If $(\mathcal{P},\mathcal{V})$ is indeed a locally convex vector space that is complete in its symmetric topology and if as required in some integrability conditions in the literature (see for example IV.10.7 in [55]) the $\mathbb{K}$-valued function φ can be approximated by a sequence of step functions converging pointwise almost everywhere towards φ and such that the sequence of integrals over these step functions is convergent in $\mathcal{P}$, then this additional requirement guarantees that the value of the integral of φ is also contained in $\mathcal{P}$ rather than in $\mathcal{P}^{**}$.

Let us discuss the convergence theorems from Section 5: Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of integrable $\mathbb{K}$-valued functions that converges pointwise θ-almost everywhere on a set $F \in \mathfrak{A}_{\mathfrak{R}}$ to a function φ in the symmetric relative topology of $\mathbb{K}$. This is the usual notion of convergence, except for the case of $\mathbb{K} = \mathbb{R}_+$ endowed with the neighborhood $\mathbb{B}_0^u$ which renders $0 \in \mathbb{R}_+$ as an isolated point. The boundedness conditions from Theorem 5.25 are as follows: We set $\varphi_{**} = \varphi_* = 0$ in all cases. For $\mathbb{K} = \mathbb{R}_+$ we require that $\varphi_n \underset{a.e.F}{\le} \varphi^*$ holds for all $n \in \mathbb{N}$ with some integrable function φ^*. For $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ with the Euclidean topology and the order we use an integrable positive-valued function φ^* and the function $f^* = \varphi^* {}_{\otimes} \mathbb{B}$ whose values are in the full cone $\mathbb{K}_{\mathcal{V}} = \{\alpha + \rho\mathbb{B} \mid \alpha \in \mathbb{K},\ \rho \ge 0\}$ to which Theorem 5.25 applies. We require that $|\varphi_n| \underset{a.e.F}{\le} \varphi^*$ holds for all $n \in \mathbb{N}$ in this case. The assumptions of Theorem 5.25 are now satisfied. Let $a_n = \int_F \varphi_n\, d\theta$, $a = \int_F \varphi\, d\theta$ and $a^* = \int_F \varphi^*\, d\theta$, or $a^* = \int_F (\varphi^* {}_{\otimes} \mathbb{K})\, d\theta$ in case $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. These integrals are in general elements of $\mathcal{P}^{**}$. The conclusion of Theorem 5.25 now states that

$$a \le \varliminf_{n\to\infty} a_n \qquad \text{and} \qquad \varlimsup_{n\to\infty} a_n \le a + \mathfrak{O}\,(a^*)$$

in $\mathcal{P}^{**}$, that is

$$a(\mu) \le \varliminf_{n\to\infty} a_n(\mu)$$

for all $\mu \in \mathcal{P}^*$, and indeed

$$a(\mu) = \lim_{n\to\infty} a_n(\mu)$$

whenever $a^*(\mu) < +\infty$. For elements $a \in \mathcal{P} \subset \mathcal{P}^{**}$ we have $a(\mu) = \mu(a)$. If $(\mathcal{P},\mathcal{V})$ is indeed a locally convex topological vector space and if $F = E \in \mathfrak{R}$, then the assumptions of Theorem 5.36 apply: The measure θ is strongly additive by Theorem 3.11. Measurability in the classical sense and boundedness below almost everywhere on the set E is sufficient for the functions φ_n, φ and $\varphi^*{}_{\otimes}\mathbb{B}$ to be $(\mathbb{K}, \mathfrak{V})$-based integrable in the sense of 5.6. The latter is indeed strongly integrable in the sense of 5.18. All integrals involved are elements of $\mathcal{P}$ and Theorem 5.36 yields

$$a = \lim_{n\to\infty} a_n$$

in the symmetric, that is the given topology of $\mathcal{P}$.

Algebra-valued measures. If $\mathcal{P}$ is a topological algebra, that is a locally convex topological vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ with a compatible multiplication, then Conditions (A) and (A*) from 6.14 read as follows:

(A) $(\theta_E)^2 = \theta_E$ *and* $\theta_E\,\theta_G = 0$ *for disjoint sets* $E, G \in \mathfrak{R}$.
(A*) $(\theta_E)^* = \theta_E$ *for all* $E \in \mathfrak{R}$.

