Chapter I
Locally Convex Cones

The purpose of this chapter is twofold: Firstly, to provide the tools and the settings for the integration theory which will be developed in Chapters II and III, and secondly, to introduce the theory of locally convex cones to a wider audience. This theory generalizes locally convex topological vector spaces and has (in the author's opinion, quite unsurprisingly) not yet received the attention that it deserves. Locally convex cones permit many more and substantially different examples and applications than locally convex vector spaces. In the aspects of the theory that have been developed so far, the increase in generality leads only to minor, if any at all, compromises with respect to the depth of its results. While some of the methods and arguments employed may at times appear rather technical and indeed counterintuitive, this is largely the consequence of the inclusion of infinity-type unbounded elements and the general non-availability of the cancellation law.

So why is it worth the effort? Endowed with suitable topologies, vector spaces yield rich and well-studied structures. Locally convex topological vector spaces permit an extensive duality theory whose study gives valuable insight into the spaces themselves. Some important mathematical settings, however, while close to the structure of vector spaces do not allow subtraction of their elements or multiplication by negative scalars. Examples are certain classes of functions that may take infinite values or are characterized through inequalities rather than equalities. They arise naturally in integration theory, potential theory and in a variety of other settings. Likewise, families of convex subsets of vector spaces which are of interest in various contexts, do not form vector spaces. If the cancellation law fails, domains of this type can not be embedded into vector spaces in order to apply the results and techniques from classical functional analysis. The inclusion of these and similar examples into an analytical theory merits the investigation of a more general structure. Apart from being useful in this sense, the theory of locally convex cones allows for some interesting and occasionally insightful and elegant mathematics.

W. Roth, *Operator-Valued Measures and Integrals for Cone-Valued Functions*,
Lecture Notes in Mathematics 1964,

The first three sections of this chapter present a review of some of the main concepts of this theory while often referring to [100] and other sources for details and proofs. A brief survey of the subject can also be found in [169]. Section 4 introduces the relative topologies of a locally convex cone and provides definitions and investigations of different types of boundedness and connectedness components. Locally convex lattice cones, quasi-full locally convex cones and cones of linear operators, are studied in Sections 5, 6 and 7, respectively. These will be used extensively in the integration theory of Chapters II and III. Some of the more specialized parts of Sections 4 to 7 are included for reference in the later stages of Chapters II and III and may be skipped at first reading.

1. Locally Convex Cones

A *cone* is a set $\mathcal{P}$ endowed with an addition $(a,b) \mapsto a+b$ and a scalar multiplication $(\alpha,a) \mapsto \alpha a$ for real numbers $\alpha \geq 0$. The addition is supposed to be associative and commutative, and there is a neutral element $0 \in \mathcal{P}$. For the scalar multiplication the usual associative and distributive properties hold, that is $\alpha(\beta a) = (\alpha\beta)a$, $(\alpha+\beta)a = \alpha a + \beta a$, $\alpha(a+b) = \alpha a + \alpha b$, $1a = a$ and $0a = 0$ for all $a,b \in \mathcal{P}$ and $\alpha,\beta \geq 0$. The *cancellation law*, stating that $a+c = b+c$ implies $a = b$, however, is not required in general. It holds if and only if the cone $\mathcal{P}$ can be embedded into a real vector space.

An *ordered cone* $\mathcal{P}$ carries a reflexive transitive relation $\leq$ such that $a \leq b$ implies $a+c \leq b+c$ and $\alpha a \leq \alpha b$ for all $a,b,c \in \mathcal{P}$ and $\alpha \geq 0$. Equality on $\mathcal{P}$ is obviously such an order. Note that anti-symmetry is not required for the relation $\leq$.

The theory of locally convex cones as developed in [100] uses order theoretical concepts to introduce a quasiuniform topological structure on an ordered cone. In a first approach, the resulting topological neighborhoods themselves will be considered to be elements of the cone. In this vein, a *full locally convex cone* $(\mathcal{P},\mathcal{V})$ is an ordered cone $\mathcal{P}$ that contains an *abstract neighborhood system* $\mathcal{V}$, that is a subset of positive elements which is directed downward, closed for addition and multiplication by strictly positive scalars. The elements v of $\mathcal{V}$ define *upper*, resp. *lower neighborhoods* for the elements of $\mathcal{P}$ by

$$v(a) = \{\, b \in \mathcal{P} \mid b \leq a+v \,\} \quad \text{resp.} \quad (a)v = \{\, b \in \mathcal{P} \mid a \leq b+v \,\},$$

Their intersection $v^s(a) = v(a) \cap (a)v$ is the corresponding *symmetric neighborhood* of a. These neighborhoods create the *upper, lower* and *symmetric topologies* on $\mathcal{P}$, respectively. All elements of $\mathcal{P}$ are supposed to be *bounded below*, that is for every $a \in \mathcal{P}$ and $v \in \mathcal{V}$ we have $0 \leq a + \lambda v$ for some $\lambda \geq 0$.

Finally, a *locally convex cone* $(\mathcal{P},\mathcal{V})$ is a subcone of a full locally convex cone not necessarily containing the abstract neighborhood system $\mathcal{V}$. Every locally convex ordered topological vector space is a locally convex cone in this sense, as it can be canonically embedded into a full locally convex cone (see Example 1.4(c) below, or Example I.2.7 in [100]).

A subset $\mathcal{V}_0$ of the neighborhood system $\mathcal{V}$ is called a *basis* for $\mathcal{V}$ if for every $v \in \mathcal{V}$ there is $v_0 \in \mathcal{V}_0$ and $\alpha > 0$ such that $\alpha v_0 \le v$.

An element a of a locally convex cone $(\mathcal{P},\mathcal{V})$ is called *bounded (above)* if for every $v \in \mathcal{V}$ there is $\lambda \ge 0$ such that $a \le \lambda v$. All invertible elements of $\mathcal{P}$ are bounded. Indeed, if $-a \in \mathcal{P}$ for some $a \in \mathcal{P}$, then given $v \in \mathcal{V}$ there is $\lambda \ge 0$ such that $0 \le (-a) + \lambda v$ since all elements of $\mathcal{P}$ are required to be bounded below. This yields $a \le \lambda v$.

For later reference we shall list a few basic properties of locally convex cone topologies. We shall use the following standard notations: A subset A of $\mathcal{P}$ is called

decreasing if $b \in A$ whenever $b \le a$ for some $a \in A$,
increasing if $b \in A$ whenever $b \ge a$ for some $a \in A$, or
order convex if $b \in A$ whenever $a \le b \le c$ for some $a, c \in A$.
balanced if $b \in A$ whenever $b = \lambda a$ or $b + \lambda a = 0$
for some $a \in A$ and $0 \le \lambda \le 1$.

The last of these definitions is of course derived from corresponding one for real vector spaces, that is the requirement that $\lambda a \in A$ whenever $a \in A$ and $-1 \le \lambda \le 1$.

Proposition 1.1. *Let $(\mathcal{P},\mathcal{V})$ be a locally convex cone. The upper (lower or symmetric) topology of $\mathcal{P}$ satisfies the following:*

(i) Every element of $\mathcal{P}$ admits a basis of convex and decreasing (increasing or order convex) neighborhoods. The symmetric neighborhoods in the basis for $0 \in \mathcal{P}$ are also balanced.
(ii) The mapping $(a,b) \mapsto a+b : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ is continuous.
(iii) The mapping $(\alpha, a) \mapsto \alpha a : [0,+\infty) \times \mathcal{P} \to \mathcal{P}$ is continuous at all points $(\alpha, a) \in [0,+\infty) \times \mathcal{P}$ such that $a \in \mathcal{P}$ is bounded.

Proof. Clearly, for $a \in \mathcal{P}$ and $v \in \mathcal{V}$ the neighborhoods $v(a)$, $(a)v$ or $v^s(a)$ are convex and decreasing, increasing or order convex, respectively. The symmetric neighborhoods of $0 \in \mathcal{P}$ are also balanced. Indeed, let $v \in \mathcal{V}$ and let $a \in v^s(0)$. Then $a \le v$ and $0 \le a + v$. Let $0 \le \lambda \le 1$. Then $\lambda a \in v^s(0)$ follows from the convexity of $v^s(0)$ since $\lambda a = \lambda a + (1-\lambda)0$. If on the other hand $b + \lambda a = 0$ for $b \in \mathcal{P}$, then

$$b \le b + \lambda(a+v) = (b+\lambda a) + v = v \qquad \text{and} \qquad 0 = b + \lambda a \le b + v$$

Hence $b \in v(0)$ holds in this case as well.

For property (ii), let $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$ and set $u = (1/2)v \in \mathcal{V}$. Then for $c \in u(a)$ and $d \in u(b)$, that is $c \le a + u$ and $d \le b + u$ we have $c+d \le (a+b)+v$, hence $c+d \in v(a+b)$. This shows continuity of the addition with respect to the upper topology. Likewise, $c \in (a)u$ and $d \in (b)u$, that is $a \le c+u$ and $b \le d+u'$ implies that $a+b \le (c+d)+v$, hence $c+d \in (a+b)v$. This yields continuity of the addition with respect to the lower topology. Combining the preceding arguments, we realize that $c \in u^s(a)$ and $d \in u^s(b)$ implies $c+d \in v^s(a+b)$, which proves continuity with respect to the symmetric topology.

For Part (iii) let $(\alpha, a) \in [0, +\infty) \times \mathcal{P}$ for a bounded element $a \in \mathcal{P}$ and let $v \in \mathcal{V}$. There is $\lambda > 0$ such that both $0 \le a + \lambda v$ and $a \le \lambda v$. Set $\varepsilon = \min\{1,\ 1/(2\lambda),\ 1/(2\alpha+2)\} > 0$ and $\alpha_0 = \max\{\alpha - \varepsilon,\ 0\}$ and $\alpha_1 = \alpha + \varepsilon$. The interval $[\alpha_0, \alpha_1]$ then is a neighborhood for α in $[0, +\infty)$. For every $\alpha_0 \le \beta \le \alpha$ we observe that

$$\alpha a = \beta a + (\alpha - \beta)a \le \beta a + (\alpha - \beta)\lambda v \le \beta a + \varepsilon \lambda v \le \beta a + \frac{1}{2}v$$

and

$$\beta a \le \beta a + (\alpha - \beta)(a + \lambda v) = \alpha a + (\alpha - \beta)\lambda v \le \alpha a + \frac{1}{2}v.$$

Likewise, for $\alpha \le \beta \le \alpha_1$ we have

$$\alpha a \le \alpha a + (\beta - \alpha)(a + \lambda v) = \beta a + (\beta - \alpha)\lambda v \le \alpha a + \frac{1}{2}v$$

and

$$\beta a = \alpha a + (\beta - \alpha)a \le \alpha a + (\beta - \alpha)\lambda v \le \beta a + \frac{1}{2}v.$$

Thus

$$\alpha a \le \beta a + \frac{1}{2}v \qquad \text{and} \qquad \beta a \le \alpha a + \frac{1}{2}v$$

holds for all $\beta \in [\alpha_0, \alpha_1]$. Now let $u = \varepsilon v \in \mathcal{V}$. Then for every $b \in u(a)$ and every $\beta \in [\alpha_0, \alpha_1]$ we have

$$\begin{aligned}\beta b \le \beta(a+u) &= \beta a + \varepsilon \beta v \\ &\le \left(\alpha a + \frac{1}{2}v\right) + \varepsilon(\alpha + \varepsilon)v \le \alpha a + \frac{1}{2}v + \frac{1}{2}v \le \alpha a + v\end{aligned}$$

by our construction of $\varepsilon > 0$. Thus $\beta b \in v(\alpha a)$. This shows continuity of the mapping $(\alpha, a) \mapsto \alpha a : [0, +\infty) \times \mathcal{P} \to \mathcal{P}$ at (α, a) with respect to the upper topology of $\mathcal{P}$. Likewise, for $b \in (a)u$ and $\beta \in [\alpha_0, \alpha_1]$ we infer using the above

$$\alpha a \le \beta a + \frac{1}{2}v \le \beta(b+u) + \frac{1}{2}v \le \beta b + \varepsilon(\alpha + \varepsilon)v + \frac{1}{2}v \le \beta b + v.$$

This yields $\beta b \in (\alpha a)v$ and continuity of the mapping $(\alpha, a) \mapsto \alpha a$: $[0, +\infty) \times \mathcal{P} \to \mathcal{P}$ at (α, a) with respect to the lower topology of $\mathcal{P}$. Combining the preceding arguments, we realize that $b \in u^s(a)$ and $\beta \in [\alpha_0, \alpha_1]$

implies $\beta b \in v^s(\alpha a)$, which proves continuity with respect to the symmetric topology. □

On the subcone $\mathcal{P}_0$ of all invertible elements in a locally convex cone $(\mathcal{P}, V)$ the scalar multiplication can be canonically extended to all real numbers if we set $\alpha a = (-\alpha)(-a)$ for $\alpha < 0$ and $a \in \mathcal{P}_0$. Proposition 1.1 then yields

Corollary 1.2. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone and let $\mathcal{P}_0$ be the subcone of all invertible elements of $\mathcal{P}$. The mapping $(\alpha, a) \mapsto \alpha a : \mathbb{R} \times \mathcal{P}_0 \to \mathcal{P}_0$ is continuous with respect to the symmetric topology of $\mathcal{P}$.*

Proof. First we observe that $a \in v^s(b)$ if and only if $-a \in v^s(-b)$ for $a, b \in \mathcal{P}_0$ and $v \in \mathcal{V}$. Thus $a_i \to a$ for a net $(a_i)_{i \in \mathcal{I}}$ in $\mathcal{P}_0$ implies that $(-a_i) \to (-a)$. Next suppose that $\alpha_i \to \alpha \in \mathbb{R}$ for $0 \leq \alpha_i \in \mathbb{R}$ and $a_i \to a$ for $a_i, a \in \mathcal{P}_0$. Then $\alpha_i a_i \to \alpha a$ by 1.1(iii) since every invertible element is bounded. Now finally, let $\alpha_i \to \alpha$ in $\mathbb{R}$ and $a_i \to a$ for $a_i, a \in \mathcal{P}_0$. Let $\beta_i = \alpha_i \vee 0$ and $\gamma_i = -(\alpha_i \wedge 0)$. Then $\beta_i, \gamma_i \geq 0$ and $\alpha_i = \beta_i - \gamma_i$. We have $\beta_i a_i \to \beta a$ and $\gamma_i(-a_i) \to \gamma(-a)$, where $\beta = \alpha \vee 0$ and $\gamma = -(\alpha \wedge 0)$, by the preceding. Thus

$$\alpha_i a_i = \beta_i a_i + \gamma_i(-a_i) \ \to \ \beta a + \gamma(-a) = \alpha a,$$

again by 1.1(ii), as claimed. □

1.3 Locally Convex Cones via Convex Quasiuniform Structures. As a subcone of a full locally convex cone, a locally convex cone $(\mathcal{P}, \mathcal{V})$ inherits both its order, algebraic structure and neighborhood system from the former. While this approach elegantly permits the use of the order structure of the full cone to describe the topologies of $\mathcal{P}$, it is not always very practical, because for concrete examples such a full cone may be difficult to access. Quite frequently, the topology of a locally convex cone is more visible as a *convex quasiuniform structure* as described in I.5 of [100]. This is a straightforward generalization of the uniform structures that define the topologies of locally convex topological vector spaces. In this vein, a neighborhood is a convex subset v of $\mathcal{P}^2$, where $\mathcal{P}$ is an ordered cone, satisfying the following conditions:

(U1) If $a \leq b$ for $a, b \in \mathcal{P}$, then $(a, b) \in v$.
(U2) If $(a, b) \in \lambda v$ and $(b, c) \in \rho v$ for $a, b, c \in \mathcal{P}$ and $\lambda, \rho > 0$, then $(a, c) \in (\lambda + \rho)v$.
(U3) For every $a \in \mathcal{P}$ there is $\lambda \geq 0$ such that $(0, a) \in \lambda v$.

If a family $\mathcal{V}$ of such neighborhoods fulfills the usual conditions for a quasiuniform structure (see [135]), that is

(U4) For $u, v \in \mathcal{V}$ there is $w \in \mathcal{V}$ such that $w \subset u \cap v$,
(U5) If $v \in \mathcal{V}$ and $\lambda > 0$, then $\lambda v \in \mathcal{V}$,

then a straightforward procedure (see I.5 in [100]) allows the embedding of $\mathcal{P}$ and $\mathcal{V}$ into a full locally convex cone $(\widehat{\mathcal{P}}, \widehat{\mathcal{V}})$ whose neighborhood system $\widehat{\mathcal{V}}$ is generated by the elements of $\mathcal{V}$, and such that $(a, b) \in v$ for $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$ means $a \leq b + v$ in $\widehat{\mathcal{P}}$. Convex quasiuniform structures therefore yield an equivalent approach to locally convex cones.

Examples 1.4. (a) In the extended real number system $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ we consider the usual order and algebraic operations, in particular $a + \infty = +\infty$ for all $a \in \overline{\mathbb{R}}$, $\alpha \cdot (+\infty) = +\infty$ for all $\alpha > 0$ and $0 \cdot (+\infty) = 0$. Endowed with the neighborhood system $\mathcal{V} = \{\varepsilon \in \mathbb{R} \mid \varepsilon > 0\}$, $\overline{\mathbb{R}}$ is a full locally convex cone. For $a \in \mathbb{R}$ the intervals $(-\infty, a + \varepsilon]$ are the upper and the intervals $[a - \varepsilon, +\infty]$ are the lower neighborhoods, while for $a = +\infty$ the entire cone $\overline{\mathbb{R}}$ is the only upper neighborhood, and $\{+\infty\}$ is open in the lower topology. The symmetric topology is the usual topology on $\mathbb{R}$ with $+\infty$ as an isolated point. It is finer than the usual topology of $\overline{\mathbb{R}}$, where the intervals $[a, +\infty]$ are the neighborhoods of $+\infty$.

(b) For the subcone $\overline{\mathbb{R}}_+ = \{a \in \overline{\mathbb{R}} \mid a \geq 0\}$ of $\overline{\mathbb{R}}$ we may also consider the singleton neighborhood system $\mathcal{V} = \{0\}$. The elements of $\overline{\mathbb{R}}_+$ are obviously bounded below even with respect to the neighborhood $v = 0$, hence $\overline{\mathbb{R}}_+$ is a full locally convex cone. For $a \in \overline{\mathbb{R}}$ the intervals $(-\infty, a]$ and $[a, +\infty]$ are the only upper and lower neighborhoods, respectively. The symmetric topology is the discrete topology on $\overline{\mathbb{R}}_+$.

(c) Let $(E, \leq)$ be a locally convex ordered topological vector space. Recall that equality is an order relation, hence this example will cover locally convex spaces in general. In order to interpret E as a locally convex cone we shall embed it into a larger full cone. This is done in a canonical way: Let $\mathcal{P} = Conv(E)$ be the cone of all non-empty convex subsets of E, endowed with the usual addition and multiplication of sets by non-negative scalars, that is $\alpha A = \{\alpha a \mid a \in A\}$ and $A + B = \{a + b \mid a \in A \text{ and } b \in B\}$ for $A, B \in \mathcal{P}$ and $\alpha \geq 0$. We define the order on $\mathcal{P}$ by

$$A \leq B \qquad \text{if} \qquad A \subset \downarrow B,$$

where $\downarrow B = \{x \in E \mid x \leq b \text{ for some } b \in B\}$ is the *decreasing hull* of the set B in E. Note that $\downarrow B$ is again a convex subset of E. The requirements for an ordered cone are easily checked. The neighborhood system in $\mathcal{P}$ is given by a basis $\mathcal{V} \subset \mathcal{P}$ of convex and balanced neighborhoods of the origin in E. That is

$$A \leq B + V \qquad \text{if} \qquad A \subset \downarrow (B + V)$$

for $A, B \in \mathcal{P}$ and $V \in \mathcal{V}$. We observe that for every $A \in \mathcal{P}$ and $V \in \mathcal{V}$ there is $\rho > 0$ such that $\rho V \cap A \neq \emptyset$. This yields $0 \in A + \rho V$. Therefore $\{0\} \leq A + \rho V$, and every element $A \in \mathcal{P}$ is indeed bounded below. Thus $(\mathcal{P}, \mathcal{V})$ is a full locally convex cone.

Via the embedding $x \mapsto \{x\} : E \to \mathcal{P}$ of its elements onto singleton subsets, the locally convex ordered topological vector space E itself may be considered as a subcone of $\mathcal{P}$. This embedding preserves the order of E, and on its image in $\mathcal{P}$, the upper or lower topologies of $\mathcal{P}$ reflect the order structure of E in the following sense: All upper or lower neighborhoods are decreasing or increasing, respectively, that is for elements $a, b \in E$ and a neighborhood $V \in \mathcal{V}$ we have

$$a \leq b + V \qquad \text{if} \qquad a - b \in {\downarrow}V.$$

For a linear operator $T : E \to E$ in particular, continuity with respect to either the induced upper or lower topology requires that T is monotone (see Section 2 below). The symmetric topology of $\mathcal{P}$, on the other hand induces a locally convex vector space topology on E in the usual sense. It coincides with the given topology of E if the neighborhoods $V \in \mathcal{V}$ are also order convex, that is if $c \in V$ whenever $a \leq c \leq b$ for $a, b \in V$ and $V \in \mathcal{V}$. If the given order on E is indeed the equality, then the upper, lower and symmetric topologies of $\mathcal{P}$ all coincide on E with the given topology since $a \leq b + V$ for $a, b \in E$ and $V \in \mathcal{V}$ means that $a - b \in V$ in this case, and since the neighborhoods in $\mathcal{V}$ were supposed to be balanced. In this way, every locally convex ordered topological vector space, endowed with a basis $\mathcal{V}$ of balanced, convex and order convex neighborhoods, is a locally convex cone, but not a full cone.

Other subcones of $\mathcal{P}$ that merit further investigation are those of all closed, closed and bounded, or compact convex sets in $\mathcal{P}$, respectively. Note that closed and bounded convex sets satisfy the cancellation law. Details on those and further related examples can be found in [100], I.1.7, I.2.7 and I.2.8.

This example can be further generalized if we replace the vector space E by a locally convex cone.

(d) If $(\mathcal{P}, \mathcal{V})$ is a locally convex cone and if $\mathcal{P}$ is indeed a vector space over $\mathbb{R}$, that is the scalar multiplication in $\mathcal{P}$ is extended to all reals, then all elements of $\mathcal{P}$ are obviously bounded, as boundedness from above for the element $a \in \mathcal{P}$ follows from boundedness from below for the element $-a \in \mathcal{P}$. We have $a \in v(b)$ for $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$ in this case if and only if $a - b \in v(0)$. While the multiplication by negative scalars is in general not continuous with respect to the upper and lower topologies on $\mathcal{P}$, the symmetric topology generated by the neighborhoods of the origin

$$v^s(0) = \{a \in \mathcal{P} \mid a \leq v \ \text{ and } \ -a \leq v\}$$

is a locally convex vector space topology in the usual sense (see Corollary 1.2).

If $\mathcal{P}$ is indeed a vector space over $\mathbb{C}$, then we need to consider the *modular symmetric topology* instead (see Section 2 in [168]). It is generated by the neighborhoods of the origin

$$v^{sm}(0) = \{a \in \mathcal{P} \mid \gamma a \leq v \ \text{ for all } \ \gamma \in \Gamma\},$$

where $\Gamma = \{\gamma \in \mathbb{C} \mid |\gamma| = 1\}$ denotes the unit circle of $\mathbb{C}$. It is easy to verify that these sets are convex, balanced and absorbing. The modular symmetric topology is therefore a locally convex vector space topology in the usual sense and yields continuity for the multiplication by all scalars in $\mathbb{C}$. Thus endowed with the modular neighborhoods $\mathcal{V}_m = \{v^{s_m} \mid v \in \mathcal{V}\}$ and the equality as its order, $(\mathcal{P}, \mathcal{V}_m)$ is again a locally convex cone, and we have $a \leq b + v^{s_m}$ for $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$ if $\gamma(a-b) \leq v$ for all $\gamma \in \Gamma$.

In the sequel, we shall say that a locally convex cone $(\mathcal{P}, \mathcal{V})$ is a locally convex topological vector space over $\mathbb{R}$ or $\mathbb{C}$ if $\mathcal{P}$ is a vector space over $\mathbb{R}$ or $\mathbb{C}$, endowed with the equality as order and a system $\mathcal{V}$ of neighborhoods such that $v(0) = v^{s_m}(0)$ holds for all $v \in \mathcal{V}$. The subsets $v(0)$ of $\mathcal{P}$ then are convex, balanced and absorbing, and $\mathcal{P}$ carries its modular symmetric topology.

(e) Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone, X a set and let $\mathcal{F}(X, \mathcal{P})$ be the cone of all $\mathcal{P}$-valued functions on X, endowed with the pointwise operations and order. If $\widehat{\mathcal{P}}$ is a full cone containing both $\mathcal{P}$ and $\mathcal{V}$, then we may identify the elements $v \in \mathcal{V}$ with the constant functions $\hat{v}$ on X, that is $x \mapsto v$ for all $x \in X$. Hence $\widehat{\mathcal{V}} = \{\hat{v} \mid v \in \mathcal{V}\}$ is a subset and a neighborhood system for $\mathcal{F}(X, \widehat{\mathcal{P}})$. A function $f \in \mathcal{F}(X, \widehat{\mathcal{P}})$ is uniformly bounded below, if for every $\hat{v} \in \widehat{\mathcal{V}}$ there is $\rho \geq 0$ such that $0 \leq f + \rho\hat{v}$. These functions form a full locally convex cone $\big(\mathcal{F}_b(X, \widehat{\mathcal{P}}), \widehat{\mathcal{V}}\big)$, carrying the topology of uniform convergence. As a subcone, $\big(\mathcal{F}_b(X, \mathcal{P}), \widehat{\mathcal{V}}\big)$ is a locally convex cone. Alternatively, a more general neighborhood system $\widehat{\mathcal{V}}$ for $\mathcal{F}(X, \mathcal{P})$ may be created using a family of $\overline{\mathcal{V}}$-valued functions on X, where $\overline{\mathcal{V}} = \mathcal{V} \cup \{0, \infty\}$ consists of the neighborhood system $\mathcal{V}$ for $\mathcal{P}$ augmented by $0 \in \mathcal{P}$ and a maximal element ∞. (We use $a + \infty = v + \infty = \alpha \cdot \infty = \infty$ and $a \leq \infty$ for all $a \in \mathcal{P}$, $v \in \mathcal{V}$ and $\alpha > 0$.) The neighborhoods $\hat{v} \in \widehat{\mathcal{V}}$ are defined for functions $f, g \in \mathcal{F}(X, \mathcal{P})$ as

$$f \leq g + \hat{v} \qquad \text{if} \qquad f(x) \leq g(x) + \hat{v}(x) \quad \text{for all} \quad x \in X.$$

In this case we consider the subcone $\mathcal{F}_{\widehat{\mathcal{V}}_b}(X, \mathcal{P})$ of all functions in $\mathcal{F}(X, \mathcal{P})$ that are bounded below relative to the functions in $\widehat{\mathcal{V}}$, that is $f \in \mathcal{F}_{\widehat{\mathcal{V}}_b}(X, \mathcal{P})$ if for every $\hat{v} \in \widehat{\mathcal{V}}$ there is $\lambda \geq 0$ such that $0 \leq f + \lambda\hat{v}$. In this way $\big(\mathcal{F}_{\widehat{\mathcal{V}}_b}(X, \mathcal{P}), \widehat{\mathcal{V}}\big)$ forms a locally convex cone. Of particular interest is the case when $\widehat{\mathcal{V}}$ is generated by a suitable family $\mathcal{Y}$ of subsets Y of X and the $\overline{\mathcal{V}}$-valued functions $\hat{v}_Y(x) = v$ for $x \in Y$ and $\hat{v}_Y(x) = \infty$, else, corresponding to some $v \in \mathcal{V}$ and $Y \in \mathcal{Y}$. In this case $\big(\mathcal{F}_{\widehat{\mathcal{V}}_b}(X, \mathcal{P}), \widehat{\mathcal{V}}\big)$ carries the topology of uniform convergence on the sets in $\mathcal{Y}$.

If X is a topological space, then suitable subcones for further investigation are those of continuous functions with respect to any of the given (upper, lower or symmetric) topologies on $\mathcal{P}$. We shall explore different notions of continuity for cone-valued functions and discuss an even wider range of suitable locally convex cone topologies in Chapter III.

Occasionally in applications of this type (see for example the proof of Proposition 5.37 and the construction of the standard lattice completion in 5.57 below) the family $\widehat{\mathcal{V}}$ of $\overline{\mathcal{V}}$-valued functions under consideration is not naturally closed for addition, and including all pointwise sums of the functions in $\widehat{\mathcal{V}}$ might not be desirable. This situation can often be remedied if we consider $\widehat{\mathcal{V}}$ as a system of abstract neighborhoods instead, with a suitably modified addition $\oplus$ for which $\widehat{\mathcal{V}}$ is closed and which is compatible with the scalar multiplication. The neighborhoods $\hat{v} \in \mathcal{F}(X,\mathcal{P})$ are defined as above using associated $\overline{\mathcal{V}}$-valued functions which for simplicity we also denote by $\hat{v}$. The latter amounts to a slight abuse of notation, since for this concept to work we need to allow that the association between neighborhoods and $\overline{\mathcal{V}}$-valued functions is not one-to-one. In order to create a convex quasiuniform structure in the sense of 1.3, hence a locally convex cone $\big(\mathcal{F}_{\widehat{\mathcal{V}}_b}(X,\mathcal{P}),\widehat{\mathcal{V}}\big)$, we require that $\hat{u} \oplus \hat{v} \geq \hat{u} + \hat{v}$ holds for all $\hat{u}, \hat{v} \in \widehat{\mathcal{V}}$, where $+$ stands for the pointwise sum of the associated $\overline{\mathcal{V}}$-valued functions. We shall use this approach in 5.37 and 5.57 below.

(f) For $x \in \overline{\mathbb{R}}$ denote $x^+ = \max\{x, 0\}$ and $x^- = -\min\{x, 0\}$. For $1 \leq p \leq +\infty$ and a sequence $(x_i)_{i\in\mathbb{N}}$ in $\overline{\mathbb{R}}$ let $\|(x_i)\|_p$ denote the usual l^p norm, that is $\|(x_i)\|_p = \big(\sum_{i=1}^{\infty} |x_i|^p\big)^{(1/p)} \in \overline{\mathbb{R}}$ for $p < +\infty$ and $\|(x_i)\|_\infty = \sup\{|x_i| \mid i \in \mathbb{N}\} \in \overline{\mathbb{R}}$. Now let $\overline{l^p}$ be the cone of all sequences $(x_i)_{i\in\mathbb{N}}$ in $\overline{\mathbb{R}}$ such that $\|(x_i^-)\|_p < +\infty$. We use the pointwise order in $\overline{l^p}$ and the neighborhood system $\mathcal{V}_p = \{\rho v_p \mid \rho > 0\}$, where

$$(x_i)_{i\in\mathbb{N}} \leq (y_i)_{i\in\mathbb{N}} + \rho v_p$$

means that $\|(x_i - y_i)^+\|_p \leq \rho$. (In this expression the l^p norm is evaluated only over the indexes $i \in \mathbb{N}$ for which $y_i < +\infty$.) It can be easily verified that $(\overline{l^p}, \mathcal{V}_p)$ is a locally convex cone. In fact $(\overline{l^p}, \mathcal{V}_p)$ can be embedded into a full cone following a procedure analogous to that in 1.4(c). The case for $p = +\infty$ is of course already covered by Example 1.4(e).

2. Continuous Linear Operators and Functionals

For cones $\mathcal{P}$ and $\mathcal{Q}$ a mapping $T : \mathcal{P} \to \mathcal{Q}$ is called a *linear operator* if

$$T(a+b) = T(a) + T(b) \qquad \text{and} \qquad T(\alpha a) = \alpha T(a)$$

holds for all $a, b \in \mathcal{P}$ and $\alpha \geq 0$. If both $\mathcal{P}$ and $\mathcal{Q}$ are indeed vector spaces over $\mathbb{R}$, then $0 = T(a-a) = T(a) + T(-a)$ implies that such an operator is linear over $\mathbb{R}$. If both $\mathcal{P}$ and $\mathcal{Q}$ are ordered, then T is called *monotone,* if $a \leq b$ implies $T(a) \leq T(b)$. If both $(\mathcal{P}, \mathcal{V})$ and $(\mathcal{Q}, \mathcal{W})$ are locally convex cones, the operator T is called *(uniformly) continuous* if for every $w \in \mathcal{W}$ one can find $v \in \mathcal{V}$ such that $T(a) \leq T(b) + w$ whenever $a \leq b + v$ for $a, b \in \mathcal{P}$. A family $\mathfrak{T}$ of linear operators is called *equicontinuous* if the above condition holds for every $w \in \mathcal{W}$ with the same $v \in \mathcal{V}$ for all $T \in \mathfrak{T}$.

Uniform continuity is not just continuity. It is immediate from the definition that it implies and combines continuity for the operator $T : \mathcal{P} \to \mathcal{Q}$ with respect to the upper, lower and symmetric topologies on $\mathcal{P}$ and $\mathcal{Q}$, respectively.

A *linear functional* on $\mathcal{P}$ is a linear operator $\mu : \mathcal{P} \to \overline{\mathbb{R}}$. The *dual cone* $\mathcal{P}^*$ of a locally convex cone $(\mathcal{P}, \mathcal{V})$ consists of all continuous linear functionals on $\mathcal{P}$ and is the union of all *polars* v° of neighborhoods $v \in \mathcal{V}$, where $\mu \in v^\circ$ means that $\mu(a) \le \mu(b) + 1$, whenever $a \le b + v$ for $a, b \in \mathcal{P}$. Continuity implies that a linear functional μ is monotone, and for a full cone $\mathcal{P}$ it requires just that $\mu(v) \le 1$ holds for some $v \in \mathcal{V}$ in addition. Continuous linear functionals can take only finite values on bounded elements. Indeed, let $\mu \in v^\circ$ for some $v \in \mathcal{V}$ and let $a \in \mathcal{P}$ be a bounded element. Then $a \le \lambda v$ for some $\lambda \ge 0$, hence $\mu(a) \le \lambda$ as claimed. We endow $\mathcal{P}^*$ with the canonical algebraic operations and the topology $w(\mathcal{P}^*, \mathcal{P})$ of pointwise convergence on the elements of $\mathcal{P}$, considered as functions on $\mathcal{P}^*$ with values in $\overline{\mathbb{R}}$ with its usual topology. As in locally convex topological vector spaces, the polar v° of a neighborhood $v \in \mathcal{V}$ is seen to be $w(\mathcal{P}^*, \mathcal{P})$-compact and convex ([100], Theorem II.2.4).

Examples 2.1. Revisiting the preceding Examples 1.4 we observe the following:

(a) The dual cone $\overline{\mathbb{R}}^*$ of $\overline{\mathbb{R}}$ (see 1.4(a)) consists of all positive reals (via the usual multiplication), and the singular functional $\bar{0}$ such that $\bar{0}(a) = 0$ for all $a \in \mathbb{R}$ and $\bar{0}(+\infty) = +\infty$.

(b) Likewise, in 1.4(b), the continuous linear functionals on $\overline{\mathbb{R}}_+$, endowed with the neighborhood system $\mathcal{V} = \{0\}$, are the positive reals together with $\bar{0}$, but further include the element $+\infty$, acting as $+\infty(0) = 0$ and $+\infty(a) = +\infty$ for all $0 \ne a \in \overline{\mathbb{R}}_+$. This functional is obviously contained in the polar of the neighborhood $0 \in \mathcal{V}$.

(c) If both $(\mathcal{P}, \mathcal{V})$ and $(\mathcal{Q}, \mathcal{W})$ are locally convex cones and ordered vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, let us also consider the modular symmetric topologies on $\mathcal{P}$ and $\mathcal{Q}$ which are defined by the modular symmetric neighborhoods v^{s_m} and w^{s_m} corresponding to the given neighborhoods $v \in \mathcal{V}$ and $w \in \mathcal{W}$, respectively. Recall from 1.4(d) that $a \le b + v^{s_m}$ for $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$ means that $\gamma(a - b) \le v$ for all $\gamma \in \mathbb{K}$ such that $|\gamma| = 1$. The modular topologies were seen to be locally convex vector space topologies. If a linear operator $T : \mathcal{P} \to \mathcal{Q}$ is continuous with respect to the given locally convex cone topologies and indeed linear over $\mathbb{K}$, then it is straightforward to verify that T is also continuous with respect to the respective modular symmetric topologies of $\mathcal{P}$ and $\mathcal{Q}$. The converse does not hold true in general. For $\mathcal{Q} = \overline{\mathbb{R}}$, however, that is for linear functionals, we have the following: If $\mathcal{P}^*$ denotes the given dual of $\mathcal{P}$, and if $\mathcal{P}_m^*$ denotes the dual of $\mathcal{P}$ if endowed with the modular symmetric neighborhoods, then $\mathcal{P}^* \subset \mathcal{P}_m^*$ since the latter topology is finer that the given one. According to Theorem 3.3 in [168], for every linear functional $\mu \in \mathcal{P}_m^*$ there are $\mu_i \in \mathcal{P}^*$ for $i = 1, 2$ in the real or $i = 1, 2, 3, 4$ in the complex case such that

$$\mu(a) = \mu_1(a) + \mu_2(-a) \quad \text{or} \quad \mu(a) = \mu_1(a) + \mu_2(-a) + \mu_3(ia) + \mu_4(-ia)$$

for all $a \in \mathcal{P}$, respectively.

(d) Let $(\mathcal{P}, \mathcal{V})$ be a locally convex vector space over $\mathbb{K}$, that is a locally convex cone which is a vector space over $\mathbb{K}$ and carries the modular symmetric topology. The functionals in the dual cone $\mathcal{P}^*$ of $\mathcal{P}$ are real-valued, but there exists a canonical correspondence between the dual cone $\mathcal{P}^*$ and the usual dual space $\mathcal{P}^*_{\mathbb{K}}$ of $\mathcal{P}$ as a locally convex topological vector space. $\mathcal{P}^*_{\mathbb{K}}$ consists of all $\mathbb{K}$-valued continuous $\mathbb{K}$-linear functionals on $\mathcal{P}$. In the real case this correspondence is obvious, as $\mathcal{P}^*$ and $\mathcal{P}^*_{\mathbb{K}}$ coincide. (If both $a, -a \in \mathcal{P}$, then $\mu(a) + \mu(-a) = 0$ for every $\mu \in \mathcal{P}^*$, hence μ is linear over $\mathbb{R}$.) In the complex case there is an established correspondence between $\mathcal{P}^*$ and $\mathcal{P}^*_{\mathbb{K}}$: The real part μ of every continuous complex linear functional $\mu_{\mathbb{K}}$ on $\mathcal{P}$ is in $\mathcal{P}^*$ and, conversely, for every $\mu \in \mathcal{P}^*$, the mapping $a \mapsto \mu(a) - i\,\mu(ia)$ defines a continuous complex linear functional $\mu_{\mathbb{K}} \in \mathcal{P}^*_{\mathbb{K}}$. $\mathcal{P}^*_{\mathbb{K}}$ is again a vector space over $\mathbb{K}$, and for $\mu_{\mathbb{K}}$ and $\alpha \in \mathbb{K}$ the respective projections μ and $(\alpha\mu)$ into $\mathcal{P}^*$ of the functionals $\mu_{\mathbb{K}}$ and $\alpha\mu_{\mathbb{K}}$ relate as

$$\big(\alpha\mu\big)(a) = \mathfrak{Re}\big((\alpha\mu_{\mathbb{K}})(a)\big) = \mathfrak{Re}\big(\mu_{\mathbb{K}}(\alpha a)\big) = \mu(\alpha a).$$

The above formula effectively extends the multiplication by non-negative reals in $\mathcal{P}^*$ to all scalars in $\mathbb{K}$ in such a way that the mapping $\mu_{\mathbb{K}} \to \mu : \mathcal{P}^*_{\mathbb{K}} \to \mathcal{P}^*$ becomes a vector space isomorphism. Similarly, every element $\varphi_{\mathbb{K}}$ of the (algebraic) second vector space dual $\mathcal{P}^{**}_{\mathbb{K}}$ of $\mathcal{P}_{\mathbb{K}}$ corresponds to a real-linear functional φ on the dual cone $\mathcal{P}^*$ by

$$\varphi(\mu) = \mathfrak{Re}\big(\varphi_{\mathbb{K}}(\mu_{\mathbb{K}})\big)$$

for $\mu \in \mathcal{P}^*$. On the other hand, every functional φ on $\mathcal{P}^*$ that is linear with respect to the non-negative reals corresponds to a $\mathbb{K}$-valued linear functional $\varphi_{\mathbb{K}} \in \mathcal{P}^{**}_{\mathbb{K}}$ on $\mathcal{P}^*_{\mathbb{K}}$ by

$$\varphi_{\mathbb{K}}(\mu_{\mathbb{K}}) = \varphi(\mu) \qquad \text{or} \qquad \varphi_{\mathbb{K}}(\mu_{\mathbb{K}}) = \varphi(\mu) - i\varphi(i\mu)$$

for $\mu_{\mathbb{K}} \in \mathcal{P}^*_{\mathbb{K}}$ in the real or complex case, respectively. Here we use the above defined extension of the scalar multiplication in $\mathcal{P}^*$. $\mathbb{K}$-linearity for $\varphi_{\mathbb{K}}$ is easily checked. Indeed, additivity is obvious for $\varphi_{\mathbb{K}}$. For compatibility with the scalar multiplication, the real case follows from $\varphi(\mu) + \varphi\big((-1)\mu\big) = \varphi(0) = 0$, hence $\varphi\big((-1)\mu\big) = -\varphi(\mu)$. In the complex case we calculate for $\mu_{\mathbb{K}} \in \mathcal{P}^*_{\mathbb{K}}$ and $\alpha = x + iy \in \mathbb{C}$

$$\begin{aligned}
\varphi_{\mathbb{K}}\big((x+iy)\mu_{\mathbb{K}}\big) &= \varphi\big((x+iy)\mu\big) - i\varphi\big((x+iy)i\mu\big) \\
&= (x+iy)\varphi(\mu) + (y-ix)\varphi(i\mu) \\
&= (x+iy)\big(\varphi(\mu) - i\varphi(i\mu)\big) \\
&= (x+iy)\varphi_{\mathbb{K}}(\mu_{\mathbb{K}}).
\end{aligned}$$

Thus there is also a canonical correspondence between $\mathcal{P}^{**}$, the cone of all real-valued linear functionals on $\mathcal{P}^*$ (see 7.3(i) below) and the second vector space dual $\mathcal{P}_{\mathbb{K}}^{**}$ of $\mathcal{P}_{\mathbb{K}}$.

(e) In 1.4(c) and (e) on the other hand, due to the generality of the settings, a complete description for the respective dual cones is not immediately available. We may, however, identify some of their elements: In 1.4(c), let μ be a continuous monotone linear function on the locally convex ordered topological vector space $(E, \leq)$. Then the mapping

$$A \mapsto \sup\{\mu(a) \mid a \in A\} : Conv(E) \to \overline{\mathbb{R}}$$

is seen to be an element of $Conv(E)^*$.

(f) In 1.4(e), if $\mu \in v^\circ \subset \mathcal{P}^*$ for some $v \in \mathcal{V}$, and if $\hat{v}(x) \leq v$ for some $v \in \widehat{\mathcal{V}}$ and $x \in X$, then the mapping $\mu_x : \mathcal{F}_{\widehat{\mathcal{V}}_b}(X, \mathcal{P}) \to \overline{\mathbb{R}}$ such that

$$\mu_x(f) = \mu\big(f(x)\big) \qquad \text{for all} \quad f \in \mathcal{F}_{\widehat{\mathcal{V}}_b}(X, \mathcal{P})$$

is a continuous linear functional on $\mathcal{F}_{\widehat{\mathcal{V}}_b}(X, \mathcal{P})$; more precisely $\mu_x \in \hat{v}^\circ$.

(g) In 1.4(g), for $p < +\infty$ the dual cone of $\overline{l^p}$ consists of all sequences $(y_i)_{i \in \mathbb{N}}$ such that $y_i \geq 0$ for all $i \in \mathbb{N}$ and $\|(y_i)\|_q < +\infty$, where q is the conjugate index of p.

2.2 Embeddings. We have intuitively used the term embedding before. Let us now establish a precise definition: Let $(\mathcal{P}, \mathcal{V})$ and $(\mathcal{Q}, \mathcal{W})$ be locally convex cones. A linear operator $\Phi : \mathcal{P} \to \mathcal{Q}$ is called an *embedding* of $(\mathcal{P}, \mathcal{V})$ into $(\mathcal{Q}, \mathcal{W})$ if it can be extended to a mapping $\Phi : (\mathcal{P} \cup \mathcal{V}) \to (\mathcal{Q} \cup \mathcal{W})$ such that $\Phi(\mathcal{V}) = \mathcal{W}$ and

$$a \leq b + v \qquad \text{holds if and only if} \qquad \Phi(a) \leq \Phi(b) + \Phi(w)$$

for all $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$.

This condition implies that Φ is continuous, and in case that Φ is one-to-one, that the inverse operator $\Phi^{-1} : \Phi(\mathcal{P}) \to \mathcal{P}$ is also continuous. It is easily verified that the composition of two embeddings is again an embedding in this sense. Embeddings are meant to preserve not just the topological structure, but also the particular neighborhood system of a locally convex cone.

Lemma 2.3. *Let $(\mathcal{P}, \mathcal{V})$ and $(\mathcal{Q}, \mathcal{W})$ be locally convex cones and let $\Phi : \mathcal{P} \to \mathcal{Q}$ be an embedding of $(\mathcal{P}, \mathcal{V})$ into $(\mathcal{Q}, \mathcal{W})$. If the symmetric topology of $\mathcal{P}$ is Hausdorff, then Φ is one-to-one.*

Proof. Under the assumptions of the Lemma, suppose that $\Phi(a) = \Phi(b)$ holds for $a, b \in \mathcal{P}$. Then $a \leq b + v$ and $b \leq a + v$, hence $a \in v^s(b)$ for all $v \in \mathcal{V}$ follows from 2.2. This yields $a = b$ since the symmetric topology of $\mathcal{P}$ is supposed to be Hausdorff. $\square$

An embedding Φ of $(\mathcal{P},\mathcal{V})$ into $(\mathcal{Q},\mathcal{W})$ is called an *isomorphism* if the mapping $\Phi : (\mathcal{P}\cup\mathcal{V}) \to (\mathcal{Q}\cup\mathcal{W})$ is invertible. Then Φ^{-1} is an embedding of $(\mathcal{Q},\mathcal{W})$ into $(\mathcal{P},\mathcal{W})$.

Hahn-Banach type extension and separation theorems for linear functionals are most important for the development of a powerful duality theory for locally convex cones. We shall mention a few results from [100] and [172]. A *sublinear functional* on a cone $\mathcal{P}$ is a mapping $p : \mathcal{P} \to \overline{\mathbb{R}}$ such that

$$p(\alpha a) = \alpha p(a) \qquad \text{and} \qquad p(a+b) \le p(a) + p(b)$$

holds for all $a,b \in \mathcal{P}$ and $\alpha \ge 0$. Likewise, an *extended superlinear functional* on $\mathcal{P}$ is a mapping $q : \mathcal{P} \to \overline{\overline{\mathbb{R}}} = \mathbb{R}\cup\{+\infty,-\infty\}$ such that

$$q(\alpha a) = \alpha q(a) \qquad \text{and} \qquad q(a+b) \ge q(a) + q(b)$$

holds for all $a,b \in \mathcal{P}$ and $\alpha \ge 0$. (We set $\alpha + (-\infty) = -\infty$ for all $\alpha \in \overline{\overline{\mathbb{R}}}$, $\alpha \cdot (-\infty) = -\infty$ for all $\alpha > 0$ and $0 \cdot (-\infty) = 0$ in this context.) We cite Theorem 3.1 from [172]:

Sandwich Theorem 2.4. *Let $(\mathcal{P},\mathcal{V})$ be a locally convex cone, and let $v \in \mathcal{V}$. For a sublinear functional $p : \mathcal{P} \to \overline{\mathbb{R}}$ and an extended superlinear functional $q : \mathcal{P} \to \overline{\overline{\mathbb{R}}}$ there exists a linear functional $\mu \in v^\circ$ such that $q \le \mu \le p$ if and only if $q(a) \le p(b) + 1$ holds whenever $a \le b + v$ for $a,b \in \mathcal{P}$.*

This theorem is the basic tool for the development of a duality theory for locally convex cones. It leads to a variety of Hahn-Banach type extension and separation results, the most general ones being Theorems 4.1 and 4.4 in [172]. For future use we shall quote both of these:

Extension Theorem 2.5. *Let $(\mathcal{P},\mathcal{V})$ be a locally convex cone, C and D non-empty convex subsets of $\mathcal{P}$, and let $v \in \mathcal{V}$. Let $p : \mathcal{P} \to \overline{\mathbb{R}}$ be a sublinear and $q : \mathcal{P} \to \overline{\overline{\mathbb{R}}}$ an extended superlinear functional. For a convex function $f : C \to \overline{\mathbb{R}}$ and a concave function $g : D \to \overline{\overline{\mathbb{R}}}$ there exists a monotone linear functional $\mu \in v^\circ$ such that*

$$q \le \mu \le p\,, \qquad g \le \mu \quad \text{on } D \qquad \text{and} \qquad \mu \le f \quad \text{on } C$$

if and only if

$$q(a) + \rho g(d) \le p(b) + \sigma f(c) + 1 \qquad \text{whenever} \qquad a + \rho d \le b + \sigma c + v$$

for $a,b \in \mathcal{P}$, $c \in C$, $d \in D$ and $\rho,\sigma \ge 0$.

In the context of this theorem (Theorem 4.1 in [172]), an $\overline{\mathbb{R}}$-valued function f defined on a convex subset $\mathcal{C}$ of an ordered cone $\mathcal{P}$ is called *convex* if

$$f\big(\lambda c_1 + (1-\lambda)c_2\big) \;\le\; \lambda f(c_1) + (1-\lambda) f(c_2)$$

holds for all $c_1, c_2 \in \mathcal{C}$ and $\lambda \in [0,1]$. Likewise, $f : \mathcal{C} \to \overline{\mathbb{R}}$ is called *concave* if

$$f\big(\lambda c_1 + (1-\lambda)c_2\big) \;\geq\; \lambda f(c_1) + (1-\lambda) f(c_2)$$

holds for all $c_1, c_2 \in \mathcal{C}$ and $\lambda \in [0,1]$. An *affine* function $f : \mathcal{C} \to \overline{\mathbb{R}}$ is both convex and concave.

The generality of Theorem 2.5 leads to a wide variety of applications and special cases. An extension theorem in the true meaning of the words can be obtained by identifying the convex sets C and D and the functions f and g. For the following (still very general) corollary we shall also leave out (by setting them equal to $+\infty$ and $-\infty$ outside $0 \in \mathcal{P}$, respectively) the functionals p and q.

Corollary 2.6. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone, C a non-empty convex subsets of $\mathcal{P}$, and let $v \in \mathcal{V}$. For an affine function $f : C \to \overline{\mathbb{R}}$ there exists a monotone linear functional $\mu \in v^\circ$ such that $\mu = f$ on C if and only if*

$$\rho f(d) \leq \sigma f(c) + 1 \qquad \textit{whenever} \qquad \rho d \leq \sigma c + v$$

for $c, d \in C$, and $\rho, \sigma \geq 0$.

If C is indeed a subcone of $\mathcal{P}$, that is $(C, \mathcal{V})$ is a locally convex subcone of $(\mathcal{P}, \mathcal{V})$, then the condition of Corollary 2.6 reduces to: $f(0) = 0$ and $f(d) \leq f(c) + 1$ holds whenever $d \leq c + v$ for $c, d \in C$. But this means that the affine function f is indeed a linear functional on $\mathcal{C}$ and contained in the polar of the neighborhood $v \in V$. This observation leads to the following most frequently used consequence of Theorem 2.5 (see also Theorem II.2.9 in [100]).

Corollary 2.7. *Let $(\mathcal{N}, \mathcal{V})$ be a subcone of the locally convex cone $(\mathcal{P}, \mathcal{V})$. Every continuous linear functional on $\mathcal{N}$ can be extended to a continuous linear functional on $\mathcal{P}$; more precisely: For every $\mu \in v^\circ_{\mathcal{N}}$ there is $\hat{\mu} \in v^\circ_{\mathcal{P}}$ such that $\hat{\mu}$ coincides with μ on $\mathcal{N}$.*

Theorem 4.4 in [172] deals with the separation of convex sets by continuous linear functionals, a result that can also be obtained by special insertions in Theorem 2.5.

Separation Theorem 2.8. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone, C and D non-empty convex subsets of $\mathcal{P}$, and let $v \in \mathcal{V}$. For $\alpha \in \mathbb{R}$ there exists a monotone linear functional $\mu \in v^\circ$ such that*

$$\mu(c) \leq \alpha \leq \mu(d) \qquad \textit{for all } c \in C \textit{ and } d \in D$$

if and only if

$$\alpha\rho \leq \alpha\sigma + 1 \qquad \textit{whenever} \qquad \rho d \leq \sigma c + v$$

for $c \in C$, $d \in D$ and $\rho, \sigma \geq 0$.

In a special case, this leads to a separation result for points and closed convex sets (see Corollary 4.6 in [172]):

Corollary 2.9. *Let A be a non-empty convex subset of a locally convex cone $(\mathcal{P},\mathcal{V})$ such that $0\in A$.*

(i) If A is closed with respect to the lower topology on $\mathcal{P}$, then for every element $b\notin A$ in $\mathcal{P}$ there exists a monotone linear functional $\mu\in\mathcal{P}^*$ such that

$$\mu(a)\le 1\le \mu(b)\qquad \text{for all } a\in A$$

and indeed $1<\mu(b)$ if b is bounded above.

(ii) If A is closed with respect to the upper topology on $\mathcal{P}$, then for every element $b\notin A$ in $\mathcal{P}$ there exists a monotone linear functional $\mu\in\mathcal{P}^*$ such that

$$\mu(b)<-1\le \mu(a)\qquad \text{for all } a\in A.$$

In view of the corresponding separation results for locally convex topological vector spaces, Corollary 2.9 is not entirely satisfying, in particular since it requires that $0\in A$. A stronger and more suitable separation statement will be derived in Section 4 (Theorem 4.30). It will make use of the relative topologies of a locally convex cone which are to be introduced below.

We shall quote and make use of another result from [172]. The Range Theorem (Theorem 5.1 in [172]) describes the scope of all linear functionals whose existence is guaranteed by the Sandwich Theorem. It is a powerful and indeed non-trivial consequence even in the special case of vector spaces, where its formulation can however be considerably simplified.

Range Theorem 2.10. *Let $(\mathcal{P},\mathcal{V})$ be a locally convex cone. Let p and q be sublinear and extended superlinear functionals on $\mathcal{P}$ and suppose that there is at least one linear functional $\mu\in\mathcal{P}^*$ satisfying $q\le\mu\le p$. Then for all $a\in\mathcal{P}$*

$$\sup_{\substack{\mu\in\mathcal{P}^*\\ q\le\mu\le p}}\mu(a) \;=\; \sup_{v\in\mathcal{V}}\inf\,\{p(b)-q(c)\mid b,c\in\mathcal{P},\;\; q(c)\in\mathbb{R},\;\; a+c\le b+v\},$$

and for all $a\in\mathcal{P}$ such that $\mu(a)$ is finite for at least one $\mu\in\mathcal{P}^*$ satisfying $q\le\mu\le p$

$$\sup_{\substack{\mu\in\mathcal{P}^*\\ q\le\mu\le p}}\mu(a) \;=\; \inf_{v\in\mathcal{V}}\sup\,\{q(c)-p(b)\mid b,c\in\mathcal{P},\;\; p(b)\in\mathbb{R},\;\; c\le a+b+v\}.$$

3. Weak Local and Global Preorders

In addition to the given order $\le$ on a locally convex cone, we shall frequently use the *weak (global) preorder* $\preccurlyeq$ (for details, see [175] and Section 4 below) which is slightly weaker then the given order and defined for $a, b \in \mathcal{P}$ by

$$a \preccurlyeq b \qquad \text{if} \qquad a \leq \gamma b + \varepsilon v$$

for all $v \in \mathcal{V}$ and $\varepsilon > 0$ with some $1 \leq \gamma \leq 1 + \varepsilon$. This order represents a closure of the given order with respect to the linear and topological structures of $\mathcal{P}$. It is obviously coarser than the given order, that is $a \leq b$ implies $a \preccurlyeq b$ for $a, b \in \mathcal{P}$. In the preceding Examples 1.4(a) and (b), however, both orders coincide. In 1.4(e) this depends on the order in $\mathcal{P}$ and the neighborhood-valued functions in $\hat{\mathcal{V}}$. If $\mathcal{P} = \overline{\mathbb{R}}$ and if for every $x \in X$ there is $\hat{v} \in \widehat{\mathcal{V}}$ such that $\hat{v}(x) < +\infty$, then the given and the weak preorder coincide. In 1.4(c), on the other hand, we have $A \preccurlyeq B$ if $A \subset \overline{\downarrow B}$, where $\overline{\downarrow B}$ denotes the topological closure in E of the decreasing hull $\downarrow B$ of B. Note that $\overline{\downarrow B}$ is again a convex subset of E. In 1.4(d), that is the case of a vector space $\mathcal{P}$ over $\mathbb{R}$ or $\mathbb{C}$, the weak preorder is given by $a \preccurlyeq b$ if $a - b \in v(0)$ for all $v \in \mathcal{V}$. In this way $(\mathcal{P}, \preccurlyeq)$ becomes a locally convex ordered topological vector space in the usual sense if endowed with the (modular) symmetric topology resulting from the neighborhood system.

The weak preorder on $\mathcal{P}$ is again compatible with the algebraic operations, as Lemma 4.1 below will imply. In Corollary 4.31 below (see also Theorem 3.1 in [175]) we shall establish that the weak preorder on a locally convex cone $\mathcal{P}$ is entirely determined by its dual cone $\mathcal{P}^*$, that is $a \preccurlyeq b$ holds for $a, b \in \mathcal{P}$ if and only if $\mu(a) \leq \mu(b)$ for all $\mu \in \mathcal{P}^*$. The weak preorder may also be used in a full cone containing $\mathcal{P}$ and $\mathcal{V}$. Consequently, the respective relation involving the neighborhoods in $\mathcal{V}$ is defined for elements $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$ as

$$a \preccurlyeq b + v \qquad \text{if} \qquad a \leq \gamma(b + v) + \varepsilon u$$

for all $u \in \mathcal{V}$ and $\varepsilon > 0$ with some $1 \leq \gamma \leq 1 + \varepsilon$. This condition can be slightly simplified:

Lemma 3.1. *Let $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$. We have $a \preccurlyeq b + v$ if and only if for every $\varepsilon > 0$ there is $1 \leq \gamma \leq 1 + \varepsilon$ such that $a \leq \gamma b + (1 + \varepsilon)v$.*

Proof. Let $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$. Suppose that $a \preccurlyeq b + v$ and let $\varepsilon > 0$. According to the preceding definition of the weak preorder involving neighborhoods, for $u = v$ and $\varepsilon/2$ in place of ε, there is $1 \leq \gamma \leq 1 + \varepsilon/2$ such that $a \leq \gamma(b+v)+(\varepsilon/2)v \leq \gamma b+\varepsilon v$. For the reverse implication suppose that the condition of the Lemma holds, and let $u \in \mathcal{V}$ and $\varepsilon > 0$. There is $\lambda \geq 0$ such that $0 \leq b + \lambda u$. Choose $0 < \delta \leq \varepsilon/\lambda$. Then there is $1 \leq \gamma \leq 1 + \delta$ such that

$$\begin{aligned} a &\leq \gamma b + (1 + \delta)v \\ &\leq \gamma b + (1 + \delta)v + (1 + \delta - \gamma)(b + \lambda u) \\ &\leq (1 + \delta)(b + v) + \delta\lambda\, u \\ &\leq (1 + \delta)(b + v) + \varepsilon u. \end{aligned}$$

This shows $a \preccurlyeq b + v$. $\square$

Endowed with the weak preorder $(\mathcal{P}, \mathcal{V})$ forms again a locally convex cone. For details we refer to [175]. In Corollary 4.34 below (see also Theorem 3.2 in [175]) we shall demonstrate that for $a, b \in \mathcal{P}$ and a neighborhood $v \in \mathcal{V}$, we have $a \preccurlyeq b + v$ if and only if $\mu(a) \le \mu(b) + 1$ holds for all $\mu \in v^\circ$. The neighborhoods with respect to the weak preorder in $\mathcal{P}$ are therefore entirely determined by their polars.

Given a neighborhood $v \in \mathcal{V}$ the *weak local preorder* (see [175]) $\preccurlyeq_v$ on $\mathcal{P}$ is the weak (global) preorder with respect to the neighborhood subsystem $\mathcal{V}_v = \{\alpha v \mid \alpha > 0\}$. That is, for $a, b \in \mathcal{P}$ we have

$$a \preccurlyeq_v b \qquad \text{if} \qquad a \le \gamma b + \varepsilon v$$

for all $\varepsilon > 0$ with some $1 \le \gamma \le 1 + \varepsilon$. Corollary 4.31 below (see also Theorem 3.1 in [175]) states that $a \preccurlyeq_v b$ if and only if $\mu(a) \le \mu(b)$ holds for all $\mu \in v^\circ$.

Lemma 3.2. *Let* $a, b \in \mathcal{P}$.

(a) $a \preccurlyeq b$ *if and only if for every* $v \in \mathcal{V}$ *and* $\varepsilon > 0$ *there is* $1 \le \gamma \le 1 + \varepsilon$ *such that* $a \preccurlyeq \gamma b + \varepsilon v$.

(b) $a \preccurlyeq_v b$ *for* $v \in \mathcal{V}$ *if and only if for every* $\varepsilon > 0$ *there is* $1 \le \gamma \le 1 + \varepsilon$ *such that* $a \preccurlyeq \gamma b + \varepsilon v$.

Proof. Part (a) follows from Part (b) as $a \preccurlyeq b$ holds if and only if $a \preccurlyeq_v b$ for all $v \in \mathcal{V}$. For Part (b) let $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$ such that the second condition in (b) holds. Given $\varepsilon > 0$ set $\delta = \min\{\varepsilon/3, 1\}$. Then $a \preccurlyeq \gamma b + \delta v$ holds with some $1 \le \gamma \le 1 + \delta$. We infer from Lemma 3.1 that there is $1 \le \gamma' \le 1 + \delta$ such that $a \le (\gamma'\gamma) b + (1 + \delta)\delta v$. Since $(1 + \delta)\delta \le \varepsilon$ and $1 \le \gamma'\gamma \le (1 + \delta)^2 \le 1 + \varepsilon$, and since $\varepsilon > 0$ was arbitrarily chosen, we conclude that $a \preccurlyeq_v b$. The reverse implication is trivial since $a \le \gamma b + \varepsilon v$ implies that $a \preccurlyeq \gamma b + \varepsilon v$. □

Lemma 3.2 shows in particular that the second iteration of the weak preorder, that is the second weak preorder generated by the first one does indeed coincide with the first one.

We observe that for a linear operator T between locally convex cones $(\mathcal{P}, \mathcal{V})$ and $(\mathcal{Q}, \mathcal{W})$, continuity with respect to the given orders implies continuity and monotonicity with respect to the respective weak preorders on $\mathcal{P}$ and $\mathcal{Q}$. Indeed, suppose that for $v \in \mathcal{V}$ and $w \in \mathcal{W}$ we have $T(a) \le T(b) + w$ whenever $a \le b + v$ for $a, b \in \mathcal{P}$. Let $a \preccurlyeq b + v$ and let $\varepsilon > 0$. According to Lemma 3.1 there is $1 \le \gamma \le 1 + \varepsilon$ such that $a \le \gamma b + (1 + \varepsilon)v$. Thus $T(a) \le \gamma T(b) + (1 + \varepsilon)w$. Since $\varepsilon > 0$ was arbitrarily chosen, we conclude that $T(a) \preccurlyeq T(b) + w$, thus establishing our claim.

The weak preorder may also be used to establish a representation for a locally convex cone $(\mathcal{P}, \mathcal{V})$ as a cone of continuous $\overline{\mathbb{R}}$-valued functions on some topological space and as a cone of convex subsets of some locally convex topological vector space, respectively. We shall cite Theorem 4.1 from [175]. Recall the definition of an embedding from 2.2.

Theorem 3.3. *Every locally convex cone $(\mathcal{P},\mathcal{V})$ can be embedded with respect to its weak preorder into*

(i) a locally convex cone of continuous $\overline{\mathbb{R}}$-valued functions on some topological space X, endowed with the pointwise order and operations and the topology of uniform convergence on a family of compact subsets of X.

(ii) a locally convex cone of convex subsets of a locally convex topological vector space, endowed with the usual addition and multiplication by scalars, the set inclusion as order and the neighborhoods inherited from the vector space.

4. Boundedness and the Relative Topologies

While all elements of a locally convex cone are bounded below by definition, they need not to be bounded above. Given a neighborhood $v \in \mathcal{V}$, an element a of a locally convex cone $(\mathcal{P},\mathcal{V})$ is called *v-bounded (above)* (see [100], I.2.3) if there is $\lambda \geq 0$ such that $a \leq \lambda v$. The subset $\mathcal{B}_v \subset \mathcal{P}$ of all v-bounded elements is a subcone and even a face of $\mathcal{P}$. Correspondingly, by $\mathcal{B} = \bigcap_{v\in\mathcal{V}} \mathcal{B}_v$ we denote the subcone (and face) of all bounded elements of $\mathcal{P}$ (see Section 1 and Proposition 4.11 below). All invertible elements of $\mathcal{P}$ were seen to be bounded, and continuous linear functionals take only finite values on bounded elements (see Section 2).

The presence of unbounded elements constitutes a significant difference between locally convex cones and locally convex topological vector spaces. It tends to make matters more interesting, but also considerably more complicated. If, for example, the element $a \in \mathcal{P}$ is not bounded, then the mapping $\alpha \mapsto \alpha a : [0,+\infty) \to \mathcal{P}$, is not necessarily continuous if we consider the usual topology of $[0,+\infty)$ and any of the given (upper, lower or symmetric) topologies on $\mathcal{P}$ $\big($see Proposition 1.1(iii)$\big)$. Hence these topologies appear to be rather restrictive. For similar reasons, our upcoming definition of measurability for $\mathcal{P}$-valued functions in Chapter II would turn out to be very limiting if applied to the given topologies of a locally convex cone. We shall therefore introduce slightly coarser neighborhoods on $\mathcal{P}$ which take unbounded elements suitably into account. Given a neighborhood $v \in \mathcal{V}$ and $\varepsilon > 0$, we define the corresponding *upper* and *lower relative neighborhoods* $v_\varepsilon(a)$ and $(a)v_\varepsilon$ for an element $a \in \mathcal{P}$ by

$$\begin{aligned} v_\varepsilon(a) &= \{\, b \in \mathcal{P} \mid b \leq \gamma a + \varepsilon v \quad \text{for some} \quad 1 \leq \gamma \leq 1+\varepsilon \,\} \\ (a)v_\varepsilon &= \{\, b \in \mathcal{P} \mid a \leq \gamma b + \varepsilon v \quad \text{for some} \quad 1 \leq \gamma \leq 1+\varepsilon \,\}. \end{aligned}$$

Their intersection $v^s_\varepsilon(a) = v_\varepsilon(a) \cap (a)v_\varepsilon$ is the corresponding *symmetric relative neighborhood*. These are of course convex subsets of $\mathcal{P}$. Note that for a positive element $a \in \mathcal{P}$ the above expressions somewhat simplify. Since $\gamma a \leq (1+\varepsilon)a$ in this case, we have $v_\varepsilon(a) = \{\, b \in \mathcal{P} \mid b \leq (1+\varepsilon)a + \varepsilon v\}$ and

$(a)v_\varepsilon = \{ b \in \mathcal{P} \mid a \leq (1+\varepsilon)b + \varepsilon v \}$. We shall frequently use the following observations:

Lemma 4.1. *Let* $a, b, c, a_i, b_i \in \mathcal{P}$, $v \in \mathcal{V}$, $\lambda \geq 0$ *and* $\varepsilon, \delta > 0$.

(a) If $a \in v_\varepsilon(b)$ *and* $b \in v_\delta(c)$, *then* $a \in v_{(\varepsilon+\delta+\varepsilon\delta)}(c)$.
(b) If $a \in v_\varepsilon(b)$ *and* $0 \leq b + \lambda v$, *then* $a \leq (1+\varepsilon)b + \varepsilon(1+\lambda)v$.
(c) If $a \in v_\varepsilon(b)$ *and* $0 \leq a + \lambda v$, *then* $a \leq (1+\varepsilon)b + \varepsilon(1+\lambda+\varepsilon)v$ *and* $0 \leq b + (\lambda+\varepsilon)v$.
(d) If $a_i \in v_\varepsilon(b_i)$ *and if* $0 \leq b_i + \lambda v$ *for* $i = 1, \ldots, n$, *then* $(a_1 + \ldots + a_n) \in v_{\varepsilon n(1+\lambda)}(b_1 + \ldots + b_n)$.

Proof. For (a), let $a \in v_\varepsilon(b)$ and $b \in v_\delta(c)$, that is $a \leq \gamma b + \varepsilon v$ and $b \leq \lambda c + \delta v$ for some $1 \leq \gamma \leq 1+\varepsilon$ and $1 \leq \lambda \leq 1+\delta$. Then $a \leq \gamma\lambda c + (\gamma\delta+\varepsilon)v$. As

$$\gamma\delta + \varepsilon \leq (1+\varepsilon)\delta + \varepsilon = \varepsilon + \delta + \varepsilon\delta$$

and

$$1 \leq \gamma\lambda \leq (1+\varepsilon)(1+\delta) = 1 + \varepsilon + \delta + \varepsilon\delta,$$

we have $a \in v_{(\varepsilon+\delta+\varepsilon\delta)}(c)$. For (b), let $a \in v_\varepsilon(b)$, that is $a \leq \gamma b + \varepsilon v$ for some $1 \leq \gamma \leq 1+\varepsilon$. If $0 \leq b + \lambda v$, then

$$a \leq \gamma b + \varepsilon v + (1+\varepsilon-\gamma)(b+\lambda v) \leq (1+\varepsilon)b + (\varepsilon+\varepsilon\lambda)v.$$

For (c), let $a \in v_\varepsilon(b)$ and $\lambda \geq 0$ such that $0 \leq a + \lambda v$. Then $a \leq \gamma b + \varepsilon v$ with some $1 \leq \gamma \leq 1+\varepsilon$, hence $0 \leq \gamma b + (\varepsilon+\lambda)v$, and indeed $0 \leq b + \frac{\varepsilon+\lambda}{\gamma} v \leq b + (\varepsilon+\lambda)v$. Part (b) yields $a \leq (1+\varepsilon)b + \varepsilon(1+\lambda+\varepsilon)v$. For (d), let $a_i \in v_\varepsilon(b_i)$ and $0 \leq b_i + \lambda v$. Then $a_i \leq (1+\varepsilon)b_i + \varepsilon(1+\lambda)v$ by Part (b). This yields

$$a_1 + \ldots + a_n \leq (1+\varepsilon)(b_1 + \ldots + b_n) + n\varepsilon(1+\lambda)v,$$

hence our claim. □

Property 4.1(a) implies in particular that $v_\varepsilon(a) \subset v_{3\varepsilon}(c)$ whenever $a \in v_\varepsilon(b)$ and $b \in v_\varepsilon(c)$ for $a, b, c \in \mathcal{P}$ and $0 < \varepsilon \leq 1$. Similar statements as in Lemma 4.1 hold for the lower and for the symmetric relative neighborhoods.

For elements $a, b \in \mathcal{P}$ the weak local and global preorders on $\mathcal{P}$ as defined in Section 3 can be recovered as

$$a \preccurlyeq_v b \qquad \text{if} \qquad a \in v_\varepsilon(b)$$

for some $v \in \mathcal{V}$ and all $\varepsilon > 0$, and

$$a \preccurlyeq b \qquad \text{if} \qquad a \in v_\varepsilon(b)$$

for all $\varepsilon > 0$ and $v \in \mathcal{V}$. Lemma 4.1(d) implies that these orders are compatible with the algebraic operations in $\mathcal{P}$.

For varying $v \in \mathcal{V}$ and $\varepsilon > 0$ the neighborhoods $v_\varepsilon(\cdot)$, $(\cdot)v_\varepsilon$ and $v^s_\varepsilon(\cdot)$ create the *upper, lower* and *symmetric relative topologies* on $\mathcal{P}$, respectively.

We notice that $a \le b + v$ for $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$ implies that $a \preccurlyeq b + v$, and for a given $\varepsilon > 0$, with $\delta = \min\{1, \varepsilon/2\}$, we notice that $a \preccurlyeq b + \delta v$ implies $a \le \gamma b + (1 + \delta)\delta v \le \gamma b + \varepsilon v$ for some $1 \le \gamma \le 1 + \delta$ (see Lemma 3.1), hence $b \in v_\varepsilon(a)$. This observation demonstrates that the given upper, lower and symmetric topologies on $\mathcal{P}$ are finer than those induced by the same neighborhood system using the weak preorder, and that in turn these topologies are finer than the above defined relative topologies.

However, while the relative neighborhoods form convex subsets of $\mathcal{P}$, they do in general not create a locally convex cone topology. Indeed, the sets $\big\{(a, b) \mid a \in v_\varepsilon(b)\big\}$ are not necessarily convex in $\mathcal{P}^2$, hence do not establish a convex semiuniform structure on $\mathcal{P}$ in the sense of 1.3.

For later reference we shall list some further properties of the relative topologies and use the earlier introduced standard notations for subsets of $\mathcal{P}$ (see 1.1).

Proposition 4.2. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone. The upper (lower or symmetric) relative topology of $\mathcal{P}$ is coarser than the given upper (lower or symmetric) topology and satisfies the following:*

(i) Every element of $\mathcal{P}$ admits a basis of convex and decreasing (increasing or order convex) neighborhoods. The symmetric relative neighborhoods in the basis for $0 \in \mathcal{P}$ are also balanced.
(ii) The mapping $(a, b) \mapsto a + b : \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ is continuous.
(iii) The mapping $(\alpha, a) \mapsto \alpha a : [0, +\infty) \times \mathcal{P} \to \mathcal{P}$ is continuous at all points $(\alpha, a) \in [0, +\infty) \times \mathcal{P}$ such that either $\alpha > 0$ or $a \in \mathcal{P}$ is bounded.
(iv) For bounded elements of $\mathcal{P}$ the neighborhoods in the upper (lower or symmetric) relative topology are equivalent to the neighborhoods in the given upper (lower or symmetric) topology.

Proof. We observed before that the relative topologies are coarser than the given topologies on $\mathcal{P}$. Clearly, for $a \in \mathcal{P}$, $v \in \mathcal{V}$ and $\varepsilon > 0$ the relative neighborhoods $v_\varepsilon(a)$, $(a)v_\varepsilon$ or $v^s_\varepsilon(a)$ are convex and decreasing, increasing or order convex, respectively. The symmetric relative neighborhoods of $0 \in \mathcal{P}$ are also balanced. Indeed, let $v \in \mathcal{V}$ and $\varepsilon > 0$ and let $a \in v^s_\varepsilon(0)$. Then $a \le \varepsilon v$ and $0 \le \gamma a + \varepsilon v$ for some $1 \le \gamma \le 1 + \varepsilon$. Thus $0 \le a + (\varepsilon/\gamma)v \le a + \varepsilon v$. Let $0 \le \lambda \le 1$. Then $\lambda a \in v^s_\varepsilon(0)$ follows from the convexity of $v^s_\varepsilon(0)$ since $\lambda a = \lambda a + (1 - \lambda)0$. If on the other hand $b + \lambda a = 0$ for $b \in \mathcal{P}$, then

$$b \le b + \lambda(a + \varepsilon v) \le \varepsilon v \qquad \text{and} \qquad 0 = b + \lambda a \le b + \varepsilon v$$

Hence $b \in v_\varepsilon(0)$ holds in this case as well.

For property (ii), let $a, b \in \mathcal{P}$, $v \in \mathcal{V}$ and $\varepsilon > 0$. There is $\lambda \ge 0$ such that $0 \le a + \lambda v$ and $0 \le b + \lambda v$. Choose $\delta = \varepsilon/(2\lambda + 4)$. Then for $c \in v_\delta(a)$ and $d \in v_\delta(b)$ we have $c + d \in v_{2\delta(1+\lambda)}(a + b) = v_\varepsilon(a + b)$ by Lemma 4.1(d). This shows continuity of the addition with respect to the upper relative topology. Next with the same choice for δ, let $c \in (a)v_\delta$ and $d \in (b)v_\delta$, that is $a \in v_\delta(c)$ and $b \in v_\delta(d)$. Then we have $a \le \gamma c + \delta v$ for some $1 \le \gamma \le 1+\delta$, hence $0 \le \gamma c + (\lambda+\delta)v$ and $0 \le c + (\lambda+1)v$. Likewise, $0 \le d + (\lambda+1)v$. Now 4.1(d) yields $a + b \in v_{2\delta(2+\lambda)}(c+d) \subset v_\varepsilon(c+d)$. Thus $c + d \in (a+b)v_\varepsilon$. This shows continuity of the addition with respect to the lower relative topology. Combining the preceding arguments, we realize that $c \in v^s_\delta(a)$ and $d \in v^s_\delta(b)$ yields $c + d \in v^s_\varepsilon(a+b)$, which proves continuity with respect to the symmetric relative topology.

Next we shall argue Part (iv): Let $a \in \mathcal{P}$ be a bounded element, let $v \in \mathcal{V}$ and $\varepsilon > 0$. There is $\lambda \ge 0$ such that $a \le \lambda v$. We shall verify that

$$(\varepsilon v)(a) \subset v_\varepsilon(a) \subset (\rho v)(a) \qquad \text{and} \qquad (a)(\varepsilon v) \subset (a)v_\varepsilon(a) \subset (a)(\rho v).$$

with $\rho = \varepsilon(1+\lambda)$. Indeed, the inclusions $(\varepsilon v)(a) \subset v_\varepsilon(a)$ and $(\varepsilon v)(a) \subset v_\varepsilon(a)$ are obvious. Moreover, for $b \in v_\varepsilon(a)$ we have $b \le \gamma a + \varepsilon v$ with some $1 \le \gamma \le 1+\varepsilon$. Then $\gamma a = a + (\gamma-1)a \le a + \varepsilon\lambda v$ implies that $b \le a + \varepsilon(1+\lambda)v = a + \rho v$, hence $b \in (\rho v)(a)$. For $b \in (a)v_\varepsilon$ on the other hand, we have $a \le \gamma b + \varepsilon v$ with $1 \le \gamma \le 1+\varepsilon$. Then $\gamma a \le a + \varepsilon\lambda v$ implies $\gamma a \le \gamma b + \varepsilon(1+\lambda)v = \gamma b + \rho v$, hence $a \le b + (\rho/\gamma)v \le b + \rho v$, and therefore $b \in (a)(\rho v)$.

For the first case in Part (iii) let $(\alpha, a) \in (0,+\infty) \times \mathcal{P}$. For $v \in \mathcal{V}$ and $\varepsilon > 0$ let $\lambda \ge 0$ such that $0 \le a + \lambda v$. For $0 < \delta < \min\{1, \varepsilon/3, \varepsilon/(2\alpha(1+\lambda))\}$ we consider the neighborhoods $u_\delta(\alpha) = [\alpha/(1+\delta), \alpha(1+\delta)]$ of α in $[0,+\infty)$ and $v_\delta(a)$ of a in $\mathcal{P}$. For every $b \in v_\delta(a)$ we have $b \le (1+\delta)a + \delta(1+\lambda)v$ by 4.1(b). For $\beta \in u_\delta(\alpha)$ we set $\gamma = \beta(1+\delta)/\alpha$ and estimate

$$\beta b \le \beta(1+\delta)a + \beta\delta(1+\lambda)v = \gamma(\alpha a) + \beta\delta(1+\lambda)v.$$

Now $\alpha/(1+\delta) \le \beta \le \alpha(1+\delta)$ and our choice for δ implies $1 \le \gamma \le (1+\delta)^2 \le 1+\varepsilon$ as well as $\beta\delta(1+\lambda) \le \alpha(1+\delta)\delta(1+\lambda) \le 2\alpha\delta(1+\lambda) \le \varepsilon$. Thus $\beta b \in v_\varepsilon(\alpha a)$. This shows continuity for the scalar multiplication at (α, a) with respect to the upper relative topology. For the lower topology, with the same choice for δ, let $b \in (a)v_\delta$ and $\beta \in u_\delta(\alpha)$. Then $a \le (1+\delta)b + \delta(2+\lambda)v$ by 4.1(c). We set $\gamma = \alpha(1+\delta)/\beta$ and obtain

$$\alpha a \le \alpha(1+\delta)b + \alpha\delta(2+\lambda)v = \gamma(\beta b) + \alpha\delta(2+\lambda)v.$$

We verify $1 \le \gamma \le 1+\varepsilon$ and $\alpha\delta(2+\lambda) \le \varepsilon$ and infer that $\alpha a \in (\beta b)v_\varepsilon$, hence $\beta b \in (\alpha a)v_\varepsilon$. This shows continuity with respect to the lower relative topology. The combination of both arguments yields continuity with respect to the symmetric relative topology.

The second case of Part (iii), that is the continuity of the scalar multiplication at $(\alpha, a) \in [0,+\infty) \times \mathcal{P}$ for a bounded element $a \in \mathcal{P}$, follows directly from Part (iv) and from Part (iii) of Proposition 1.1. Indeed, the given and the relative upper (lower or symmetric) topologies coincide locally at $a \in \mathcal{P}$ by (iv), thus continuity with respect to any of the given topologies which was established in Proposition 1.1(iii) implies continuity with respect to the corresponding relative topology. □

For $\mathcal{P} = \overline{\mathbb{R}}$, in particular, Part (iv) of the preceding proposition implies that the given and the relative topologies coincide on all reals. They also coincide on the element $+\infty$, thus everywhere, as can be easily verified (for details on this, see Example 4.37(a) below).

Part (iv) together with Corollary 1.2 also yields:

Corollary 4.3. *Let* $(\mathcal{P}, \mathcal{V})$ *be a locally convex cone and let* $\mathcal{P}_0$ *be the subcone of all invertible elements of* $\mathcal{P}$*. The mapping* $(\alpha, a) \mapsto \alpha a : \mathbb{R} \times \mathcal{P}_0 \to \mathcal{P}_0$ *is continuous with respect to the symmetric relative topology of* $\mathcal{P}$*.*

We observe that the given upper (lower or symmetric) topologies do of course satisfy the properties listed in Proposition 4.2 with the exception of 4.2(iii). More precisely, we take note:

Proposition 4.4. *Let* $(\mathcal{P}, \mathcal{V})$ *be a locally convex cone. The upper (or lower) relative topology is the finest topology on* $\mathcal{P}$ *which is coarser than the given upper (or lower) topology and satisfies property* (iii) *from Proposition 4.2.*

Proof. Let τ be any topology on $\mathcal{P}$ which is finer than the upper (or lower) topology and satisfies property (iii) from Proposition 4.2. Let $a \in \mathcal{P}$ and let $U(a)$ be a neighborhood in τ for a. We shall show that $U(a)$ contains some upper (or lower) relative neighborhood of a. The mapping $(\alpha, b) \mapsto \alpha b : [0, +\infty) \times \mathcal{P} \to \mathcal{P}$ is continuous with respect to τ at the point $(1, a)$. Thus there is a neighborhood $V(a)$ in τ and $0 < \varepsilon \le 1$ such that $\beta b \in U(a)$ for all $b \in V(a)$ and $\beta \in [1-\varepsilon, 1+\varepsilon]$. Moreover, since τ is coarser than the upper (or lower) topology of $\mathcal{P}$ there is $v \in \mathcal{V}$ such that $v(a) \subset V(a)$ $\big($or $(a)v \subset V(a)\big)$. In the case of the upper topology, then for every $c \in v_\varepsilon(a)$ we have $c \le \gamma a + \varepsilon v$ for some $1 \le \gamma \le 1 + \varepsilon$. Thus $d \le a + (\varepsilon/\gamma)v \le a + v$ for $d = (1/\gamma)c$. We infer that $d \in v(a) \subset V(a)$, hence $c = \gamma d \in U(a)$ since $\gamma \in [1-\varepsilon, 1+\varepsilon]$. This shows $v_\varepsilon(a) \subset U(a)$. Likewise, in the case of the lower topology, for $c \in (a)v_\varepsilon$ we have $a \le \gamma c + \varepsilon v$ for some $1 \le \gamma \le 1 + \varepsilon$, hence $d = \gamma c \in (a)(v) \subset V(a)$. This yields $c = (1/\gamma)d \in U(a)$ since $(1/\gamma) \in [1-\varepsilon, 1+\varepsilon]$. We conclude that $(a)v_\varepsilon \subset U(a)$ in this case. □

Proposition 4.5. *Let* $(\mathcal{P}, \mathcal{V})$ *and* $(\mathcal{Q}, \mathcal{W})$ *be locally convex cones. A continuous linear operator* $T : \mathcal{P} \to \mathcal{Q}$ *is also continuous if both* $\mathcal{P}$ *and* $\mathcal{Q}$ *are endowed with either their respective upper, lower or symmetric relative topologies.*

Proof. Let $T : \mathcal{P} \to \mathcal{Q}$ be a continuous linear operator. Given $w \in \mathcal{W}$, there is $v \in \mathcal{V}$ such that $a \le b + v$ implies $T(a) \le T(b) + w$ for elements $a, b \in \mathcal{P}$. Thus $a \in v_\varepsilon(b)$, that is $a \le \gamma b + \varepsilon v$ with some $1 \le \gamma \le 1 + \varepsilon$, implies $T(a) \le \gamma T(b) + \varepsilon w$, hence $T(a) \in w_\varepsilon\big(T(b)\big)$. A similar argument shows continuity with respect to either the lower or symmetric relative topologies of $\mathcal{P}$ and $\mathcal{Q}$. □

For $\mathcal{Q} = \overline{\mathbb{R}}$, in particular, we remarked earlier (see also Example 4.37(a) below) that the given and the relative topologies coincide. A linear functional $\mu \in \mathcal{P}^*$ is therefore also continuous if we endow $\mathcal{P}$ with either of its relative and $\overline{\mathbb{R}}$ with the corresponding given topology.

We shall also use the *(upper, lower, symmetric) relative v-topologies* on $\mathcal{P}$, generated by the relative neighborhoods for a fixed $v \in \mathcal{V}$. The symmetric relative v-topology, in particular, is induced by the pseudometric

$$d_v(a,b) = \inf\left\{1, \sqrt{\varepsilon} \mid a \in v^s_\varepsilon(b)\right\}.$$

The properties of a pseudometric (see Section 2.1 in [198]) are readily checked for this expression: We obviously have $d_v(a,b) \geq 0$, $d_v(a,a) = 0$ and $d_v(a,b) = d_v(b,a)$ for $a,b \in \mathcal{P}$. The triangular inequality, namely $d_v(a,c) \leq d_v(a,b) + d_v(b,c)$ for $a,b,c \in \mathcal{P}$, holds trivially true if either $d_v(a,b) = 1$ or $d_v(b,c) = 1$. Otherwise, if $d_v(a,b) < \varepsilon < 1$ and $d_v(b,c) < \delta < 1$, then $a \in v^s_{\varepsilon^2}(b)$ and $b \in v^s_{\delta^2}(c)$ implies by Lemma 4.1(a) that $a \in v^s_\rho(c)$, where $\rho = \varepsilon^2 + \delta^2 + \varepsilon^2\delta^2 \leq (\varepsilon+\delta)^2$. Thus $d_v(a,c) \leq \varepsilon + \delta$, hence the triangular inequality holds. As a consequence of the availability of a pseudometric for the symmetric relative v-topology, arbitrary subsets of separable subsets of $\mathcal{P}$ remain separable (see 16G in [198]). We shall use this fact in Chapter II.