According to 6.14, Condition (A) guarantees the multiplicativity of the integral as an operator from $\mathcal{E}_{\mathfrak{R}}(X, \mathbb{K}) = \mathcal{F}_{\mathfrak{R}}(X, \mathbb{K})$ into $\mathcal{P}^{**}$, that is $\int_X (fg)\, d\theta \in \left(\int_X f\, d\theta\right) \cdot \left(\int_X g\, d\theta\right)$, Condition (A*) the compatibility with an involution, that is $\int_X f^*\, d\theta = \left(\int_X f\, d\theta\right)^*$ for $\mathbb{K}$-valued functions $f, g \in \mathcal{F}_{\mathfrak{R}}(X, \mathbb{K})$.

Lattice-valued measures. Now suppose that $\mathcal{P}$ is a lattice cone over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{R}_+$ in the sense of 6.16, that is a quasi-full locally convex cone containing suprema for any two of its elements and satisfying the properties specified in 6.16. For the integral to determine a $\vee$-semilattice homomorphism from $\mathcal{E}_{\mathfrak{R}}(X, \mathbb{K}) = \mathcal{F}_{\mathfrak{R}}(X, \mathbb{K})$ into $\mathcal{P}^{**}$ in the sense of 6.16 we need to require that the measure θ satisfies Condition (L), that is $\theta_E(a) \vee \theta_E(b) = \theta_E(a \vee b)$ and $\theta_E(a) \vee \theta_G(b) = \theta_E(a) + \theta_G(b)$ holds for all $0 \le a, b \in \mathbb{K}$ and disjoint sets $E, G \in \mathfrak{R}$. As $\theta_E(a) = a\theta_E$ in this case, Condition (L) reads as follows:

(L) $\theta_E \ge 0$ *and* $\theta_E \vee \theta_G = \theta_E + \theta_G$ *for disjoint sets* $E, G \in \mathfrak{R}$,

that is the elements $\theta_E \in \mathcal{P}$ are positive and mutually disjoint for disjoint sets $E, G \in \mathfrak{R}$. According to 6.16, Condition (L) guarantees that the integral as an operator from $\mathcal{S}^{\sigma}_{\mathfrak{R}}(X, \mathbb{K}) = \mathcal{F}_{\mathfrak{R}}(X, \mathbb{K})$ into $\mathcal{P}^{**}$ is a $\vee$-semilattice homomorphism, that is $\int_X (f \vee g)\, d\theta = \left(\int_X f\, d\theta\right) \vee \left(\int_X g\, d\theta\right)$ for functions $f, g \in \mathcal{F}_{\mathfrak{R}}(X, \mathbb{K})$.

6.24 Positive Linear Operators on Cones of $\overline{\mathbb{R}}$-Valued Functions. Let $\mathcal{P} = \overline{\mathbb{R}}$, let X and $\mathfrak{R}$ be as before, and let $\mathcal{W}$ be a neighborhood system for $\mathcal{F}(X, \overline{\mathbb{R}})$, consisting of non-negative functions $w \in \mathcal{F}(X, \overline{\mathbb{R}})$. Let $\mathcal{Q} = \mathcal{F}_{\mathcal{W}}(X, \overline{\mathbb{R}})$ be the subcone of functions in $\mathcal{F}(X, \overline{\mathbb{R}})$ that are bounded below with respect to $\mathcal{W}$. Then $\left(\mathcal{F}_{\mathcal{W}}(X, \overline{\mathbb{R}}), \mathcal{W}\right)$ is a full locally convex complete lattice cone, provided that for every $x \in X$ there is $w \in \mathcal{W}$ such that $w(x) < +\infty$ (see Example I.5.7(c)). There are two distinct types of continuous linear operators from $\overline{\mathbb{R}}$ into $\mathcal{F}_{\mathcal{W}}(X, \overline{\mathbb{R}})$. Firstly, for a non-negative real-valued function φ such that both $\varphi, -\varphi \in \mathcal{F}_{\mathcal{W}}(X, \overline{\mathbb{R}})$, let $T_{\varphi}(a) = a\varphi$

for $a \in \overline{\mathbb{R}}$. (In particular, this means $T_\varphi(+\infty)(x) = +\infty$ for all $x \in X$ such that $\varphi(x) > 0$ and $T_\varphi(+\infty)(x) = 0$ else.) Secondly, for a function $\psi \in \mathcal{F}_{\mathcal{W}}(X, \overline{\mathbb{R}})$ that takes only the values 0 and $+\infty$, set $T^0_\psi(a) = 0$ for $a \in \mathbb{R}$ and $T^0_\psi(+\infty) = \psi$. Then every linear operator $T \in \mathfrak{L}\big(\overline{\mathbb{R}}, \mathcal{F}_{\mathcal{W}}(X, \overline{\mathbb{R}})\big)$ can be expressed as $T = T_\varphi + T^0_\psi$ with some $\varphi, \psi \in \mathcal{F}_{\mathcal{W}}(X, \overline{\mathbb{R}})$ as above. Consequently, an $\mathfrak{L}\big(\overline{\mathbb{R}}, \mathcal{F}_{\mathcal{W}}(X, \overline{\mathbb{R}})\big)$-valued measure θ on $\mathfrak{R}$ can be expressed as a sum of two $\mathcal{F}_{\mathcal{W}}(X, \overline{\mathbb{R}})$-valued measures θ^1 and θ^0, both yielding functions in $\mathcal{F}_{\mathcal{W}}(X, \overline{\mathbb{R}})$, and such that for each $E \in \mathfrak{R}$ the function θ^1_E is positive and both $\theta^1_E, -\theta^1_E \in \mathcal{F}_{\mathcal{W}}(X, \overline{\mathbb{R}})$, and the function θ^0_E takes only the values 0 and $+\infty$. For a step function

$$h = \sum_{i=1}^{n} \chi_{E_i} a_i \;\in\; \mathcal{S}_{\mathfrak{R}}(X, \overline{\mathbb{R}}),$$

where $a_1, \ldots, a_n \in \overline{\mathbb{R}}$, we have in particular

$$\int_X h\, d\theta = \sum_{i=1}^{n} a_i \theta^1_{E_i} + \sum_{\substack{i=1,\ldots,n \\ \text{s.th.} a_i = +\infty}} \theta^0_{E_i} \;\in\; \mathcal{F}_{\mathcal{W}}(X, \overline{\mathbb{R}}).$$

On $\mathcal{F}_{\mathfrak{R}}(X, \overline{\mathbb{R}})$, the mapping

$$f \mapsto \int_X f\, d\theta \;:\; \mathcal{F}_{\mathfrak{R}}(X, \overline{\mathbb{R}}) \to \mathcal{F}_{\mathcal{W}}(X, \overline{\mathbb{R}})$$

defines a linear operator, continuous with respect to the locally convex cone topologies induced by the neighborhood system $\mathcal{W}$, that is

$$\int_X f\, d\theta \le \int_X g\, d\theta + w \qquad \text{whenever} \qquad f \le g + \mathfrak{v}_w,$$

for $f, g \in \mathcal{F}_{\mathfrak{R}}(X, \overline{\mathbb{R}})$. Recall from Section 4 that $\mathfrak{v}_w$ consists of all step functions $s = \sum_{i=1}^n \chi_{E_i} \otimes a_i$ for $0 < a_i \in \mathbb{R}$ such that $\int_X s\, d\theta = \sum_{i=1}^n a_i \theta^1_{E_i} \le w$.

According to 6.17, the linear operator determined by the integral is indeed a $\vee$-semilattice homomorphism, if Condition (L) holds, that is if $\theta_E(a) \vee \theta_E(b) = \theta_E(a \vee b)$ and $\theta_E(a) \vee \theta_G(b) = \theta_E(a) + \theta_G(b)$ for all $a, b \ge 0$ in $\overline{\mathbb{R}}$ and disjoint sets $E, G \in \mathfrak{R}$. The first part of this condition holds always true for an $\mathfrak{L}\big(\overline{\mathbb{R}}, \mathcal{F}_{\mathcal{W}}(X, \overline{\mathbb{R}})\big)$-valued measure θ as introduced above, since the operators involved, T_φ and T^0_ψ, are defined using non-negative functions φ and ψ. Let us investigate the second part of the condition in (L): For disjoint sets $E, G \in \mathfrak{R}$ let $\theta_E = T_{\varphi_E} + T^0_{\psi_E}$ and $\theta_G = T_{\varphi_G} + T^0_{\psi_G}$. Then (L) requires that the functions φ_E and φ_G are orthogonal, that is $\varphi_E(x)\varphi_G(x) = 0$ for all $x \in X$. (There are no additional conditions for the functions ψ_E and ψ_G.) If this condition is satisfied, then we have