The (upper, lower, symmetric) relative topologies on $\mathcal{P}$ are the common refinements of all (upper, lower, symmetric) relative v-topologies.

4.6 The Weak Topology $\boldsymbol{\sigma(\mathcal{P},\mathcal{P}^*)}$. The *weak topology* $\sigma(\mathcal{P},\mathcal{P}^*)$ on a locally convex cone $(\mathcal{P},\mathcal{V})$ is generated by its dual cone in the following way: For an element $a \in \mathcal{P}$ an upper neighborhood $\mathcal{V}_\Upsilon(a)$, corresponding to a finite subset $\Upsilon = \{\mu_1,\ldots,\mu_n\}$ of $\mathcal{P}^*$, is given by

$$\mathcal{V}_\Upsilon(a) = \left\{b \in \mathcal{P} \;\mid \mu_i(b) \leq \mu_i(a) + 1 \quad \text{for all } \mu_i \in \Upsilon\right\}.$$

Endowed with these neighborhoods, $\mathcal{P}$ forms again a locally convex cone (see Section II.3 in [100]). We are mostly interested in the resulting symmetric topology $\sigma(\mathcal{P},\mathcal{P}^*)$ which is generated by the symmetric neighborhoods

$$\mathcal{V}^s_\Upsilon(a) = \left\{ b \in \mathcal{P} \;\middle|\; \begin{array}{rll} |\mu_i(b) - \mu_i(a)| \leq & 1\,, & \text{if} \quad \mu_i(a) < +\infty \\ \mu_i(b) = & +\infty\,, & \text{if} \quad \mu_i(a) = +\infty \end{array} \right\}$$

In this way *weak convergence* for a net $(a_i)_{i\in\mathcal{I}}$ in $(\mathcal{P},\mathcal{V})$ means that $\big(\mu(a_i)\big)_{i\in\mathcal{I}}$ converges towards $\mu(a)$ in $\overline{\mathbb{R}}$ (with respect to the symmetric locally convex cone topology of $\overline{\mathbb{R}}$) for every continuous linear functional $\mu \in \mathcal{P}^*$.

While the relative topologies of a locally convex cone are generally coarser than the given ones, we observe from the preceding definition that the relative upper, lower and symmetric weak topologies do indeed coincide with the given upper, lower and symmetric weak topologies on $\mathcal{P}$.

Lemma 4.7. *The weak topology $\sigma(\mathcal{P},\mathcal{P}^*)$ on a locally convex cone $(\mathcal{P},\mathcal{V})$ is coarser than the symmetric relative topology.*

Proof. For this, let $a \in \mathcal{P}$, let Υ be a finite subset of $\mathcal{P}^*$ and consider the weak neighborhood $\mathcal{V}^s_\Upsilon(a)$ from above. Choose $v \in \mathcal{V}$ such that $\mu_i \in v^\circ$ for all $i = 1,\ldots,n$. We shall show that for a suitable $\varepsilon > 0$ the symmetric neighborhood $v^s_\varepsilon(a)$ is contained in $\mathcal{V}^s_\Upsilon(a)$. Indeed, let $b \in v^s_\varepsilon(a)$, that is

$$b \le \gamma a + \varepsilon v \qquad \text{and} \qquad a \le \gamma' b + \varepsilon v$$

for some $1 \le \gamma, \gamma' \le 1+\varepsilon$. Thus

$$\mu(b) \le \gamma\mu(a) + \varepsilon \qquad \text{and} \qquad \mu(a) \le \gamma'\mu(b) + \varepsilon$$

for all $\mu \in \Upsilon$. If $\mu(a) = +\infty$, then $\mu(a) = +\infty$. Moreover, $\varepsilon > 0$ may be chosen such that the above implies $|\mu(b) - \mu(a)| \le 1$ for all $\mu \in \Upsilon$ such that $\mu(a) < +\infty$. This shows $b \in \mathcal{V}^s_\Upsilon(a)$. □

Proposition 4.8. *Let $(\mathcal{P},\mathcal{V})$ be a locally convex cone. The following statements are equivalent:*

(i) The symmetric relative topology on $\mathcal{P}$ is Hausdorff.
(ii) The weak topology on $\mathcal{P}$ is Hausdorff.
(iii) The weak preorder on $\mathcal{P}$ is antisymmetric.

Proof. Clearly, (ii) implies (i), since the symmetric relative topology is finer than $\sigma(\mathcal{P},\mathcal{P}^*)$. If $a \preccurlyeq b$ and $b \preccurlyeq a$ for $a,b \in \mathcal{P}$, then $a \in v^s_\varepsilon(b)$ for all $v \in \mathcal{V}$ and $\varepsilon > 0$. If the symmetric relative topology is Hausdorff, then this implies $a = b$. Thus (i) implies (iii). If the weak preorder is antisymmetric, then for distinct elements $a,b \in \mathcal{P}$ we have either $a \not\preccurlyeq b$ or $b \not\preccurlyeq a$, thus $a \not\preccurlyeq b+v$ or $b \not\preccurlyeq a+v$ for some $v \in \mathcal{V}$ by Lemma 3.2. Then there exists a linear functional $\mu \in v^\circ$ such that $\mu(a) > \mu(b)+1$ or $\mu(b) > \mu(a)+1$, respectively (see Section 3 and Corollary 4.34 below). The weak neighborhoods $\mathcal{V}^s_{\{(1/3)\mu\}}(a)$ and $\mathcal{V}^s_{\{(1/3)\mu\}}(b)$ are therefore seen to be disjoint. Thus (iii) implies (ii) as well. □

4.9 Boundedness Components. For an element $a \in \mathcal{P}$ we define the *upper* and *lower boundedness components* of a as

$$\mathcal{B}(a) = \bigcap_{v\in\mathcal{V}} \bigcup_{\varepsilon>0} v_\varepsilon(a) \qquad \text{and} \qquad (a)\mathcal{B} = \bigcap_{v\in\mathcal{V}} \bigcup_{\varepsilon>0} (a)v_\varepsilon,$$

respectively. The elements of $\mathcal{B}(a)$ are called *bounded above relative to* a. Correspondingly, the elements of $(a)\mathcal{B}$ are called *bounded below relative to* a. By the definition of a locally convex cone we have $0 \in \mathcal{B}(a)$ for all $a \in \mathcal{P}$, and $\mathcal{B}(0) = \mathcal{B}$ consists of all bounded elements of $\mathcal{P}$. We shall first list a few basic properties of the upper boundedness components.

Proposition 4.10. *Let* $a, b, \in \mathcal{P}$. *The following are equivalent:*

(i) $b \in \mathcal{B}(a)$.
(ii) $\mathcal{B}(b) \subset \mathcal{B}(a)$.
(iii) For every $v \in \mathcal{V}$ *there are* $\alpha, \beta \geq 0$ *such that* $b \leq \alpha a + \beta v$.
(iv) The mapping

$$\alpha \mapsto a + \alpha b \ : \ [0, +\infty) \to \mathcal{P}$$

is continuous with respect to the symmetric relative topology of $\mathcal{P}$.
(v) For all $\mu \in \mathcal{P}^*$, $\mu(a) < +\infty$ *implies* $\mu(b) < +\infty$.

Proof. Let $a, b \in \mathcal{P}$. We shall first establish the equivalence of (i), (ii) and (iii): Suppose that $b \in \mathcal{B}(a)$ and let $c \in \mathcal{B}(b)$. Then for every $v \in \mathcal{V}$ there are $\varepsilon, \delta > 0$ such that $c \in v_\varepsilon(b)$ and $b \in v_\delta(a)$. Following Lemma 4.1(a), this implies $c \in v_{(\varepsilon+\delta+\varepsilon\delta)}(a)$. We conclude that $c \in \mathcal{B}(a)$, hence $\mathcal{B}(b) \subset \mathcal{B}(a)$, and (i) implies (ii). If $\mathcal{B}(b) \subset \mathcal{B}(a)$, then $b \in \mathcal{B}(a)$ since $b \in \mathcal{B}(b)$ trivially holds. Thus for every $v \in \mathcal{V}$ there is $\varepsilon > 0$ such that $b \in v_\varepsilon(a)$, that is $b \leq \alpha a + \beta v$ for some $\alpha, \beta \geq 0$. Therefore (ii) implies (iii). If, on the other hand, for some $v \in \mathcal{V}$ we have $b \leq \alpha a + \beta v$ for $\alpha, \beta \geq 0$, we choose $\lambda \geq 0$ such that $0 \leq a + \lambda v$. Then

$$b \leq (\alpha a + \beta v) + (a + \lambda v) = (1 + \alpha) a + (\beta + \lambda) v,$$

hence $b \in v_\varepsilon(a)$ for every $\varepsilon > \max\{\alpha, \beta + \lambda\}$. If this argument can be made for all $v \in \mathcal{V}$, then we have $b \in \mathcal{B}(a)$, hence (iii) implies (i) as well, and the Conditions (i), (ii) and (iii) are seen to be equivalent.

Next we shall verify that (iii) implies (iv): Following Proposition 4.2(iii), for any choice of $b \in \mathcal{P}$ the mapping $\alpha \mapsto \alpha b$ is continuous with respect to the symmetric relative topology of $\mathcal{P}$ on the open interval $(0, +\infty)$. Likewise, of course, is the constant mapping $\alpha \mapsto a$. Thus by the continuity of the addition in $\mathcal{P}$ (see Proposition 4.2(ii)), the mapping $f : [0, +\infty) \to \mathcal{P}$ such that

$$f(\alpha) = a + \alpha b$$

is also continuous on $(0, +\infty)$. In case that (iii) holds, we shall verify continuity at $\alpha = 0$ as well: Given $v \in \mathcal{V}$ and $\varepsilon > 0$ there is $\lambda > 0$ such that $0 \leq b + \lambda v$, and by (iii) there are $\gamma, \rho \geq 0$ such that $b \leq \gamma a + \rho v$. Then for $\delta = \min\{\varepsilon/\gamma, \varepsilon/\rho, \varepsilon/\lambda\}$ and all $\alpha \in [0, \delta)$ we have

$$a + \alpha b \leq a + \alpha(\gamma a + \rho v) \leq (1 + \alpha\gamma) a + \alpha\rho v.$$

Since our choice of δ guarantees that both $\alpha\gamma \leq \varepsilon$ and $\alpha\rho \leq \varepsilon$, we infer that $f(\alpha) \in v_\varepsilon\big(f(0)\big)$. Similarly, one observes that

$$a \leq a + \alpha(b + \lambda v) \leq (a + \alpha b) + \alpha\lambda v$$

holds for all $\alpha \geq 0$. If indeed $\alpha \in [0, \delta)$, then our choice for δ guarantees that $\alpha\lambda \leq \varepsilon$. This shows $f(0) \in v_\varepsilon\big(f(\alpha)\big)$, that is $f(\alpha) \in \big(f(0)\big)v_\varepsilon$, and together with the above, $f(\alpha) \in v_\varepsilon^s\big(f(0)\big)$ for all $\alpha \in [0, \delta)$. We infer continuity for the function f at $\alpha = 0$ with respect to the symmetric relative topology of $\mathcal{P}$.

Next suppose that (iv) holds. Then for every linear functional $\mu \in \mathcal{P}^*$ the mapping $\varphi : [0, +\infty) \to \overline{\mathbb{R}}$ such that

$$\varphi(\alpha) = \mu(a + \alpha b) = \mu(a) + \alpha\mu(b)$$

is also continuous at $\alpha = 0$ (see the remark after Proposition 4.5) if we consider $\overline{\mathbb{R}}$ in its symmetric topology, for which $= \infty$ is an isolated point (see Example 4.37(a) below). Therefore $\mu(b)$ is finite whenever $\varphi(0) = \mu(a)$ is finite, and we infer that (iv) implies (v).

Finally, suppose that Condition (iii) fails for the element b. Given a neighborhood $v \in \mathcal{V}$, we define a corresponding functional μ_v on $\mathcal{P}$ setting $\mu_v(c) = 0$ for all $c \in \mathcal{P}$ such that $c \leq \alpha a + \beta v$ for some $\alpha, \beta \geq 0$, and $\mu_v(c) = +\infty$, else. It is straightforward to check that μ_v is linear. Indeed, if $\mu_v(c) = \mu_v(d) = 0$, that is $c \leq \alpha a + \beta v$ and $c \leq \gamma + \delta v$ for some $\alpha, \beta, \gamma, \delta \geq 0$, then $c+d \leq (\alpha+\gamma)a + (\beta+\delta)v$, hence $\mu_v(c+d) = 0$ as well. If, on the other hand, $\mu_v(c+d) = 0$, that is $c+d \leq \alpha a+\beta v$ for some $\alpha, \beta \geq 0$, we choose $\lambda \geq 0$ such that $0 \leq d+\lambda v$ and have $c \leq c+d+\lambda v \leq \alpha a+(\beta+\lambda)v$. This shows $\mu_v(c) = 0$. Similarly, one verifies that $\mu_v(d) = 0$. Moreover, we realize that μ_v is an element of the polar v° of v, as for $c \leq d + v$, we have $\mu_v(c) = 0$ whenever $\mu_v(d) = 0$, hence $\mu_v(c) \leq \mu_v(d) + 1$ holds in any case. Using this construction, we proceed with our argument: If (iii) fails for b, then there is a neighborhood $v \in \mathcal{V}$ such that $b \not\leq \alpha a + \beta v$ for all choices of $\alpha, \beta \geq 0$, hence $\mu_v(b) = +\infty$, while $\mu_v(a) = 0$. Thus Condition (v) does not hold either. This in turn shows that (v) implies (iii) and completes our argument. □

Proposition 4.11. *Let $a, b, c \in \mathcal{P}$. Then*

(a) $\mathcal{B}(a)$ is a subcone of $\mathcal{P}$, and $\mathcal{B} \subset \mathcal{B}(a)$.
(b) $\mathcal{B}(a)$ is a face in $\mathcal{P}$, that is $b + c \in \mathcal{B}(a)$ implies both $b, c \in \mathcal{B}(a)$.
(c) $\mathcal{B}(\alpha a) = \mathcal{B}(a)$ for $\alpha > 0$, and $\mathcal{B}(a) + \mathcal{B}(b) \subset \mathcal{B}(a + b)$.
(d) $\mathcal{B}(a)$ is closed in $\mathcal{P}$ with respect to the lower relative topology of $\mathcal{P}$.

Proof. Part (a) is obvious from Proposition 4.10(iii), since $b \leq \alpha a + \beta v$ and $c \leq \gamma a + \delta v$ for $v \in \mathcal{V}$ and $\alpha, \beta, \gamma, \delta \geq 0$ implies that $b + c \leq (\alpha + \gamma)a + (\beta + \delta)v$ and $\lambda b \leq \lambda\alpha a + \lambda\beta v$ for $\lambda \geq 0$. Moreover, since $0 \in \mathcal{B}(a)$, Proposition 4.10(ii) yields that $\mathcal{B} = \mathcal{B}(0) \subset \mathcal{B}(a)$.

For (b), let $b+c \in \mathcal{B}(a)$, that is, given $v \in \mathcal{V}$, we have $b+c \leq \alpha a+\beta v$ for some $\alpha, \beta \geq 0$. Because all elements of a locally convex cone are bounded below, there is $\lambda \geq 0$ such that $0 \leq c + \lambda v$. Thus $b \leq b + c + \lambda v \leq \alpha a + (\beta + \lambda)v$. Hence $b \in \mathcal{B}(a)$. Similarly, one verifies that $c \in \mathcal{B}(a)$.

The first statement of (c) is obvious from 4.10(iii). For the second statement, let $c \in \mathcal{B}(a)$, $d \in \mathcal{B}(b)$ and $v \in \mathcal{V}$. Then $c \leq \alpha a+\beta v$ and $d \leq \gamma b+\delta v$ for some $\alpha, \beta, \gamma, \delta \geq 0$. Let $\lambda \geq 0$ such that both $0 \leq a+\lambda v$ and $0 \leq b+\lambda v$.

In case that $\alpha \le \gamma$, this yields $c \le \alpha a + \beta v + (\gamma - \alpha)(a + \lambda v) = \gamma a + \rho v$, where $\rho = \beta + (\gamma - \alpha)\lambda$. Thus $c + d \le \gamma(a + b) + (\rho + \delta)v$. In case that $\alpha > \gamma$, a similar argument leads to $c + d \le \alpha(a + b) + (\rho' + \beta)v$, where $\rho' = \delta + (\alpha - \gamma)\lambda$. This verifies $c + d \in \mathcal{B}(a + b)$.

Finally, for Part (c), we remarked before that a linear functional $\mu \in \mathcal{P}^*$ is a continuous mapping from $\mathcal{P}$ into $\overline{\mathbb{R}}$ if we endow $\mathcal{P}$ with either its upper, lower or symmetric relative topology, and $\overline{\mathbb{R}}$ with either its given upper, lower or symmetric topology, respectively. We shall use this observation for the functionals $\mu_v \in \mathcal{P}^*$ for $v \in \mathcal{V}$, that we constructed in the argument for the implication (v) $\Rightarrow$ (iii) in the proof of Proposition 4.10, that is $\mu_v(c) = 0$ if $c \le \alpha a + \beta v$ for some $\alpha, \beta \ge 0$, and $\mu(c) = +\infty$, else. Because $\mathbb{R}$ is a closed subset of $\overline{\mathbb{R}}$ in the lower topology of $\overline{\mathbb{R}}$ (see Example 1.4(a)), its inverse image $\mu_v^{-1}(\mathbb{R})$ under μ_v is closed in the lower relative topology of $\mathcal{P}$. We have $\mathcal{B}(a) = \bigcap_{v \in \mathcal{V}} \mu_v^{-1}(\mathbb{R})$ by Proposition 4.10(v). Thus $\mathcal{B}(a)$ is indeed closed in the lower relative topology of $\mathcal{P}$. $\square$

We proceed to identify the corresponding properties of the lower boundedness components.

Proposition 4.12. *Let $a, b, \in \mathcal{P}$. The following are equivalent:*

(i) $b \in (a)\mathcal{B}$.
(ii) $a \in \mathcal{B}(b)$.
(iii) $\mathcal{B}(a) \subset \mathcal{B}(b)$.
(iv) $(b)\mathcal{B} \subset (a)\mathcal{B}$.
(v) For every $v \in \mathcal{V}$ there are $\alpha, \beta > 0$ such that $\alpha a \le b + \beta v$.
(vi) The mapping

$$\alpha \mapsto \alpha a + b \;:\; [0, +\infty) \to \mathcal{P}$$

is continuous with respect to the symmetric relative topology of $\mathcal{P}$.
(vii) For all $\mu \in \mathcal{P}^*$, $\mu(a) = +\infty$ implies $\mu(b) = +\infty$.

Proof. Let $a, b \in \mathcal{P}$. First we observe that $b \in (a)\mathcal{B}$ holds for $a, b \in \mathcal{P}$ if and only if for every $v \in \mathcal{V}$ there is $\varepsilon > 0$ such that $b \in (a)v_\varepsilon$, that is $a \in v_\varepsilon(b)$. The latter means that $a \in \mathcal{B}(b)$. Hence (i) and (ii) are indeed equivalent.

The equivalence of (ii) and (iii) follows from the corresponding one in Proposition 4.10: We have $b \in (a)\mathcal{B}$ if and only if $a \in \mathcal{B}(b)$ by the preceding argument, and the latter holds if and only if $\mathcal{B}(a) \subset \mathcal{B}(b)$ by 4.10.

Now suppose that $\mathcal{B}(a) \subset \mathcal{B}(b)$ holds and let $c \in (b)\mathcal{B}$. Then $b \in \mathcal{B}(c)$, hence $\mathcal{B}(a) \subset \mathcal{B}(b) \subset \mathcal{B}(c)$ by 4.10(ii). Thus $a \in \mathcal{B}(c)$, hence $c \in (a)\mathcal{B}$. This shows $(b)\mathcal{B} \subset (a)\mathcal{B}$. For the converse suppose that $(b)\mathcal{B} \subset (a)\mathcal{B}$. This implies $b \in (a)\mathcal{B}$, hence $a \in \mathcal{B}(b)$ and $\mathcal{B}(a) \subset \mathcal{B}(b)$ by 4.10(ii). Therefore (iii) and (iv) are also equivalent.

Next suppose that for every $v \in \mathcal{V}$ there are $\alpha, \beta > 0$ such that $\alpha a \le b + \beta v$. Then $a \le (1/\alpha)b + (\beta/\alpha)v$, hence $a \in \mathcal{B}(b)$ by 4.10(iii). For the converse, let $a \in \mathcal{B}(b)$, that is $a \in v_\varepsilon(b)$ for every $v \in \mathcal{V}$ with some $\varepsilon > 0$. This yields

$a \le \gamma b + \varepsilon v$ for some $1 \le \gamma \le 1+\varepsilon$, hence $(1/\gamma)a \le b + (\varepsilon/\gamma)v$, as claimed. We infer that (ii) and (v) are also equivalent. The remaining parts of this proof require only little effort if we use the already established equivalence of (i) and (ii) and the corresponding results for the upper boundedness components in Proposition 4.10:

The equivalence of (ii) and (vi) follows from the equivalence of (i) and (iv) in Proposition 4.10. The equivalence of Conditions (i) and (v) from 4.10, on the other hand, yields that $a \in \mathcal{B}(b)$ if and only if $\mu(b) < +\infty$ implies $\mu(a) < +\infty$ for every $\mu \in \mathcal{P}^*$. But the latter is equivalent to the formulation of Condition (vii) in the present proposition. □

Proposition 4.13. *Let $a, b, c \in \mathcal{P}$. Then*

(a) If $b \in (a)\mathcal{B}$ and $c \in \mathcal{P}$, then $\beta b + c \in (a)\mathcal{B}$ for all $\beta > 0$.
(b) $(\alpha a)\mathcal{B} = (a)\mathcal{B}$ for $\alpha > 0$, and $(a+b)\mathcal{B} = (a)\mathcal{B} \cap (b)\mathcal{B}$.
(c) $(a)\mathcal{B}$ is closed in $\mathcal{P}$ with respect to the upper relative topology of $\mathcal{P}$.

Proof. For Part (a), let $b \in (a)\mathcal{B}$, that is $a \in \mathcal{B}(b)$, let $c \in \mathcal{P}$ and $\beta > 0$. 4.11(c) shows that $a \in \mathcal{B}(\beta b)$, hence $a \in \mathcal{B}(\beta b) + \mathcal{B}(c) \subset \mathcal{B}(\beta b + c)$. Thus $\beta b + c \in (a)\mathcal{B}$.

The first part of (b) is obvious from 4.12(v). For the second part let $c \in (a+b)\mathcal{B}$. Then $a + b \in \mathcal{B}(c)$, hence both $a \in \mathcal{B}(c)$ and $b \in \mathcal{B}(c)$, since $\mathcal{B}(c)$ is a face in $\mathcal{P}$ by Proposition 4.11(b). Thus $c \in (a)\mathcal{B} \cap (b)\mathcal{B}$. This argument is indeed reversible: If $c \in (a)\mathcal{B} \cap (b)\mathcal{B}$, then both $a \in \mathcal{B}(c)$ and $b \in \mathcal{B}(c)$. This implies $a + b \in \mathcal{B}(c)$, since $\mathcal{B}(c)$ is a subcone of $\mathcal{P}$ (see 4.11(a)). Thus $c \in (a+b)\mathcal{B}$.

For Part (c) we recall that the singleton set $\{+\infty\}$ is closed in the upper topology of $\overline{\mathbb{R}}$, hence its inverse image $\mu^{-1}(\{+\infty\})$ under any linear functional $\mu \in \mathcal{P}^*$ is closed with respect to the upper relative topology of $\mathcal{P}$. Following Proposition 4.12(vii), $(a)\mathcal{B}$ is the intersection of the sets $\mu^{-1}(\{+\infty\})$ for all $\mu \in \mathcal{P}^*$ such that $\mu(a) = +\infty$, hence $(a)\mathcal{B}$ is indeed closed for the upper relative topology. □

The sets

$$\mathcal{B}^s(a) = \mathcal{B}(a) \cap (a)\mathcal{B}$$

are called the *symmetric boundedness components* of $\mathcal{P}$. The elements of $\mathcal{B}^s(a)$ are called *bounded relative to* a. The symmetric boundedness components are of particular interest, since they will provide a natural partition of a locally convex cone into boundedness equivalence classes. Before establishing this feature, we shall list a few properties of the symmetric boundedness components:

Proposition 4.14. *Let $a, b, \in \mathcal{P}$. The following are equivalent:*

(i) $b \in \mathcal{B}^s(a)$.
(ii) $a \in \mathcal{B}^s(b)$.
(iii) $\mathcal{B}(b) = \mathcal{B}(a)$.

(iv) $(b)\mathcal{B} = (a)\mathcal{B}$.
(v) $\mathcal{B}^s(b) = \mathcal{B}^s(a)$.
(vi) For every $v \in \mathcal{V}$ *there are* $\alpha, \beta \geq 0$ *such that both*

$$b \leq \alpha a + \beta v \qquad \textit{and} \qquad a \leq \alpha b + \beta v.$$

(vii) The mapping

$$\alpha \mapsto \alpha a + (1-\alpha)b \ : \ [0,1] \to \mathcal{P}$$

is continuous with respect to the symmetric relative topology of $\mathcal{P}$.
(viii) For all $\mu \in \mathcal{P}^*$, $\mu(a) = +\infty$ *if and only if* $\mu(b) = +\infty$.

Proof. Let $a, b \in \mathcal{P}$. If $b \in \mathcal{B}^s(a) = \mathcal{B}(a) \cap (a)\mathcal{B}$, then $b \in \mathcal{B}(a)$ and $a \in \mathcal{B}(b)$. This implies $\mathcal{B}(a) = \mathcal{B}(b)$ by 4.10(ii). On the other hand, if $\mathcal{B}(a) = \mathcal{B}(b)$, then $b \in \mathcal{B}(a)$ and $a \in \mathcal{B}(b)$, hence $b \in \mathcal{B}(a) \cap (a)\mathcal{B}$. This yields the equivalence of (i) and (iii).

The equivalence of (iii) and (iv) follows immediately from 4.12(iii) and (iv).

Conditions (iii) and (iv) are symmetric in a and b and therefore also equivalent to (ii).

Conditions (iii) and (iv) imply (v), which in turn obviously renders (i), since $\mathcal{B}^s(b) = \mathcal{B}^s(a)$ implies $b \in \mathcal{B}^s(b) = \mathcal{B}^s(a)$.

Clearly, (vi) implies (i), since by Proposition 4.10(iii) it yields $b \in \mathcal{B}(a)$ and $a \in \mathcal{B}(b)$, hence $b \in \mathcal{B}^s(a)$. On the other hand, if $b \in \mathcal{B}^s(a)$, then $b \in \mathcal{B}(a)$ and $a \in \mathcal{B}(b)$, and by 4.10(iii), given $v \in \mathcal{V}$, there are $\alpha', \alpha'', \beta', \beta'' \geq 0$ such that

$$b \leq \alpha' a + \beta' v \qquad \text{and} \qquad a \leq \alpha'' b + \beta'' v.$$

There is $\lambda \geq 0$ such that both $0 \leq a + \lambda v$ and $0 \leq b + \lambda v$. Set $\alpha = \max\{\alpha', \alpha''\}$ and $\beta = \max\{\,\beta' + (\alpha - \alpha')\lambda,\, \beta'' + (\alpha - \alpha'')\lambda\,\}$. Then

$$\begin{aligned} b &\leq (\alpha' a + \beta' v) + (\alpha - \alpha')(a + \lambda v) \\ &\leq \alpha a + \big(\beta' + (\alpha - \alpha')\lambda\big)v \leq \alpha a + \beta v, \end{aligned}$$

and, likewise,

$$\begin{aligned} a &\leq (\alpha'' a + \beta'' v) + (\alpha - \alpha'')(a + \lambda v) \\ &\leq \alpha a + \big(\beta'' + (\alpha - \alpha'')\lambda\big)v \leq \alpha a + \beta v. \end{aligned}$$

Therefore (i) implies (vi) as well.

Condition (viii) of this proposition is the combination of the corresponding conditions in Propositions 4.10 and 4.12 and therefore also equivalent to Conditions (i) to (vi). All left to show is that (vii) is equivalent to the rest. First let us verify that (vii) implies (viii). If the mapping

$$\alpha \mapsto \alpha a + (1-\alpha)b \ : \ [0,1] \to \mathcal{P}$$

is continuous with respect to the symmetric relative topology of $\mathcal{P}$, then for every linear functional $\mu \in \mathcal{P}^*$ the mapping $\varphi : [0,1] \to \overline{\mathbb{R}}$ such that

$$\varphi(\alpha) = \mu\big(\alpha a + (1-\alpha)b\big) = \alpha\mu(a) + (1-\alpha)\mu(b)$$

is also continuous (see the remark after Proposition 4.5) if we consider $\overline{\mathbb{R}}$ in its symmetric topology, for which $= \infty$ is an isolated point. Therefore $\varphi(0) = \mu(b)$ is finite if and only if $\varphi(1) = \mu(a)$ is finite. Hence (vii) implies (viii). Finally, we shall demonstrate how the other conditions imply (vii). Following Proposition 4.2(iii), for any choice of $a, b \in \mathcal{P}$ the mappings $\alpha \mapsto \alpha a$ and $\alpha \mapsto (1-\alpha)b$ are continuous with respect to the symmetric relative topology of $\mathcal{P}$ on the open intervals $(0, +\infty)$ and $(-\infty, 1)$, respectively. Thus by Proposition 4.2(ii), that is the continuity of the addition in $\mathcal{P}$, the mapping $f : [0,1] \to \mathcal{P}$ such that

$$f(\alpha) = \alpha a + (1-\alpha)b$$

is continuous on the interval $(0,1)$. In case that $b \in \mathcal{B}^s(a)$, we shall verify continuity at the endpoints $\alpha = 0$ and $\alpha = 1$ as well: Proposition 4.12(vi), if applied to the element $(1/2)b \in (a)\mathcal{B}$, states that the mapping

$$\alpha \mapsto \alpha a + \frac{1}{2}b \ : \ \big[0, +\infty\big) \to \mathcal{P}$$

is continuous at $\alpha = 0$. The mapping

$$\alpha \mapsto \left(\frac{1}{2} - \alpha\right) b \ : \ \left(-\infty, \frac{1}{2}\right] \to \mathcal{P},$$

on the other hand, is continuous at 0 by 4.2(iii). Thus the sum of these mappings, that is the function f, is also continuous at 0. A similar argument holds for $\alpha = 1$. Following Propositions 4.10(iv) and 4.2(iii), respectively, the mappings

$$\alpha \mapsto \frac{1}{2}a + (1-\alpha)b \ : \ \big(-\infty, 1\big]$$

and

$$\alpha \mapsto \left(\alpha - \frac{1}{2}\right) a \ : \ \left[\frac{1}{2}, +\infty\right) \to \mathcal{P}$$

are continuous at $\alpha = 1$. So is their sum, the function f. This concludes our argument. □

Proposition 4.15. *Let* $a, b, c \in \mathcal{P}$. *Then*

(a) If $b, c \in \mathcal{B}^s(a)$, *then* $\beta b + \gamma c \in \mathcal{B}^s(a)$ *for all* $\beta, \gamma > 0$.
(b) $\mathcal{B}^s(\alpha a) = \mathcal{B}^s(a)$ *for* $\alpha > 0$, *and* $\mathcal{B}^s(a+b) \supset \mathcal{B}^s(a) \cap \mathcal{B}^s(b)$.
(c) $\mathcal{B}^s(a)$ *is closed in* $\mathcal{P}$ *with respect to the symmetric relative topology of* $\mathcal{P}$.

Proof. Part (a) follows directly from Propositions 4.11(a) and 4.13(a). The first part of (b) follows from the first parts of 4.11(c) and 4.13(b). The same sources yield the second part of (b) as well, since the relations $(a+b)\mathcal{B} = (a)\mathcal{B} \cap (b)\mathcal{B}$ and $\mathcal{B}(a+b) \supset \mathcal{B}(a) + \mathcal{B}(b) \supset \mathcal{B}(a) \cap \mathcal{B}(b)$ imply that

$$\mathcal{B}^s(a+b) = \mathcal{B}(a+b) \cap (a+b)\mathcal{B} \supset \mathcal{B}(a) \cap \mathcal{B}(b) \cap (a)\mathcal{B} \cap (b)\mathcal{B} = \mathcal{B}^s(a) \cap \mathcal{B}^s(b).$$

Finally, by Propositions 4.11(d) and 4.13(c), the sets $\mathcal{B}(a)$ and $(a)\mathcal{B}$ are closed in the lower and upper relative topologies of $\mathcal{P}$, respectively. Consequently, both of these sets as well as their intersection, that is $\mathcal{B}^s(a)$, are also closed in the symmetric relative topology of $\mathcal{P}$, which is finer than both the upper and the lower relative topologies. □

Proposition 4.16. *The symmetric boundedness components satisfy a version of the cancellation law, that is* $a+c \preccurlyeq b+c$ *for elements* a, b *and* c *of the same boundedness component implies that* $a \preccurlyeq b$.

Proof. Suppose that the elements $a, b, c \in \mathcal{P}$ are bounded relative to each other and that $a+c \preccurlyeq b+c$. Given $v \in \mathcal{V}$ there is $\lambda \geq 0$ such that $0 \leq c + \lambda v$. Thus $a + (c+\lambda v) \preccurlyeq b + (c + \lambda v)$. As we observed before, $(\mathcal{P}, \mathcal{V})$ endowed with the weak preorder $\preccurlyeq$ forms again a locally convex cone. Following Lemma I.4.2 in [100], if applied to this order and the positive element $(a+\lambda v)$ of a full cone containing $\mathcal{P}$, the above implies $a \preccurlyeq b + \varepsilon(c+\lambda v)$ for all $\varepsilon > 0$. By our assumption, there are $\alpha, \beta \geq 0$ such that $c \leq \alpha b + \beta v$. Now combining the above yields

$$a \preccurlyeq b + \varepsilon\big(\alpha b + (\beta+\lambda)v\big) = (1+\varepsilon\alpha)b + \varepsilon(\beta+\lambda)v$$

for all $\varepsilon > 0$. This shows $a \preccurlyeq_v b$ by our definition of the weak local preorder in Section 3. Finally, because $a \preccurlyeq_v b$ holds for all $v \in \mathcal{V}$, we infer that $a \preccurlyeq b$. □

Proposition 4.17. *The symmetric boundedness components furnish a partition of* $\mathcal{P}$ *into disjoint convex subsets that are closed and connected in the symmetric relative topology.*

Proof. Proposition 4.15(a) implies that the symmetric boundedness components are convex subset of $\mathcal{P}$. They are closed in the symmetric relative topology by 4.15(c). Moreover, the equivalence of (i) and (v) in Proposition 4.14 shows that any two symmetric boundedness components of $\mathcal{P}$ either coincide or are disjoint. For connectedness, let $a \in \mathcal{P}$, and let $b, c \in \mathcal{B}^s(a)$. Then $\mathcal{B}^s(a) = \mathcal{B}^s(b) = \mathcal{B}^s(b)$ by Proposition 4.14(v), and by the equivalent condition in 4.14(vii), the mapping $f : [0,1] \to \mathcal{B}^s(a)$ such that

$$f(\alpha) = \alpha b + (1-\alpha)c$$

is continuous with respect to the symmetric relative topology of $\mathcal{P}$. As $f(0) = c$ and $f(1) = b$, this shows that $\mathcal{B}^s(a)$ is pathwise connected, hence connected in the symmetric relative topology of $\mathcal{P}$ (see Theorem 27.2 in [198]). $\square$

We shall also consider the local boundedness components of a locally convex cone $\mathcal{P}$ that arise if we endow $\mathcal{P}$ with the neighborhood subsystem $\mathcal{V}_v = \{\alpha v \mid \alpha > 0\}$ consisting of the multiples of a single neighborhood $v \in \mathcal{V}$. For an element $a \in \mathcal{P}$ and a neighborhood $v \in \mathcal{V}$, we define the *(local) upper, lower* and *symmetric v-boundedness components* of a as

$$\mathcal{B}_v(a) = \bigcup_{\varepsilon>0} v_\varepsilon(a), \quad (a)\mathcal{B}_v = \bigcup_{\varepsilon>0} (a)v_\varepsilon, \quad \text{and} \quad \mathcal{B}_v^s(a) = \mathcal{B}_v(a) \cap (a)\mathcal{B}_v,$$

respectively. The elements of $\mathcal{B}_v(a)$ are called *v-bounded above relative to a*. $\mathcal{B}_v(0) = \mathcal{B}_v$ consists of all v-bounded elements of $\mathcal{P}$. The global boundedness components may be recovered as

$$\mathcal{B}(a) = \bigcap_{v\in\mathcal{V}} \mathcal{B}_v(a), \qquad (a)\mathcal{B} = \bigcap_{v\in\mathcal{V}} (a)\mathcal{B}_v \qquad \text{and} \qquad \mathcal{B}^s(a) = \bigcap_{v\in\mathcal{V}} \mathcal{B}_v^s(a),$$

respectively. Obviously, the statements of Propositions 4.10 to 4.17 apply also to the local boundedness components, since we may replace the given neighborhood system $\mathcal{V}$ by the subsystem $\mathcal{V}_v$ and consider the locally convex cone $(\mathcal{P}, \mathcal{V}_v)$ for this purpose. The cancellation law in Proposition 4.16 holds with the weak local preorder $\preccurlyeq_v$ in this case. The dual cone $\mathcal{P}^*$ of $(\mathcal{P}, \mathcal{V}_v)$ consists only of the multiples of the functionals in v°, and the relative topologies of $\mathcal{P}$ are the relative v-topologies.

The main benefit in considering the local boundedness components as compared to the global ones, is the following: We shall proceed to verify that the disjoint partition of $\mathcal{P}$ into symmetric local boundedness components provides indeed a topological partition as well.

Proposition 4.18. *Let $a \in \mathcal{P}$ and $v \in \mathcal{V}$.*

(a) $\mathcal{B}_v(a)$ is open in $\mathcal{P}$ with respect to the upper, closed with respect to the lower and both open and closed with respect to the symmetric relative v-topology of $\mathcal{P}$.
(b) $(a)\mathcal{B}_v$ is closed in $\mathcal{P}$ with respect to the upper, open with respect to the lower and both open and closed with respect to the symmetric relative v-topology of $\mathcal{P}$.

Proof. Let $a \in \mathcal{P}$ and $v \in \mathcal{V}$ Proposition 4.11(d) states that $\mathcal{B}_v(a)$ is closed in the lower relative v-topology of $\mathcal{P}$. Let $b \in \mathcal{B}_v(a)$, that is $b \le \alpha b + \beta v$ for some $\alpha, \beta \ge 0$, and let $v_\varepsilon(b)$ be a lower neighborhood of b. Then for $c \in v_\varepsilon(b)$ we have $c \le \gamma b + \varepsilon v$ with some $1 \le \gamma \le 1 + \varepsilon$, and therefore $c \le (\alpha\gamma)a + (\beta\gamma + \varepsilon)v$. This shows $c \in \mathcal{B}_v(a)$, hence $v_\varepsilon(b) \subset \mathcal{B}_v(a)$, and $\mathcal{B}_v(a)$ is seen to be open in the lower relative v-topology of $\mathcal{P}$. Moreover, because the symmetric relative v-topology is the common refinement of the upper and lower topologies, $\mathcal{B}_v(a)$ is indeed both open and closed in this topology. This completes Part (a).

The argument for Part (b) is similar: Proposition 4.13(c) states that $(a)\mathcal{B}_v$ is closed in the upper relative v-topology of $\mathcal{P}$. Let $b \in (a)\mathcal{B}_v$, that is $\alpha a \leq b + \beta v$ for some $\alpha, \beta > 0$, and let $(b)v_\varepsilon$ be a lower neighborhood of b. Then for $c \in (b)v_\varepsilon$ we have $b \leq \gamma c + \varepsilon v$ with some $1 \leq \gamma \leq 1 + \varepsilon$, and therefore $\alpha a \leq \gamma c + (\varepsilon + \delta)v$, hence $(\alpha/\gamma)a \leq c + (\varepsilon + \delta)/\gamma v$. This shows $c \in (a)\mathcal{B}_v$, hence $(b)v_\varepsilon \subset (a)\mathcal{B}_v$, and $(a)\mathcal{B}_v$ is seen to be open in the lower relative v-topology of $\mathcal{P}$. Hence $(a)\mathcal{B}_v$ is both open and closed in the symmetric relative v-topology. □

Propositions 4.18 and 4.17 now yield a topological and algebraic partition of a locally convex cone into local boundedness components.

Proposition 4.19. *For every neighborhood $v \in \mathcal{V}$, the symmetric v-boundedness components furnish a partition of $\mathcal{P}$ into disjoint convex subsets that are open, closed and connected in the symmetric relative v-topology.*

A subset of $\mathcal{P}$ that is open or closed in any of the relative v-topologies is of course also open or closed in the corresponding (global) relative topology of $\mathcal{P}$. The same statement does however not hold for connectedness.

4.20 Connectedness. Topological vector spaces are connected and all of their elements are bounded. This does not hold for locally convex cones in general. However, Propositions 4.17 and 4.19 suggest relations between the boundedness and the connectedness components of a locally convex cone. Let us recall some of the relevant concepts from topology: The *quasi-component of a point* x in a topological space X is the intersection of all closed and open subsets of X which contain x. The quasi-components constitute a decomposition of X into pairwise disjoint and closed subsets (see VIII.26 in [198] or VI.1 in [59]). The *component of a point* $x \in X$, on the other hand is the largest connected subset of X which contains the point x. The components are subsets of the quasi-components and constitute a decomposition of X into pairwise disjoint, connected and closed subsets. A topological space is locally connected, if each of its points has a basis of connected neighborhoods. In locally connected spaces the quasi-components and components coincide and are both open and closed (see Corollary 27.10 in [198]).

Proposition 4.21. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone.*

(a) In the symmetric relative topology of $\mathcal{P}$ the components, quasi-components and the symmetric boundedness components coincide.
(b) For every neighborhood $v \in \mathcal{V}$ and the symmetric relative v-topology, $\mathcal{P}$ is locally connected and the components, quasi-components and the symmetric v-boundedness components coincide.

Proof. (a) For an element $a \in \mathcal{P}$ Proposition 4.17 implies that $\mathcal{B}^s(a)$ is contained in its (connectedness) component. On the other hand, $\mathcal{B}^s(a)$ is the intersection of the sets $\mathcal{B}_v^s(a)$ for all $v \in \mathcal{V}$, all of which are open and closed in the respective symmetric relative v-topologies, hence in the symmetric relative topology of $\mathcal{P}$ by Proposition 4.19. This shows that the quasi-component of a is contained in $\mathcal{B}^s(a)$. Hence these three components coincide.

For Part (b) let $v \in \mathcal{V}$ and $a \in \mathcal{P}$. The v-boundedness component $\mathcal{B}_v^s(a)$ of a contains all the neighborhoods $v_\varepsilon^s(a)$ for $\varepsilon > 0$. Convexity then guarantees (see the corresponding argument in the proof of Proposition 4.17) that these neighborhoods are pathwise connected in the symmetric relative v-topology, hence $\mathcal{P}$ is locally connected. The components, quasi-components and the symmetric v-boundedness components of $\mathcal{P}$ coincide by Part (a) if we endow $\mathcal{P}$ with the neighborhood subsystem $\mathcal{V}_v = \{\alpha v \mid \alpha > 0\}$. □

Proposition 4.22. *A locally convex cone* $(\mathcal{P}, \mathcal{V})$ *is locally connected in its symmetric relative topology if and only if every point* $a \in \mathcal{P}$ *has a basis of symmetric relative neighborhoods that are contained in* $\mathcal{B}^s(a)$.

Proof. Let $a \in \mathcal{P}$. The argument in the proof of Proposition 4.17 shows that every convex subset of $\mathcal{B}^s(a)$ is pathwise connected, hence connected in the symmetric relative topology. On the other hand, every connected subset of $\mathcal{P}$ containing the element a is a subset of $\mathcal{B}^s(a)$, the component of a by 4.21(a). Because the symmetric relative neighborhoods of a are convex, our claim follows. □

4.23 Locally Convex Cones with Uniform Boundedness Components. We shall say that a locally convex cone $(\mathcal{P}, \mathcal{V})$ has *uniform boundedness components* if the boundedness components of $\mathcal{P}$ for all neighborhoods coincide, that is if $\mathcal{B}_v^s(a) = \mathcal{B}^s(a)$ for all $v \in \mathcal{V}$ and $a \in \mathcal{P}$. Locally convex topological vector spaces are obviously of this type as all their elements are bounded with respect to every neighborhood. Also, any locally convex cone whose neighborhood system consists of the multiples of a single neighborhood, has uniform boundedness components. Proposition 4.22 yields that a locally convex cone with uniform boundedness components is locally connected. Its global boundedness components are both open and closed in the each of the symmetric relative v-topologies (Proposition 4.19), hence also in the (global) symmetric relative topology.

Similar and related notions of boundedness components in locally convex cones had previously been established in [170] and [176].

4.24 Bounded Subsets. We shall also use notions of boundedness for subsets corresponding to those for elements of a locally convex cone $(\mathcal{P}, \mathcal{V})$. A subset A of $\mathcal{P}$ is called

(i) *bounded below* if for every $v \in \mathcal{V}$ there is $\lambda \geq 0$ such that $0 \leq a + \lambda v$ for all $a \in A$;
(ii) *bounded above* if for every $v \in \mathcal{V}$ there is $\lambda \geq 0$ such that $a \leq \lambda v$ for all $a \in A$;

(iii) *bounded* if it is both bounded below and above;
(iv) *bounded above relative to* $b \in \mathcal{P}$ if for every $v \in \mathcal{V}$ there are $\lambda, \rho \geq 0$ such that $a \leq \rho b + \lambda v$ for all $a \in A$;
(v) *relatively bounded above* if it is bounded above relative to some element of $\mathcal{P}$; and
(vi) *relatively bounded* if it is both bounded below and relatively bounded above, that is if there is $b \in \mathcal{P}$ such that for every $v \in \mathcal{V}$ there are $\lambda, \rho \geq 0$ such that $0 \leq a + \lambda v$ and $a \leq \rho b + \lambda v$ for all $a \in A$.

All these notions do of course coincide in a locally convex topological vector space. Similar concepts may be used to define local boundedness, that is boundedness relative to a specific neighborhood $v \in \mathcal{V}$, for subsets of $\mathcal{P}$.

Note that a continuous linear operator $T : \mathcal{P} \to \mathcal{Q}$, where both $(\mathcal{P}, \mathcal{V})$ and $(\mathcal{Q}, \mathcal{W})$ are locally convex cones, maps bounded subsets of one of the above types in $\mathcal{P}$ into bounded subsets of the same type in $\mathcal{Q}$.

A Uniform-Boundedness-type theorem from [172] allows relative boundedness for subsets of a locally convex cone $\mathcal{P}$ to be characterized in terms of its dual cone $\mathcal{P}^*$.

Proposition 4.25. *Let A be a subset of a locally convex cone $(\mathcal{P}, \mathcal{V})$, and let $b \in \mathcal{P}$. If for every linear functional $\mu \in \mathcal{P}^*$ such that $\mu(b) < +\infty$ the set $\mu(A)$ is bounded in $\overline{\mathbb{R}}$, then A is bounded above relative to b.*

Proof. Let A be a subset of $\mathcal{P}$ which is not bounded above relative to the element $b \in \mathcal{P}$. Then there is $v \in \mathcal{V}$ such that the condition in 4.24(iv) does not hold for this neighborhood. We define a monotone sublinear functional $p : \mathcal{P} \to \overline{\mathbb{R}}$ by

$$p(a) = \inf\{\lambda + \rho \mid \lambda, \rho \geq 0, \;\; a \leq \rho b + \lambda v\}$$

and observe that: (i) Let $c \leq d + v$ for $c, d \in \mathcal{P}$. Then $d \leq \rho b + \lambda v$ for $\lambda, \rho \geq 0$ implies that $c \leq \rho b + (\lambda + 1)v$. Thus $p(c) \leq p(d) + 1$, and the functional p is seen to be continuous with respect to v in the sense of Theorem 3.4 in [172]; (ii) p is unbounded on A. Assume to the contrary that there is $M > 0$ such that $p(a) < M$ for all $a \in A$. Let $\sigma \geq 0$ such that $0 \leq b + \sigma v$. Then for every $a \in A$ there are $\lambda, \rho \geq 0$ such that $a \leq \rho b + \lambda v$ and $\lambda + \rho \leq M$. Then

$$a \leq (\rho b + \lambda v) + (M - \rho)(b + \sigma v) \leq Mb + M(1 + \sigma)v,$$

contradiction our assumption that $A \subset \mathcal{P}$ is not bounded above relative to b. Now Theorem 3.4 from [172] yields the existence of a continuous linear functional $\mu \in v^\circ$ such that $\mu(c) \leq p(c)$ for all $c \in \mathcal{P}$, that is $\mu(b) \leq 1$ in particular, and such that μ is unbounded on the set A. $\square$

Similar notions of boundedness will be used for nets in a locally convex cone, that is a net $(a_i)_{i \in \mathcal{I}}$ in $\mathcal{P}$ will be called *bounded (below, above, relative* *to an element,...)* if the corresponding requirements 4.24(i) to (vi) hold for the set $\{a_i \mid i \geq i_0\}$ for some $i_0 \in \mathcal{I}$.

4.26 Closed Convex Sets. We shall proceed making some observations regarding subsets of a locally convex cone which are closed either with respect to the lower or the upper relative topology.

Lemma 4.27. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone. Every subset of $\mathcal{P}$ that is closed with respect to the lower (or the upper) relative topology is decreasing (or increasing) with respect to the weak preorder.*

Proof. Indeed, suppose that $A \subset \mathcal{P}$ is closed with respect to the lower relative topology and let $b \preccurlyeq a$ for some $b \in \mathcal{P}$ and $a \in A$. Then $b \in v_\varepsilon(a)$, thus $a \in (b)v_\varepsilon$ and $(b)v_\varepsilon \cap A \neq \emptyset$ for all $v \in \mathcal{V}$ and $\varepsilon > 0$. Thus b is in the closure of A with respect to the lower relative topology which coincides with A. Similarly one argues for a subset of $\mathcal{P}$ which is closed with respect to the upper relative topology. □

For a subset A of a locally convex cone $(\mathcal{P}, \mathcal{V})$ we denote by $\overline{A}^{(l)}$ and $\overline{A}^{(u)}$ its closure with respect to the lower and the upper relative topology of $\mathcal{P}$, respectively.

Proposition 4.28. *Let A be a subset of a locally convex cone $(\mathcal{P}, \mathcal{V})$.*

(a) The set $\overline{A}^{(l)}$ consists of all elements $b \in \mathcal{P}$ such that for every $v \in \mathcal{V}$ and $\varepsilon > 0$ there is some $a \in A$ such that $b \in v_\varepsilon(a)$.
(b) The set $\overline{A}^{(l)}$ is convex whenever A is convex.
(c) The set $\overline{A}^{(l)}$ is bounded above whenever A is bounded above.

Proof. (a) We have $b \in \overline{A}^{(l)}$ if and only if $(b)v_\varepsilon \cap A \neq \emptyset$ for all $v \in \mathcal{V}$ and $\varepsilon > 0$, that is if there is $a \in A$ such that $b \in v_\varepsilon(a)$. For Part (b) suppose that A is convex and let $b, b' \in \overline{A}^{(l)}$ and $b'' = \alpha b + (1-\alpha)b'$ for some $0 \leq \alpha \leq 1$. Given $v \in \mathcal{V}$ and $\varepsilon > 0$, by Part (i) there are $a, a' \in A$ such that $b \in v_\varepsilon(a)$ and $b' \in v_\varepsilon(a')$. Then $b \leq \gamma a + \varepsilon v$ and $b' \leq \gamma' a' + \varepsilon v$ for some $1 \leq \gamma, \gamma' \leq 1 + \varepsilon$. Set $\gamma'' = (\alpha\gamma + (1-\alpha)\gamma')$. Then

$$a'' = \frac{\alpha\gamma}{\gamma''}a + \frac{(1-\alpha)\gamma'}{\gamma''}a' \in A$$

and $b'' \leq \gamma'' a'' + (1+\varepsilon)v$. Since $1 \leq \gamma'' \leq 1 + \varepsilon$, this demonstrates that $b'' \in \overline{A}^{(l)}$, and therefore this set is also convex. For Part (c) suppose that A is bounded above in the sense of 4.24(ii). Let $v \in \mathcal{V}$ and suppose that there is $\lambda \geq 0$ such that $a \leq \lambda v$ for all $a \in A$. Then for every $b \in \overline{A}^{(l)}$ there is $a \in A$ such that $b \in v_1(a)$. This means $b \leq \gamma a + v$ for some $1 \leq \gamma \leq 2$, hence $b \leq (\gamma\lambda + 1)v \leq (2\lambda + 1)v$. Our claim follows. □

In a similar way one proves corresponding statements for the closure of a set with respect to the upper relative topology.

Proposition 4.29. *Let A be a subset of a locally convex cone $(\mathcal{P},\mathcal{V})$.*

(a) The set $\overline{A}^{(u)}$ consists of all elements $b\in\mathcal{P}$ such that for every $v\in\mathcal{V}$ and $\varepsilon>0$ there is some $a\in A$ such that $a\in v_\varepsilon(b)$.
(b) The set $\overline{A}^{(u)}$ is convex whenever A is convex.
(c) The set $\overline{A}^{(u)}$ bounded below whenever A is bounded below.

Proposition 4.28(a) implies in particular that for a singleton set $\{a\}$ we have $b\in\overline{\{a\}}^{(l)}$ if and only if $b\in v_\varepsilon(a)$ for all $v\in\mathcal{V}$ and $\varepsilon>0$, that is $b\preccurlyeq a$. Thus

$$\overline{\{a\}}^{(l)}=\{b\in\mathcal{P}\mid b\preccurlyeq a\}.$$

Likewise, Proposition 4.29(a) yields

$$\overline{\{a\}}^{(u)}=\{b\in\mathcal{P}\mid a\preccurlyeq b\}.$$

Theorem 4.30. *Let A be a convex subset of a locally convex cone $(\mathcal{P},\mathcal{V})$ and let $b\in\mathcal{P}$. Then*

(a) $b\in\overline{A}^{(l)}$ if and only if $\mu(b)\le\sup\{\mu(a)\mid a\in A\}$ for all $\mu\in\mathcal{P}^*$.
(b) $b\in\overline{A}^{(u)}$ if and only if $\mu(b)\ge\inf\{\mu(a)\mid a\in A\}$ for all $\mu\in\mathcal{P}^*$.

Proof. Let $A\subset\mathcal{P}$ be convex and let $b\in\mathcal{P}$. We may assume that $A\neq\emptyset$, because for $A=\emptyset$ our claim is trivial. (As usual, we set $\inf\emptyset=+\infty$ and $\sup\emptyset=-\infty$ and use the fact that for every $a\in\mathcal{P}$ there is some $\mu\in\mathcal{P}^*$ such that $\mu(a)<+\infty$.) For Part (a), let $b\in\overline{A}^{(l)}$ and let $\mu\in\mathcal{P}^*$, that is $\mu\in v^\circ$ for some $v\in\mathcal{V}$. Given $\varepsilon>0$, according to Proposition 4.28(a) there is $a\in A$ and $1\le\gamma\le1+\varepsilon$ such that $b\le\gamma a+\varepsilon v$, hence $\mu(b)\le\gamma\mu(a)+\varepsilon\le\gamma\sup\{\mu(a)\mid a\in A\}+\varepsilon$. This shows $\mu(a)\le\sup\{\mu(a)\mid a\in A\}$. The proof of the converse implication will however require some advanced Hahn-Banach type arguments that had been established in the [172] and quoted earlier in Section 2: For a fixed number $\beta\in\mathbb{R}$ consider the sublinear functional p on $\mathcal{P}$ defined for $x\in\mathcal{P}$ as

$$p(x)=\inf\{\lambda\beta\mid x=\lambda a\quad\text{for some}\quad a\in A\quad\text{and}\quad\lambda\ge0\},$$

together with the extended superlinear functional $q(0)=0$ and $q(x)=-\infty$ for $x\neq0$. Following Theorem 2.4 (a quote of Theorem 3.1 in [172]) there is a linear functional $\mu\in\mathcal{P}^*$ such that $q\le\mu\le p$ if and only if we can find a neighborhood $v\in\mathcal{V}$ such that $q(x)\le p(y)+1$ whenever $x\le y+v$ for $x,y\in\mathcal{P}$; that is in our particular case $0\le\lambda\beta+1$ whenever $0\le\lambda a+v$ for some $a\in A$ and $\lambda\ge0$. For this we shall have to distinguish two cases: (i) If for every $v\in\mathcal{V}$ there is $a\in A$ such that $0\le a+v$, then we have to require that $\beta\ge0$. (ii) If there is $v\in\mathcal{V}$ such that $0\not\le a+v$ for all $a\in A$, then for $\varepsilon>0$ the condition $0\le\lambda a+\varepsilon v$ can hold only for $\lambda<\varepsilon$.

Thus we can choose any $\beta = -(1/\varepsilon)$ for the neighborhood $\varepsilon v \in \mathcal{V}$. In other words, in case (ii) for every $\beta \in \mathbb{R}$ we can find a neighborhood in V such that the above condition is satisfied. Now we shall use Theorem 2.10 (a quote of Theorem 4.23 in [172]), which describes the range of all linear functionals $\mu \in \mathcal{P}^*$ such that $q \le \mu \le p$ on a fixed element $b \in \mathcal{P}$. It states that if there is at least one such linear functional μ, then

$$\sup_{\substack{\mu \in \mathcal{P}^* \\ q \le \mu \le p}} \mu(b) \;=\; \sup_{v \in \mathcal{V}} \inf \{p(x) - q(y) \mid x, y \in \mathcal{P},\ \ q(y) \in \mathbb{R},\ \ b + y \le x + v\}.$$

With the particular insertions for p and q from above we need to consider only the choice of $y = 0$ and obtain

$$\sup_{\substack{\mu \in \mathcal{P}^* \\ q \le \mu \le p}} \mu(b) \;=\; \sup_{v \in \mathcal{V}} \inf \{\lambda\beta \mid \lambda \ge 0,\ \ b \le \lambda a + v \quad \text{for some} \quad a \in A\}.$$

Now let us assume that $\mu(b) \le \sup\{\mu(a) \mid a \in A\}$ holds for all $\mu \in \mathcal{P}^*$. As $q \le \mu \le p$ implies that $\sup\{\mu(a) \mid a \in A\} \le \beta$, this yields

$$\sup_{v \in \mathcal{V}} \inf \{\lambda\beta \mid \lambda \ge 0,\ \ b \le \lambda a + v \quad \text{for some} \quad a \in A\} \;\le\; \beta$$

for all admissible values of β. We shall use 4.28(a) to derive $b \in \overline{A}^{(l)}$ from this. Let $v \in \mathcal{V}$ and $\varepsilon > 0$. We choose $\beta = 1$ in the above and observe that there is $a \in A$ and $\lambda \ge 0$ such that

$$b \le \lambda a + \frac{\varepsilon}{2} v \qquad \text{and} \qquad \lambda \le 1 + \varepsilon.$$

If $1 \le \lambda$, this satisfies the criterion in 4.28(a). Otherwise we proceed distinguishing the above cases: In case (i) there is $a' \in A$ such that $0 \le a' + (\varepsilon/2)v$. Thus

$$b \le \lambda a + \frac{\varepsilon}{2} v + (1 - \lambda)\left(a' + \frac{\varepsilon}{2} v\right) \le a'' + \varepsilon v$$

with $a'' = \lambda a + (1 - \lambda)a' \in A$, satisfying the requirement from 4.28(a). In case (ii) we may use the above inequality for $\beta = -1$ as well. There is $\rho > 0$ such that $0 \le b + \rho v$. Set $\delta = \min\{1/2, \varepsilon/(4\rho + 2\varepsilon)\}$. We find $a' \in A$ and $\lambda' \ge 0$ such that

$$b \le \lambda' a' + \frac{\varepsilon}{2} v \qquad \text{and} \qquad -\lambda' \le -1 + \delta,$$

that is $\lambda' \ge 1 - \delta$. Next we choose $0 \le \alpha \le 1$ such that $1 - \delta \le \lambda'' \le 1$ holds for $\lambda'' = \alpha\lambda + (1 - \alpha)\lambda'$. (Recall that we are considering the case that $\lambda < 1$, therefore such a choice of α is possible.) Then

$$b \le \alpha\left(\lambda a + \frac{\varepsilon}{2} v\right) + (1 - \alpha)\left(\lambda' a' + \frac{\varepsilon}{2} v\right) \le \lambda'' a'' + \frac{\varepsilon}{2} v$$

with

$$a'' = \frac{\alpha\lambda}{\lambda''}\, a + \frac{(1-\alpha)\lambda'}{\lambda''}\, a' \in A.$$

From $0 \le b + \rho v$ we infer that $0 \le \lambda'' a'' + (\rho + \varepsilon/2)v$. Our assumption that $\delta \le 1/2$ guarantees $1/2 \le \lambda'' \le 1$ and $1 - \lambda'' \le \delta$. Using this we infer

$$\begin{aligned} 0 &\le (1-\lambda'')a'' + \frac{(1-\lambda'')(2\rho+\varepsilon)}{2\lambda''} v \\ &\le (1-\lambda'')a'' + \delta(2\rho+\varepsilon)v \\ &\le (1-\lambda'')a'' + \frac{\varepsilon}{2} v \end{aligned}$$

since $\delta \le \varepsilon/(4\rho + 2\varepsilon)$. Now combining the above yields

$$b \le \lambda'' a'' + \frac{\varepsilon}{2} v + \Big((1-\lambda'')a'' + \frac{\varepsilon}{2} v\Big) \le a'' + \varepsilon v,$$

again satisfying the requirement from 4.28(a). We conclude that $b \in \overline{A}^{(l)}$, as claimed.

The argument for Part (b) of the Theorem follows similar lines, but is sufficiently different from the preceding one to be presented here too: If $b \in \overline{A}^{(u)}$ and if $\mu \in \mathcal{P}^*$, then a similar argument than before using Proposition 4.29(a) yields $\mu(a) \ge \inf\{\mu(a) \mid a \in A\}$. For the converse implication we will again employ Theorem 2.10. For fixed numbers $0 \le \alpha \in \overline{\mathbb{R}}$ and $\beta \in \mathbb{R}$ consider the sublinear functional p on $\mathcal{P}$ defined for $x \in \mathcal{P}$ as $p(x) = \rho\alpha$ if $x = \rho b$ and $p(x) = +\infty$ else, together with the extended superlinear functional

$$q(x) = \sup\{\lambda\beta \mid x = \lambda a \quad \text{for some} \quad a \in A \quad \text{and} \quad \lambda \ge 0\}.$$

There is $\mu \in \mathcal{P}^*$ such that $q \le \mu \le p$ if and only if there is $v \in \mathcal{V}$ such that $q(x) \le p(y) + 1$ whenever $x \le y + v$ for $x, y \in \mathcal{P}$; that is in our particular case $\lambda\beta \le \rho\alpha + 1$ whenever $\lambda a \le \rho b + v$ for some $a \in A$ and $\lambda, \rho \ge 0$. For this we shall again have to distinguish two cases:

(i) If for every $v \in \mathcal{V}$ and $\varepsilon > 0$ there is $a \in A$ and $0 \le \delta \le \varepsilon$ such that $a \le \delta b + v$, then we have to require that $\beta \le 0$.

(ii) If there are $v \in \mathcal{V}$ and $\varepsilon > 0$ such that $a \not\le \delta b + v$ for all $a \in A$ and $0 \le \delta \le \varepsilon$, then the above condition holds for this neighborhood v with any $\beta \in \mathbb{R}$, provided that $\alpha \ge 1/\varepsilon$. Indeed, assume that $\lambda a \le \rho b + v$ for some $a \in A$ and $\lambda, \rho \ge 0$, but $\lambda\beta > \rho\alpha + 1$. Then

$$a \le \frac{\rho}{\lambda} b + \frac{1}{\lambda} v \le \rho b + v.$$

This shows $\rho/\lambda > \varepsilon$, hence

$$\rho > \varepsilon\lambda > \varepsilon\rho\alpha + \varepsilon \ge \rho + \varepsilon,$$

a contradiction. In order to apply Theorem 2.10 we need to guarantee that $\mu(b) < +\infty$ for at least one $\mu \in \mathcal{P}^*$ satisfying $q \le \mu \le p$. Our insertions for p and q imply that $\beta \le \inf\{\mu(a) \mid a \in a\}$ and $\mu(b) \le \alpha$. We shall use $\alpha = +\infty$ in 2.10, but the preceding discussion involving different choices for α demonstrates that there is $\mu \in \mathcal{P}^*$ such that $q \le \mu \le p$ and $\mu(b) < +\infty$ in case (i), for any choice of $\beta \le 0$ and in case (ii) for any choice of $\beta \in \mathbb{R}$. Thus we may use Theorem 2.10 for

$$\inf_{\substack{\mu\in\mathcal{P}^*\\ q\le\mu\le p}} \mu(b) \;=\; \inf_{v\in\mathcal{V}} \sup\,\{q(x) - p(y) \mid x, y \in \mathcal{P},\;\; p(y) \in \mathbb{R},\;\; x \le b + y + v\},$$

With the particular insertions for p and q from above we need to consider only the choice of $y = 0$ and obtain

$$\inf_{\substack{\mu\in\mathcal{P}^*\\ q\le\mu\le p}} \mu(b) \;=\; \inf_{v\in\mathcal{V}} \sup\,\{\lambda\beta \mid \lambda \ge 0,\;\; \lambda a \le b + v \quad \text{for some} \quad a \in A\}.$$

Now let us assume that $\mu(b) \ge \inf\{\mu(a) \mid a \in A\}$ holds for all $\mu \in \mathcal{P}^*$. This yields

$$\inf_{v\in\mathcal{V}} \sup\,\{\lambda\beta \mid \lambda \ge 0,\;\; \lambda a \le b + v \quad \text{for some} \quad a \in A\} \;\ge\; \beta$$

for all admissible values of β. We shall use 4.29(a) to derive $b \in \overline{A}^{(u)}$ from this. Let $v \in \mathcal{V}$ and $\varepsilon > 0$. There is $\rho > 0$ such that $0 \le b + \rho v$. We choose $\beta = -1$ in the above and observe that there is $a \in A$ and $\lambda \ge 0$ such that

$$\lambda a \le b + \frac{\varepsilon}{2} v \quad \text{and} \quad -\lambda \ge -1 - \frac{\varepsilon}{2\rho} \quad \text{that is} \qquad \lambda \le 1 + \frac{\varepsilon}{2\rho}.$$

If $1 \le \lambda + 1 + (\varepsilon/2\rho)$, we proceed as follows:

$$\lambda a \le b + \frac{\varepsilon}{2} v + (\lambda - 1)(b + \rho v) \le \lambda b + \varepsilon v,$$

since

$$\frac{\varepsilon}{2} + (\lambda - 1)\rho \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2\rho}\rho = \varepsilon.$$

Thus

$$a \le b + \frac{\varepsilon}{\lambda} v \le b + \varepsilon v,$$

demonstrating that $a \in v_\varepsilon(b)$ as required in 4.29(b). Otherwise, that is if $\lambda < 1$, we continue to distinguish the above cases: In case (i) we set $\delta = \varepsilon/(2 - 2\lambda)$ and according to this case can find $a' \in A$ such that $a' \le \delta' b + \delta v$ for some $0 \le \delta' \le \delta$. Thus

$$a'' = \lambda a + (1 - \lambda) a' \le \Big((1 + (1 - \lambda)\delta')\Big) b + \Big(\frac{\varepsilon}{2} + (1 - \lambda)\delta\Big)\, v.$$

Since $a'' \in A$, and since $(1 - \lambda)\delta = \varepsilon/2$ and

$$1 \le 1+(1-\lambda)\delta' \le 1+(1-\lambda)\delta \le 1+(\varepsilon/2),$$

this shows $a'' \in v_\varepsilon(b)$ as required in 4.29(b). In case (ii) we may use the above inequality for $\beta = +1$ as well. For $\sigma = \max\{1/2,\, 1/(1+\varepsilon)\} < 1$ we find $a' \in A$ and $\lambda' \ge 0$ such that

$$\lambda' a' \le b + \frac{\varepsilon}{2} v \qquad \text{and} \qquad \lambda' \ge \sigma.$$

We can choose $0 \le \alpha \le 1$ such that $\sigma \le \lambda'' \le 1$ holds for $\lambda'' = \alpha\lambda + (1-\alpha)\lambda'$. (Recall that we are considering the case that $\lambda < 1$, therefore such a choice of α is possible.) With

$$a'' = \frac{\alpha\lambda}{\lambda''}\, a + \frac{(1-\alpha)\lambda'}{\lambda''}\, a' \in A$$

we have

$$\lambda'' a'' \;\le\; b + \frac{\varepsilon}{2}\, v,$$

hence

$$a'' \le \frac{1}{\lambda''}\, b + + \frac{\varepsilon}{2\lambda''}\, v.$$

Because $1 \le 1/\lambda'' \le 1/\sigma \le 1+\varepsilon$, and because $\varepsilon/(2\lambda'') \le \varepsilon/2\sigma \le \varepsilon$ we infer that $a'' \in v_\varepsilon(b)$, again satisfying the requirement from 4.29(a). We conclude that $b \in \overline{A}^{(u)}$, as claimed. □

Theorem 4.30 is a generalization of Theorem 3.1 in [175] as the following corollary will show.

Corollary 4.31. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone. Then $a \preccurlyeq b$ holds for $a, b \in \mathcal{P}$ if and only if $\mu(a) \le \mu(b)$ for all $\mu \in \mathcal{P}^*$.*

Proof. Let $a, b \in \mathcal{P}$. We have $a \preccurlyeq b$ if and only if $a \in \overline{\{b\}}^{(l)}$, and if and only if $b \in \overline{\{a\}}^{(u)}$. By Theorem 4.30, Parts (a) and (b), each of these statements holds if and only if $\mu(a) \le \mu(b)$ for all $\mu \in \mathcal{P}^*$. □

We proceed to define neighborhoods for subsets of a locally convex cone $(\mathcal{P}, \mathcal{V})$. For a subset $A \subset \mathcal{P}$, a neighborhood $v \in \mathcal{V}$ we define *upper* and *lower relative neighborhoods* as subsets of $\mathcal{P}$ by

$$v(A) = \left\{ b \in \mathcal{P} \;\middle|\; \begin{array}{l} \text{for every } \varepsilon > 0 \text{ there is } a \in A \text{ and } 1 \le \gamma \le 1+\varepsilon \\ \text{such that } b \le \gamma a + (1+\varepsilon) v \end{array} \right\}$$

and

$$(A)v = \left\{ b \in \mathcal{P} \;\middle|\; \begin{array}{l} \text{for every } \varepsilon > 0 \text{ there is } a \in A \text{ and } 1 \le \gamma \le 1+\varepsilon \\ \text{such that } a \le \gamma b + (1+\varepsilon) v \end{array} \right\}.$$

Note that this notation is consistent with the one earlier introduced for elements of $\mathcal{P}$, as we have $a \preccurlyeq b+v$ if and only if $a \in v\big(\{b\}\big)$ (see Lemma 3.1) and if and only if $b \in \big(\{a\}\big)v$.

Lemma 4.32. *Let A be a subset of a locally convex cone $(\mathcal{P},\mathcal{V})$ and let $v \in \mathcal{V}$.*

(a) The upper neighborhood $v\big(A\big)$ is closed in $\mathcal{P}$ with respect to the lower relative topology.
(b) The lower neighborhood $\big(A\big)v$ is closed in $\mathcal{P}$ with respect to the upper relative topology.
(c) If A is convex, then both $v\big(A\big)$ and $(A)v$ are convex.

Proof. Let $A \subset \mathcal{P}$ and let $b \in \overline{v\big(A\big)}^{(l)}$. Given $\varepsilon > 0$, set $\delta = \min\{1, \varepsilon/4\}$. According to 4.28(a) there is $c \in v\big(A\big)$ such that $b \in v_\delta(c)$, that is $b \le \gamma c + \delta v$ with some $1 \le \gamma \le 1+\delta$. Moreover, we have $c \le \gamma' a + (1+\delta)v$ for some $a \in A$ and $1 \le \gamma' \le 1+\delta$. Thus

$$b \le (\gamma\gamma')a + \big(\gamma(1+\delta)+\delta\big)v \le \gamma\gamma' a + \big((1+\delta)^2+\delta\big)v.$$

Since both $1 \le \gamma\gamma' \le (1+\delta)^2 \le 1+\varepsilon$ and $(1+\delta)^2+\delta \le 1+\varepsilon$, we infer that $b \in v\big(A\big)$. Similarly one verifies Part (b) of the Lemma. For Part (c) suppose that A is convex and let $b, b' \in v\big(A\big)$ and $b'' = \alpha b + (1-\alpha)b'$ for some $0 \le \alpha \le 1$. Given $v \in \mathcal{V}$ and $\varepsilon > 0$ there are $a, a' \in A$ such that $b \le \gamma a + (1+\varepsilon)v$ and $b' \le \gamma' a' + (1+\varepsilon)v$ for some $1 \le \gamma, \gamma' \le 1+\varepsilon$. Set $\gamma'' = (\alpha\gamma + (1-\alpha)\gamma')$. Then

$$a'' = \frac{\alpha\gamma}{\gamma''}a + \frac{(1-\alpha)\gamma'}{\gamma''}a' \in A$$

and $b'' \le \gamma'' a'' + (1+\varepsilon)v$. Since $1 \le \gamma'' \le 1+\varepsilon$, this demonstrates that $b'' \in v\big(A\big)$, and therefore this set is also convex. Similarly one argues for the lower neighborhood $\big(A\big)v$. □

Theorem 4.33. *Let A be a convex subset of a locally convex cone $(\mathcal{P},\mathcal{V})$, let $v \in \mathcal{V}$ and $b \in \mathcal{P}$. Then*

(a) $b \in v\big(A\big)$ if and only if $\mu(b) \le \sup\{\mu(a) \mid a \in A\} + 1$ for all $\mu \in v^\circ$.
(b) $b \in \big(A\big)v$ if and only if $\mu(b) \ge \inf\{\mu(a) \mid a \in A\} - 1$ for all $\mu \in v^\circ$.

Proof. We may again assume that $A \ne \emptyset$. For Part (a), suppose that $b \in v\big(A\big)$ and let $\mu \in v^\circ$. Given $\varepsilon \ge 0$ there is $a \in A$ such that $b \le \gamma a + (1+\varepsilon)v$, hence

$$\mu(b) \le \gamma\,\mu(a) + (1+\varepsilon) \le \gamma \sup\{\mu(a) \mid a \in A\} + (1+\varepsilon)$$

for some $1 \le \gamma \le 1+\varepsilon$. This shows $\mu(b) \le \sup\{\mu(a) \mid a \in A\} + 1$ since $\varepsilon > 0$ was arbitrarily chosen. In order to prove the converse implication we consider a full locally convex cone $(\widehat{\mathcal{P}},\mathcal{V})$ containing both $\mathcal{P}$ and the neighborhood system $\mathcal{V}$. Then $A \subset \widehat{\mathcal{P}}$. The lower neighborhood $\hat{v}(A)$ formed in $\widehat{\mathcal{P}}$ is larger than $v(A)$ formed in $\mathcal{P}$, but we have $v(A) = \hat{v}(A) \cap \mathcal{P}$. Thus if $b \notin v(A)$ for $b \in \mathcal{P}$ we also have $b \notin \hat{v}(A)$. Since $\hat{v}(A)$ is a convex subset of $\widehat{\mathcal{P}}$ and closed with respect to the lower relative topology, according to Theorem 4.30 there is $\mu \in \widehat{\mathcal{P}}^*$ such that $\mu(b) > \sup\{\mu(c) \mid c \in v(A)\}$. This implies in particular that $\sup\{\mu(c) \mid c \in v(A)\}$ is finite, and since $a+v \in v(A)$ whenever $a \in A$ we have $\mu(a+v) = \mu(a)+\mu(v) < +\infty$, hence $\mu(v) < +\infty$. If $\mu(v) = 0$, then $\lambda\mu \in v^\circ$ for all $\lambda \geq 0$, and we may choose λ such that

$$(\lambda\mu)(b) > \sup\{(\lambda\mu)(c) \mid c \in v(A)\} + 1 \geq \sup\{(\lambda\mu)(a) \mid a \in A\} + 1.$$

If $\mu(v) > 0$, we set $\lambda = 1/\mu(v)$ and have again $\lambda\mu \in v^\circ$. Then for every $a \in A$ we have $a + v \in v(A)$, hence

$$(\lambda\mu)(b) > (\lambda\mu)(a+v) = (\lambda\mu)(a) + 1$$

and therefore

$$(\lambda\mu)(b) > \sup\{(\lambda\mu)(a) \mid a \in A\} + 1.$$

Since the restriction of the functional $\lambda\mu \in \widehat{\mathcal{P}}^*$ to $\mathcal{P}$ is an element of $v^\circ \subset \mathcal{P}^*$, this proves our claim for Part (a). The argument for Part (b) uses the easily verified fact that $b \notin (A)v$ implies that $b + v \notin \overline{A}^{(u)}$. Indeed, if $b + v \in \overline{A}^{(u)}$, then by 4.29(b) for every $\varepsilon > 0$ there is $a \in A$ such that $a \leq \gamma(b+v) + (\varepsilon/2)v$ with some $1 \leq \gamma \leq 1 + (\varepsilon/2)$. This shows

$$a \leq \gamma b + \big(\gamma + (\varepsilon/2)\big)v \leq a \leq \gamma b + (1+\varepsilon)v,$$

hence $b \in (A)v$. The remainder of the argument is similar to that in Part (a). □

Theorem 4.33 is a generalization of Theorem 3.2 in [175] as the following corollary will show.

Corollary 4.34. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone. Then $a \preccurlyeq b + v$ holds for $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$ if and only if $\mu(a) \leq \mu(b) + 1$ for all $\mu \in v^\circ$.*

Proof. Let $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$ We have $a \preccurlyeq b + v$ if and only if $a \in v(\{b\})$ and if and only if $b \in (\{a\})v$. By Theorem 4.33, Parts (a) and (b), each of these statements holds if and only if $\mu(a) \leq \mu(b) + 1$ for all $\mu \in \mathcal{P}^*$. □

Corollary 4.35. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone. Let $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$ such that the element a is v-bounded. Then $a \preccurlyeq b + v$ holds if and only if $\mu(a) \leq \mu(b) + 1$ for all extreme points μ of v°.*

Proof. All left to show is the following: Let $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$ such that a is v-bounded. Then $\mu(a) < +\infty$ for all $\mu \in v^\circ$. Thus the function

$$\mu \mapsto \big(\mu(b) - \mu(a)\big) \,:\, v^\circ \to \overline{\mathbb{R}}$$

is affine and continuous with respect to the topology $w(\mathcal{P}^*, \mathcal{P})$ (see Section 2). According to Lemmas II.4.4 and II.4.5 in [100] this function attains its minimum value at some extreme point of v°. If $a \not\preccurlyeq b + v$, then according to Corollary 4.34 this minimum value is less than -1. Our claim follows. □

Remark 4.36. The following counterexample will demonstrate that the statement of Theorem 4.30 does in general not hold true for convex subsets $A \subset \mathcal{P}$ which are closed for the given lower or upper topologies rather than for the (coarser) upper or lower relative topologies. For this, let $(\mathcal{P}, \mathcal{V})$ be the locally convex cone of all continuous real-valued and bounded below functions on the interval $[0, +\infty)$, endowed with the positive constant functions $v > 0$ as its neighborhood system $\mathcal{V}$. $\big($see Example 1.4(e)$\big)$. Let the subset $A \in \mathcal{P}$ consist of all functions in $g \in \mathcal{P}$ with the following properties: (i) $g(x) \leq x$ for all $x \in [0, +\infty)$, and (ii) there is $M \geq 0$ and $\alpha < 1$ such that $g(x) \leq \alpha x$ for all $x \in [M, +\infty)$. We claim that A is closed in the given lower topology. Indeed, let $f \in \mathcal{P}$ be in the closure of A. Then, given $v > 0$ there is $g \in (f)v \cap A$, that is $f \leq g + v$, hence $f(x) \leq g(x) + v \leq x + v$ for all $x \in [0, +\infty)$. Thus $f(x) \leq x$ for all $x \in [0, +\infty)$, since $v > 0$ was arbitrarily chosen, hence (i) holds for f. For (ii) let $g \in A$ such that $f(x) \leq g(x) + 1$ for all $x \in [0, +\infty)$, and let $M \geq 0$ and $\alpha < 1$ such that $g(x) \leq \alpha x$ for all $x \in [M, +\infty)$. Choose $N = \max\{M, 2/(1-\alpha)\}$. Then for all $x \in [N, +\infty)$ we have $2/(1-\alpha) \leq x$, hence $1 \leq (1-\alpha)x/2$ and

$$f(x) \leq g(x) + 1 \leq \alpha x + \frac{(1-\alpha)}{2} x \leq \frac{(1+\alpha)}{2} x.$$

Since $(1+\alpha)/2 < 1$, this shows $f \in A$, confirming that A is closed with respect to the lower topology. The set $A \subset \mathcal{P}$ is however not closed with respect to the coarser lower relative topology as the function $f(x) = x$ is contained in $\overline{A}^{(l)}$. Indeed, given $v > 0$ and $\varepsilon > 0$, set $\alpha = 1/(1+\varepsilon) < 1$. Then

$$f(x) = x = (1+\varepsilon)\alpha x \leq (1+\varepsilon)\alpha x + \varepsilon v \qquad \text{for all} \quad x \in [0, +\infty).$$

This shows $g \in (f)v_\varepsilon \cap A \neq \emptyset$, where $g(x) = \alpha x$. We therefore have $\mu(f) \leq \sup\{\mu(g) \mid g \in A\}$ for all $\mu \in \mathcal{P}^*$, but $f \notin A$.

Examples 4.37. (a) Let $\mathcal{P} = \overline{\mathbb{R}}$, endowed with the neighborhood system $\mathcal{V} = \{\varepsilon \in \mathbb{R} \mid \varepsilon > 0\}$ $\big($see Example 1.4(a)$\big)$. For the neighborhood $v = 1$ and $\varepsilon > 0$ the relative neighborhoods of an element $a \in \overline{\mathbb{R}}$ are

$$v_\varepsilon(a) = \big(-\infty,\, (1+\varepsilon)a + \varepsilon\big] \qquad \text{or} \qquad v_\varepsilon(a) = \big(-\infty,\, a + \varepsilon\big]$$

if $a \geq 0$ or if $a < 0$, respectively. Thus

$$(a)v_\varepsilon = \left[\tfrac{a-\varepsilon}{1+\varepsilon},\, +\infty\right] \qquad \text{or} \qquad (a)v_\varepsilon = \left[a-\varepsilon,\, +\infty\right]$$

if $a \geq \varepsilon$ or if $a < \varepsilon$, respectively. This yields

$$v^s_\varepsilon(a) = \left[\tfrac{a-\varepsilon}{1+\varepsilon},\ (1+\varepsilon)a+\varepsilon\right], \qquad v^s_\varepsilon(a) = \left[a-\varepsilon,\ (1+\varepsilon)a+\varepsilon\right],$$

or

$$v^s_\varepsilon(a) = \left[a-\varepsilon, a+\varepsilon\right]$$

if $a \geq \varepsilon$, if $0 \leq a < \varepsilon$, or if $a < 0$, respectively. The upper, lower and symmetric relative topologies of $\overline{\mathbb{R}}$ therefore coincide with the corresponding given topologies. (see 1.4(a)). The symmetric relative topology, in particular, is the usual topology on $\mathbb{R}$ with $+\infty$ as an isolated point.

(b) Let $\mathcal{P} = \overline{\mathbb{R}}_+ = \{a \in \overline{\mathbb{R}} \mid a \geq 0\}$, endowed with the neighborhood system $\mathcal{V} = \{0\}$ (see Example 1.4(b)). For the only neighborhood $v = 0 \in \mathcal{V}$ and $\varepsilon > 0$ the relative neighborhoods of an element $a \in \overline{\mathbb{R}}_+$ are

$$v_\varepsilon(a) = \left[0,\, (1+\varepsilon)a\right], \quad (a)v_\varepsilon = \left[\tfrac{a}{1+\varepsilon},\, +\infty\right] \quad \text{and} \quad v^s_\varepsilon(a) = \left[\tfrac{a}{1+\varepsilon},\ (1+\varepsilon)a\right].$$

The symmetric relative topology therefore coincides with the Euclidean topology on $(0,+\infty)$, but renders $0 \in \mathcal{P}$ and $+\infty \in \mathcal{P}$ as isolated points. Recall from Example 1.4(b) that the symmetric given topology on $\overline{\mathbb{R}}_+$, in contrast, is the discrete topology. For the boundedness components of $\overline{\mathbb{R}}_+$ we have

$$\mathcal{B}(a) = [0,+\infty), \qquad (a)\mathcal{B} = (0,+\infty] \qquad \text{and} \qquad \mathcal{B}^s(a) = (0,+\infty)$$

for $a \in (0,+\infty)$,

$$\mathcal{B}(0) = \{0\}, \qquad (0)\mathcal{B} = [0,+\infty] \qquad \text{and} \qquad \mathcal{B}^s(0) = \{0\},$$

and

$$\mathcal{B}(+\infty) = [0,+\infty], \qquad (+\infty)\mathcal{B} = \{\infty\} \qquad \text{and} \qquad \mathcal{B}^s(+\infty) = \{\infty\}.$$

As claimed, the symmetric boundedness components furnish a partition of $\mathcal{P} = \overline{\mathbb{R}}_+$ into disjoint subsets that are both open and closed in the symmetric relative topology.

(c) Let us consider Example 1.4(e) with the special insertions for $\mathcal{P} = \overline{\mathbb{R}}$ and the neighborhood system $\widehat{\mathcal{V}}$ generated by a family $\mathcal{Y}$ of subsets of the domain X as elaborated in 1.4(e). Recall that $\widehat{\mathcal{V}}$ is spanned by the $\overline{\mathbb{R}}$-valued functions $\hat{v}_Y \in \widehat{\mathcal{V}}$, corresponding to some $Y \in \mathcal{Y}$, and such that $\hat{v}_Y(x) = 1$ for $x \in Y$ and $\hat{v}_Y(x) = +\infty$, else. Thus $\big(\mathcal{F}_{\widehat{\mathcal{V}}_b}(X,\overline{\mathbb{R}}), \widehat{\mathcal{V}}\big)$ is the locally convex cone of all bounded below (on the sets in $\mathcal{Y}$) $\overline{\mathbb{R}}$-valued functions on X, carrying the topology of uniform convergence on the sets in $\mathcal{Y}$. For a function $f \in \mathcal{F}_{\widehat{\mathcal{V}}_b}(X,\overline{\mathbb{R}})$ and a neighborhood $\hat{v}_Y \in \widehat{\mathcal{V}}$, the $\hat{v}_Y$-boundedness component $\mathcal{B}^s_{\hat{v}_Y}(f)$ consists of all $g \in \mathcal{F}_{\widehat{\mathcal{V}}_b}(X, \overline{\mathbb{R}})$ such that

$$\alpha f(x) - \beta \leq g(x) \leq \gamma f(x) + \delta$$

holds with some constants $\alpha, \beta, \gamma, \delta > 0$ for all $x \in Y$. Thus, obviously, $(\hat{v}_Y)^s_\varepsilon(g) \subset \mathcal{B}^s_{\hat{v}_Y}(f)$ for all $\varepsilon > 0$ whenever $g \in \mathcal{B}^s_{\hat{v}_Y}(f)$. This observation confirms that the component $\mathcal{B}^s_{\hat{v}_Y}(f)$ is both open and closed in the symmetric relative $\hat{v}_Y$-topology, which is the topology of uniform convergence on Y. Yet the (global) boundedness component $\mathcal{B}^s(f) = \bigcap_{Y \in \mathcal{Y}} \mathcal{B}^s_{\hat{v}_Y}(f)$ is in general only closed in the symmetric relative topology, which is the topology of uniform convergence on all sets $Y \in \mathcal{Y}$. However, if the set X itself is contained in $\mathcal{Y}$, then the multiples of the neighborhood $\hat{v}_X$ form already a basis for $\hat{\mathcal{V}}$, and the $\hat{v}_X$-boundedness components coincide with the global ones. Following Proposition 4.22, $\mathcal{F}_{\widehat{\mathcal{V}}_b}(X, \overline{\mathbb{R}})$ is locally connected in this case. Its boundedness components therefore coincide with the components and quasi-components in the symmetric relative topology (Proposition 4.21) and are both open and closed. If, for another special case, $\mathcal{Y}$ consists of all finite subsets of X, then for $Y \in \mathcal{Y}$ the above condition yields that two functions $f, g \in \mathcal{F}_{\widehat{\mathcal{V}}_b}(X, \overline{\mathbb{R}})$ are contained in the same $\hat{v}_Y$-boundedness component if and only if they take the value $+\infty$ at exactly the same points of Y. The symmetric relative $\hat{v}_Y$-topology is the topology of pointwise convergence on the set Y in this case. Correspondingly, the global boundedness components consist of functions that take the value $+\infty$ at exactly the same points of X, and the symmetric relative topology is the topology of pointwise convergence on X. If X itself is an infinite set, then the global boundedness components are seen to be closed but not open in this topology.