$$\int_X (f \vee g)\, d\theta = \left(\int_X f\, d\theta\right) \vee \left(\int_X g\, d\theta\right)$$

for all $f, g \in \mathcal{F}_{\mathfrak{R}}(X, \overline{\mathbb{R}})$.

6.25 Bounded Linear Operators on Spaces of Real- or Complex-Valued Functions. Now let $\mathcal{P} = \mathbb{K}$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, endowed with the equality as order and the usual topology, that is $\mathcal{V} = \{\rho\mathbb{B} \mid \rho > 0\}$, and $a \le b + \rho\mathbb{B}$ if $|a-b| \le \rho$ for $a, b \in \mathbb{K}$. Let X and $\mathfrak{R}$ be as before. Let $\mathcal{W}$ be a system of nonnegative $\overline{\mathbb{R}}$-valued functions on X, closed for addition and multiplication by (strictly) positive scalars and directed downward. Suppose that for every $x \in X$ there is $v \in \mathcal{V}$ such that $v(x) < +\infty$. Let $\mathcal{Q}_0 = \mathcal{F}_{\mathcal{W}}(X, \mathbb{K})$ be the vector space over $\mathbb{K}$ of all functions $f \in \mathcal{F}(X, \mathbb{K})$ that are bounded with respect to the functions in $\mathcal{W}$, that is for every $w \in \mathcal{W}$ there is $\lambda \ge 0$ such that $|f(x)| \le \lambda w(x)$ for all $x \in X$. The above condition on $\mathcal{W}$ guarantees that for every $x \in X$ the point evaluation ε_x is contained in the vector space dual $\mathcal{Q}^*_{0\mathbb{K}}$ of $\mathcal{Q}_0$. Let $\mathcal{Q}$ be the standard lattice completion of Q_0. We shall consider an $\mathfrak{L}(\mathbb{K}, \mathcal{Q}_0)$-valued measure θ such that for all $E \in \mathfrak{R}$ the operators $\theta_E \in \mathfrak{L}(\mathbb{K}, \mathcal{Q}_0)$ are linear over $\mathbb{K}$. According to 6.12(iii) then the operator

$$f \mapsto \int_X f\, d\theta \,:\, \mathcal{F}_{\mathfrak{R}}(X, \mathbb{K}) \to \mathcal{Q}$$

is linear over $\mathbb{K}$ in the sense that

$$\left(\int_X af\, d\theta \right)(\mu) = \left(\int_X f\, d\theta \right)(a\mu)$$

for every $f \in \mathcal{F}_{(F,\theta)}(X, \mathcal{P})$, $\mu \in \mathcal{Q}^*_0$ and $a \in \mathbb{K}$. If we set

$$\left(\int_F f\, d\theta \right)(x) \equiv \left(\int_F f\, d\theta \right)(\varepsilon_x) - i \left(\int_F f\, d\theta \right)(i\varepsilon_x)$$

for $x \in X$, then these integrals may be reinterpreted as $\mathbb{K}$-valued functions on X and the integral is a $\mathbb{K}$-linear operator from $\mathcal{F}_{\mathfrak{R}}(X, \mathbb{K})$ into $\mathcal{F}_{\mathcal{W}}(X, \mathbb{K})$. Moreover,

$$\left| \int_F f\, d\theta \right| \le w \qquad \text{holds whenever} \qquad f \le \mathfrak{v}_w$$

for $f \in \mathcal{F}_{\mathfrak{R}}(X, \mathbb{K})$ and $w \in \mathcal{W}$. (Recall that $\mathfrak{v}_w$ consists of all step functions $s = \sum_{i=1}^n \alpha_i \chi_{E_i \otimes \mathbb{B}}$ for $0 < \alpha_i \in \mathbb{R}$ such that $\int_X s\, d\theta = \sum_{i=1}^n \alpha_i |\theta|(E_i, \mathbb{B}) \le w$.)