(d) Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone and let $\mathcal{Q}$ be the family of all non-empty convex subsets of $\mathcal{P}$ which are closed with respect to the lower relative topology. (See 4.26 to 4.35 before.) If we use the standard multiplication for sets by non-negative scalars and a slightly modified addition, that is

$$A \oplus B = \overline{(A+B)}^{(l)} \qquad \text{for} \quad A, B \in \mathcal{Q},$$

then $\mathcal{Q}$ becomes a cone. Indeed, since the set $A + B$ is obviously again convex, so is its closure with respect to the lower relative topology by Proposition 4.28(b). The neutral element of $\mathcal{Q}$ is given by $\overline{\{0\}}^{(l)} = \{b \in \mathcal{P} \mid b \preccurlyeq 0\}$. We use the set inclusion as the order on $\mathcal{Q}$ and define neighborhoods corresponding to those in $\mathcal{P}$: We set

$$A \leq B \oplus v \qquad \text{if} \quad A \subset v(B)$$

for $A, B \in \mathcal{Q}$ and $v \in \mathcal{V}$, that is if for every $a \in A$ and $\varepsilon > 0$ there is $b \in B$ such that $a \leq \gamma b + (1+\varepsilon)v$ for some $1 \leq \gamma \leq 1+\varepsilon$. First we observe that for every $A \in \mathcal{Q}$ and a fixed element $a \in A$ $v \in \mathcal{V}$ there is $\lambda \geq 0$ such that $0 \leq a + \lambda v$. Since $\overline{\{0\}}^{(l)} = \{b \in \mathcal{P} \mid b \preccurlyeq 0\}$, this yields $\overline{\{0\}}^{(l)} \leq A \oplus (\lambda+1)v$. Indeed, for every $b \preccurlyeq 0$, we have $b \leq v$, hence

$b \le a+(\lambda+1)v$. Thus every element $A \in \mathcal{Q}$ is seen to be bounded below and $(\mathcal{Q},\mathcal{V})$ satisfies the requirements for a locally convex cone. Next we observe that the weak preorder on $(\mathcal{Q},\mathcal{V})$ coincides with the given order. Indeed, suppose that $A \preccurlyeq B$, and let $a \in A$. Given $v \in \mathcal{V}$ and $\varepsilon > 0$ we set $\delta = \min\{\varepsilon/3, 1\}$ and have $A \le \gamma B \oplus \delta v$ for some $1 \le \gamma \le 1+\delta$. According to Lemma 3.1 there is $1 \le \gamma' \le 1+\delta$ such that $a \le (\gamma'\gamma)b + (1+\delta)\delta v$ for some $b \in B$. Since $(1+\delta)\delta \le \varepsilon$, this yields $a \le (\gamma'\gamma)b + \varepsilon v$, and since $1 \le \gamma\gamma' \le (1+\delta)^2 \le 1+\varepsilon$, we have $a \in v_\varepsilon(b)$ and infer from (i) that $a \in \overline{B}^{(l)} = B$, hence $A \le B$. Therefore $A \preccurlyeq B$ holds if and only if $A \le B$. A similar argument shows that $A \preccurlyeq B \oplus v$ holds for $A, B \in \mathcal{Q}$ and $v \in \mathcal{V}$ if and only if $A \le B \oplus v$. An element $A \in \mathcal{Q}$ is bounded above in $\mathcal{Q}$ if for every $v \in \mathcal{V}$ there is $\lambda \ge 0$ such that $A \le \lambda v$, that is $a \le (\lambda+1)v$ holds for all $a \in A$, that is if the set $A \subset \mathcal{P}$ is bounded above in $\mathcal{P}$ in the sense of 4.25(ii).

(e) Similarly, but less intuitively we may consider the family $\mathcal{Q}$ of all convex subsets of a locally convex cone $\mathcal{P}$ which are closed with respect to the upper relative topology and bounded below in the sense of 4.25(i). (See 4.26 to 4.35 before.) We use the standard multiplication for sets by non-negative scalars and the addition

$$A \oplus B = \overline{(A+B)}^{(u)} \qquad \text{for} \quad A, B \in \mathcal{Q}.$$

Since the sum of two bounded below convex subsets of $\mathcal{P}$ is obviously again bounded below and convex, Proposition 4.29(b) and (c) guarantees that the set $A \oplus B$ is indeed also bounded below and convex. Thus $\mathcal{Q}$ is a cone with the neutral element $\overline{\{0\}}^{(u)} = \{b \in \mathcal{P} \mid 0 \preccurlyeq b\}$. In this example we use the inverse set inclusion as the order on $\mathcal{Q}$, that is

$$A \le B \qquad \text{if} \quad B \subset A$$

and define neighborhoods corresponding to those in $\mathcal{P}$ by

$$A \le B \oplus v \qquad \text{if} \quad B \subset \big(A\big)v$$

for $A, B \in \mathcal{Q}$ and $v \in \mathcal{V}$, that is if for every $b \in B$, and $\varepsilon > 0$ there is $a \in A$ such that $a \le \gamma b + (1+\varepsilon)v$ for some $1 \le \gamma \le 1+\varepsilon$. Because for every $A \in \mathcal{Q}$ and $v \in \mathcal{V}$ there is $\lambda > 0$ such that $0 \le a + \lambda v$ for all $a \in A$, we have $\overline{\{0\}}^{(u)} \le A \oplus \lambda v$, and every element $A \in \mathcal{Q}$ is bounded below. Hence $(\mathcal{Q},\mathcal{V})$ is a locally convex cone. A similar argument than in (d) yields that $(\mathcal{Q},\mathcal{V})$ carries its weak preorder. Note that other than in (d) the empty set is a member of $\mathcal{Q}$, indeed its maximal element. We set $A \oplus \emptyset = \emptyset$, $\alpha \cdot \emptyset = \emptyset$ and $0 \cdot \emptyset = \overline{\{0\}}^{(u)}$ for all $A \in \mathcal{Q}$ and $\alpha > 0$. An element $A \in \mathcal{Q}$ is bounded above in $\mathcal{Q}$ if for every $v \in \mathcal{V}$ there is $\lambda \ge 0$ such that $A \le \lambda v$, that is there is $a \in A$ such that $a \le \lambda v$.

Note that in both Examples (d) and (e) the given locally convex cone $\mathcal{P}$ may be considered as a subcone of $\mathcal{Q}$ via the embedding $a \mapsto \overline{\{a\}}$. This is an embedding of $(\mathcal{P},\mathcal{V})$ into $(\mathcal{Q},\mathcal{V})$ in the sense of 2.2, provided that $\mathcal{P}$ is endowed with the weak preorder, that is $\overline{\{a\}} \le \overline{\{b\}} + v$ holds if and only if $a \preccurlyeq b+v$ for $a,b \in \mathcal{P}$ and $v \in \mathcal{V}$ $\big($see also 2.2(iii)$\big)$.

Remarks 4.38. (a) As a consequence of the last observation in 4.37(a) and of Proposition 4.5 we infer that a continuous linear functional μ on a locally convex cone $(\mathcal{P},\mathcal{V})$ is still continuous if we endow $\mathcal{P}$ with its relative topologies. More precisely: Let $\mu \in v^\circ$, that is the polar of some neighborhood $v \in \mathcal{V}$. Then μ is a continuous linear operator from $(\mathcal{P},\mathcal{V}_0)$ to $\overline{\mathbb{R}}$, where $\mathcal{V}_0$ consists of the multiples of the single neighborhood v. As shown in 4.37(a), the relative topologies on $\overline{\mathbb{R}}$ coincide with the given ones as described in Example 1.4(a). Thus according to 4.5, the functional μ is also continuous if we endow $\mathcal{P}$ with either the upper, lower or symmetric relative v-topology and, correspondingly, $\overline{\mathbb{R}}$ with its given upper, lower or symmetric topology.

(b) We noted earlier that for a locally convex cone $(\mathcal{P},\mathcal{V})$ the mapping

$$(\alpha,a) \mapsto \alpha a \ : \ [0,+\infty) \times \mathcal{P} \to \mathcal{P},$$

is generally not continuous with respect to any of the given topologies of $\mathbb{R}$ and $\mathcal{P}$. However, if we endow $\mathcal{P}$ with either of the relative topologies, this mapping is continuous at all points $(\alpha,a) \in [0,+\infty) \times \mathcal{P}$ such that either $\alpha > 0$ or $a \in \mathcal{P}$ is bounded. This was established in Proposition 4.2(iii). Now using 4.37(b) we realize that this mapping is continuous at all points of $[0,+\infty) \times \mathcal{P}$ if we consider the symmetric relative topology of $\overline{\mathbb{R}}_+$ $\big($see 4.37(b)$\big)$ and any of the relative topologies on $\mathcal{P}$ instead. Indeed, the symmetric relative topology of $\overline{\mathbb{R}}_+$ coincides with the usual topology of $\mathbb{R}$ on $(0,+\infty)$, hence continuity at all points $(\alpha,a) \in [0,+\infty) \times \mathcal{P}$ such that $\alpha > 0$ follows from 4.2(iii). Continuity at the points $(0,a)$ for all $a \in \mathcal{P}$, on the other hand is obvious, since 0 is an isolated element in the symmetric relative topology of $\overline{\mathbb{R}}_+$.

5. Locally Convex Lattice Cones

Our upcoming integration theory for cone-valued functions in Chapter II deals with locally convex cones that contain suprema and infima for sufficiently many of their subsets. Let us recall the classical concepts: A *topological vector lattice* is a vector lattice and a locally convex topological vector space E over $\mathbb{R}$ that possesses a neighborhood base of solid sets. (See for example Chapter V.7 in [185], also [132] or [184]. Recall that a subset A of E is called *solid* if $b \in A$ whenever $|b| \le |a|$ for $b \in E$ and $a \in A$.) Some of the following definitions and results are adaptations of the corresponding classical ones. The presence of unbounded elements and the general unavailability of negatives in locally convex cones, however, requires a more delicate approach.

5.1 Locally Convex Lattice Cones. We shall say that a locally convex cone $(\mathcal{P},\mathcal{V})$ is a *locally convex* $\vee$*-semilattice cone* if its order is antisymmetric and if for any two elements $a,b\in\mathcal{P}$ their supremum $a\vee b$ exists in $\mathcal{P}$ and if

($\vee$1) $(a+c)\vee(b+c)=a\vee b+c$ *holds for all* $a,b,c\in\mathcal{P}$.
($\vee$2) $a\le c+v$ *and* $b\le c+w$ *for* $a,b,c\in\mathcal{P}$ *and* $v,w\in\mathcal{V}$ *implies that* $a\vee b\le c+(v+w)$.

Likewise, $(\mathcal{P},\mathcal{V})$ is a *locally convex* $\wedge$*-semilattice cone* if its order is antisymmetric and if for any two elements $a,b\in\mathcal{P}$ their infimum $a\wedge b$ exists in $\mathcal{P}$ and if

($\wedge$1) $(a+c)\wedge(b+c)=a\wedge b+c$ *holds for all* $a,b,c\in\mathcal{P}$.
($\wedge$2) $c\le a+v$ *and* $c\le b+w$ *for* $a,b,c\in\mathcal{P}$ *and* $v,w\in\mathcal{V}$ *implies that* $c\le a\wedge b+(v+w)$.

If both sets of the above conditions hold, then $(\mathcal{P},\mathcal{V})$ is called a *locally convex lattice cone.* In case that $(\mathcal{P},\mathcal{V})$ is indeed a locally convex topological vector space, the existence of suprema implies the existence of infima and vice versa, as $a\wedge b=-\big((-a)\vee(-b)\big)$. Conditions ($\vee$1) and ($\vee$2) then are equivalent to ($\wedge$1) and ($\wedge$2) and consistent with the above mentioned definition of a topological vector lattice. Indeed, $a\le c+v$ and $b\le c+w$ means that $a\le c+s$ $b\le c+t$ in this case, for some elements s and t of the neighborhoods v and w, respectively. Because these neighborhoods are supposed to be solid, we have $s\vee 0\le v$ and $t\vee 0\le w$ as well. Now $a\le c+s\vee 0+t\vee 0$ and $b\le c+s\vee 0+t\vee 0$ implies

$$a\vee b\le c+s\vee 0+t\vee 0\le c+(v+w)$$

as required in ($\vee$1).

Proposition 5.2. *Let* $(\mathcal{P},\mathcal{V})$ *be a locally convex* $\vee$*- (or* $\wedge$*-) semilattice cone. The lattice operation* $(a,b)\mapsto a\vee b$ *(or* $(a,b)\mapsto a\wedge b$*) is a continuous mapping from* $\mathcal{P}\times\mathcal{P}$ *to* $\mathcal{P}$ *if* $\mathcal{P}$ *is endowed with the symmetric relative topology.*

Proof. Suppose that $(\mathcal{P},\mathcal{V})$ is a locally convex $\vee$-semilattice cone, and let $a\in v_\varepsilon(b)$ and $c\in v_\varepsilon(d)$ for $a,b,c,d\in\mathcal{P}$, $v\in\mathcal{V}$ and $\varepsilon>0$. There is $\lambda\ge 0$ such that both $0\le b+\lambda v$ and $0\le d+\lambda v$. Then $a\le(1+\varepsilon)b+\varepsilon(1+\lambda)v$ and $c\le(1+\varepsilon)d+\varepsilon(1+\lambda)v$ by Lemma 4.1(b). Thus

$$a\le(1+\varepsilon)(b\vee d)+\varepsilon(1+\lambda)v \qquad\text{and}\qquad c\le(1+\varepsilon)(b\vee d)+\varepsilon(1+\lambda)v,$$

hence

$$a\vee c\le(1+\varepsilon)(b\vee d)+2\varepsilon(1+\lambda)v$$

by ($\vee$2). This shows $a\vee c\in v_{(2\varepsilon(1+\lambda))}(b\vee d)$. Similarly, using 4.1(c) one verifies that $a\in(b)v_\varepsilon$ and $c\in(d)v_\varepsilon$ implies $a\vee c\in(b\vee d)v_{(2\varepsilon(1+\lambda+\varepsilon))}(b\vee d)$.

Combining these observations for both the upper and lower relative neighborhoods then demonstrates that $a \in v^s_\varepsilon(b)$ and $c \in v^s_\varepsilon(d)$ implies $a \vee c \in v^s_{(2\varepsilon(1+\lambda+\varepsilon))}(b \vee d)$, hence our claim. A similar argument yields our claim for locally convex $\wedge$-semilattice cones. $\square$

Proposition 5.3. *Let* $(\mathcal{P}, \mathcal{V})$ *be a locally convex lattice cone. Then* $a + b = a \vee b + a \wedge b$ *for all* $a, b \in \mathcal{P}$.

Proof. We observe that

$$a + b \leq \inf \big\{ a + a \vee b \,,\, b + a \vee b \big\} = a \wedge b + a \vee b$$

by ($\wedge$1), and by ($\vee$1)

$$a \vee b + a \wedge b = \sup \big\{ a + a \wedge b \,,\, b + a \wedge b \big\} \leq a + b.$$

As the order of $\mathcal{P}$ is supposed to by antisymmetric, this yields our claim. $\square$

Proposition 5.3 implies in particular that $a = a \vee 0 + a \wedge 0$ for all elements a of a locally convex lattice cone.

Examples of locally convex lattice cones include classical topological vector lattices and the cones $\overline{\mathbb{R}}$ and $\overline{\mathbb{R}}_+$ from Examples 1.4(a) and (b). If $(\mathcal{P}, \mathcal{V})$ is a locally convex $\vee$- or $\wedge$-semilattice cone, if X is a set, and if $\widehat{\mathcal{V}}$ is a neighborhood system consisting of $\big(\mathcal{V} \cup \{\infty\}\big)$-valued functions on X, then the locally convex cone $\big(\mathcal{F}_{\widehat{\mathcal{V}}_b}(X, \mathcal{P}), \hat{\mathcal{V}}\big)$ of $\mathcal{P}$-valued functions from Example 1.4(e) is also a semilattice cone of the same type. Suprema and infima are formed pointwise in this case. The cones $(\overline{l^p}, \mathcal{V}_p)$ from 1.4(f) are locally convex lattice cones. The locally convex cone of all non-empty convex subsets of some locally convex topological vector space E $\big($see Example 1.4(c)$\big)$ is antisymmetrically ordered by set inclusion (we assume that equality is the order in E) and indeed a $\vee$-semilattice cone. The supremum of two convex subsets of E is the convex hull of their union while infima, that is intersections, do not always exist. Requirements ($\vee$1) and ($\vee$2) are readily checked.

5.4 Locally Convex Complete Lattice Cones. Later in this text, in particular when developing our integration theory, we shall consider substantially stronger properties concerning the lattice operations of a locally convex cone. We shall require the existence of suprema and infima for bounded and bounded below subsets, respectively. This assumption corresponds to the notion of order completeness for ordered vector spaces. Moreover, the upper or lower neighborhoods are supposed to be closed for suprema or infima of their subsets, respectively. This requirement corresponds to the properties of M-topologies in locally convex vector lattices.

We shall say that a locally convex cone $(\mathcal{P}, \mathcal{V})$ is a *locally convex* $\bigvee$-*semilattice cone* if $\mathcal{P}$ carries the weak preorder (that is the given order coincides with the weak preorder for the elements and the neighborhoods in $\mathcal{P}$), this order is antisymmetric and if

$(\bigvee 1)$ *Every non-empty subset* $A \subset \mathcal{P}$ *has a supremum* $\sup A \in \mathcal{P}$ *and* $\sup(A + b) = \sup A + b$ *holds for all* $b \in \mathcal{P}$.

$(\bigvee 2)$ *Let* $\emptyset \neq A \subset \mathcal{P}$, $b \in \mathcal{P}$ *and* $v \in \mathcal{V}$. *If* $a \leq b + v$ *for all* $a \in A$, *then* $\sup A \leq b + v$.

In particular, every $\bigvee$-semilattice cone $\mathcal{P}$ contains a largest element, that is $+\infty = \sup \mathcal{P}$, which can be adjoined as a maximal element to any locally convex cone with the convention that $a + \infty = +\infty$, $\alpha \cdot (+\infty) = +\infty$, $0 \cdot (+\infty) = 0$ and that $a \leq +\infty$ for all $a \in \mathcal{P}$ and $\alpha > 0$. Likewise, $(\mathcal{P}, \mathcal{V})$ is said to be a *locally convex* $\bigwedge$*-semilattice cone* if $\mathcal{P}$ carries the weak preorder, this order is antisymmetric and if

$(\bigwedge 1)$ *Every subset* $A \subset \mathcal{P}$ *that is bounded below has an infimum* $\inf A \in \mathcal{P}$ *and* $\inf(A + b) = \inf A + b$ *holds for all* $b \in \mathcal{P}$.

$(\bigwedge 2)$ *Let* $A \subset \mathcal{P}$ *be bounded below,* $b \in \mathcal{P}$ *and* $v \in \mathcal{V}$. *If* $b \leq a + v$ *for all* $a \in A$, *then* $b \leq \inf A + v$.

These requirements are obviously stronger then the corresponding ones in 4.23, so every locally convex $\bigvee$- (or $\bigwedge$-) semilattice cone is also a $\vee$- (or $\wedge$-) semilattice cone. The assumptions $(\bigvee 2)$ and $(\bigwedge 2)$ signify that the upper or lower neighborhoods in $\mathcal{P}$ are closed for suprema or infima of their subsets, respectively. If $(\mathcal{P}, \mathcal{V})$ is a full cone, then $(\bigvee 2)$ is evident, and $(\bigwedge 2)$ follows from $(\bigwedge 1)$. Recall from our convention in 4.24(i) that a subset A of $\mathcal{P}$ is said to be *bounded below* if for every $v \in \mathcal{V}$ there is $\lambda \geq 0$ such that $0 \leq a + \lambda v$ for all $a \in A$. This condition does in general not imply the existence of a lower bound in $\mathcal{P}$. However, if A has a lower bound $b \in \mathcal{P}$, that is $b \leq a$ for all $a \in A$, then A is bounded below in the above sense. Indeed, for every $v \in \mathcal{V}$ there is $\lambda \geq 0$ such that $0 \leq b + \lambda v$, hence $0 \leq a + \lambda v$ holds for all $a \in A$. Note that the empty set $\emptyset \subset \mathcal{P}$ is bounded below, and we have $\inf \emptyset = +\infty$ (see the remark above).

Combining both of the above notions, we shall say that a locally convex cone $(\mathcal{P}, \mathcal{V})$ is a *locally convex complete lattice cone* if $\mathcal{P}$ is both a $\bigvee$-semilattice cone and a $\bigwedge$-semilattice cone.

Corresponding to a family $\{A_i\}_{i \in \mathcal{I}}$ of non-empty subsets of a locally convex $\bigvee$-semilattice cone $\mathcal{P}$ we denote the subset

$$\bigvee_{i \in \mathcal{I}} A_i = \Big\{ \bigvee_{i \in \mathcal{I}} a_i \;\Big|\; (a_i)_{i \in \mathcal{I}} \in \prod_{i \in \mathcal{I}} A_i, \Big\} \subset \mathcal{P}.$$

Lemma 5.5. *Let* $(\mathcal{P}, \mathcal{V})$ *be a locally convex* $\bigvee$*-semilattice cone. Let* A, B *and* $\{A_i\}_{i \in \mathcal{I}}$ *be non-empty subsets of* $\mathcal{P}$. *Then*

(a) $\sup(A + B) = \sup A + \sup B$.

(b) $\sup\Big(\bigcup_{i \in \mathcal{I}} A_i\Big) = \sup\Big(\bigvee_{i \in \mathcal{I}} A_i\Big) = \sup_{i \in \mathcal{I}} \big\{ \sup A_i \mid i \in \mathcal{I} \big\}$.

Proof. For Part (b) we observe that for every $a \in \bigcup_{i \in \mathcal{I}} A_i$ there is $(a_i)_{i \in \mathcal{I}} \in \prod_{i \in \mathcal{I}} A_i$ such that a is one of the projections of $(a_i)_{i \in \mathcal{I}}$ onto the factor spaces A_i. This yields $a \le \bigvee_{i\in\mathcal{I}} a_i$, hence

$$\sup\Big(\bigcup_{i\in\mathcal{I}} A_i\Big) \le \sup\Big(\bigvee_{i\in\mathcal{I}} A_i\Big).$$

For every $(a_i)_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} A_i$ on the other hand, we have $a_i \le \sup A_i$ for all $i\in\mathcal{I}$, hence $\bigvee_{i\in\mathcal{I}} a_i \le \sup_{i\in\mathcal{I}}\big\{\sup A_i \mid i\in\mathcal{I}\big\}$ and

$$\sup\Big(\bigvee_{i\in\mathcal{I}} A_i\Big) \le \sup_{i\in\mathcal{I}}\big\{\sup A_i \mid i\in\mathcal{I}\big\}.$$

Finally, since $\sup A_i \le \sup\Big(\bigcup_{i\in\mathcal{I}} A_i\Big)$ holds for all $i\in\mathcal{I}$, we infer that

$$\sup_{i\in\mathcal{I}}\big\{\sup A_i \mid i\in\mathcal{I}\big\} \le \sup\Big(\bigcup_{i\in\mathcal{I}} A_i\Big).$$

Our claim in Part (b) now follows from the requirement that the order in $\mathcal{P}$ is antisymmetric.

For Part (a) we argue as follows: If A and B are non-empty subsets of $\mathcal{P}$, then we use Part (b) and $(\bigvee 1)$ for

$$\begin{aligned}\sup(A+B) &= \sup\Big(\bigcup_{b\in B}(A+b)\Big)\\ &= \sup_{b\in B}\big\{\sup(A+b) \mid b\in B\big\}\\ &= \sup_{b\in B}\big\{\sup A + b \mid b\in B\big\}\\ &= \sup A + \sup B. \quad \square\end{aligned}$$

Similarly, for a family $\{A_i\}_{i\in\mathcal{I}}$ of subsets of a locally convex $\bigwedge$-semilattice cone such that $\bigcup_{i\in\mathcal{I}} A_i$ is bounded below in $\mathcal{P}$ we denote

$$\bigwedge_{i\in\mathcal{I}} A_i = \Big\{\bigwedge_{i\in\mathcal{I}} a_i \;\Big|\; (a_i)_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} A_i,\Big\} \subset \mathcal{P}$$

and obtain in analogy to Lemma 5.5:

Lemma 5.6. *Let $(\mathcal{P},\mathcal{V})$ be a locally convex $\bigwedge$-semilattice cone. Let A, B and $\{A_i\}_{i\in\mathcal{I}}$ be bounded below subsets of $\mathcal{P}$ and suppose that $\bigcup_{i\in\mathcal{I}} A_i$ is also bounded below. Then*

(a) $\inf(A+B) = \inf A + \inf B$.
(b) $\inf\Big(\bigcup_{i\in\mathcal{I}} A_i\Big) = \inf\Big(\bigwedge_{i\in\mathcal{I}} A_i\Big) = \inf_{i\in\mathcal{I}}\big\{\inf A_i \mid i\in\mathcal{I}\big\}$.

Remarks and Examples 5.7. (a) Every locally convex $\bigwedge$-semilattice cone $\mathcal{P}$ contains also suprema for all of its non-empty subsets. Indeed, the set of all upper bounds for a non-empty subset A of $\mathcal{P}$ is bounded below, and its infimum is the supremum of A in $\mathcal{P}$. Requirement $\big(\bigvee 1\big)$ does however not necessarily follow (see Example (e) below). Likewise, every locally convex $\bigvee$-semilattice cone has infima for subsets with lower bounds in $\mathcal{P}$. (Recall the before mentioned subtle difference between "bounded below" and "having a lower bound".) But again, requirement $\big(\bigwedge 1\big)$ does not follow (see (d) below).

(b) The locally convex cones $\overline{\mathbb{R}}$ and $\overline{\mathbb{R}}_+$ $\big($Examples 1.4(a) and (b)$\big)$ are of course complete lattices.

(c) If $(\mathcal{P},\mathcal{V})$ is a locally convex $\bigvee$-semilattice (or $\bigwedge$-semilattice) lattice cone, if X is a set, and if $\widehat{\mathcal{V}}$ is a neighborhood system consisting of $\big(\mathcal{V}\cup\{\infty\}\big)$-valued functions $\big($see Example 1.4(e)$\big)$, then the locally convex cone $\big(\mathcal{F}_{\widehat{\mathcal{V}}_b}(X,\mathcal{P}),\hat{\mathcal{V}}\big)$ of $\mathcal{P}$-valued functions from 1.4(e) is also a locally convex $\bigvee$-semilattice (or $\bigwedge$-semilattice) lattice cone, provided that for every $x\in X$ and $v\in\mathcal{V}$ there is $\hat{v}\in\widehat{\mathcal{V}}$ such that $\hat{v}(x)\le v$. (Using Lemma 3.2, this condition guarantees that $\big(\mathcal{F}_{\widehat{\mathcal{V}}_b}(X,\mathcal{P}),\hat{\mathcal{V}}\big)$ carries its weak preorder.) Suprema and infima are formed pointwise. For $\mathcal{P}=\overline{\mathbb{R}}$ in particular, $\big(\mathcal{F}_{\widehat{\mathcal{V}}_b}(X,\overline{\mathbb{R}}),\hat{\mathcal{V}}\big)$ is a locally convex complete lattice cone, provided that for each $x\in X$ there is $\hat{v}\in\widehat{\mathcal{V}}$ such that $\hat{v}(x)<+\infty$.

(d) Example 4.37(d) yields a locally convex $\bigvee$-semilattice cone. The cone $(Q,\mathcal{V})$ of all non-empty closed (with respect to the lower relative topology) convex subsets of a locally convex cone $(\mathcal{P},\mathcal{V})$ is ordered by set inclusion and carries the weak preorder which is antisymmetric $\big($see 4.37(d)$\big)$. For a non-empty family $\mathcal{A}\subset\mathcal{Q}$ its supremum is given by

$$\sup\mathcal{A}=\overline{conv\Big(\bigcup_{A\in\mathcal{A}}A\Big)}^{(l)},$$

where $conv(C)$ denotes the convex hull of a set $C\subset\mathcal{P}$, and the closure is meant with respect to the lower relative topology of $\mathcal{P}$. Condition $(\bigvee 1)$ can be readily checked: Let $B\in\mathcal{P}$. Clearly $A\oplus B\subset\sup\mathcal{A}\oplus B$ for all $A\in\mathcal{A}$, hence $\sup\{A\oplus B\mid A\in\mathcal{A}\}\le\sup\mathcal{A}\oplus B$. For the converse inequality let $c\in\sup\mathcal{A}\oplus B=\overline{\big(conv\big(\bigcup_{A\in\mathcal{A}}A\big)+B\big)}^{(l)}$. Then for every lower relative neighborhood $(c)v_\varepsilon$ there is $d\in(c)v_\varepsilon\cap\big(conv\big(\bigcup_{A\in\mathcal{A}}A\big)+B\big)$. This means $d=\sum_{i=1}^n\alpha_ia_i+b$ for some $a_i\in A_i\in\mathcal{A}$, $b\in B$ and $0\le\alpha_i$ such that $\sum_{i=1}^n\alpha_i=1$. Thus $d=\sum_{i=1}^n\alpha_i(a_i+b)\in\sup\{A\oplus B\mid A\in\mathcal{A}\}$. This implies $c\in\sup\{A\oplus B\mid A\in\mathcal{A}\}$ as well, since this set is closed in the lower topology. Our claim follows.

(e) A similar argument shows that Example 4.37(e) yields a locally convex $\bigwedge$-semilattice cone. In this case $\mathcal{Q}$ consists of all bounded below closed (with respect to the upper relative topology) convex subsets of $\mathcal{P}$ and is ordered by the inverse set inclusion. For a bounded below family $\mathcal{A}\subset\mathcal{Q}$ its infimum is given by

$$\inf\mathcal{A}=\overline{conv\Big(\bigcup_{A\in\mathcal{A}}A\Big)}^{(u)},$$

where the closure is meant with respect to the upper relative topology of $\mathcal{P}$. It is easily checked that for such a bounded below family $\mathcal{A} \subset \mathcal{Q}$ the convex hull of its union is again a bounded below subset of $\mathcal{P}$, hence by Proposition 4.28 also the closure of the latter with respect to the upper relative topology.

(f) Let X be a topological space, and let $\mathcal{P}$ be the cone of all $\overline{\mathbb{R}}$-valued lower semicontinuous functions on X, where $\overline{\mathbb{R}}$ is endowed with the usual, that is the one-point compactification topology. $\mathcal{P}$ is endowed with the pointwise operations and order and neighborhoods $v \in \mathcal{V}$ for $\mathcal{P}$ are given by the strictly positive constant functions. Because the pointwise infimum of any two functions as well as the pointwise supremum of any non-empty family of functions in $\mathcal{P}$ is again an $\overline{\mathbb{R}}$-valued and lower semicontinuous function, $(\mathcal{P}, \mathcal{V})$ forms a locally convex lattice as well as a $\bigvee$-semilattice cone, however in general not a locally convex complete lattice cone. Similarly, the cone of all $\overline{\mathbb{R}}$-valued bounded below upper semicontinuous functions on X forms a locally convex lattice and $\bigwedge$-semilattice cone.

5.8 Zero Components. Throughout the following we shall assume that $(\mathcal{P}, \mathcal{V})$ is a locally convex $\bigwedge$-semilattice cone. We define the *zero component* of an element a of a locally convex $\bigwedge$-semilattice cone $\mathcal{P}$ by

$$\mathfrak{O}(a) = \inf \{ b \geq 0 \mid a \in \mathcal{B}(b) \}.$$

This expression is well defined, and $\mathfrak{O}(a) \geq 0$ for all $a \in \mathcal{P}$. Recall from Proposition 4.10 that $a \in \mathcal{B}(b)$ if and only if for every $v \in \mathcal{V}$ there are $\alpha, \beta \geq 0$ such that $a \leq \alpha b + \beta v$. If $(a)\mathcal{B}$ does not contain a positive element, then $\mathfrak{O}(a) = \inf \emptyset = +\infty \in \mathcal{P}$.

The introduction of zero components is especially useful for the investigation of variations of the cancellation law in $\bigwedge$-semilattice cones.

Proposition 5.9. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex $\bigwedge$-semilattice cone, and let $a, b, c \in \mathcal{P}$ and $v \in \mathcal{V}$. If $a + c \preccurlyeq_v b + c$, then $a \preccurlyeq_v b + \mathfrak{O}(c)$.*

Proof. Let $a, b, c \in \mathcal{P}$ and $v \in \mathcal{V}$ and suppose that $a + c \preccurlyeq_v b + c$. As we observed before, the weak local preorder $\preccurlyeq_v$ is compatible with the algebraic operations in $\mathcal{P}$. Following Lemma I.4.1 in [100] if applied to this order, the above implies $a + \rho c \preccurlyeq b + \rho c$ for all $\rho > 0$. There is $\lambda > 0$ such that both $0 \leq b + \lambda v$ and $0 \leq c + \lambda v$. Thus $0 \leq (b + \rho c) + 2\lambda v$ for all $0 < \rho \leq 1$. Next we recall that $a + \rho c \preccurlyeq_v b + \rho c$ means that $a + \rho c \in v_\varepsilon(b + \rho c)$ for all $\varepsilon > 0$. Using Lemma 4.1(b) we infer that

$$a + \rho c \leq (1 + \varepsilon)(b + \rho c) + \varepsilon(1 + 2\lambda)v$$

holds for all $\varepsilon > 0$ and $0 < \rho \leq 1$. Thus

$$a \leq a + \rho(c + \lambda v) \leq (1 + \varepsilon)(b + \rho c) + (\varepsilon + 2\varepsilon\lambda + \rho\lambda)v.$$

Let $d \geq 0$ such that $c \in \mathcal{B}(d)$. Then $c \leq \alpha d + \beta v$ holds for some $\alpha, \beta \geq 0$. Consequently, for all $\rho > 0$ such that $\rho \leq \max \{ (\varepsilon/\lambda), (1/\alpha), (2\varepsilon/\beta) \}$ we have

$$\rho c \leq (\rho\alpha)d + (\rho\beta)v \leq d + 2\varepsilon v$$

since $d \geq 0$, and

$$(\varepsilon + 2\varepsilon\lambda + \rho\lambda)v \leq 2\varepsilon(1+\lambda)v,$$

hence

$$a \leq (1+\varepsilon)(b+\rho c) + 2\varepsilon(1+\lambda)v \leq (1+\varepsilon)(b+d) + 2\varepsilon(2+\lambda)v.$$

Now we may use rules $(\bigwedge 1)$ and $(\bigwedge 2)$ and take the infimum over the right-hand side of this inequality with respect to all $d \geq 0$ such that $c \in \mathcal{B}(d)$. This yields

$$a \leq (1+\varepsilon)\big(b + \mathfrak{O}(c)\big) + 2\varepsilon(2+\lambda)v.$$

This last inequality holds true for all $\varepsilon > 0$ and therefore demonstrates

$$a \preccurlyeq_v b + \mathfrak{O}(c). \qquad \square$$

Proposition 5.10. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex $\bigwedge$-semilattice cone, and let $a, b, c \in \mathcal{P}$.*

(a) If $a + c \leq b + c$, then $a \leq b + \mathfrak{O}(c)$.
(b) If $a \in \mathcal{B}(b)$, then $\mathfrak{O}(a) \leq \mathfrak{O}(b)$
(c) If a is bounded, then $\mathfrak{O}(a) = 0$.

Proof. Let $a, b, c \in \mathcal{P}$. Recall that $a \preccurlyeq b$, that is $a \leq b$ in the case of a completely ordered locally convex cone which is supposed to carry its weak global preorder, means that $a \preccurlyeq_v b$ holds for all $v \in \mathcal{V}$. This yields Part (a) as an immediate consequence of 5.9. For Part (b) suppose that $a \in \mathcal{B}(b)$. Then for every $c \geq 0$ such that $b \in \mathcal{B}(c)$ we have $\mathcal{B}(b) \subset \mathcal{B}(c)$ by 4.10(ii), hence $a \in \mathcal{B}(c)$ as well. This yields $\mathfrak{O}(a) \leq \mathfrak{O}(b)$. Part (c) follows from Part (b) with $b = 0$. $\square$

Proposition 5.11. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex $\bigwedge$-semilattice cone, and let $a, b, \in \mathcal{P}$. Then*

(a) $\mathfrak{O}(a+b) = \mathfrak{O}(a) + \mathfrak{O}(b)$.
(b) $\mathfrak{O}(\alpha a) = \alpha\mathfrak{O}(a) = \mathfrak{O}(a)$ for all $\alpha > 0$.
(c) If $\alpha a = a$ for all $\alpha > 0$, then $\mathfrak{O}(a) = a$.

Proof. Let $a, b, \in \mathcal{P}$. Part (b) is obvious since for every $\alpha > 0$ and every $c \in \mathcal{P}$ we have $\alpha a \in \mathcal{B}(c)$ if and only if $a \in \mathcal{B}(c)$ by 4.11(a). For Part (a) let $a \in \mathcal{B}(c)$ and $b \in \mathcal{B}(d)$ for $c, d \geq 0$. Then $a + b \in \mathcal{B}(c+d)$ by 4.11(c). This shows $\mathfrak{O}(a+b) \leq \mathfrak{O}(a) + \mathfrak{O}(b)$. For the converse, given $v \in \mathcal{V}$ there is $\lambda \geq 0$ such that $0 \leq b + \lambda v$. Hence $a \leq (a+b) + \lambda v$, and we infer that $a \in \mathcal{B}(a+b)$. Thus $\mathfrak{O}(a) \leq \mathfrak{O}(a+b)$ by 5.10(b), and likewise $\mathfrak{O}(b) \leq \mathfrak{O}(a+b)$. This yields

$$\mathfrak{O}(a) + \mathfrak{O}(b) \leq 2\mathfrak{O}(a+b) = \mathfrak{O}(a+b).$$

For Part (c) let $a \in \mathcal{P}$ such that $\alpha a = a$ for all $\alpha \geq 0$. For every $v \in \mathcal{V}$ there is $\lambda > 0$ such that $0 \leq a + \lambda v$, hence $0 \leq (1/\lambda)a + v = a + v$. This shows $0 \leq a$, since $\mathcal{P}$ carries the weak preorder. Thus $\mathfrak{O}(a) \leq a$. If on the other hand $a \in \mathcal{B}(c)$ for some $c \geq 0$, then there are $\alpha, \beta \geq 0$ such that $a \leq \alpha c + \beta v$. Since $\varepsilon \alpha c \leq c$ for all $0 < \varepsilon \leq 1/(\alpha+1)$, this implies

$$a = \varepsilon a \leq \varepsilon \alpha c + \varepsilon \beta v \leq c + \varepsilon \beta v$$

for all such ε. This yields $a \leq c$ since $\mathcal{P}$ carries the weak preorder, and we also have $a \leq \mathfrak{O}(a)$. □

Proposition 5.11(b) implies in particular that a linear functional $\mu \in \mathcal{P}^*$ can attain only the values 0 or $+\infty$ at a zero component.

Some additional properties can be derived if $\mathcal{P}$ contains also suprema of its elements, that is if $(\mathcal{P}, \mathcal{V})$ is also a locally convex lattice or indeed a locally convex complete lattice cone (see Example 5.7(f)).

Lemma 5.12. *Suppose $(\mathcal{P}, \mathcal{V})$ is a locally convex lattice and $\bigwedge$-semilattice cone. Then the zero component of an element $a \in \mathcal{P}$ can be alternatively expressed as*

$$\mathfrak{O}(a) = \inf_{\varepsilon > 0} \big\{ \varepsilon\,(a \vee 0) \big\}.$$

Proof. Let $a \in \mathcal{P}$. Then $0 \leq a \vee 0$ and $a \leq a \vee b$. Thus $a \in \mathcal{B}(a \vee b)$. This implies $a \in \mathcal{B}\big(\varepsilon\,(a \vee 0)\big)$ for all $\varepsilon > 0$ by 4.11(c). Hence $\inf\big\{ b \geq 0 \mid a \in \mathcal{B}(b) \big\} \leq \inf_{\varepsilon > 0} \big\{ \varepsilon\,(a \vee 0) \big\}$. For the converse inequality let $b \geq 0$ such that $a \in \mathcal{B}(b)$. Given $v \in \mathcal{V}$ and $\varepsilon > 0$ there are $\alpha, \beta \geq 0$ such that $a \leq \alpha b + \beta v$ (see 4.10(iii)). Condition ($\vee$2) then yields $a \vee 0 \leq \alpha b + 2\beta v$. Thus for $0 < \delta \leq \min\{\frac{\varepsilon}{2\beta+1}, \frac{1}{\alpha+1}\}$ we have

$$\delta(a \vee 0) \leq \delta \alpha b + 2\delta\beta v \leq b + \varepsilon v,$$

since $b \geq 0$ and $\delta\alpha \leq 1$ implies $(\delta\alpha)b \leq b$. This shows $\inf_{\varepsilon > 0} \big\{ \varepsilon\,(a \vee 0) \big\} \leq b + \varepsilon v$, hence

$$\inf_{\varepsilon > 0} \big\{ \varepsilon\,(a \vee 0) \big\} \leq \inf \big\{ b \geq 0 \mid a \in \mathcal{B}(b) \big\} + \varepsilon v$$

by $\big(\bigwedge 2\big)$. Because this holds for all $v \in \mathcal{V}$ and for all $\varepsilon > 0$, and because $\mathcal{P}$ carries the weak preorder, we conclude that

$$\inf_{\varepsilon > 0} \big\{ \varepsilon\,(a \vee 0) \big\} \leq \inf \big\{ b \geq 0 \mid a \in \mathcal{B}(b) \big\}.$$

□

Proposition 5.13. *Suppose $(\mathcal{P}, \mathcal{V})$ is a locally convex lattice and $\bigwedge$-semilattice cone. Let $a, b, c \in \mathcal{P}$ and $v \in \mathcal{V}$.*

(a) If $a \in \mathcal{B}_v(b)$, then $\mathfrak{O}(a) \preccurlyeq_v \mathfrak{O}(b)$ and $b + \mathfrak{O}(a) \preccurlyeq_v b$.
(b) If a is v-bounded, then $\mathfrak{O}(a) \preccurlyeq_v 0$.

Proof. Let $a,b,c \in \mathcal{P}$ and $v \in \mathcal{V}$. For Part (a) , suppose that $a \in \mathcal{B}_v(b)$. There are $\alpha,\beta > 0$ such that $a \le \alpha b + \beta v$ and $\lambda \ge 0$ such that $0 \le a + \lambda v$, hence also $0 \le a \wedge 0 + \lambda v$ by $(\bigwedge 2)$. Then

$$\begin{aligned} b + \mathfrak{O}(a) &\le b + \varepsilon\,(a \vee 0) \\ &\le b + \varepsilon\,(a \vee 0) + \varepsilon\big((a \wedge 0) + \lambda v\big) = b + \varepsilon a + \varepsilon\lambda v \\ &\le (1+\varepsilon\alpha)b + \varepsilon(\beta+\lambda)v. \end{aligned}$$

for all $\varepsilon > 0$ by Lemma 5.12. This shows $b + \mathfrak{O}(a) \preccurlyeq_v b$. Furthermore, using the cancellation rule from Proposition 5.9 for the element b in $\mathfrak{O}(a) + b \preccurlyeq_v 0 + b$ yields $\mathfrak{O}(a) \preccurlyeq_v \mathfrak{O}(b)$ as claimed. Part (b) follows from Part (a) with $b = 0$. □

Proposition 5.14. *Suppose $(\mathcal{P},\mathcal{V})$ is a locally convex lattice and $\bigwedge$-semilattice cone. Then $b + \mathfrak{O}(a) = b$ holds for all $a,b \in \mathcal{P}$ whenever $a \in \mathcal{B}(b)$.*

Proof. Let $a,b \in \mathcal{P}$ such that $a \in \mathcal{B}(b)$. Then $a \in \mathcal{B}_v(b)$, hence $b + \mathfrak{O}(a) \preccurlyeq_v b$ by Proposition 5.13, for all $v \in \mathcal{V}$. Thus $b + \mathfrak{O}(a) \preccurlyeq b$. Since $\mathcal{P}$ carries the weak preorder which is supposed to be antisymmetric, and since $b \preccurlyeq b + \mathfrak{O}(a)$ is evident, our claim follows. □

Proposition 5.15. *Let $(\mathcal{P},\mathcal{V})$ be a locally convex complete lattice cone, and let A, B be non-empty subsets of $\mathcal{P}$. Then*

(a) $\inf(A \vee B) = \inf A \vee \inf B$ *if both A and B are bounded below.*
(b) $\sup(A \wedge B) \le \sup A \wedge \sup B \le \sup(A \wedge B) + \mathfrak{O}\left(\sup(A \vee B)\right).$

Proof. We first observe that

$$\inf A \vee \inf B \le a \vee b \qquad \text{and} \qquad a \wedge b \le \sup A \wedge \sup B$$

holds for all $a \in A$ and $b \in B$. Thus

$$\inf A \vee \inf B \le \inf(A \vee B) \qquad \text{and} \qquad \sup(A \wedge B) \le \sup B \wedge \sup A.$$

For Part (a) we assume that both sets A and B are bounded below and use Proposition 5.3 for

$$\begin{aligned} \inf(A \vee B) + \inf(A \wedge B) &\le \inf\{a \vee b + a \wedge b \mid a \in a,\ b \in B\} \\ &= \inf(A + B) \\ &= \inf A + \inf B \\ &= \inf A \vee \inf B + \inf A \wedge \inf B. \end{aligned}$$

As $\inf(A \wedge B) = \inf A \wedge \inf B$, the cancellation law in Proposition 5.10(a) yields

$$\inf(A \vee B) \le \inf A \,\vee\, \inf B + \mathfrak{O}\big(\inf(A \wedge B)\big).$$

Similarly, one obtains

$$\sup A \wedge \sup B \leq \sup(A \wedge B) + \mathfrak{O}\left(\sup(A \vee B)\right),$$

that is Part (b). Finally, as $\inf(A \wedge B) = \inf A \wedge \inf B \leq \inf A \vee \inf B$, Proposition 5.14 shows

$$\inf A \vee \inf B + \mathfrak{O}\big(\inf A \wedge \inf B\big) = \inf A \vee \inf B.$$

This completes our proof of Part (a). □

We proceed to refine the cancellation rules in Proposition 5.10 further. Let $(\mathcal{P}, \mathcal{V})$ be a locally convex lattice and $\bigwedge$-semilattice cone. We define the *zero component* of an element $a \in \mathcal{P}$ *relative to* $b \in \mathcal{P}$ by

$$\mathfrak{O}(a \smallsetminus b) = \inf\big\{c \geq 0 \mid c + \mathfrak{O}(b) \geq \mathfrak{O}(a)\big\}.$$

Obviously, $\mathfrak{O}(a \smallsetminus 0) = \mathfrak{O}(a)$. Also, $\mathfrak{O}(\alpha a \smallsetminus \beta b) = \alpha\mathfrak{O}(a \smallsetminus b) = \mathfrak{O}(a \smallsetminus b)$ holds for all $\alpha, \beta > 0$.

Proposition 5.16. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex lattice and $\bigwedge$-semilattice cone, and let $a, b, c \in \mathcal{P}$.*

(a) $0 \leq \mathfrak{O}(a \smallsetminus b) \leq \mathfrak{O}(a) \leq \mathfrak{O}(a \smallsetminus b) + \mathfrak{O}(b)$.
(b) If $a + c \leq b + c$, then $a \leq b + \mathfrak{O}(c \smallsetminus b)$.
(c) If $a \in \mathcal{B}(b)$, then $\mathfrak{O}(a \smallsetminus b) = 0$ and $b + \mathfrak{O}(c) = b + \mathfrak{O}(c \smallsetminus a)$.

Proof. Part (a) follows directly from the definition of $\mathfrak{O}(a \smallsetminus b)$ together with $(\bigwedge 2)$. For (b) we recall that $a + c \leq b + c$ implies $a \leq b + \mathfrak{O}(c)$ by 5.10(a). As $\mathfrak{O}(c) \leq \mathfrak{O}(c \smallsetminus b) + \mathfrak{O}(b)$ by Part (a) and $b + \mathfrak{O}(b) = b$ by 5.14, our claim follows. For (c), let $a \in \mathcal{B}(b)$. Then $\mathfrak{O}(a) \leq \mathfrak{O}(b)$ by 5.10(b), and we may use $c = 0$ in the definition of $\mathfrak{O}(a \smallsetminus b)$. Thus indeed $\mathfrak{O}(a \smallsetminus b) = 0$. Furthermore, we have

$$b + \mathfrak{O}(c) \leq b + \big(\mathfrak{O}(c \smallsetminus a) + \mathfrak{O}(a)\big) = \big(b + \mathfrak{O}(a)\big) + \mathfrak{O}(c \smallsetminus a) = b + \mathfrak{O}(c \smallsetminus a)$$

by Part (a) and 5.14. □

Examples 5.17. (a) If $\mathcal{P} = \overline{\mathbb{R}}$ or $\mathcal{P} = \overline{\mathbb{R}}_+$ $\big($see 1.4(a) and 1.4(b)$\big)$, then $\mathfrak{O}(a) = 0$ for all $a < +\infty$, and $\mathfrak{O}(+\infty) = +\infty$.

(b) Consider $\big(\mathcal{F}_{\widehat{\mathcal{V}}_b}(X, \mathcal{P}), \hat{\mathcal{V}}\big)$, where $(\mathcal{P}, \mathcal{V})$ is a locally convex $\bigwedge$-semilattice cone, X a set, and $\widehat{\mathcal{V}}$ is a neighborhood system consisting of $\big(\mathcal{V} \cup \{\infty\}\big)$-valued functions $\big($see Example 1.4(e)$\big)$ such that for every $x \in X$ and $v \in \mathcal{V}$ there is $\hat{v} \in \widehat{\mathcal{V}}$ such that $\hat{v}(x) \leq v$ $\big($see 5.7(c)$\big)$. For $f \in \mathcal{F}_{\widehat{\mathcal{V}}_b}(X, \mathcal{P})$ then $\mathfrak{O}(f)$ is the mapping $x \mapsto \mathfrak{O}\big(f(x)\big)$. For $\mathcal{P} = \overline{\mathbb{R}}$, in particular, the zero component of an $\overline{\mathbb{R}}$-valued function $f \in \mathcal{F}_{\widehat{\mathcal{V}}_b}(X, \overline{\mathbb{R}})$ is the mapping $\mathfrak{O}\big(f\big)(x) = 0$ if $f(x) < +\infty$, and $\mathfrak{O}\big(f\big)(x) = +\infty$ else. The same observation applies to the second part of Example 5.7(f), that is the cone of $\overline{\mathbb{R}}$-valued bounded below upper semicontinuous functions on a topological space with the positive constants as neighborhoods. This was seen to be an example of a locally convex lattice and $\bigwedge$-semilattice cone.

(c) Let us consider Example 4.37(e) $\big($see also 5.7(e)$\big)$, that is the locally convex $\bigwedge$-semilattice cone $(\mathcal{Q},\mathcal{V})$ of all convex subsets locally convex cone $(\mathcal{P},\mathcal{V})$, which are bounded below and closed with respect to the upper relative topology. Recall that the order in $\mathcal{Q}$ is the inverse set inclusion and the neighborhoods are given by $A \le B \oplus v$ for $A, B \in \mathcal{Q}$ and $v \in \mathcal{V}$, if for every $b \in B$, and $\varepsilon > 0$ there is $a \in A$ such that $a \le \gamma b + (1+\varepsilon)v$ for some $1 \le \gamma \le 1+\varepsilon$. The closed convex subsets (including the empty set) of $\overline{\{0\}}^{(u)} = \{b \in \mathcal{P} \mid 0 \preccurlyeq b\}$ are the positive elements in $\mathcal{Q}$. We claim that for an element $A \in \mathcal{Q}$ we have

$$\mathfrak{D}(A) = \{b \succcurlyeq 0 \mid \mathcal{B}_v(b) \cap A \neq \emptyset \quad \text{for all} \quad v \in \mathcal{V}\}.$$

We shall argue for this using the following steps: Let B denote the set on the right-hand side of the above equation.

(i) The set $B \subset \mathcal{P}$ is convex. Indeed, let $b_1, b_2 \in B$, $0 \le \lambda_1, \lambda_2 \le 1$ such that $\lambda_1 + \lambda_2 = 1$ and $b = \lambda_1 b_1 + \lambda_2 b_2$. Given $v \in \mathcal{V}$ there are $a_1 \in \mathcal{B}_v(b_1) \cap A$ and $a_2 \in \mathcal{B}_v(b_2) \cap A$. Set $a = \lambda_1 a_1 + \lambda_2 a_2 \in A$ and choose $\alpha_1, \alpha_2, \beta, \rho \ge 0$ such that

$$a_1 \le \alpha_1 b_1 + \beta v, \qquad a_2 \le \alpha_2 b_2 + \beta v, \qquad 0 \le b_1 + \rho v \quad \text{and} \quad 0 \le b_2 + \rho v.$$

Setting $\alpha = \max\{\alpha_1, \alpha_2\}$ we have

$$a_1 \le \big(\alpha_1 b_1 + \beta v\big) + (\alpha - \alpha_1)(b_1 + \rho v) + \alpha_1 \rho v = \alpha b_1 + (\beta + \alpha\rho) v$$

and, likewise

$$a_2 \le \big(\alpha_2 b_1 + \beta v\big) + (\alpha - \alpha_2)(b_2 + \rho v) + \alpha_2 \rho v = \alpha b_2 + (\beta + \alpha\rho) v.$$

Thus

$$a \le \lambda_1\big(\alpha b_1 + (\beta + \alpha\rho)v\big) + \lambda_2\big(\alpha b_1 + (\beta + \alpha\rho)v\big) = \alpha b + (\beta + \alpha\rho) v.$$

We infer that $a \in \mathcal{B}_v(b) \cap A$, hence $\mathcal{B}_v(b) \cap A \neq \emptyset$. Since this holds for all $v \in \mathcal{V}$ and since $b \succcurlyeq 0$ is evident from $b_1, b_2 \succcurlyeq 0$, we conclude that $b \in B$.

(ii) The set $B \subset \mathcal{P}$ is closed with respect to the upper topology. Indeed, let $c \in \overline{B}^{(u)}$ and let $v \in \mathcal{V}$. There is $b \in v_1(c) \cap B$, that is $b \le \gamma c + v$ for some $1 \le \gamma \le 2$. There is $a \in \mathcal{B}_v(b) \cap A$, that is $a \le \alpha b + \beta v$ for some $\alpha, \beta \ge 0$. Combining these yields $a \le \alpha\gamma c + (\alpha + \beta) v$. This shows $\mathcal{B}_v(c) \cap A \neq \emptyset$ for all $v \in \mathcal{V}$. Furthermore, since $B \subset \overline{\{0\}}^{(u)} = \{b \in \mathcal{P} \mid 0 \preccurlyeq b\}$ which is closed with respect to the upper relative topology, we have $c \in \overline{\{0\}}^{(u)}$ as well, hence $c \succcurlyeq 0$. Together with the above this yields $c \in B$. Since $B \subset \mathcal{P}$ is obviously bounded below (we have $0 \leq b+v$ for all $b \in \mathcal{P}$), we conclude from (i) and (ii) that $B \in \mathcal{Q}$.

(iii) We have $A \in \mathcal{B}\big(\overline{\{b\}}^{(u)}\big)$ for all $b \in B$. Indeed, let $v \in \mathcal{V}$. Given $b \in B$ there is some $a \in \mathcal{B}_v(b) \cap A$, that is there are $\alpha, \beta, \lambda \geq 0$ such that $a \leq \alpha b + \beta v$ and $0 \leq b + \lambda v$. Then for every $c \in \overline{\{b\}}^{(u)}$, that is $b \preccurlyeq c$, we have $b \in v_1(c)$, hence $b \leq 2c + (2+\lambda)v$ (see Lemma 4.1(c) with $\varepsilon = 1$). This yields $a \leq 2\alpha c + (2\alpha + \lambda\alpha + \beta)v$ and $A \leq 2\alpha\, \overline{\{b\}}^{(u)} \oplus (2\alpha + \lambda\alpha + \beta)v$, hence $A \in \mathcal{B}\big(\overline{\{b\}}^{(u)}\big)$. Consequently,

$$\mathfrak{O}(A) \leq \inf\big\{\overline{\{b\}}^{(u)} \mid b \in B\big\} = \overline{conv\big(\bigcup_{b \in B} \overline{\{b\}}^{(u)}\big)}^{(u)} = B.$$

(iv) On the other hand, let $C \in \mathcal{Q}$ such that $C \geq 0$, that is $C \subset \overline{\{0\}}^{(u)}$, and $A \in \mathcal{B}(C)$. Let $c \in C$. Given $v \in \mathcal{V}$ there are $\alpha, \beta \geq 0$ such that $A \leq \alpha C \oplus \beta v$. According to our definition of the neighborhoods in $\mathcal{Q}$ (see 4.37(e)), for $\varepsilon = 1$ we find $a \in A$ such that $a \leq \gamma(\alpha c) + 2(\beta v)$ with some $1 \leq \gamma \leq 2$. This yields $\mathcal{B}_v(b) \cap A \neq \emptyset$ for all $v \in \mathcal{V}$, hence $c \in B$ since $c \succcurlyeq 0$. Thus

$$C = \overline{conv\big(\bigcup_{c \in C} \overline{\{c\}}^{(u)}\big)}^{(u)} \subset B.$$

This shows $\mathfrak{O}(A) \subset B$, that is $\mathfrak{O}(A) \geq B$, and our claim follows.

In particular, we have $\mathfrak{O}(A) = \overline{\{0\}}^{(u)}$ if and only if $\mathcal{B}_v(0) \cap A \neq \emptyset$ for all $v \in \mathcal{V}$, that is if and only if for every $v \in \mathcal{V}$ there are $a \in A$ and $\lambda \geq 0$ such that $a \leq \lambda v$, that is if and only if the element $A \in \mathcal{Q}$ is bounded above (see 4.37(e)).

For a concrete example let $\mathcal{P}$ be the cone of all real-valued bounded below continuous functions on the open interval $(0,1)$, endowed with the positive constants as neighborhoods (see 1.4(e)) and let $\mathcal{Q}$ be as before. Consider the subset

$$C = \left\{ f \in \mathcal{P} \;\middle|\; f(x) \geq \frac{1}{x} - 2 \;\text{ for all }\; x \in (0,1) \right\}.$$

This set is convex, bounded below and closed with respect to the upper relative topology, hence $C \in \mathcal{Q}$. For a function $g \geq 0$ in $\mathcal{P}$, we have $\mathcal{B}(g) \cap C \neq \emptyset$ if and only if there are $\alpha, \beta \geq 0$ such that $1/x \leq \alpha g(x) + \beta$ for all $x \in (0,1)$, that is if and only if the inferior limit of $xg(x)$ at 0 is greater than 0. Thus

$$\mathfrak{O}(C) = \{g \in \mathcal{P} \,\big|\, g \geq 0 \;\text{ and }\; \varliminf_{x \to 0} xg(x) > 0\}.$$

Now according to the cancellation rule in Proposition 5.10(a), if $A, B \in \mathcal{Q}$ such that $A + C \leq B + C$, that is $B + C \subset A + C$, then $A \leq B + \mathfrak{O}(c)$, that is $B + \mathfrak{O}(C) \subset A$.

5.18 Order Convergence. We proceed to define order convergence for nets in a locally convex complete lattice cone $(\mathcal{P},\mathcal{V})$. A net $(a_i)_{i\in\mathcal{I}}$ in $\mathcal{P}$ is called *bounded below* if there is $i_0\in\mathcal{I}$ such that the set $\{a_i \mid i\geq i_0\}$ is bounded below in the sense of 4.24(i). We define the superior and inferior limits of a bounded below net $(a_i)_{i\in\mathcal{I}}$ in $\mathcal{P}$ by

$$\underline{\lim}_{i\in\mathcal{I}}\, a_i=\sup_{i\in\mathcal{I}}\Big(\inf_{k\geq i} a_k\Big) \qquad \text{and} \qquad \overline{\lim}_{i\in\mathcal{I}}\, a_k=\inf_{i\in\mathcal{I}}\Big(\sup_{k\geq i} a_k\Big).$$

Because the order of $\mathcal{P}$ is supposed to be antisymmetric, both limits are uniquely defined. Obviously, $\underline{\lim}_{i\in\mathcal{I}}\, a_i\leq\overline{\lim}_{i\in\mathcal{I}}\, a_i$. If $\underline{\lim}_{i\in\mathcal{I}}\, a_i$ and $\overline{\lim}_{i\in\mathcal{I}}\, a_i$ coincide, we shall denote their common value by $\lim_{i\in\mathcal{I}} a_i$ and say that the net $(a_i)_{i\in\mathcal{I}}$ is *order convergent.* Obviously, every increasing or decreasing bounded below net is order convergent in this sense, converging towards the supremum or the infimum of the set of its elements, respectively.

Lemma 5.19. *Let $(\mathcal{P},\mathcal{V})$ be a locally convex complete lattice cone, and let $(a_i)_{i\in\mathcal{I}}$ and $(b_i)_{i\in\mathcal{I}}$ be bounded below nets in $\mathcal{P}$. Then*

$$\underline{\lim}_{i\in\mathcal{I}} a_i+\underline{\lim}_{i\in\mathcal{I}} b_i\leq\underline{\lim}_{i\in\mathcal{I}}(a_i+b_i)\leq\overline{\lim}_{i\in\mathcal{I}} a_i+\underline{\lim}_{i\in\mathcal{I}} b_i\leq\overline{\lim}_{i\in\mathcal{I}}(a_i+b_i)\leq\overline{\lim}_{i\in\mathcal{I}} a_i+\overline{\lim}_{i\in\mathcal{I}} b_i.$$

Proof. For any bounded below net $(c_i)_{i\in\mathcal{I}}$ in $\mathcal{P}$, for $i\in\mathcal{I}$, let

$$s_i^{(c)}=\inf_{k\geq i} c_k \qquad \text{and} \qquad S_i^{(c)}=\sup_{k\geq i} c_k.$$

The nets $(s_i^{(c)})_{i\in\mathcal{I}}$ and $(S_i^{(c)})_{i\in\mathcal{I}}$ are increasing and decreasing, respectively, and

$$\underline{\lim}_{i\in\mathcal{I}} c_i=\sup_{i\in\mathcal{I}} s_i^{(c)} \qquad \text{and} \qquad \overline{\lim}_{i\in\mathcal{I}} c_i=\inf_{i\in\mathcal{I}} S_i^{(c)}.$$

Now, using the nets $(a_i)_{i\in\mathcal{I}}$, $(b_i)_{i\in\mathcal{I}}$ and $(a_i+b_i)_{i\in\mathcal{I}}$ in place of $(c_i)_{i\in\mathcal{I}}$ we observe that

$$s_i^{(a+b)}\geq s_i^{(a)}+s_i^{(b)} \qquad \text{and} \qquad S_i^{(a+b)}\leq S_i^{(a)}+S_i^{(b)}$$

for all $i\in\mathcal{I}$. For every $k\in\mathcal{I}$ we have by $(\bigvee 1)$

$$s_k^{(a)}+\sup_{i\in\mathcal{I}} s_i^{(b)}=\sup_{i\in\mathcal{I}}(s_k^{(a)}+s_i^{(b)})\leq\sup_{l\in\mathcal{I}}(s_l^{(a)}+s_l^{(b)}),$$

as $s_k^{(a)}+s_i^{(b)}\leq s_l^{(a)}+s_l^{(b)}$ whenever $i,k\leq l$. This shows

$$\underline{\lim}_{i\in\mathcal{I}} a_i+\underline{\lim}_{i\in\mathcal{I}} b_i=\sup_{k\in\mathcal{I}} s_k^{(a)}+\sup_{i\in\mathcal{I}} s_k^{(b)}\leq\sup_{l\in\mathcal{I}}(s_l^{(a)}+s_l^{(b)})=\underline{\lim}_{i\in\mathcal{I}}(a_i+b_i),$$

the first part of our claim. A similar argument using the decreasing nets $(S_i^{(c)})_{i\in\mathcal{I}}$ yields

$$\overline{\lim_{i\in\mathcal{I}}}\, a_i + \overline{\lim_{i\in\mathcal{I}}}\, b_i = \inf_{k\in\mathcal{I}} S_k^{(a)} + \inf_{i\in\mathcal{I}} S_k^{(b)} \geq \inf_{l\in\mathcal{I}} (S_l^{(a)} + S_l^{(b)}) = \overline{\lim_{i\in\mathcal{I}}}(a_i + b_i).$$

Finally, for all $i, l \in \mathcal{I}$ and $j \geq i, l$ we have

$$s_i^{(a+b)} = \inf_{k\geq i}(a_k + b_k) \leq \inf_{k\geq j}(S_l^{(a)} + b_k) = S_l^{(a)} + \inf_{k\geq j} b_k \leq S_l^{(a)} + \underline{\lim}_{i\in\mathcal{I}}\, b_i,$$

hence

$$\underline{\lim}_{i\in\mathcal{I}}(a_i + b_i) = \sup_{i\in\mathcal{I}} s_i^{(a+b)} \leq \inf_{l\in\mathcal{I}} S_l^{(a)} + \underline{\lim}_{i\in\mathcal{I}}\, b_i = \overline{\lim_{i\in\mathcal{I}}}\, a_i + \underline{\lim}_{i\in\mathcal{I}}\, b_i.$$

A similar argument shows that

$$\overline{\lim_{i\in\mathcal{I}}}\, a_i + \underline{\lim}_{i\in\mathcal{I}}\, b_i \leq \overline{\lim_{i\in\mathcal{I}}}(a_i + b_i).$$

□

Note that Lemma 5.19 implies in particular that

$$\underline{\lim}_{i\in\mathcal{I}}(a + b_i) = a + \underline{\lim}_{i\in\mathcal{I}}\, b_i \qquad \text{and} \qquad \overline{\lim_{i\in\mathcal{I}}}(a + b_i) = a + \overline{\lim_{i\in\mathcal{I}}}\, b_i$$

holds for $a \in \mathcal{P}$ and a bounded below net $(b_i)_{i\in\mathcal{I}}$. We shall use Conditions $(\bigvee 2)$ and $(\bigwedge 2)$ for a comparison of the inferior and superior limits of nets:

Lemma 5.20. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex complete lattice cone, let $(a_i)_{i\in\mathcal{I}}$ and $(b_j)_{j\in\mathcal{J}}$ be nets in $\mathcal{P}$, and let $v \in \mathcal{V}$.*

(a) If for every $i_0 \in \mathcal{I}$ there is $j_0 \in \mathcal{J}$ such that for every $j \geq j_0$ there is $i \geq i_0$ such that $a_i \leq b_j + v$, then $\underline{\lim}_{i\in\mathcal{I}}\, a_i \leq \underline{\lim}_{j\in\mathcal{J}}\, b_j + v$.
(b) If for every $j_0 \in \mathcal{J}$ there is $i_0 \in \mathcal{I}$ such that for every $i \geq i_0$ there is $j \geq j_0$ such that $a_i \leq b_j + v$, then $\overline{\lim}_{i\in\mathcal{I}}\, a_i \leq \overline{\lim}_{j\in\mathcal{J}}\, b_j + v$.
(c) If $\mathcal{I} = \mathcal{J}$ and if there is $i_0 \in \mathcal{I}$ such that $a_i \leq b_i + v$ for all $i \geq i_0$, then $\underline{\lim}_{i\in\mathcal{I}}\, a_i \leq \underline{\lim}_{i\in\mathcal{I}}\, b_i + v$ and $\overline{\lim}_{i\in\mathcal{I}}\, a_i \leq \overline{\lim}_{i\in\mathcal{I}}\, b_i + v$.
(d) If $(a_{i_l})_{l\in\mathcal{L}}$ is a subnet of $(a_i)_{i\in\mathcal{I}}$, then $\underline{\lim}_{i\in\mathcal{I}}\, a_i \leq \underline{\lim}_{l\in\mathcal{L}}\, a_{i_l}$ and $\overline{\lim}_{l\in\mathcal{L}}\, a_{i_l} \leq \overline{\lim}_{i\in\mathcal{I}}\, a_i$.

Proof. (a) Given $i_0 \in \mathcal{I}$ choose $j_0 \in \mathcal{J}$ as in the assumption of Part (a). Then $\inf_{i\geq i_0} a_i \leq b_j + v$ for all $j \geq j_0$, hence

$$\inf_{i\geq i_0} a_i \leq \inf_{j\geq j_0} b_j + v \leq \underline{\lim}_{j\in\mathcal{J}}\, b_j + v$$

by $(\bigwedge 2)$. Thus by $(\bigvee 2)$ we have $\underline{\lim}_{i\in\mathcal{I}}\, a_i \leq \underline{\lim}_{j\in\mathcal{J}}\, b_j + v$ as well. The argument for Part (b) is similar. The assumptions for Part (c) yield those for Parts (a) and (b) with $j_0 = i_0$ and $j = i$. Part (d) follows from (a) and (b) if we set $\mathcal{J} = \mathcal{L}$ and $b_l = a_{i_j}$. □

Lemma 5.20(c) yields in particular that $\lim_{i\in\mathcal{I}} a_i \leq \lim_{i\in\mathcal{I}} b_i$ for order convergent nets $(a_i)_{i\in\mathcal{I}}$ and $(b_i)_{i\in\mathcal{I}}$ whenever $a_i \leq b_i$ for all $i \in \mathcal{I}$. Part (d) implies that every subnet of an order convergent net is again order convergent with the same limit.

Lemma 5.21. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex complete lattice cone. Let $(a_i)_{i\in\mathcal{I}}$ be a bounded below net in $\mathcal{P}$, and let $(\alpha_i)_{i\in\mathcal{I}}$ be a bounded net of non-negative reals such that $\lim\limits_{i\in\mathcal{I}} \alpha_i > 0$. Then*

$$\Big(\lim_{i\in\mathcal{I}} \alpha_i\Big)\Big(\underline{\lim}_{i\in\mathcal{I}} a_i\Big) \leq \underline{\lim}_{i\in\mathcal{I}}(\alpha_i a_i) \leq \overline{\lim}_{i\in\mathcal{I}}(\alpha_i a_i) \leq \Big(\lim_{i\in\mathcal{I}} \alpha_i\Big)\Big(\overline{\lim}_{i\in\mathcal{I}} a_i\Big).$$

Proof. Obviously the net $(\alpha_i a_i)_{i\in\mathcal{I}}$ is also bounded below in $\mathcal{P}$. We set $\alpha = \lim\limits_{i\in\mathcal{I}} \alpha_i > 0$. Given $v \in \mathcal{V}$ there is $\lambda \geq 0$ such that $0 \leq a_i + \lambda v$ for all $i \in \mathcal{I}$. For $\varepsilon > 0$ we set $\gamma = 1 + \varepsilon$ and find $i_0 \in \mathcal{I}$ such that $(1/\gamma)\alpha \leq \alpha_i \leq \gamma\alpha$ for all $i \geq i_0$. Thus

$$\alpha_i a_i + \frac{\alpha}{\gamma}\lambda v \leq \alpha_i(a_i + \lambda v) \leq \gamma\alpha(a_i + \lambda v),$$

hence, using the cancellation law for positive elements (see Lemma I.4.2 in [100])

$$\alpha_i a_i \leq \gamma\alpha a_i + \alpha\lambda\Big(\gamma - \frac{1}{\gamma}\Big)v + \varepsilon v \leq \gamma\alpha a_i + \varepsilon(2\alpha\lambda + 1)v.$$

Using Lemma 5.20(c) we infer that

$$\overline{\lim}_{i\in\mathcal{I}} \alpha_i a_i \leq \gamma\Big(\alpha\,\overline{\lim}_{i\in\mathcal{I}} a_i\Big) + \varepsilon(2\alpha\lambda + 1)v.$$

Since the latter holds for all $\varepsilon > 0$ and since $\mathcal{P}$ carries the weak preorder, we conclude that

$$\overline{\lim}_{i\in\mathcal{I}} \alpha_i a_i \leq \alpha\,\overline{\lim}_{i\in\mathcal{I}} a_i.$$

The first part of the inequality in our claim follows in a similar fashion. □

Proposition 5.22. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex complete lattice cone. Let $(a_i)_{i\in\mathcal{I}}$ and $(b_i)_{i\in\mathcal{I}}$ be order convergent nets in $\mathcal{P}$, and let $(\alpha_i)_{i\in\mathcal{I}}$ be a bounded net of non-negative reals such that $\lim\limits_{i\in\mathcal{I}} \alpha_i > 0$. Then*

$$\lim_{i\in\mathcal{I}}(a_i + b_i) = \lim_{i\in\mathcal{I}} a_i + \lim_{i\in\mathcal{I}} b_i \qquad \textit{and} \qquad \lim_{i\in\mathcal{I}}(\alpha_i a_i) = \Big(\lim_{i\in\mathcal{I}} \alpha_i\Big)\Big(\lim_{i\in\mathcal{I}} a_i\Big).$$

The latter is an obvious consequence of our previous results 5.19 and 5.21. Note that the requirement that $\lim_{i\in\mathcal{I}} \alpha_i > 0$ can not be omitted if the elements of the net $(a_i)_{i\in\mathcal{I}}$ are not bounded in $\mathcal{P}$: In the locally convex complete lattice cone $\overline{\mathbb{R}}$ choose $a_n = +\infty$ and $\alpha_n = (1/n)$. Then $\lim\limits_{n\to\infty}(\alpha_n a_n) = +\infty$, but $\big(\lim\limits_{n\to\infty} \alpha_n\big)\big(\lim\limits_{n\to\infty} a_n\big) = 0$.

The following will provide a useful criterion for the convergence of a given net.

Proposition 5.23. *Let* $(\mathcal{P},\mathcal{V})$ *be a locally convex complete lattice cone, and let* $(a_i)_{i\in\mathcal{I}}$ *be a bounded below net in* $\mathcal{P}$*. If for every* $v\in\mathcal{V}$ *there is a convergent net* $(b_i)_{i\in\mathcal{I}}$ *in* $\mathcal{P}$ *such that* $(a_i+b_i)_{i\in\mathcal{I}}$ *is convergent and the limit of* $(b_i)_{i\in\mathcal{I}}$ *is* v*-bounded, then the net* $(a_i)_{i\in\mathcal{I}}$ *is also convergent.*

Proof. Let $(a_i)_{i\in\mathcal{I}}$ be a net in $\mathcal{P}$ and for $v\in\mathcal{V}$ let $(b_i)_{i\in\mathcal{I}}$ be as stated. We use Lemma 5.19 for

$$\overline{\lim_{i\in\mathcal{I}}}\, a_i + \lim_{i\in\mathcal{I}} b_i \le \lim_{i\in\mathcal{I}}(a_i+b_i) \le \underline{\lim_{i\in\mathcal{I}}}\, a_i + \lim_{i\in\mathcal{I}} b_i .$$

As $b=\lim_{i\in\mathcal{I}} b_i$ is v-bounded, following Proposition 5.13(b) we have $\mathfrak{O}(b)\le\varepsilon v$ for all $\varepsilon>0$, hence

$$\overline{\lim_{i\in\mathcal{I}}}\, a_i \le \underline{\lim_{i\in\mathcal{I}}}\, a_i + \varepsilon v.$$

by Proposition 5.10(a). Because this holds for all $v\in\mathcal{V}$ and $\varepsilon>0$ and because $\mathcal{P}$ is a complete lattice cone, we infer that

$$\overline{\lim_{i\in\mathcal{I}}}\, a_i \le \underline{\lim_{i\in\mathcal{I}}}\, a_i$$

holds as claimed. □

Proposition 5.24. *Let* $(\mathcal{P},\mathcal{V})$ *be a locally convex complete lattice cone, and let* $(a_i)_{i\in\mathcal{I}}$ *be a bounded below net in* $\mathcal{P}$*. Then*

$$\underline{\lim_{i\in\mathcal{I}}}\,\mathfrak{O}(a_i) \le \mathfrak{O}\big(\underline{\lim_{i\in\mathcal{I}}}\, a_i\big) \qquad \text{and} \qquad \overline{\lim_{i\in\mathcal{I}}}\,\mathfrak{O}(a_i) \le \mathfrak{O}\big(\overline{\lim_{i\in\mathcal{I}}}\, a_i\big).$$

Proof. Let $(a_i)_{i\in\mathcal{I}}$ be a bounded below net, let $v\in\mathcal{V}$ and $\lambda\ge 0$ such that $0\le a_i+\lambda v$ for all $i\ge i_0\in\mathcal{I}$. Then $\mathfrak{O}(a_i)\le\varepsilon(a_i+\lambda v)$ for all $i\ge i_0$ and $\varepsilon>0$, hence

$$\underline{\lim_{i\in\mathcal{I}}}\,\mathfrak{O}(a_i) \le \varepsilon\ \underline{\lim_{i\in\mathcal{I}}}\, a_i + \varepsilon\lambda v \le \varepsilon\ \sup\Big\{\underline{\lim_{i\in\mathcal{I}}}\, a_i\,,\,0\Big\} + \varepsilon\lambda v$$

by 5.20(a). Taking the infimum over all $\varepsilon>0$ on the right-hand side we obtain

$$\underline{\lim_{i\in\mathcal{I}}}\,\mathfrak{O}(a_i) \le \mathfrak{O}\Big(\underline{\lim_{i\in\mathcal{I}}}\, a_i\Big) + \mathfrak{O}(\lambda v) \le \mathfrak{O}\Big(\underline{\lim_{i\in\mathcal{I}}}\, a_i\Big) + v.$$

Because this last inequality holds for all $v\in\mathcal{V}$ and because $\mathcal{P}$ carries the weak preorder, we conclude that

$$\underline{\lim_{i\in\mathcal{I}}}\,\mathfrak{O}(a_i) \le \mathfrak{O}\Big(\underline{\lim_{i\in\mathcal{I}}}\, a_i\Big)$$

holds as claimed. A similar argument demonstrates the same inequality for the superior limits. □

A simple example can show that equality does in general not hold in the expressions of Proposition 5.24: The locally convex cone $\mathcal{P} = \overline{\mathbb{R}}$ is a complete lattice. Order convergence in $\overline{\mathbb{R}}$ means convergence with respect to its usual one-point compactification topology, which at the point $+\infty$ differs from the symmetric topology of $\overline{\mathbb{R}}$ as a locally convex cone. For each $n \in \mathbb{N}$ let $a_n = n \in \overline{\mathbb{R}}$. Then $\lim_{n\to\infty} a_n = +\infty$ with respect to order convergence (but not with respect to the symmetric topology). We therefore have $\mathfrak{O}(a_n) = 0$ for all $n \in \mathbb{N}$, but $\mathfrak{O}\big(\lim_{n\to\infty} a_n\big) = +\infty$.

We proceed to investigate continuity of the lattice operations with respect to order convergence (c.f. Proposition 5.2).

Proposition 5.25. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex complete lattice cone and let $(a_i)_{i\in\mathcal{I}}$ and $(b_i)_{i\in\mathcal{I}}$ be convergent nets in $\mathcal{P}$. Then*

(a) $\lim_{i\in\mathcal{I}}(a_i \vee b_i) = \Big(\lim_{i\in\mathcal{I}} a_i\Big) \vee \Big(\lim_{i\in\mathcal{I}} b_i\Big)$.

(b) $\overline{\lim}_{i\in\mathcal{I}}(a_i \wedge b_i) \le \Big(\lim_{i\in\mathcal{I}} a_i\Big) \wedge \Big(\lim_{i\in\mathcal{I}} b_i\Big) \le \underline{\lim}_{i\in\mathcal{I}}(a_i \wedge b_i) + \mathfrak{O}\Big(\lim_{i\in\mathcal{I}}(a_i \vee b_i)\Big)$.

Proof. (a) Let $(a_i)_{i\in\mathcal{I}}$ and $(b_i)_{i\in\mathcal{I}}$ be convergent nets. Then

$$\overline{\lim_{i\in\mathcal{I}}}(a_i \vee b_i) = \inf_{i\in\mathcal{I}}\Big(\sup_{l\ge i}(a_l \vee b_l)\Big) \le \inf_{i\in\mathcal{I}}\Big(\big(\sup_{l\ge i} a_l\big) \vee \big(\sup_{j\ge i} b_j\big)\Big).$$

Because for any choice of $i, k \in \mathcal{I}$ and any $p \in \mathcal{I}$ such that both $i \le p$ and $k \le p$ we have

$$\big(\sup_{l\ge p} a_l\big) \vee \big(\sup_{j\ge p} b_j\big) \le \big(\sup_{l\ge i} a_l\big) \vee \big(\sup_{j\ge k} b_j\big),$$

we realize that

$$\inf_{i\in\mathcal{I}}\Big(\big(\sup_{l\ge i} a_l\big) \vee \big(\sup_{j\ge i} b_j\big)\Big) \le \inf_{i,k\in\mathcal{I}}\Big(\big(\sup_{l\ge i} a_l\big) \vee \big(\sup_{j\ge k} b_j\big)\Big).$$

Now we use Proposition 5.15(a) for

$$\inf_{i,k\in\mathcal{I}}\Big(\big(\sup_{l\ge i} a_l\big) \vee \big(\sup_{j\ge k} b_j\big)\Big) = \inf_{i\in\mathcal{I}}\Big(\sup_{l\ge i} a_l\Big) \vee \inf_{k\in\mathcal{I}}\Big(\sup_{j\ge k} b_j\Big) = \Big(\overline{\lim_{i\in\mathcal{I}}} a_i\Big) \vee \Big(\overline{\lim_{i\in\mathcal{I}}} b_i\Big).$$

Both nets $(a_i)_{i\in\mathcal{I}}$ and $(b_i)_{i\in\mathcal{I}}$ are supposed to be convergent. So we have

$$\Big(\overline{\lim_{i\in\mathcal{I}}} a_i\Big) \vee \Big(\overline{\lim_{i\in\mathcal{I}}} b_i\Big) = \Big(\underline{\lim}_{i\in\mathcal{I}} a_i\Big) \vee \Big(\underline{\lim}_{i\in\mathcal{I}} b_i\Big) \le \underline{\lim}_{i\in\mathcal{I}}(a_i \vee b_i).$$

Summarizing, the above yields

$$\overline{\lim_{i\in\mathcal{I}}}(a_i \vee b_i) \le \Big(\lim_{i\in\mathcal{I}} a_i\Big) \vee \Big(\lim_{i\in\mathcal{I}} b_i\Big) \le \underline{\lim}_{i\in\mathcal{I}}(a_i \vee b_i)$$

as claimed in Part (a). Similarly, one verifies Part (b): The inequality

$$\overline{\lim_{i\in\mathcal{I}}}(a_i \wedge b_i) \leq \Big(\overline{\lim_{i\in\mathcal{I}}}\, a_i\Big) \wedge \Big(\overline{\lim_{i\in\mathcal{I}}}\, b_i\Big) = \Big(\lim_{i\in\mathcal{I}} a_i\Big) \wedge \Big(\lim_{i\in\mathcal{I}} b_i\Big)$$

is obvious. Next we use Part (a), Proposition 5.3 and the limit rules from Lemma 5.17 for

$$\begin{aligned}\Big(\lim_{i\in\mathcal{I}} a_i\Big) \wedge \Big(\lim_{i\in\mathcal{I}} b_i\Big) + \lim_{i\in\mathcal{I}}(a_i \vee b_i) &= \Big(\lim_{i\in\mathcal{I}} a_i\Big) \wedge \Big(\lim_{i\in\mathcal{I}} b_i\Big) + \Big(\lim_{i\in\mathcal{I}} a_i\Big) \vee \Big(\lim_{i\in\mathcal{I}} b_i\Big)\\ &= \lim_{i\in\mathcal{I}} a_i + \lim_{i\in\mathcal{I}} b_i = \lim_{i\in\mathcal{I}}(a_i + b_i)\\ &= \lim_{i\in\mathcal{I}}(a_i \wedge b_i + a_i \vee b_i)\\ &\leq \underline{\lim_{i\in\mathcal{I}}}(a_i \wedge b_i) + \overline{\lim_{i\in\mathcal{I}}}(a_i \vee b_i)\\ &= \underline{\lim_{i\in\mathcal{I}}}(a_i \wedge b_i) + \lim_{i\in\mathcal{I}}(a_i \vee b_i).\end{aligned}$$

Now the cancellation rule from Lemma 5.9(a) yields the remaining part of (b). □

5.26 Series. A series $\sum_{i=1}^{\infty} a_i$ with terms a_i in a locally convex complete lattice cone $(\mathcal{P}, \mathcal{V})$ is said to be convergent with limit $s \in \mathcal{P}$ if the sequence $s_n = \sum_{i=1}^{n} a_i$ of its partial sums is order convergent to s. We write $\sum_{i=1}^{\infty} a_i = s$ in this case. Convergence of a series requires in particular that the sequence of its partial sums is bounded below (see 5.18).

Proposition 5.27. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex complete lattice cone and let $a_i, b_i \in \mathcal{P}$ for $i \in \mathbb{N}$. If the series $\sum_{i=1}^{\infty} a_i$ is convergent and if $a_i \leq b_i$ for all $i \in \mathbb{N}$, then the series $\sum_{i=1}^{\infty} b_i$ is also convergent.*

Proof. Let $a_i, b_i \in \mathcal{P}$ such that $a_i \leq b_i$ for all $i \in \mathbb{N}$. Let $s_n = \sum_{i=1}^{n} a_i$ and $r_n = \sum_{i=1}^{n} b_i$ be the partial sums of the series $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$, and let $s = \sum_{i=1}^{\infty} a_i$. Then $s_n \leq r_n$ for all $n \in \mathbb{N}$, hence $s \leq \underline{\lim}_{n\to\infty} r_n$. For $m \geq n$ we have

$$r_n + s_m = r_n + s_n + \sum_{i=n+1}^{m} a_i \leq r_n + s_n + \sum_{i=n+1}^{m} b_i = r_m + s_n.$$

For a fixed $n \in \mathbb{N}$ and $m \to \infty$ this leads to

$$r_n + s = r_n + \lim_{m\to\infty} s_m = \lim_{m\to\infty}(r_n + s_m) \leq \underline{\lim}_{m\to\infty}(s_n + r_m) = \underline{\lim}_{m\to\infty} r_m + s_n.$$

Now we let $n \to \infty$ and obtain

$$\begin{aligned}\overline{\lim}_{n\to\infty} r_n + s &= \overline{\lim}_{n\to\infty}(r_n + s) \leq \overline{\lim}_{n\to\infty}\Big(\underline{\lim}_{m\to\infty} r_m + s_n\Big)\\ &= \underline{\lim}_{m\to\infty} r_m + \overline{\lim}_{n\to\infty} s_n = \underline{\lim}_{m\to\infty} r_m + s.\end{aligned}$$

The cancellation law from Proposition 5.10(a) now yields

$$\overline{\lim_{n\to\infty}}\, r_n \le \underline{\lim_{n\to\infty}}\, r_n + \mathfrak{O}(s).$$

But $s \le \underline{\lim}_{n\to\infty} r_n$, as we observed before, and therefore $\underline{\lim}_{n\to\infty} r_n + \mathfrak{O}(s) = \underline{\lim}_{n\to\infty} r_n$ by Proposition 5.19. This yields

$$\overline{\lim_{n\to\infty}}\, r_n \le \underline{\lim_{n\to\infty}}\, r_n,$$

hence convergence of the sequence $(r_n)_{n\in\mathbb{N}}$, that is the partial sums of the series $\sum_{i=1}^{\infty} b_i$. □

We shall say that a series $\sum_{i=1}^{\infty} A_i$ of non-empty subsets A_i of a locally convex complete lattice cone $\mathcal{P}$ is *convergent* if the series $\sum_{i=1}^{\infty} \inf A_i$ converges in $\mathcal{P}$. In this case, all series $\sum_{i=1}^{\infty} a_i$, for any choice of elements $a_i \in A_i$, are convergent by Proposition 5.27, and we shall denote the set of all limits of these series by $\sum_{i=1}^{\infty} A_i$.