Obviously, $\mathbb{K}$-linear operators in $\mathfrak{L}\big(\mathbb{K}, \mathcal{F}_{\mathcal{W}}(X, \mathbb{K})\big)$ correspond to functions $\varphi \in \mathcal{F}_{\mathcal{W}}(X, \mathbb{K})$. They operate as

$$T_\varphi(a) = a\varphi \qquad \text{for} \quad a \in \mathbb{K}.$$

An $\mathfrak{L}\big(\mathbb{K}, \mathcal{F}_{\mathcal{W}}(X, \mathbb{K})\big)$-valued measure θ on $\mathfrak{R}$ may therefore be considered as an $\mathcal{F}_{\mathcal{W}}(X, \mathbb{K})$-valued set function on $\mathfrak{R}$. Boundedness means that for every $E \in \mathfrak{R}$ and $w \in \mathcal{W}$ there is $\rho \ge 0$ such that

$$|\theta|(E,\mathbb{B}) = \sup\left\{ \sum_{i=1}^{n} |\theta_{E_i}| \;\middle|\; E_i \in \mathfrak{R} \text{ disjoint subsets of } E \right\} \le \rho w.$$

Measurability for a function in $\mathcal{F}(X,\mathbb{K})$ in the sense of Section 1 coincides with measurability in the usual sense.

Both $\mathcal{P} = \mathbb{K}$ and $\mathcal{Q}_0 = \mathcal{F}_{\mathcal{W}}(X,\mathbb{K})$ are indeed topological algebras. Thus according to 6.14, the integral is an algebra homomorphism if Condition (A) holds, that is if $\theta_E(a)\theta_E(b) = \theta_E(ab)$ and $\theta_E(a)\theta_G(b) = 0$ holds for all $a,b \in \mathbb{K}$ and disjoint sets $E,G \in \mathfrak{R}$. The first part of this condition means that the function $\theta_E \in \mathcal{F}_{\mathcal{W}}(X,\mathbb{K})$ takes only the values 0 or 1, i.e. is the characteristic function of some subset $\Phi(E)$ of X. The second part of (A) requires that for disjoint sets $E,G \in \mathfrak{R}$ the functions θ_E and θ_G are orthogonal, that is $\theta_E(x)\theta_G(x) = 0$ for all $x \in X$, that is the sets $\Phi(E)$ and $\Phi(G)$ are disjoint. If this condition is satisfied, then we have

$$\int_X (fg)\, d\theta = \left(\int_X f\, d\theta\right)\left(\int_X g\, d\theta\right)$$

for all $f,g \in \mathcal{F}_{\mathfrak{R}}(X,\mathbb{K})$. The extension of the multiplication from $\mathcal{Q}_0$ to $\mathcal{Q}$ that was introduced in 6.14 implies pointwise multiplication for the corresponding $\mathbb{K}$-valued functions. Thus, under Condition (A) the integral defines a $\mathbb{K}$-linear bounded and multiplicative operator from $\mathcal{F}_{\mathfrak{R}}(X,\mathbb{K})$ into $\mathcal{F}_{\mathcal{W}}(X,\mathbb{K})$. It also preserves the involution since Condition (A*) is obviously implied by (A).

7. Extended Integrability

We can further extend integrability to a wider class of functions $f \in \mathcal{F}(X,\mathcal{P})$. Obviously, if there is $g \in \mathcal{F}_{(F,\theta)}(X,\mathcal{P})$ such that both $f+g$ and g are contained in $\mathcal{F}_{(F,\theta)}(X,\mathcal{P})$ and if the element $\int_F g\, d\theta$ is invertible in $\mathcal{Q}$, then we may set

$$\int_F f\, d\theta = \int_F (f+g)\, d\theta - \int_F g\, d\theta.$$

The class of these functions $f \in \mathcal{F}(X,\mathcal{P})$ will be denoted by $\mathfrak{F}_{(F,\theta)}(X,\mathcal{P})$.

The following is straightforward to verify:

Theorem 7.1. *$\mathfrak{F}_{(F,\theta)}$ is a subcone of $\mathcal{F}(X,\mathcal{P})$ containing $\mathcal{F}_{(F,\theta)}(X,\mathcal{P})$. If $f,g \in \mathfrak{F}_{(F,\theta)}$ and $0 \le \alpha \in \mathbb{R}$, then*

(a) $\int_F (\alpha f)\, d\theta = \alpha \int_F f\, d\theta$

(b) $\int_F (f+g)\, d\theta = \int_F f\, d\theta + \int_F g\, d\theta$

(c) $\int_F f\, d\theta \le \int_F g\, d\theta + w$ *whenever* $f \le g + \mathfrak{v}_w$ *for* $w \in \mathcal{W}$.

8. Notes and Remarks

The beginnings of modern measure theory date back to the late 19th century, some of the foundations being laid by Riemann, Harnack, Peano, Jordan, Borel, Baire, Lebesgue, Carathéodory and Radon, to name just a few of the mathematicians involved. Excellent expositions about the early history of measure theory can be found in the works of Lebesgue [114], [115] and [116], Carathéodory [30], Hahn and Rosenthal [80], Halmos [83] and Saks [182]. Vector-valued measure theory originated in the first half of the twentieth century in treatises by Clarkson, Bochner, Dunford, Morse, Pettis and Gelfand among others. Since its appearance in 1977 the book by Diestel and Uhl [43] about vector measures has become a standard reference on the subject and is also often cited in this text. It contains various sections with detailed surveys of the history of the field. There is also an extensive literature on finitely additive measures. The books by Dunford and Schwartz [55], [56], [57] and Diestel and Uhl [43] contain some sections about these. However, finitely additive measures appear to be less suitable for analytic purposes, and we therefore do not address them in this text.

The *(total) variation* of a Banach space-valued measure θ on a σ-field $\mathfrak{R}$ is usually defined as the positive $\overline{\mathbb{R}}$-valued set-function $|\theta|$ by

$$|\theta|(E) = \sup\left\{ \sum_{i=1}^{n} \|\theta(E_i)\| \;\middle|\; E_i \imath \mathfrak{R} \text{ disjoint subsets of } E \right\}$$

for $E \in \mathfrak{R}$ (See III.1.4 in [55] or I.1.4 in [43]). The *semivariation* of a vector-valued measure was introduced by Gowurin [74] and is given by

$$\|\theta\|(E) = \sup\left\{ \left\| \sum_{i=1}^{n} \gamma_i \theta(E_i) \right\| \;\middle|\; |\gamma_i| \leq 1, \quad E_i \in \mathfrak{R} \text{ disjoint subsets of } E \right\},$$

Clearly $\|\theta\|(E) \leq |\theta|(E)$, and every countably additive vector measure is known to be bounded, that is $|\theta|(X) < +\infty$ (see IV.10.2 in [55]). The set-function $|\theta|$ is seen to be σ-additive, whereas $\|\theta\|$ is generally only subadditive. On the other hand, the definition of the modulus of θ from Section 3.2, if applied to this situation, reads as

$$|\theta|(E, \mathbb{B}) = \sup\left\{ \sum_{i=1}^{n} \gamma_i \theta_{E_i} \;\middle|\; |\gamma_i| \leq 1,\ E_i \in \mathfrak{R} \text{ disjoint subsets of } E \right\},$$

where $\mathbb{B}$ denotes the unit ball of $\mathcal{P} = \mathbb{R}$ or $\mathcal{P} = \mathbb{C}$. Recall that $|\theta|(E, \mathbb{B})$ is an element of the standard lattice completion of the given Banach space, that is an $\overline{\mathbb{R}}$-valued function on its dual unit ball $\mathbb{B}^*$. Since this function is non-negative, it cannot be considered as an element of the second dual of this Banach space. However, its supremum norm $\big\| |\theta|(E,\mathbb{B})\big\|$ as a function on $\mathbb{B}^*$ is the semivariation of the measure. Indeed,