Proposition 5.28. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex complete lattice cone and let $\sum_{i=1}^{\infty} A_i$ be a convergent series of non-empty subsets of $\mathcal{P}$. Then*

(a) $\sum_{i=1}^{\infty} \sup A_i = \sup\left\{\sum_{i=1}^{\infty} A_i\right\}$.
(b) $\sum_{i=1}^{\infty} \inf A_i \le \inf\left\{\sum_{i=1}^{\infty} A_i\right\} \le \sum_{i=1}^{\infty} \inf A_i + \mathfrak{O}\left(\inf\left\{\sum_{i=1}^{\infty} A_i\right\}\right)$.

Proof. Let $\sum_{i=1}^{\infty} A_i$ be a convergent series of non-empty subsets of $\mathcal{P}$. We shall consider Parts (a) and (b) simultaneously. Let $S_i = \sup A_i$ and $s_i = \inf A_i$ for all $i \in \mathbb{N}$. By our assumption on the series $\sum_{i=1}^{\infty} A_i$ of sets, and following Proposition 5.27, all the series $\sum_{i=1}^{\infty} s_i$, $\sum_{i=1}^{\infty} S_i$ and $\sum_{i=1}^{\infty} a_i$ for any choice of $a_i \in A_i$ are convergent in $\mathcal{P}$. Moreover, for any choice of elements $a_i \in A_i$, for $i \in \mathbb{N}$, as $\sum_{i=1}^{n} s_i \le \sum_{i=1}^{n} a_i \le \sum_{i=1}^{n} S_i$ holds for all $n \in \mathbb{N}$, we have

$$\sum_{i=1}^{\infty} s_i \le \sum_{i=1}^{\infty} a_i \le \sum_{i=1}^{\infty} S_i,$$

and therefore

$$\sum_{i=1}^{\infty} s_i \le \inf\left\{\sum_{i=1}^{\infty} A_i\right\} \le \sup\left\{\sum_{i=1}^{\infty} A_i\right\} \le \sum_{i=1}^{\infty} S_i.$$

Thus all left to show is that

$$\sum_{i=1}^{\infty} S_i \le \sup\left\{\sum_{i=1}^{\infty} A_i\right\}$$

and

$$\inf\left\{\sum_{i=1}^{\infty} A_i\right\} \le \sum_{i=1}^{\infty} s_i + \mathfrak{O}\left(\inf\left\{\sum_{i=1}^{\infty} A_i\right\}\right).$$

For this, let us fix $n \in \mathbb{N}$ and choose arbitrary elements $a_i, b_i \in A_i$. We set $c_i = b_i$ for $i = 1, \ldots, n$ and $c_i = a_i$, else. Then, obviously, for every $m \geq n$ we have

$$\sum_{i=1}^{n} b_i + \sum_{i=1}^{m} a_i = \sum_{i=1}^{m} c_i + \sum_{i=1}^{n} a_i.$$

We let m tend to infinity and obtain

$$\sum_{i=1}^{n} b_i + \sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} c_i + \sum_{i=1}^{n} a_i.$$

As $c_i \in A_i$ for all $i \in \mathbb{N}$, this yields

$$\inf\left\{\sum_{i=1}^{\infty} A_i\right\} + \sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} b_i + \sum_{i=1}^{\infty} a_i \leq \sup\left\{\sum_{i=1}^{\infty} A_i\right\} + \sum_{i=1}^{n} a_i.$$

As $\sup\{\sum_{i=1}^{n} A_i\} = \sum_{i=1}^{n} S_i$ and $\inf\{\sum_{i=1}^{n} A_i\} = \sum_{i=1}^{n} s_i$ by Lemma 5.6(a), variation of the elements $b_1, \ldots, b_n$ yields

$$\inf\left\{\sum_{i=1}^{\infty} A_i\right\} + \sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} s_i + \sum_{i=1}^{\infty} a_i$$

and
$$\sum_{i=1}^{n} S_i + \sum_{i=1}^{\infty} a_i \leq \sup\left\{\sum_{i=1}^{\infty} A_i\right\} + \sum_{i=1}^{n} a_i.$$

Now we let n tend to infinity and infer that

$$\inf\left\{\sum_{i=1}^{\infty} A_i\right\} + \sum_{i=1}^{\infty} a_i \leq \sum_{i=1}^{\infty} s_i + \sum_{i=1}^{\infty} a_i$$

and
$$\sum_{i=1}^{\infty} S_i + \sum_{i=1}^{\infty} a_i \leq \sup\left\{\sum_{i=1}^{\infty} A_i\right\} + \sum_{i=1}^{\infty} a_i.$$

Finally, we take the infimum over all choices for the elements $a_i \in A_i$ in this last pair of inequalities and obtain

$$\inf\left\{\sum_{i=1}^{\infty} A_i\right\} + \inf\left\{\sum_{i=1}^{\infty} A_i\right\} \leq \sum_{i=1}^{\infty} s_i + \inf\left\{\sum_{i=1}^{\infty} A_i\right\}$$

and
$$\sum_{i=1}^{\infty} S_i + \inf\left\{\sum_{i=1}^{\infty} A_i\right\} \leq \sup\left\{\sum_{i=1}^{\infty} A_i\right\} + \inf\left\{\sum_{i=1}^{\infty} A_i\right\}.$$

Now the cancellation rule in Proposition 5.10(a) yields

$$\inf\left\{\sum_{i=1}^{\infty} A_i\right\} \le \sum_{i=1}^{\infty} s_i + \mathfrak{O}\left(\inf\left\{\sum_{i=1}^{\infty} A_i\right\}\right)$$

and

$$\sum_{i=1}^{\infty} S_i \le \sup\left\{\sum_{i=1}^{\infty} A_i\right\} + \mathfrak{O}\left(\inf\left\{\sum_{i=1}^{\infty} A_i\right\}\right).$$

As $\inf\{\sum_{i=1}^{\infty} A_i\} \le \sup\{\sum_{i=1}^{\infty} A_i\}$, Proposition 5.14 yields

$$\sup\left\{\sum_{i=1}^{\infty} A_i\right\} + \mathfrak{O}\left(\inf\left\{\sum_{i=1}^{\infty} A_i\right\}\right) = \sup\left\{\sum_{i=1}^{\infty} A_i\right\}.$$

This demonstrates our claim. □

5.29 Order Continuous Linear Operators and Functionals. Let $(\mathcal{P},\mathcal{V})$ and $(\mathcal{Q},\mathcal{W})$ be locally convex complete lattice cones. We shall say that a continuous linear operator $T:\mathcal{P}\to\mathcal{Q}$ is *order continuous* if it is continuous with respect to order convergence, that is if

$$T\Big(\lim_{i\in\mathcal{I}} a_i\Big) = \lim_{i\in\mathcal{I}} T(a_i)$$

holds for every order convergent net $(a_i)_{i\in\mathcal{I}}$ in $\mathcal{P}$. The limits refer to order convergence in $\mathcal{P}$ and $\mathcal{Q}$, respectively. Sums and non-negative multiples of order continuous linear operators are again order continuous. We are particularly interested in *order continuous linear functionals* in $\mathcal{P}^*$, that is order continuous linear operators from $\mathcal{P}$ into the locally convex complete lattice cone $\overline{\mathbb{R}}$. They form a subcone of $\mathcal{P}^*$. For every bounded below net $(a_i)_{i\in\mathcal{I}}$ in $\mathcal{P}$ and every order continuous linear operator $T:\mathcal{P}\to\mathcal{Q}$ we have

$$T\Big(\underline{\lim}_{i\in\mathcal{I}}\, a_i\Big) = T\Big(\lim_{i\in\mathcal{I}}\inf_{k\ge i} a_k\Big) = \lim_{i\in\mathcal{I}} T\Big(\inf_{k\ge i} a_k\Big) \le \lim_{i\in\mathcal{I}}\inf_{k\ge i} T(a_k) = \underline{\lim}_{i\in\mathcal{I}}\, T(a_i)$$

and, likewise

$$T\Big(\overline{\lim}_{i\in I}\, a_i\Big) = T\Big(\lim_{i\in\mathcal{I}}\sup_{k\ge i} a_k\Big) = \lim_{i\in\mathcal{I}} T\Big(\sup_{k\ge i} a_k\Big) \ge \lim_{i\in\mathcal{I}}\sup_{k\ge i} T(a_k) = \overline{\lim}_{i\in\mathcal{I}}\, T(a_i),$$

that is

$$T\Big(\underline{\lim}_{i\in\mathcal{I}}\, a_i\Big) \le \underline{\lim}_{i\in\mathcal{I}}\, T(a_i) \le \overline{\lim}_{i\in\mathcal{I}}\, T(a_i) \le T\Big(\overline{\lim}_{i\in\mathcal{I}}\, a_i\Big).$$

5.30 Lattice Homomorphisms. Let both $(\mathcal{P},\mathcal{V})$ and $(\mathcal{Q},\mathcal{W})$ be locally convex $\vee$- (or $\wedge$-)semilattice cones. A continuous linear operator $T:\mathcal{P}\to\mathcal{Q}$ is called a $\vee$- (*or* $\wedge$-)*semilattice homomorphism* if it is compatible with the lattice operations in $\mathcal{P}$ and $\mathcal{Q}$, that is if

$$T(a\vee b) = T(a)\vee T(b) \qquad \big(\text{or} \quad T(a\wedge b) = T(a)\wedge T(b)\big)$$

holds for all $a,b\in\mathcal{P}$. If both $(\mathcal{P},\mathcal{V})$ and $(\mathcal{Q},\mathcal{W})$ are locally convex lattice cones and $T:\mathcal{P}\to\mathcal{Q}$ is both a $\vee$- and a $\wedge$-semilattice homomorphism, then T is called a *lattice homomorphism.* Non-negative multiples of lattice homomorphism are again lattice homomorphisms, but sums are generally not.

Linear operators that are both order continuous and lattice homomorphisms are of particular interest. Suppose that both $(\mathcal{P},\mathcal{V})$ and $(\mathcal{Q},\mathcal{W})$ are locally convex complete lattice cones. A continuous linear operator $T:\mathcal{P}\to\mathcal{Q}$ is an order continuous lattice homomorphism if and only if

$$T(\sup A)=\sup T(A) \qquad \text{and} \qquad T(\inf B)=\inf T(B)$$

holds for all non-empty subsets A and bounded below subsets B of $\mathcal{P}$, that is if and only if T preserves that lattice operations of $\mathcal{P}$ and $\mathcal{Q}$. Indeed, $\sup A$ or $\inf B$ is the limit with respect to order convergence of the net of suprema or infima of finite subsets of A or B, respectively. Since an order continuous lattice homomorphism $T:\mathcal{P}\to\mathcal{Q}$ preserves finite suprema and infima as well as order convergence, we conclude that T preserves infinite suprema and infima as well. Conversely, if T preserves the suprema and infima of subsets of $\mathcal{P}$, then we have

$$T\Big(\underline{\lim}_{i\in\mathcal{I}}\, a_i\Big)=\underline{\lim}_{i\in\mathcal{I}}\, T(a_i) \qquad \text{and} \qquad T\Big(\overline{\lim}_{i\in\mathcal{I}}\, a_i\Big)=\overline{\lim}_{i\in\mathcal{I}}\, T(a_i)$$

for every bounded below net $(a_i)_{i\in\mathcal{I}}$ in $\mathcal{P}$. Thus T maps order convergent nets in $\mathcal{P}$ into order convergent nets in $\mathcal{Q}$ and is therefore an order continuous lattice homomorphism.

Examples 5.31. (a) Theorem II.6.7 in [100] states that for every neighborhood $v\in\mathcal{V}$ in an M-type locally convex $\vee$- (or $\wedge$-)semilattice cone $(\mathcal{P},\mathcal{V})$ all the extreme points of its polar $v^\circ\subset\mathcal{P}^*$ are $\vee$- (or $\wedge$-)semilattice homomorphisms from $\mathcal{P}$ into $\overline{\mathbb{R}}$.

(b) Let $(\mathcal{P},\mathcal{V})$ be a locally convex cone with dual $\mathcal{P}^*$ and let $(\mathcal{Q},\mathcal{V})$ be the cone of all non-empty convex subsets of $\mathcal{P}$ which are closed with respect to the lower topology $\big($see Example 4.37(d)$\big)$. In 5.7(d) we showed that $(\mathcal{Q},\mathcal{V})$ is a locally convex $\bigvee$-semilattice cone ordered by the set inclusion. There is a natural embedding $\mu\mapsto\tilde{\mu}:\mathcal{P}^*\to\mathcal{Q}^*$, where

$$\tilde{\mu}(A)=\sup\{\mu(a)\mid a\in A\}$$

for $\mu\in\mathcal{P}^*$ and $A\in\mathcal{Q}$. Indeed, if $\mu\in v^\circ$ for some $v\in\mathcal{V}$, then $A\le B\oplus v$ for $A,B\in\mathcal{Q}$ means that for every $a\in A$ and $\varepsilon\ge 0$ there is $b\in B$ such that $a\le\gamma b+(1+\varepsilon)v$ $\big($see 4.37(e)$\big)$ with some $1\le\gamma\le 1+\varepsilon$. This yields

$$\mu(a)\le\gamma\mu(b)+(1+\varepsilon)\le\gamma\tilde{\mu}(b)+(1+\varepsilon)$$

for all $\varepsilon>0$, hence $\mu(a)\le\tilde{\mu}(B)+1$. We infer $\hat{\mu}(A)\le\tilde{\mu}(B)+1$, and conclude that $\tilde{\mu}\in v^\circ\subset\mathcal{Q}^*$. Moreover, $\tilde{\mu}$ is a $\vee$-semilattice homomorphism even with respect to arbitrary suprema in $\mathcal{Q}:$ Let $\mathcal{A}$ be a subset of $\mathcal{Q}$ and let c be an element of $conv\big(\bigcup_{A\in\mathcal{A}} A\big)$, the convex hull of the union of all elements of $\mathcal{A}$. Then $c=\sum_{i=1}^{n}\alpha_i a_i$ for some $a_i\in A_i\in\mathcal{A}$ and $\alpha_i\geq 0$ such that $\sum_{i=1}^{n}\alpha_i=1$. Thus

$$\mu(c)=\sum_{i=1}^{n}\alpha_i\mu(a_i)\leq\sum_{i=1}^{n}\alpha_i\tilde{\mu}(A_i)\leq\sup_{A\in\mathcal{A}}\tilde{\mu}(A).$$

Since the functional $\mu:\mathcal{P}\to\overline{\mathbb{R}}$ is also continuous with respect to the lower relative topology on $\mathcal{P}$, we conclude that

$$\begin{aligned}\tilde{\mu}(\sup\mathcal{A})&=\sup\left\{\mu(a)\;\middle|\;a\in\overline{conv\big(\bigcup_{A\in\mathcal{A}}A\big)}^{(l)}\right\}\\&=\sup\left\{\mu(a)\;\middle|\;a\in conv\big(\bigcup_{A\in\mathcal{A}}A\big)\right\}\leq\sup_{A\in\mathcal{A}}\tilde{\mu}(A).\end{aligned}$$

The converse inequality is obvious.

(c) Similarly one argues for the locally convex cone $(\mathcal{Q},\mathcal{V})$ of all bounded below convex subsets of $\mathcal{P}$ which are closed with respect to the upper topology (see Examples 4.37(e) and 5.7(e)). In this case $(\mathcal{Q},\mathcal{V})$ is a locally convex $\bigwedge$-semilattice cone, ordered by the inverse set inclusion. There is a natural embedding $\mu\mapsto\tilde{\mu}:\mathcal{P}^*\to\mathcal{Q}^*$, where

$$\tilde{\mu}(A)=\inf\{\mu(a)\mid a\in A\}$$

for $\mu\in\mathcal{P}^*$ and $A\in\mathcal{Q}$. As similar argument as in (b) shows that

$$\tilde{\mu}(\inf\mathcal{A})=\inf_{A\in\mathcal{A}}\tilde{\mu}(A)$$

holds for every bounded below family of sets $\mathcal{A}\subset\mathcal{Q}$.

(d) Let $(\mathcal{P},\mathcal{V})$ be a locally convex $\vee$- (or $\wedge$-)semilattice cone, X a set, and consider the locally convex cone $\big(\mathcal{F}_{\widehat{\mathcal{V}}_b}(X,\mathcal{P}),\widehat{\mathcal{V}}\big)$ of $\mathcal{P}$-valued functions on X, endowed with the neighborhood system $\widehat{\mathcal{V}}$ consisting of $\big(\mathcal{V}\cup\{\infty\}\big)$ -valued functions. (Example 1.4(e)). This was seen to be again a locally convex $\vee$- (or $\wedge$-)semilattice cone, provided that for every $x\in X$ and $v\in\mathcal{V}$ there is $\hat{v}\in\widehat{\mathcal{V}}$ such that $\hat{v}(x)\leq v$ (see 5.7(c)). For $\mu\in v^\circ\subset\mathcal{P}^*$, a neighborhood $\hat{v}\in\widehat{\mathcal{V}}$ and $x\in X$ such that $\hat{v}(x)\leq v$, the mapping $\mu_x:\mathcal{F}_{\widehat{\mathcal{V}}_b}(X,\mathcal{P})\to\overline{\mathbb{R}}$ such that $\mu_x(f)=\mu\big(f(x)\big)$ for all $f\in\mathcal{F}_{\widehat{\mathcal{V}}_b}(X,\mathcal{P})$ is a continuous linear functional on $\mathcal{F}_{\widehat{\mathcal{V}}_b}(X,\mathcal{P})$ (see 2.1(f)), more precisely: an element of $\hat{v}^\circ$. Moreover, if μ is a $\vee$- (or $\wedge$-)semilattice homomorphism for $\mathcal{P}$, then μ_x is a semilattice homomorphism of the same type for $\mathcal{F}_{\widehat{\mathcal{V}}_b}(X,\mathcal{P})$.

5.32 Functionals Supporting the Separation Property. Corollary 4.34 (see also the Separation Theorem 3.2 in [175]) guarantees that in a locally convex cone the neighborhoods with respect to the weak preorder are completely determined by their polars, that is $a \preccurlyeq b+v$ holds for $a,b \in \mathcal{P}$ and $v \in \mathcal{V}$ if and only if $\mu(a) \leq \mu(b)+1$ for all $\mu \in v^\circ$. In this vein, for a locally convex cone $(\mathcal{P},\mathcal{V})$ we shall say that a subset Υ of $\mathcal{P}^*$ *supports the separation property for* $\mathcal{P}$ if for $a,b \in \mathcal{P}$ and $v \in \mathcal{V}$ such that $a \not\preccurlyeq b+v$ there is $\alpha \geq 0$ and $\mu \in \Upsilon \cap (\alpha v^\circ)$ such that $\mu(a) > \mu(b)+\alpha$. This property implies in particular that the functionals in Υ determine the weak preorder of $\mathcal{P}$, that is $a \preccurlyeq b$ holds for $a,b \in \mathcal{P}$ if and only if $\mu(a) \leq \mu(b)$ for all $\mu \in \Upsilon$. Indeed, the latter implies that $a \preccurlyeq b+v$ for all $v \in \mathcal{V}$, which by Lemma 3.2(a) yields $a \preccurlyeq b$.

Examples 5.33. (a) In Examples 1(a) and (b), that is for $\mathcal{P} = \overline{\mathbb{R}}$ or $\mathcal{P} = \overline{\mathbb{R}}_+$ the dual cone contains all positive reals, and the set $\Upsilon = \{1\}$ supports the separation property.

(b) If $\mathcal{V}$ consists of the multiples of a single neighborhood v, then we may choose

$$\Upsilon = \{\mu \in \mathcal{P}^* \mid \psi_v(\mu) = 0 \ \text{ or } \ \psi_v(\mu) = 1\},$$

where $\psi_v(\mu) = \inf\{\alpha \geq 0 \mid \mu \in \alpha v^\circ\}$. (In case that $v \in \mathcal{P}$, we have $\psi_v(\mu) = \mu(v)$.) Indeed, if $a \not\preccurlyeq b+(\rho v)$ for $a,b \in \mathcal{P}$ and $\rho v \in \mathcal{P}$, then by Corollary 4.34 there is $\mu \in (\rho v)^\circ = (1/\rho)v^\circ$ such that $\mu(a) > \mu(b)+1$. This implies $\psi_v(\mu) \leq 1/\rho$. If $\psi_v(\mu) = 0$, then $\mu \in \Upsilon \cap (\rho v)^\circ$ as required. Otherwise, we set $\alpha = 1/\psi_v(\mu) > 0$ and $\nu = \alpha\mu \in \Upsilon$ and observe that both $\nu \in \alpha(\rho v)^\circ$ and $\nu(a) > \nu(b)+\alpha$, again satisfying the requirement.

If in addition all elements of $\mathcal{P}$ are bounded, that is for example, if $\mathcal{P}$ is normed vector space, then according to Corollary 4.35 we may further reduce the size of Υ and choose

$$\Upsilon = \mathit{Ex}(v^\circ),$$

that is the set of all extreme points of the $w(\mathcal{P}^*,\mathcal{P})$-compact convex set v°. $\big($Obviously, $\psi_v(\mu) = 1$ holds for every $\mu \in \mathit{Ex}(v^\circ)$.$\big)$

(c) More generally, a locally convex cone $(\mathcal{P},\mathcal{V})$ is said to be *tightly covered by its bounded elements* $\big($see II.2.13 in [100]$\big)$ if for all $a,b \in \mathcal{P}$ and $v \in \mathcal{V}$ such that $a \not\preccurlyeq b+v$ there is some bounded element $a' \in \mathcal{P}$ such that $a' \preccurlyeq a$ and $a' \not\preccurlyeq b+v$. In this case, if $\mathcal{V}_0$ is a subcollection of $\mathcal{V}$ such that every $v \in \mathcal{V}$ is a multiple of some $v_0 \in \mathcal{V}_0$, then according to Corollary II.4.7 in [100] the set

$$\Upsilon = \bigcup_{v_0 \in \mathcal{V}_0} \mathit{Ex}(v_0^\circ)$$

supports the separation property for $\mathcal{P}$.

(d) Let $(\mathcal{P},\mathcal{V})$ be a locally convex cone with dual $\mathcal{P}^*$ and let $(\mathcal{Q},\mathcal{V})$ be the cone of all non-empty convex subsets of $\mathcal{P}$ which are closed with respect to the lower topology $\big($see Example 4.37(d)$\big)$. For every $\mu \in \mathcal{P}^*$ the formula

$$\tilde{\mu}(A) = \sup\{\mu(a) \mid a \in A\} \qquad \text{for} \quad A \in \mathcal{Q}$$

defines an element $\tilde{\mu} \in \mathcal{Q}^*$, more precisely, $\tilde{\mu} \in v^\circ$ whenever $\mu \in v^\circ$ (see Example 5.31(b) before). Now Theorem 4.33 guarantees that the set

$$\Upsilon = \{\tilde{\mu} \mid \mu \in \mathcal{P}^*\} \subset \mathcal{Q}^*$$

supports the separation property for $\mathcal{Q}$. Indeed, if $A \not\leq B \oplus v$ for $A, B \in \mathcal{Q}$ and $v \in \mathcal{V}$, then there is $a \in A$ such that $a \notin v\big(B\big)$ $\big($see 4.37(d)$\big)$. Following Theorem 4.33(a) then there is $\mu \in v^\circ$, hence $\tilde{\mu} \in v^\circ$ such that

$$\mu(a) > \sup\{\mu(b) \mid b \in B\} + 1 = \tilde{\mu}(B) + 1.$$

Thus $\tilde{\mu}(A) = \sup\{\mu(a) \mid a \in A\} > \tilde{\mu}(B) + 1$.

(e) Similarly, if $(\mathcal{Q}, \mathcal{V})$ is the locally convex cone of all bounded below convex subsets of $\mathcal{P}$ which are closed with respect to the upper topology $\big($see Example 4.37(e)$\big)$, then for every $\mu \in \mathcal{P}^*$ the formula

$$\tilde{\mu}(A) = \inf\{\mu(a) \mid a \in A\} \qquad \text{for} \quad A \in \mathcal{Q}$$

defines an element $\tilde{\mu} \in \mathcal{Q}^*$, more precisely, $\tilde{\mu} \in v^\circ$ whenever $\mu \in v^\circ$ $\big($see 5.31(c)$\big)$. Then

$$\Upsilon = \{\tilde{\mu} \mid \mu \in \mathcal{P}^*\} \subset \mathcal{Q}^*$$

supports the separation property for $\mathcal{Q}$. Indeed, if $A \not\leq B \oplus v$ for $A, B \in \mathcal{Q}$ and $v \in \mathcal{V}$, then there is $b \in B$ such that $b \notin \big(A\big)v$ $\big($see 4.37(e)$\big)$. Following Theorem 4.33(b) then there is $\mu \in v^\circ$, hence $\tilde{\mu} \in v^\circ$ such that $\mu(b) < \inf\{\mu(b) \mid b \in B\} - 1 = \tilde{\mu}(A) - 1$. Thus $\tilde{\mu}(B) = \inf\{\mu(b) \mid b \in B\} < \tilde{\mu}(A) - 1$, that is $\tilde{\mu}(A) > \tilde{\mu}(B) + 1$.

(f) Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone, X a set, and consider the locally convex cone $\big(\mathcal{F}_{\widehat{\mathcal{V}}_b}(X, \mathcal{P}), \widehat{\mathcal{V}}\big)$ of $\mathcal{P}$-valued functions on X, where the neighborhood system $\widehat{\mathcal{V}}$ is generated by a family of $\big(\mathcal{V} \cup \{\infty\}\big)$-valued functions on X as elaborated in Example 1.4(e). For every $\mu \in v\circ \subset \mathcal{P}^*$ for $v \in \mathcal{V}$, and $x \in X$ such that $\hat{v}(x) \leq v$ for $\hat{v} \in \widehat{\mathcal{V}}$, the formula

$$\mu_x(f) = \mu\big(f(x)\big) \qquad \text{for} \quad f \in \mathcal{F}_{\widehat{\mathcal{V}}_b}(X, \mathcal{P})$$

defines a continuous linear functional on $\mathcal{F}_{\widehat{\mathcal{V}}_b}(X, \mathcal{P})$ $\big($see 2.1(f)$\big)$, more precisely: We have $\mu_x \in \hat{v}^\circ$. Let us denote by X_0 the subset of all $x \in X$ such that $\hat{v}(x) \neq \infty$ for at least one $\hat{v} \in \widehat{\mathcal{V}}$. If $\Upsilon \subset \mathcal{P}^*$ supports the separation property for $\mathcal{P}$, then

$$\widehat{\Upsilon} = \{\mu_x \mid \mu \in \Upsilon, \ x \in X_0\} \subset \mathcal{F}_{\widehat{\mathcal{V}}_b}(X, \mathcal{P})^*$$

supports the separation property for $\mathcal{F}_{\widehat{\mathcal{V}}_b}(X, \mathcal{P})$. Indeed, if $f \not\leq g + \hat{v}$ for $f, g \in \mathcal{F}_{\widehat{\mathcal{V}}_b}(X, \mathcal{P})$ and $\hat{v} \in \widehat{\mathcal{V}}$, then there is $x \in X$ such that $f(x) \not\leq g(x) + \hat{v}(x)$. This implies $\hat{v}(x) \neq \infty$, hence $v = \hat{v}(x) \in \mathcal{V}$. Following our assumption there is $\alpha \geq 0$ and $\mu \in \Upsilon \cap (\alpha v^\circ)$ such that $\mu\big(f(x)\big) > \mu\big(g(x)\big) + \alpha$.

We therefore have $\mu_x \in \widehat{\Upsilon} \cap (\alpha\hat{v}^\circ)$ and $\mu_x(f) > \mu_x(g) + \alpha$, as required. In case that $\mathcal{P} = \overline{\mathbb{R}}$ or $\mathcal{P} = \overline{\mathbb{R}}_+$ we may choose $\Upsilon = \{1\}$ $\big($see 5.33(a)$\big)$. Then $\widehat{\Upsilon}$ consists of all point evaluations at the points $x \in X$ such that $\hat{v}(x) < +\infty$ for at least one of the $\overline{\mathbb{R}}_+$-valued neighborhood functions $\hat{v} \in \widehat{\mathcal{V}}$.

The presence of suitable subsets of $\mathcal{P}^*$ supporting the separation property permits a strengthening of certain statements for the general case. The following Propositions 5.34 and 5.35 will improve on Proposition 5.15(b) and Propositions 5.25(b) and 5.28(b) under these circumstances. Recall that a subset A of an ordered cone $\mathcal{P}$ is said to be *directed upward* (or *downward*) if for $a, b \in A$ there is $c \in A$ such that both $a \leq c$ and $b \leq c$ (or $c \leq a$ and $c \leq b$.)

Proposition 5.34. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex complete lattice cone, and suppose that the order continuous lattice homomorphisms support the separation property for $\mathcal{P}$. Then*

(a) $\sup(A \wedge B) = \sup A \wedge \sup B$ *for non-empty subsets A, B of $\mathcal{P}$.*

(b) $\lim_{i\in\mathcal{I}}(a_i \wedge b_i) = \left(\lim_{i\in\mathcal{I}} a_i\right) \wedge \left(\lim_{i\in\mathcal{I}} b_i\right)$
for convergent nets $(a_i)_{i\in\mathcal{I}}$ and $(b_i)_{i\in\mathcal{I}}$ in $\mathcal{P}$.

Proof. Let Υ be the subset of all order continuous lattice homomorphisms in $\mathcal{P}^*$ and suppose that Υ supports the separation property for $\mathcal{P}$. For Part (a), let A, B be non-empty subsets of $\mathcal{P}$. In Proposition 5.15(b) we already demonstrated $\sup(A \wedge B) \leq \sup A \wedge \sup B$. For the converse inequality, it suffices to verify that

$$\mu(\sup A \wedge \sup B) \leq \mu\big(\sup(A \wedge B)\big)$$

holds for all $\mu \in \Upsilon$ (see 5.32). For this, let $\mu \in \Upsilon$. Then

$$\mu(\sup A \wedge \sup B) = \mu(\sup A) \wedge \mu(\sup B) = \sup\big(\mu(A)\big) \wedge \sup\big(\mu(B)\big),$$

since μ is an order continuous lattice homomorphism. We may assume that $\mu(\sup A) \leq \mu(\sup B)$. Then for every $a \in A$ and $\varepsilon > 0$ there is $b \in B$ such that $\mu(a) \leq \mu(b) + \varepsilon$. Thus also $\mu(a) \leq \mu(a \wedge b) + \varepsilon$. This shows

$$\sup\big(\mu(A)\big) \wedge \sup\big(\mu(B)\big) = \sup\big(\mu(A)\big) \leq \mu\big(\sup(A \wedge B)\big) + \varepsilon$$

and verifies our claim.

For Part (b), let $(a_i)_{i\in\mathcal{I}}$ and $(b_i)_{i\in\mathcal{I}}$ be convergent nets in $\mathcal{P}$. In the light of 5.25(b) and our assumption on Υ it suffices to verify that

$$\mu\left(\left(\lim_{i\in\mathcal{I}} a_i\right) \wedge \left(\lim_{i\in\mathcal{I}} b_i\right)\right) \leq \mu\left(\underline{\lim}_{i\in\mathcal{I}}(a_i \wedge b_i)\right),$$

that is

$$\Big(\lim_{i\in\mathcal{I}}\mu(a_i)\Big)\wedge\Big(\lim_{i\in\mathcal{I}}\mu(b_i)\Big)\le\underline{\lim}_{i\in\mathcal{I}}\Big(\mu(a_i)\wedge\mu(b_i)\Big)$$

holds for all $\mu\in\Upsilon$. For this, given a functional $\mu\in\Upsilon$, we may assume that $\lim_{i\in\mathcal{I}}\mu(a_i)\le\lim_{i\in\mathcal{I}}\mu(b_i)$. Then for every $\varepsilon>0$ there is $i_0\in\mathcal{I}$ such that $\mu(a_i)\le\mu(b_i)+\varepsilon$ for all $i\ge i_0$. This implies $\mu(a_i)\le\mu(a_i)\wedge\mu(b_i)+\varepsilon$. Thus $\lim_{i\in\mathcal{I}}\mu(a_i)\le\underline{\lim}_{i\in\mathcal{I}}\big(\mu(a_i)\wedge\mu(b_i)\big)+\varepsilon$. This yields our claim. □

Proposition 5.35. *Let $(\mathcal{P},\mathcal{V})$ be a locally convex complete lattice cone. If the order continuous lattice homomorphisms (or the order continuous linear functionals) support the separation property for $\mathcal{P}$, then*

$$\sum_{i=1}^{\infty}\inf A_i=\inf\left\{\sum_{i=1}^{\infty}A_i\right\}$$

for every convergent series $\sum_{i=1}^{\infty}A_i$ of non-empty (or non-empty directed downward) subsets of $\mathcal{P}$.

Proof. Let Υ be the subset of all order continuous lattice homomorphisms (or order continuous linear functionals) in $\mathcal{P}^*$ and suppose that Υ supports the separation property for $\mathcal{P}$. In the second case we assume in addition that the sets $A_i\subset\mathcal{P}$ are directed downward. Thus, in each of the cases for Υ we have

$$\mu\big(\inf A_i\big)=\inf\{\mu(A_i)\}$$

for all $i\in\mathbb{N}$ and $\mu\in\Upsilon$. The order continuity of the functionals μ then yields

$$\mu\left(\sum_{i=1}^{\infty}\inf A_i\right)=\sum_{i=1}^{\infty}\mu\big(\inf A_i\big)=\sum_{i=1}^{\infty}\inf\big\{\mu(A_i)\big\}.$$

Likewise, since the sets $\sum_{i=1}^{\infty}A_i$ are seen to be directed downward in the second case for Υ, we have

$$\mu\left(\inf\left\{\sum_{i=1}^{\infty}A_i\right\}\right)$$
$$=\inf\left\{\mu\left(\sum_{i=1}^{\infty}a_i\right)\;\middle|\;a_i\in A_i\right\}=\inf\left\{\sum_{i=1}^{\infty}\mu(a_i)\;\middle|\;a_i\in A_i\right\}.$$

Given $\mu\in\Upsilon$ we choose $a_i\in A_i$ such that $\mu(a_i)\le\inf\{\mu(A_i)\}+2^{-i}$. Then

$$\inf\left\{\sum_{i=1}^{\infty}\mu(a_i)\;\middle|\;a_i\in A_i\right\}\le\sum_{i=1}^{\infty}\mu(a_i)\le\sum_{i=1}^{\infty}\inf\{\mu(A_i)\}+1,$$

hence

$$\mu\left(\inf\left\{\sum_{i=1}^{\infty} A_i\right\}\right) \leq \mu\left(\sum_{i=1}^{\infty} \inf A_i\right) + 1.$$

Because Υ supports the separation property for $\mathcal{P}$, this shows

$$\inf\left\{\sum_{i=1}^{\infty} A_i\right\} \leq \sum_{i=1}^{\infty} \inf A_i + v$$

for all $v \in \mathcal{V}$, hence

$$\inf\left\{\sum_{i=1}^{\infty} A_i\right\} \leq \sum_{i=1}^{\infty} \inf A_i\,,$$

since $\mathcal{P}$ carries its weak preorder. The reverse inequality was established in Proposition 5.28(b). □

We shall say that a subcone $\mathcal{N}$ of $\mathcal{P}$ is a *locally convex lattice subcone of* $(\mathcal{P},\mathcal{V})$ if $a \vee b \in \mathcal{N}$ and $a \wedge b \in \mathcal{N}$ whenever $a, b \in \mathcal{N}$. Likewise, $\mathcal{N}$ is a *locally convex complete lattice subcone of* $(\mathcal{P},\mathcal{V})$ if $\sup A \in \mathcal{N}$ and $\inf B \in \mathcal{N}$ whenever $A, B \subset \mathcal{N}$, A is not empty and B is bounded below. The suprema and infima are taken in $\mathcal{P}$.

A family $\mathfrak{A}$ of subsets of $\mathcal{P}$ will be called *sup-bounded below* if the set $\{\sup A \mid A \in \mathfrak{A}\}$ is bounded below in $\mathcal{P}$. This implies in particular that $\emptyset \notin \mathfrak{A}$ and that $\inf\{\sup A \mid A \in \mathfrak{A}\}$ exists in $\mathcal{P}$.

Proposition 5.36. *Let $(\mathcal{P},\mathcal{V})$ be a locally convex complete lattice cone, and suppose that the order continuous lattice homomorphisms support the separation property for $\mathcal{P}$. Let $\mathcal{N}$ be a subcone of $\mathcal{P}$. The smallest locally convex complete lattice subcone of $\mathcal{P}$ that contains $\mathcal{N}$ consists of all elements $a \in \mathcal{P}$ which can be expressed in the following way:*

$$a = \inf\{\sup A \mid A \in \mathfrak{A}\},$$

where $\mathfrak{A}$ is a sup-bounded below family of subsets of $\mathcal{N}$.

Proof. Let $(\mathcal{P},\mathcal{V})$ be a locally convex complete lattice cone. Corresponding to a sup-bounded below family $\mathfrak{A}$ of non-empty subsets of $\mathcal{P}$ let us define the element $a_{\mathfrak{A}} \in \mathcal{P}$ by

$$a_{\mathfrak{A}} = \inf\{\sup A \mid A \in \mathfrak{A}\}.$$

For families $\mathfrak{A}$ and $\mathfrak{B}$ of this type and $\alpha \geq 0$ we denote $\alpha\mathfrak{A} = \{\alpha A \mid A \in \mathfrak{A}\}$ and $\mathfrak{A} + \mathfrak{B} = \{A + B \mid A \in \mathfrak{A},\ B \in \mathfrak{B}\}$. It is evident from Lemmas 5.5 and 5.6 that these are again sup-bounded below families of subsets of $\mathcal{P}$. We also use 5.5 and 5.6 for the following observations:

(i) $\alpha a_{\mathfrak{A}} = \alpha \inf\{\sup A \mid A \in \mathfrak{A}\} = \inf\{\sup \alpha A \mid A \in \mathfrak{A}\} = a_{\alpha A}.$

$$\text{(ii)} \qquad \begin{aligned} a_{\mathfrak{A}} + a_{\mathfrak{B}} &= \inf\left\{ \sup A \mid A \in \mathfrak{A}\right\} + \inf\left\{ \sup B \mid B \in \mathfrak{B}\right\} \\ &= \inf\left\{ \sup A + \sup B \mid A \in \mathfrak{A},\ B \in \mathfrak{B}\right\} \\ &= \inf\left\{ \sup(A+B) \mid A \in \mathfrak{A},\ B \in \mathfrak{B}\right\} \\ &= \inf\left\{ \sup C \mid C \in (\mathfrak{A}+\mathfrak{B})\right\} \\ &= a_{(\mathfrak{A}+\mathfrak{B})}. \end{aligned}$$

Let $\{\mathfrak{A}_i\}_{i\in\mathcal{I}}$ be a collection of sup-bounded families $\mathfrak{A}_i$ of subsets of $\mathcal{P}$. In a first instance, suppose that this collection is not empty, and let $\mathfrak{A} = \{\cup_{i\in\mathcal{I}} A_i \mid (A_i)_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} \mathfrak{A}_i\}$, that is the elements A of $\mathfrak{A}$ are all unions of the type $A = \cup_{i\in\mathcal{I}} A_i$, where $A_i \in \mathfrak{A}_i$. (The Axiom of Choice is required for this construction.) This family $\mathfrak{A}$ is also sup-bounded below. Indeed, given $v \in \mathcal{V}$ and a fixed $k \in \mathcal{I}$ there is $\lambda \geq 0$ such that $0 \leq \sup A_k + \lambda v$ for all $A_k \in \mathfrak{A}_k$. Thus for every $A \in \mathfrak{A}$ we have $A_k \subset A$ for some $A_k \in \mathfrak{A}_k$, hence $0 \leq \sup A_k + \lambda v \leq \sup A + \lambda v$. We claim that

$$\text{(iii)} \qquad \sup_{i\in\mathcal{I}} a_{\mathfrak{A}_i} = \sup_{i\in\mathcal{I}} \inf\{\sup A_i \mid A_i \in \mathfrak{A}_i\} = \inf\{\sup A \mid A \in \mathfrak{A}\} = a_{\mathfrak{A}}.$$

Indeed, for every fixed $i \in \mathcal{I}$ and every $A \in \mathfrak{A}$ there is some $A_i \in \mathfrak{A}_i$ such that $A_i \subset A$. This shows $\inf\{\sup A_i \mid A_i \in \mathfrak{A}_i\} \leq \sup A$ for all $A \in \mathfrak{A}$, hence $\inf\{\sup A_i \mid A_i \in \mathfrak{A}_i\} \leq \inf\{\sup A \mid A \in \mathfrak{A}\}$ holds for all $i \in \mathcal{I}$. This yields

$$\sup_{i\in\mathcal{I}} \inf\{\sup A_i \mid A_i \in \mathfrak{A}_i\} \leq \inf\{\sup A \mid A \in \mathfrak{A}\}.$$

For the converse inequality we will have to use the fact that the lattice operations are formed in a locally convex complete lattice cone for which the order continuous lattice homomorphisms support the separation property, that is it suffices to verify that

$$\mu\big(\inf\{\sup A \mid A \in \mathfrak{A}\}\big) \leq \mu\big(\sup_{i\in\mathcal{I}} \inf\{\sup A_i \mid A_i \in \mathfrak{A}_i\}\big)$$

holds for every order continuous lattice homomorphism $\mu \in \mathcal{P}^*$. For this assume to the contrary that there is

$$\rho < \mu\big(\inf\{\sup A \mid A \in \mathfrak{A}\}\big) = \inf\{\sup \mu(A) \mid A \in \mathfrak{A}\},$$

and that

$$\mu\big(\sup_{i\in\mathcal{I}} \inf\{\sup A_i \mid A_i \in \mathfrak{A}_i\}\big) = \sup_{i\in\mathcal{I}} \inf\{\sup \mu(A_i) \mid A_i \in \mathfrak{A}_i\} < \rho$$

holds for some order continuous lattice homomorphism $\mu \in \mathcal{P}^*$. This means $\inf\{\sup \mu(A_i) \mid A_i \in \mathfrak{A}_i\} < \rho$ for all $i \in \mathcal{I}$, hence $\sup \mu(A_i) < \rho$ for some $A_i \in \mathfrak{A}_i$. We use these sets A_i for $A = \cup_{i\in\mathcal{I}} A_i \in \mathfrak{A}$. We have $\sup \mu(A) = \sup\{\mu(A_i) \mid i \in \mathcal{I}\} \leq \rho$, contradicting the assumption that $\rho < \sup \mu(A)$ holds for all $A \in \mathfrak{A}$. This yields our claim.