$$\begin{aligned}
&\big\| |\theta|(E,\mathbb{B})\big\| \\
&\quad = \sup_{\mu\in\mathbb{B}^*}\left(\sup\left\{\sum_{i=1}^{n}\gamma_i\theta_{E_i}\;\middle|\;|\gamma_i|\le 1,\ E_i\in\mathfrak{R}\text{ disjoint subsets of }E\right\}\right)(\mu)\\
&\quad = \sup_{\mu\in\mathbb{B}^*}\sup\left\{\left(\sum_{i=1}^{n}\gamma_i\theta_{E_i}\right)(\mu)\;\middle|\;|\gamma_i|\le 1,\ E_i\in\mathfrak{R}\text{ disjoint subsets of }E\right\}\\
&\quad = \sup\left\{\sup_{\mu\in\mathbb{B}^*}\left(\sum_{i=1}^{n}\gamma_i\theta_{E_i}\right)(\mu)\;\middle|\;|\gamma_i|\le 1,\ E_i\in\mathfrak{R}\text{ disjoint subsets of }E\right\}\\
&\quad = \sup\left\{\left\|\sum_{i=1}^{n}\gamma_i\theta_{E_i}\right\|\;\middle|\;|\gamma_i|\le 1,\ E_i\in\mathfrak{R}\text{ disjoint subsets of }E\right\}\\
&\quad = \|\theta\|(E).
\end{aligned}$$

This observation establishes the relationship between the modulus of a vector-valued measure according to Section 3.6 and its classical semivariation. However, while the modulus is a countably additive set-function, the semivariation, as its norm is only subadditive. Boundedness in the sense of Section 3.6 means that $\sup_{\mu\in\mathbb{B}^*}|\theta|(X,\mathbb{B})(\mu) < +\infty$, hence that the semivariation $\|\theta\|(X)$ is finite. Boundedness guarantees that the linear operators from $\mathcal{P}$ into $\mathcal{Q}$ which are the values of the measure can be extended to linear operators from the standard full extension $\mathcal{P}_\nu$ into $\mathcal{Q}$.

In the literature there is no shortage of different concepts of integrability for scalar-valued functions with respect to vector-valued functions or measures and variations in the resulting definitions of the integral. The best known are perhaps those by Bochner [19], Pettis [144], Bartle [8], [9] and Dunford and Schwartz [55]. There are also some corresponding differences in the definition of measurability. Again, a comprehensive treatment of the relevant definitions and their implications can be found in Chapter II of Diestel and Uhl [43]. Due to its well-understood properties, the Bochner integral is probably most used in applications. Not surprisingly, our very general approach in this chapter covers many of the above-mentioned notions. This is because we are using locally convex cones in our settings and order convergence for most of our definitions and results, and order convergence is generally weaker than the originally given topological convergence. Stronger results are pointed out when possible. Since $\mathcal{Q}$, the range of the integrals, is required to be a locally convex complete lattice cone, if applied to the case of a vector space, the results of this chapter often refer to the second dual of this vector space (Section 6.5). This situation is well-understood for vector-valued measures.

It would probably be worthwhile, though demanding, to explore the Radon-Nikodým property in the settings of this chapter. This refers to a special case of the Application 6.20 from above. Let $\mathcal{P}$ and $\mathcal{Q}$ be locally convex cones satisfying our standard assumptions, and let μ be a scalar-valued (positive, real or complex-valued) measure on $\mathfrak{R}$. These scalars can be interpreted as linear operators from $\mathfrak{L}(\mathcal{P},\mathcal{Q})$ into itself. One can therefore integrate certain $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued functions with respect to μ, and the integral is evaluated in the standard completion $\widehat{\mathfrak{L}}(\mathcal{P},\mathcal{Q})$ of $\mathfrak{L}(\mathcal{P},\mathcal{Q})$. Given a suitable function φ of this type, this can be used to define an $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued set function by

$$\theta_E = \int_E \varphi \, d\mu \qquad \text{for} \quad E \in \mathfrak{R}.$$

The convergence theorems then will guarantee that θ is countably additive and indeed an $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measure. Now investigations would have to be carried out, under which conditions a given $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued measure θ can be expressed in this way using a given scalar-valued measure μ and some $\mathfrak{L}(\mathcal{P},\mathcal{Q})$-valued density function φ.