In a second instance, suppose that the set $\{a_{\mathfrak{A}_i}\}_{i\in\mathcal{I}}$ is bounded below in $\mathcal{P}$. Then the family $\mathfrak{A} = \cup_{i\in\mathcal{I}} \mathfrak{A}_i$ is also sup-bounded below. Indeed, given $v \in \mathcal{V}$

there is $\lambda \geq 0$ such that $0 \leq a_{\mathfrak{A}_i} + \lambda v$ for all $i \in \mathcal{I}$. Thus for every $A \in \mathfrak{A}$ we have $A \in \mathfrak{A}_i$ for some $i \in \mathcal{I}$ and therefore $0 \leq a_{\mathfrak{A}_i} + \lambda v \leq \sup A + \lambda v$. Now we infer that

$$\text{(iv)} \qquad \begin{aligned} \inf_{i \in \mathcal{I}} a_{\mathfrak{A}_i} &= \inf_{i \in \mathcal{I}} \big\{ \inf\{\sup A_i \mid A_i \in \mathfrak{A}_i\} \big\} \\ &= \inf \big\{ \sup A \mid A \in \cup_{i \in \mathcal{I}} \mathfrak{A}_i \big\} \\ &= a_{\mathfrak{A}} \end{aligned}$$

Now let $\mathcal{N}$ be a subcone of $\mathcal{P}$ and denote by $\widehat{\mathcal{N}}$ the subset of $\mathcal{P}$ consisting of all elements $a_{\mathfrak{A}}$, where $\mathfrak{A}$ is an sup-bounded below family of subsets of $\mathcal{N}$. The preceding arguments (i) to (iv) yield that $\widehat{\mathcal{N}}$ is a locally convex lattice subcone of $\mathcal{P}$. Since $a_{\mathfrak{A}} = a$ for every $a \in \mathcal{N}$ with $\mathfrak{A} = \big\{\{a\}\big\}$, we have $\mathcal{N} \subset \widehat{\mathcal{N}}$. On the other hand, every locally convex lattice subcone of $\mathcal{P}$ that contains $\mathcal{N}$, necessarily contains also all elements $a_{\mathfrak{A}} \in \mathcal{P}$ of this type. Thus $\widehat{\mathcal{N}}$ is indeed the smallest locally convex complete lattice subcone of $\mathcal{P}$ that contains $\mathcal{N}$. □

For the following recall the notations from Example 1.4(e). We observed before that $\big(\mathcal{F}_{\widehat{\mathcal{V}}_b}(X, \overline{\mathbb{R}}), \widehat{\mathcal{V}}\big)$ is a locally convex complete lattice cone for any choice of the set X and the neighborhood system $\widehat{\mathcal{V}}$, provided that for every $x \in X$ there is $\hat{v} \in \widehat{\mathcal{V}}$ such that $\hat{v}(x) < +\infty$ (see 5.7(c)). The point evaluations at the points $x \in X$ are order continuous lattice homomorphisms and according to Example 5.33(f) support the separation property for $\mathcal{F}_{\widehat{\mathcal{V}}_b}(X, \overline{\mathbb{R}})$. Thus for every locally convex complete lattice subcone of $\big(\mathcal{F}_{\widehat{\mathcal{V}}_b}(X, \overline{\mathbb{R}}), \widehat{\mathcal{V}}\big)$ the order continuous lattice homomorphisms support the separation property. For an inverse implication recall the definition of an embedding in 2.2:

Proposition 5.37. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex complete lattice cone. If the set Υ of all order continuous lattice homomorphisms in $\mathcal{P}^*$ supports the separation property, then $(\mathcal{P}, \mathcal{V})$ can be embedded into the locally convex complete lattice of $\big(\mathcal{F}_{\widehat{\mathcal{V}}_b}(\Upsilon, \overline{\mathbb{R}}), \widehat{\mathcal{V}}\big)$, endowed with a suitable system $\widehat{\mathcal{V}}$ of $\overline{\mathbb{R}}_+$ -valued neighborhood functions. This embedding is one-to-one and preserves the lattice operations.*

Proof. Let Υ be the set of order continuous lattice homomorphisms in $\mathcal{P}^*$. Recall that $\alpha\mu \in \Upsilon$ whenever $\mu \in \Upsilon$ and $\alpha \geq 0$. With every element $a \in \mathcal{P}$ we associate the function $\varphi_a : \Upsilon \to \overline{\mathbb{R}}$ such that

$$\varphi_a(\mu) = \mu(a) \qquad \text{for all} \quad \mu \in \Upsilon.$$

The mapping $\Phi : \mathcal{P} \to \mathcal{F}(\Upsilon, \overline{\mathbb{R}})$ such that $\Phi(a) = \varphi_a$ for all $a \in \mathcal{P}$ is obviously linear, monotone, and since $\mathcal{P}$ carries the weak preorder which is supposed to be antisymmetric, Φ is also one-to one. Since the elements of Υ are all order continuous lattice homomorphisms in $\mathcal{P}^*$, for every subset A of $\mathcal{P}$ and $\mu \in \Upsilon$ we have

$$\mu(\sup A) = \sup\{\mu(a) \mid a \in A\} = \sup\{\varphi_a(\mu) \mid a \in A\} = \big(\sup \varphi_a\big)(\mu).$$

Thus $\Phi(\sup A) = \sup \Phi(A)$. Likewise, we have $\Phi(\inf B) = \inf \Phi(B)$ for every bounded below subset B of $\mathcal{P}$. Recall that the lattice operations are carried out pointwise in $\mathcal{F}(\Upsilon, \overline{\mathbb{R}}.)$ Corresponding to the neighborhoods $v \in \mathcal{V}$ we consider the $\overline{\mathbb{R}}$-valued functions ψ_v on Υ such that

$$\psi_v(\mu) = \inf\{\alpha > 0 \mid \mu \in \alpha v^\circ\}$$

for all $\mu \in \mathcal{P}^*$. As usual, we set $\inf \emptyset = +\infty$, but observe that for every $\mu \in \Upsilon$ there is $v \in \mathcal{V}$ such that $\psi_v(\mu) < +\infty$. Note that $\psi_v = \varphi_v$ in case that $v \in \mathcal{P}$. We also note that the family of all functions ψ_v for $v \in \mathcal{V}$ is not necessarily closed for the pointwise addition of its functions. For this reason we refer to the last remark in Example 1.4(e) relating to the construction of a locally convex cone of cone-valued functions. For $\mathcal{F}(\Upsilon, \overline{\mathbb{R}})$ we use the abstract neighborhood system $\widehat{V} = \{\hat{v} \mid v \in \mathcal{V}\}$ with the addition $\oplus$ and multiplication by scalars carried over by the corresponding operations in $\mathcal{V}$, that is $\hat{u} \oplus \hat{v} = \widehat{u+v}$ and $\alpha\hat{v} = \widehat{\alpha v}$ for $u, v \in \mathcal{V}$ and $\alpha > 0$. The neighborhood system $\widehat{\mathcal{V}}$ corresponds to the family $\{\psi_v \mid v \in \mathcal{V}\}$ of $\overline{\mathbb{R}}_+$-valued neighborhood functions which define the neighborhoods $\hat{v} \in \widehat{\mathcal{V}}$ for $\mathcal{F}(\Upsilon, \overline{\mathbb{R}})$ by

$$f \le g + \hat{v} \qquad \text{if} \qquad f(\mu) \le g(\mu) + \psi_v(\mu) \quad \text{for all} \quad \mu \in \Upsilon$$

(see 1.4(e)) for functions $f, g \in \mathcal{F}(\Upsilon, \overline{\mathbb{R}})$. As required, we have $\psi_{(\alpha v)} = \alpha\psi_v$ and $\psi_{(u+v)} \ge \psi_u + \psi_v$ for all $u, v \in \mathcal{V}$ and $\alpha > 0$. The first of these claims is obvious. For the second one, let $\mu \in \Upsilon$ and let $\sigma < \psi_u(\mu)$ and $\rho < \psi_v(\mu)$. Since both $\mu \notin \sigma u^\circ$ and $\mu \notin \rho v^\circ$, there are $a, b, c, d \in \mathcal{P}$ such that $a \le b+u$ and $\mu(a) > \mu(b) + \sigma$ as well as $c \le d + v$ and $\mu(c) > \mu(d) + \sigma$. Then from $(a+c) \le (b+d)+(u+w)$ and $\mu(a+c) > \mu(b+d)+(\sigma+\rho)$ we conclude that $\mu \notin (\sigma + \rho)(u + v)^\circ$. This shows $\psi_{(u+v)}(\mu) \ge (\sigma + \rho)$, yielding our claim. Thus $\big(\mathcal{F}_{\widehat{\mathcal{V}}_b}(\Upsilon, \overline{\mathbb{R}}), \widehat{\mathcal{V}}\big)$ is a locally convex lattice cone in the sense of 1.4(e). Moreover, since $\psi_v(\mu) < +\infty$ holds for all $\mu \in \mathcal{P}^*$ with some $\hat{v} \in \widehat{\mathcal{V}}$, we established in Example 5.7 that $\big(\mathcal{F}_{\widehat{\mathcal{V}}_b}(\mathcal{P}^*, \overline{\mathbb{R}}), \widehat{\mathcal{V}}\big)$ is indeed a locally convex complete lattice cone. We claim that $\Phi(\mathcal{P})$ is contained in $\big(\mathcal{F}_{\widehat{\mathcal{V}}_b}(\Upsilon, \overline{\mathbb{R}}), \widehat{\mathcal{V}}\big)$ and that

$$a \le b + v \qquad \text{if and only if} \qquad \Phi(a) \le \Phi(b) + \hat{v}$$

holds for $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$. Indeed, suppose that $a \le b + v$. Then for every $\mu \in \Upsilon$ and $\alpha > 0$ such that $\mu \in \alpha v^\circ$ we have $\mu(a) \le \mu(b) + \alpha$, that is $\varphi_a(\mu) \le \varphi_b(\mu) + \psi_v(\mu)$, hence $\Phi(a) \le \Phi(b) + \psi_v$. Conversely, if $a \not\le b + v$, then following our assumption that Υ supports the separation property, there is $\mu \in v^\circ \cap \Upsilon$ such that $\mu(a) > \mu(b) + 1$. The former implies $\psi_v(\mu) \le 1$, hence $\varphi_a(\mu) > \varphi_b(\mu) + \psi_v(\mu)$ and therefore $\Phi(a) \not\le \Phi(b) + \psi_v$. We infer in particular that the functions $\Phi(a) \in \mathcal{F}(\Upsilon, \overline{\mathbb{R}})$ are bounded below relative to the neighborhoods in $\widehat{\mathcal{V}}$. Indeed, given $a \in \mathcal{P}$ and $v \in \mathcal{V}$ there is $\lambda \geq 0$ such that $0 \leq a + \lambda v$, hence $0 \leq \Phi(a) + \lambda \hat{v}$. Therefore the element $\Phi(a)$ is contained in $\mathcal{F}_{\widehat{\mathcal{V}}_b}(\Upsilon, \overline{\mathbb{R}})$ as claimed.

Finally we establish that the linear operator $\Phi : \mathcal{P} \to \mathcal{F}_{\widehat{\mathcal{V}}_b}(\Upsilon, \overline{\mathbb{R}})$ is an embedding in the sense of 2.2 of the locally convex complete lattice cone $(\mathcal{P}, \mathcal{V})$ into $\big(\mathcal{F}_{\widehat{\mathcal{V}}_b}(\Upsilon, \overline{\mathbb{R}}), \widehat{\mathcal{V}}\big)$. Indeed, we set $\Phi(v) = \hat{v}$ for $v \in \mathcal{V}$ towards the extension

$$\Phi : (\mathcal{P} \cup \mathcal{V}) :\to \big(\mathcal{F}_{\widehat{\mathcal{V}}_b}(\Upsilon, \overline{\mathbb{R}}) \cup \widehat{\mathcal{V}}\big).$$

Then $\Phi(\mathcal{V}) = \widehat{\mathcal{V}}$, and by the above $a \leq b + v$ holds for $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$ if and only if $\Phi(a) \leq \Phi(b) + \Phi(v)$, as required in 2.2. Moreover, since the (weak pre-)order of the locally convex complete lattice cone $\mathcal{P}$ is antisymmetric, its symmetric topology is Hausdorff by Proposition 4.8. Lemma 2.3 therefore yields that the operator $\Phi : \mathcal{P} \to \mathcal{F}_{\widehat{\mathcal{V}}_b}(\Upsilon, \overline{\mathbb{R}})$ is one-to-one, as claimed. □

We shall demonstrate in 5.57 below that every locally convex cone $(\mathcal{P}, \mathcal{V})$ can be canonically embedded into a locally convex complete lattice cone for which the set of order continuous lattice homomorphisms in $\mathcal{P}^*$ supports the separation property.

5.38 Almost Order Convergent Nets. The concept of order convergence can in some cases be meaningfully extended to nets that are not necessarily bounded below. We shall say that a net $(a_i)_{i \in \mathcal{I}}$ in a locally convex complete lattice cone $(\mathcal{P}, \mathcal{V})$ is *almost order convergent towards* $a \in \mathcal{P}$ if for every $k \in \mathcal{I}$ the net $(a_i \vee a_k)_{i \in \mathcal{I}}$ is order convergent and if

$$\lim_{k \in \mathcal{I}} \Big(\lim_{i \in \mathcal{I}} (a_i \vee a_k) \Big) = a.$$

The net $(a_i \vee a_k)_{i \in \mathcal{I}}$ is of course bounded below for any choice of $k \in \mathcal{I}$. Indeed, given $v \in \mathcal{V}$ there is $\lambda \geq 0$ such that $0 \leq a_k + \lambda v \leq (a_i \vee a_k) + \lambda v$ for all $i \in \mathcal{I}$.

Lemma 5.39. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex complete lattice cone. A bounded below net $(a_i)_{i \in \mathcal{I}}$ in $\mathcal{P}$ is order convergent if and only if it is almost order convergent with the same limit.*

Proof. Let $(a_i)_{i \in \mathcal{I}}$ be a bounded below net in $\mathcal{P}$. If $(a_i)_{i \in \mathcal{I}}$ is order convergent and $\lim_{i \in \mathcal{I}} a_i = a$, then

$$\lim_{i \in \mathcal{I}} (a_i \vee a_k) = \Big(\lim_{i \in \mathcal{I}} a_i \Big) \vee a_k = a \vee a_k$$

for all $k \in \mathcal{I}$ by Proposition 5.25(a). Therefore

$$\lim_{k \in \mathcal{I}} \Big(\lim_{i \in \mathcal{I}} (a_i \vee a_k) \Big) = \lim_{k \in \mathcal{I}} (a \vee a_k) = a \vee \Big(\lim_{k \in \mathcal{I}} a_k \Big) = a$$

again by 5.25(a), and we infer that $(a_i)_{i \in \mathcal{I}}$ is almost order convergent towards a. On the other hand, for every $b \in \mathcal{P}$ we have

$$\begin{aligned}\varliminf_{i\in\mathcal{I}}(a_i \vee b) &= \sup_{i\in\mathcal{I}}\Big(\inf_{j\geq i}(a_j \vee b)\Big) = \sup_{i\in\mathcal{I}}\Big(\big(\inf_{j\geq i} a_j\big) \vee b\Big)\\ &= \Big(\sup_{i\in\mathcal{I}}\big(\inf_{j\geq i} a_j\big)\Big) \vee b = \Big(\varliminf_{i\in\mathcal{I}} a_i\Big) \vee b\end{aligned}$$

by Proposition 5.15 and Lemma 5.5. Similarly one realizes that

$$\begin{aligned}\varlimsup_{i\in\mathcal{I}}(a_i \vee a_k) &= \inf_{i\in\mathcal{I}}\Big(\sup_{j\geq i}(a_j \vee b)\Big) = \inf_{i\in\mathcal{I}}\Big(\big(\sup_{j\geq i} a_j\big) \vee b\Big)\\ &= \Big(\inf_{i\in\mathcal{I}}\big(\sup_{j\geq i} a_j\big)\Big) \vee b = \Big(\varlimsup_{i\in\mathcal{I}} a_i\Big) \vee b\end{aligned}$$

If the net $(a_i)_{i\in\mathcal{I}}$ is almost order convergent towards $a \in \mathcal{P}$, this yields

$$\lim_{i\in\mathcal{I}}(a_i \vee a_k) = \Big(\varliminf_{i\in\mathcal{I}} a_i\Big) \vee a_k = \Big(\varlimsup_{i\in\mathcal{I}} a_i\Big) \vee a_k$$

for all $k \in \mathcal{I}$. Thus, again using the above

$$\begin{aligned}a &= \lim_{k\in\mathcal{I}}\Big(\lim_{i\in\mathcal{I}}(a_i \vee a_k)\Big)\\ &= \varliminf_{k\in\mathcal{I}}\Big(\Big(\varliminf_{i\in\mathcal{I}} a_i\Big) \vee a_k\Big)\\ &= \Big(\varliminf_{i\in\mathcal{I}} a_i\Big) \vee \Big(\varliminf_{k\in\mathcal{I}} a_k\Big) = \Big(\varliminf_{i\in\mathcal{I}} a_i\Big),\end{aligned}$$

as well as

$$\begin{aligned}a &= \lim_{k\in\mathcal{I}}\Big(\lim_{i\in\mathcal{I}}(a_i \vee a_k)\Big)\\ &= \varlimsup_{k\in\mathcal{I}}\Big(\Big(\varlimsup_{i\in\mathcal{I}} a_i\Big) \vee a_k\Big)\\ &= \Big(\varlimsup_{i\in\mathcal{I}} a_i\Big) \vee \Big(\varlimsup_{k\in\mathcal{I}} a_k\Big) = \Big(\varlimsup_{i\in\mathcal{I}} a_i\Big).\end{aligned}$$

This yields $\lim_{i\in\mathcal{I}} a_i = a.$ □

Examples 5.40. Let $\mathcal{P}$ be the cone of all bounded below $\overline{\mathbb{R}}$-valued functions on $[0,+\infty)$, endowed with the pointwise operations and order, and the positive constant functions $v > 0$ as its neighborhood system $\mathcal{V}$ $\big($see Example 1.4(e)$\big)$. $(\mathcal{P},\mathcal{V})$ is a locally convex complete lattice cone, and order convergence in $\mathcal{P}$ implies pointwise convergence on $[0,+\infty)$ for the functions involved. Pointwise convergence, on the other hand does not require that a net in $\mathcal{P}$ is bounded below and therefore does not always imply order convergence. Let us illustrate this in a simple example: For $n \in \mathbb{N}$ let $f_n \in \mathcal{P}$ such that $f_n(x) = -n$ for $0 < x \leq 1/n$, and $f_n(x) = 0$ else. The sequence $(f_n)_{n\in\mathbb{N}}$ converges pointwise to $0 \in \mathcal{P}$, but it is not bounded below in $\mathcal{P}$ and therefore not order convergent. However, for every $m \in \mathbb{N}$ the sequence

$(f_n \vee f_m)_{n\in\mathbb{N}}$ is bounded below and converges pointwise, hence in order towards $0 \in \mathcal{P}$. We infer that $(f_n)_{n\in\mathbb{N}}$ is almost order convergent towards $0 \in \mathcal{P}$. In fact, it can be easily verified that pointwise convergence coincides with almost order convergence in this example (see Proposition 5.51 below).

We proceed probing different patterns of convergence in a locally convex complete lattice cone $(\mathcal{P},\mathcal{V})$. For a net $(a_i)_{i\in\mathcal{I}}$ in $\mathcal{P}$, convergence with respect to the symmetric relative topology of $\mathcal{P}$ towards $a \in \mathcal{P}$ means that for every $v \in \mathcal{V}$ and $\varepsilon > 0$ there is $i_0 \in I$ such that $a_i \in v^s_\varepsilon(a)$ for all $i \geq i_0$. $(a_i)_{i\in\mathcal{I}}$ is a *Cauchy net* if for every $v \in \mathcal{V}$ and $\varepsilon > 0$ there is $i_0 \in I$ such that $a_i \in v_\varepsilon(a_k)$ for all $i,k \geq i_0$. Obviously, convergence implies that $(a_i)_{i\in\mathcal{I}}$ is a Cauchy net. The converse, that is topological completeness holds also true:

Proposition 5.41. *Every locally convex complete lattice cone is complete with respect to the symmetric relative topology.*

Proof. Suppose that $(a_i)_{i\in\mathcal{I}}$ is a Cauchy net in $\mathcal{P}$. We shall first demonstrate that $(a_i)_{i\in\mathcal{I}}$ is order convergent. Let $v \in \mathcal{V}$ and $0 < \varepsilon \leq 1$. There is $i_0 \in \mathcal{I}$ such that $a_i \in v_\varepsilon(a_k)$ for all $i,k \geq i_0$. Choose $\lambda \geq 0$ such that $0 \leq a_{i_0} + \lambda v$. Following Lemma 4.1(b) and (c) this implies

$$a_i \leq (1+\varepsilon)a_{i_0} + \varepsilon(1+\lambda)v \qquad \text{and} \qquad a_{i_0} \leq (1+\varepsilon)a_i + \varepsilon(2+\lambda)v$$

for all $i \geq i_0$. This shows in particular that $(a_i)_{i\in\mathcal{I}}$ is bounded below and also that

$$a_i \leq (1+\varepsilon)^2 a_k + 3\varepsilon(2+\lambda)v$$

for all $i,k \geq i_0$. This shows

$$\overline{\lim_{i\in\mathcal{I}}}\, a_i \leq (1+\varepsilon)^2 \, \underline{\lim_{k\in I}}\, a_k + 3\varepsilon(2+\lambda)v.$$

As this holds for all $v \in \mathcal{V}$ and $0 < \varepsilon \leq 1$, and as $\mathcal{P}$ carries the weak preorder which is supposed to be antisymmetric, we infer that $\overline{\lim}_{i\in\mathcal{I}}\, a_i = \underline{\lim}_{k\in I}\, a_k$, hence order convergence towards an element $a \in \mathcal{P}$. Moreover, the above shows that

$$a_i \leq (1+\varepsilon^2)a + 3\varepsilon(2+\lambda)v \qquad \text{and} \qquad a \leq (1+\varepsilon^2)a_i + 3\varepsilon(2+\lambda)v$$

holds for all $i \geq i_0$. Thus the net $(a_i)_{i\in\mathcal{I}}$ converges to a in the symmetric relative topology as well. □

In fact, we just verified that every Cauchy net, hence every convergent net in the symmetric relative topology of $(\mathcal{P},\mathcal{V})$ is indeed order convergent with the same limit. We shall formulate this as a separate proposition:

Proposition 5.42. *Let $(\mathcal{P},\mathcal{V})$ be a locally convex complete lattice cone. Convergence of a net $(a_i)_{i\in\mathcal{I}}$ in $\mathcal{P}$ towards $a \in \mathcal{P}$ in the symmetric relative topology implies order convergence towards a.*

While convergence in the symmetric relative topology implies order convergence, the converse is not necessarily true, as a simple example can show: In the locally convex complete lattice cone $\overline{\mathbb{R}}$ order convergence means convergence in the usual (one-point compactification) topology of $\overline{\mathbb{R}}$ which for the element $+\infty$ does not coincide with the symmetric relative topology of $\overline{\mathbb{R}}$. The sequence $(n)_{n\in\mathbb{N}}$, for example, is order convergent towards $+\infty \in \overline{\mathbb{R}}$, but does not converge in the symmetric relative topology, as $+\infty$ is an isolated point in this topology.

5.43 Order Topology. While order convergence in a locally convex complete lattice cone $(\mathcal{P}, \mathcal{V})$ does not necessarily correspond to a topology on $\mathcal{P}$ in the sense that order and topological convergence for nets coincide (see 1.1.9 in [132]), there is a finest topology $\mathcal{O}(\mathcal{P})$ on $\mathcal{P}$ with the following properties:

(OT1) *Every very element of $\mathcal{P}$ admits a basis of both convex and order convex neighborhoods. The neighborhoods in the basis for $0 \in \mathcal{P}$ are also balanced.*

(OT2) *The mappings $(a, b) \mapsto a + b$, $(a, b) \mapsto a \vee b$ and $(a, b) \mapsto a \wedge b$ from $\mathcal{P}^2$ into $\mathcal{P}$ are continuous.*

(OT3) *The mapping $(\alpha, a) \mapsto \alpha a : [0, +\infty) \times \mathcal{P} \to \mathcal{P}$ is continuous at all points $(\alpha, a) \in [0, +\infty) \times \mathcal{P}$ such that either $\alpha > 0$ or $a \in \mathcal{P}$ is bounded.*

(OT4) *All almost order convergent nets in $\mathcal{P}$ are topologically convergent with the same limit.*

Indeed, let $\mathfrak{T}$ be the family of all topologies on $\mathcal{P}$ with these properties. These topologies need not be Hausdorff. Therefore $\mathfrak{T}$ is not empty as it contains the discrete topology. Let $\mathcal{O}(\mathcal{P})$ be the supremum of this family in the lattice of topologies on $\mathcal{P}$. A neighborhood basis in $\mathcal{O}(\mathcal{P})$ for a point $a \in \mathcal{P}$ is generated by the intersections of finitely many neighborhoods for a taken from topologies in $\mathfrak{T}$. This shows that $\mathcal{O}(\mathcal{P})$ again satisfies (OT1) to (OT4), hence is the finest topology with these properties. We shall call $\mathcal{O}(\mathcal{P})$ the *(strong) order topology* on $\mathcal{P}$. Note that $\mathcal{O}(\mathcal{P})$ is not necessarily a locally convex cone topology. For $\mathcal{P} = \overline{\mathbb{R}}$, for example, the order topology is the usual topology of $\overline{\mathbb{R}}$ where $+\infty$ is not an isolated point.

In Proposition 4.2 we verified that the symmetric relative topology of $\mathcal{P}$ satisfies (OT1), (OT2) and (OT3), however it does not meet (OT4) in general.

Proposition 5.44. *Let $(\mathcal{P}, \mathcal{V})$ be a locally convex complete lattice cone. The order topology $\mathcal{O}(\mathcal{P})$ on $\mathcal{P}$ is coarser than the symmetric relative topology.*

Proof. We observed in Proposition 5.42 that convergence for a net in the symmetric relative topology implies order convergence, hence convergence in $\mathcal{O}(\mathcal{P})$. Since the closure in any topology of a given subset A of $\mathcal{P}$ can be described as the set of all limit points of convergent nets in this subset, Proposition 5.42 implies that the closure of A with respect to the symmetric relative topology is contained in the closure of A with respect to $\mathcal{O}(\mathcal{P})$. We infer that $\mathcal{O}(\mathcal{P})$ is generally coarser than the symmetric relative topology. □

Lemma 5.45. *Let $(\mathcal{P},\mathcal{V})$ be a locally convex complete lattice cone and let $\mathcal{P}_0$ be the subcone of all invertible elements of $\mathcal{P}$. The mapping $(\alpha,a)\mapsto \alpha a : \mathbb{R}\times\mathcal{P}_0\to\mathcal{P}_0$ is continuous with respect to the order topology $\mathcal{O}(\mathcal{P})$.*

Proof. We shall make this argument in several short steps: First suppose that $a_i\to 0$ for $a_i\in\mathcal{P}_0$ in any topology satisfying (OT1) to (OT4). Given a neighborhood U in the basis for 0 there is i_0 such that $a_i\in U$ for all $i\geq i_0$. This implies $-a_i\in U$ as well since U is supposed to be balanced by (OT1). Thus $(-a_i)\to 0$. Next suppose that $a_i\to a$ for $a_i,a\in\mathcal{P}_0$. Then $\big(a_i+(-a)\big)\to 0$ by (OT2), hence $\big((-a_i)+a\big)\to 0$ by the preceding step, and $(-a_i)\to(-a)$ by (OT2). In a third step, suppose that $\alpha_i\to\alpha\in\mathbb{R}$ for $0\leq\alpha_i\in\mathbb{R}$ and $a_i\to a$ for $a_i,a\in\mathcal{P}_0$. Then $\alpha_i a_i\to\alpha a$ by (OT3) since every invertible element is bounded. Now in the fourth and final step of our argument, let $\alpha_i\to\alpha$ in $\mathbb{R}$ and $a_i\to a$ for $a_i,a\in\mathcal{P}_0$. Let $\beta_i=\alpha_i\vee 0$ and $\gamma_i=-(\alpha_i\wedge 0)$. Then $\beta_i,\gamma_i\geq 0$ and $\alpha_i=\beta_i-\gamma_i$. We have $\beta_i a_i\to\beta a$ and $\gamma_i(-a_i)\to\gamma(-a)$, where $\beta=\alpha\vee 0$ and $\gamma=-(\alpha\wedge 0)$, by the second and third steps of our argument. Thus

$$\alpha_i a_i=\beta_i a_i+\gamma_i(-a_i)\ \to\ \beta a+\gamma(-a)=\alpha a,$$

again by (OT2), as claimed. □

Proposition 5.46. *Let $(\mathcal{P},\mathcal{V})$ and $(\mathcal{Q},\mathcal{W})$ be a locally convex complete lattice cones. An order continuous lattice homomorphism $T:\mathcal{P}\to\mathcal{Q}$ is also continuous with respect to the respective order topologies $\mathcal{O}(\mathcal{P})$ and $\mathcal{O}(\mathcal{Q})$.*

Proof. Let $T:\mathcal{P}\to\mathcal{Q}$ be an order continuous lattice homomorphism, consider the order topology $\mathcal{O}(\mathcal{Q})$ on $\mathcal{Q}$ and let τ be the initial topology induced on $\mathcal{P}$ by T, that is τ is the coarsest topology on $\mathcal{P}$ for which the mapping $T:\mathcal{P}\to\mathcal{Q}$ is continuous. The sets in τ then are just the inverse images under T of the sets in $\mathcal{O}(\mathcal{Q})$. It is straightforward to verify that τ satisfies the requirements (OT1) to (OT4): For $a\in\mathcal{P}$ the element $T(a)\in\mathcal{Q}$ admits a basis of neighborhoods in $\mathcal{O}(\mathcal{Q})$ satisfying (OT1). Their inverse images under T have the same properties and form a neighborhood basis for a in τ. Next suppose that $a_i\to a$ and $b_i\to b$ in τ. Then $T(a_i)\to T(a)$ and $T(b_i)\to T(b)$ in $\mathcal{O}(\mathcal{Q})$. Thus $T(a_i)+T(b_i)\to T(a)+T(b)=T(a+b)$ since $\mathcal{O}(\mathcal{Q})$ satisfies (OT2). Because every neighborhood of $a+b$ in τ is the inverse image under T of a neighborhood of $T(a+b)$, this shows that $(a_i+b_i)\to(a+b)$ in τ. Similarly one verifies the continuity of the mappings $(a,b)\mapsto a\vee b$, $(a,b)\mapsto a\wedge b$ and $(\alpha,a)\mapsto\alpha a$ with respect to τ. For (OT4) let $(a_i)_{i\in\mathcal{I}}$ be an almost order convergent net in $\mathcal{P}$ with limit $a\in\mathcal{P}$. Then

$$T(a)=T\left(\lim_{k\in\mathcal{I}}\left(\lim_{i\in\mathcal{I}}(a_i\vee a_k)\right)\right)=\lim_{k\in\mathcal{I}}\left(\lim_{i\in\mathcal{I}}\Big(T(a_i)\vee T(a_k)\Big)\right)$$

since T is an order continuous lattice homomorphism. The net $\big(T(a_i)\big)_{i\in\mathcal{I}}$ is therefore almost order convergent with limit $T(a)$ in $\mathcal{Q}$. As $\mathcal{O}(\mathcal{Q})$ satisfies (OT4), this implies $T(a_i) \to T(a)$ in $\mathcal{O}(\mathcal{Q})$, and therefore $a_i \to a$ in τ, since the neighborhoods of a in τ are inverse images under T of neighborhoods of $T(a)$ in $\mathcal{O}(\mathcal{Q})$. Summarizing, we have verified that the topology τ on $\mathcal{P}$ satisfies conditions (OT1) to (OT4) and is therefore coarser then the order topology $\mathcal{O}(\mathcal{P})$. Hence the operator $T : \mathcal{P} \to \mathcal{Q}$ is also continuous if we endow $\mathcal{P}$ with $\mathcal{O}(\mathcal{P})$. □

Proposition 5.47. *Let $(\mathcal{P},\mathcal{V})$ be a locally convex complete lattice cone and let $\mathcal{N}$ be a subcone of $\mathcal{P}$. Then the closure $\overline{\mathcal{N}}$ of $\mathcal{N}$ with respect to $\mathcal{O}(\mathcal{P})$ is again a subcone of $\mathcal{P}$. If $\mathcal{N}$ is a lattice subcone of $\mathcal{P}$, then $\overline{\mathcal{N}}$ is a complete lattice subcone of $\mathcal{P}$.*

Proof. The first part of the claim follows directly from (OT2) and (OT3). For the second part suppose that $\mathcal{N}$ is a lattice subcone of $\mathcal{P}$ and let $a, b \in \mathcal{N}$. There are nets $(a_i)_{i\in\mathcal{I}}$ and $(b_j)_{j\in\mathcal{J}}$ in $\mathcal{N}$ converging in the order topology towards a and b, respectively. Then the net $(a_i \vee b_j)_{(i,j)\in\mathcal{I}\times\mathcal{J}}$ in $\mathcal{N}$ converges to $a \vee b$ by (OT3). Thus $a \vee b \in \overline{\mathcal{N}}$. Similarly one shows that $a \wedge b \in \overline{\mathcal{N}}$, hence $\overline{\mathcal{N}}$ is also a lattice subcone of $\mathcal{P}$. Now let A be a non-empty subset of $\overline{\mathcal{N}}$. For every finite subset $i = \{a_1, \ldots, a_n\}$ of A set $a_i = a_1 \vee \ldots \vee a_n \in \overline{\mathcal{N}}$. Then $\sup A = \lim_{i\in\mathcal{I}} a_i$, where $\mathcal{I}$ is the collection of all finite subsets of A, ordered by set inclusion. This shows $\sup A \in \overline{\mathcal{N}}$ by (OT4). Similarly one shows that $\inf B \in \overline{\mathcal{N}}$ whenever B is a bounded below subset of $\overline{\mathcal{N}}$. Thus $\overline{\mathcal{N}}$ is indeed a complete lattice subcone of $\mathcal{P}$. □

Proposition 5.48. *Let $(\mathcal{P},\mathcal{V})$ be a locally convex complete lattice cone and let $\mathcal{N}$ be a complete lattice subcone of $\mathcal{P}$. The restriction of $\mathcal{O}(\mathcal{P})$ to $\mathcal{N}$ is coarser than the order topology $\mathcal{O}(\mathcal{N})$ of $\mathcal{N}$.*

Proof. This follows from the easily verifiable fact that the restriction of $\mathcal{O}(\mathcal{P})$ to the complete lattice subcone $\mathcal{N}$ satisfies the requirements (OT1) to (OT4). □

5.49 Weak Order Convergence. *Weak order convergence* for a net $(a_i)_{i\in\mathcal{I}}$ in a locally convex complete lattice cone $(\mathcal{P},\mathcal{V})$ means that $\big(\mu(a_i)\big)_{i\in\mathcal{I}}$ converges towards $\mu(a)$ in $\overline{\mathbb{R}}$ (with respect to order convergence) for every order continuous lattice homomorphism $\mu \in \mathcal{P}^*$. This notion of convergence results from the *weak order topology* $o(\mathcal{P},\mathcal{P}^*)$ on $\mathcal{P}$ which is generated by the (both convex and order convex) neighborhoods

$$\mathcal{V}^o_\Upsilon(a) = \left\{ b \in \mathcal{P} \;\middle|\; \begin{array}{rll} |\mu_i(b) - \mu_i(a)| \le 1\,, & \text{if} & \mu_i(a) < +\infty \\ \mu_i(b) \ge 1\,, & \text{if} & \mu_i(a) = +\infty \end{array} \right\},$$

for an element $a \in \mathcal{P}$, corresponding to a finite set $\Upsilon = \{\mu_1, \ldots, \mu_n\}$ of order continuous lattice homomorphisms in $\mathcal{P}^*$. Like the order topology $\mathcal{O}(\mathcal{P})$, this is in general not a locally convex cone topology.

Proposition 5.50. *Let $(\mathcal{P},\mathcal{V})$ be a locally convex complete lattice cone. The weak order topology $o(\mathcal{P},\mathcal{P}^*)$ on $\mathcal{P}$ is coarser than the order topology $\mathcal{O}(\mathcal{P})$ and also coarser than the weak topology $\sigma(\mathcal{P},\mathcal{P}^*)$.*

Proof. Requirements (OT1) to (OT4) from 5.40 are readily checked for the weak order topology: (OT1) and the first part of (OT2) are self evident. The second part of (OT2) follows from the easily verified fact that $\mu(a'\vee b')\leq\mu(a\vee b)+1$ holds whenever $\mu(a')\leq\mu(a)+1$ and $\mu(b')\leq\mu(b)+1$ for elements $a,a',b,b'\in\mathcal{P}$ and an order continuous lattice homomorphism $\mu\in\mathcal{P}^*$. Similarly one argues for the third part of (OT2). For (OT4) let $(a_i)_{i\in\mathcal{I}}$ be an almost order convergent net in $\mathcal{P}$ with limit $a\in\mathcal{P}$. Then

$$\mu(a)=\mu\left(\lim_{k\in\mathcal{I}}\left(\lim_{i\in\mathcal{I}}(a_i\vee a_k)\right)\right)=\lim_{k\in\mathcal{I}}\left(\lim_{i\in\mathcal{I}}\Big(\mu(a_i)\vee\mu(a_k)\Big)\right)$$

for every order continuous lattice homomorphism $\mu\in\mathcal{P}^*$. The limit on the right-hand side is taken with respect to the usual (that is the order) topology of $\overline{\mathbb{R}}$. The net $(a_i)_{i\in\mathcal{I}}$ is therefore also convergent with respect to the weak order topology. We infer that $o(\mathcal{P},\mathcal{P}^*)$ is generally coarser than the order topology $\mathcal{O}(\mathcal{P})$. The second statement of Proposition 5.50 follows immediately from a comparison of the respective neighborhoods in 4.6 and in 5.49: For $a\in\mathcal{P}$ and a finite set $\Upsilon=\{\mu_1,\ldots,\mu_n\}$ of order continuous lattice homomorphisms in $\mathcal{P}^*$ we have $\mathcal{V}^s_\Upsilon(a)\subset\mathcal{V}^o_\Upsilon(a)$. Thus $\sigma(\mathcal{P},\mathcal{P}^*)$ is indeed finer than $o(\mathcal{P},\mathcal{P}^*)$. $\square$

Proposition 5.51. *Let $(\mathcal{P},\mathcal{V})$ be a locally convex complete lattice cone, and suppose that the order continuous lattice homomorphisms support the separation property for $\mathcal{P}$. Then the order and the weak order topologies coincide on $\mathcal{P}$ and are Hausdorff. A net in $\mathcal{P}$ is convergent in the (weak) order topology if and only if it is almost order convergent.*

Proof. Let $(\mathcal{P},\mathcal{V})$ be a locally convex complete lattice cone such that the order continuous lattice homomorphisms support the separation property for $\mathcal{P}$. Let us fist argue that the weak order topology is Hausdorff. Indeed, for distinct elements $a,b\in\mathcal{P}$ we have either $a\not\leq b$ or $b\not\leq a$, since the order of $\mathcal{P}$ is supposed to be antisymmetric. Thus $a\not\leq b+v$ or $b\not\leq a+v$ for some $v\in\mathcal{V}$ by Lemma 3.2. Then there exists an order continuous linear functional $\mu\in v^\circ$ such that $\mu(a)>\mu(b)+1$ or $\mu(b)>\mu(a)+1$, respectively. For suitable $\varepsilon,\delta>0$ then the neighborhoods $\mathcal{V}^o_{\{\varepsilon\mu\}}(a)$ and $\mathcal{V}^o_{\{\delta\mu\}}(b)$ are seen to be disjoint. Next we shall verify the last statement of our claim: Let $(a_i)_{i\in\mathcal{I}}$ be a net in $\mathcal{P}$. If $(a_i)_{i\in\mathcal{I}}$ is almost order convergent, then it is convergent with the same limit in $\mathcal{O}(\mathcal{P})$ by (OT4), hence weakly order convergent since the weak order topology is coarser than $\mathcal{O}(\mathcal{P})$. For the converse suppose that $(a_i)_{i\in\mathcal{I}}$ is weakly order convergent toward $a\in\mathcal{P}$. Then for every $b\in\mathcal{P}$ and every order continuous lattice homomorphism $\mu\in\mathcal{P}^*$ we have

$$\begin{aligned}\mu\Big(\overline{\lim_{i\in\mathcal{I}}}(a_i\vee b)\Big) &= \overline{\lim_{i\in\mathcal{I}}}\Big(\mu(a_i)\vee\mu(b)\Big)\\ &= \Big(\overline{\lim_{i\in\mathcal{I}}}\,\mu(a_i)\Big)\vee\mu(b)\\ &= \mu(a)\vee\mu(b)\\ &= \mu(a\vee b).\end{aligned}$$

This shows $\overline{\lim}_{i\in\mathcal{I}}(a_i\vee b)=(a\vee b)$ since the weak order topology was seen to be Hausdorff. Similarly one verifies that $\underline{\lim}_{i\in\mathcal{I}}(a_i\vee b)=(a\vee b)$, hence

$$\lim_{i\in\mathcal{I}}(a_i\vee b)=(a\vee b).$$

For $b=a_k$ in particular, this renders $\lim_{i\in\mathcal{I}}(a_i\vee a_k)=(a\vee a_k)$ for every $k\in\mathcal{I}$. Repeating this argument with $b=a$ and a_k in place of a_i then yields

$$\lim_{k\in\mathcal{I}}(a\vee a_k)=(a\vee a)=a.$$

We thus verified that the net $(a_i)_{i\in\mathcal{I}}$ is almost order convergent towards $a\in\mathcal{P}$. This completes our argument for convergent nets and also implies the first part of our claim. Indeed, since the closed sets in any given topology can be described in terms of limits of convergence nets alone, having the same notion of convergence for nets means that the topologies involved coincide. □

Proposition 5.52. *Let $(\mathcal{P},\mathcal{V})$ be a locally convex complete lattice cone such that the order continuous lattice homomorphisms support the separation property for $\mathcal{P}$, and let $\mathcal{N}$ be a complete lattice subcone of $\mathcal{P}$. Then $\mathcal{N}$ is closed in $\mathcal{O}(\mathcal{P})$. The order topology $\mathcal{O}(\mathcal{N})$ of $\mathcal{N}$ coincides with the restriction of $\mathcal{O}(\mathcal{P})$ to $\mathcal{N}$.*

Proof. Let $(\mathcal{P},\mathcal{V})$ be a locally convex complete lattice cone and let $\mathcal{N}$ be a complete lattice subcone of $\mathcal{P}$. Because the restriction to $\mathcal{N}$ of an order continuous lattice homomorphism on $\mathcal{P}$ is an order continuous lattice homomorphism on $\mathcal{N}$, under the assumptions of the Proposition these functionals support the separation property for both $\mathcal{P}$ and $\mathcal{N}$. The conclusions of Proposition 5.51 therefore apply to both of these cones. Let $(a_i)_{i\in\mathcal{I}}$ be a net in $\mathcal{N}$. We observe the following: If $(a_i)_{i\in\mathcal{I}}$ is almost order convergent as a net in $\mathcal{N}$ with limit $a\in\mathcal{N}$, then it is also almost order convergent as a net in $\mathcal{P}$ with the same limit. Conversely, if $(a_i)_{i\in\mathcal{I}}$ is almost order convergent in as a net in $\mathcal{P}$ with limit $a\in\mathcal{P}$, then $a\in\mathcal{N}$, and $(a_i)_{i\in\mathcal{I}}$ is also almost order convergent as a net in $\mathcal{N}$ with the same limit. This is an immediate consequence of the fact that the subcone $\mathcal{N}$ contains the infima and suprema of its sets as elements, hence the limits of its order convergent nets. Now both of our claims follow, since the convergent nets in the order topologies of $\mathcal{P}$ and of $\mathcal{N}$ coincide with the almost order convergent nets in $\mathcal{P}$ and $\mathcal{N}$, respectively. □

In Proposition 5.37 we established that every locally convex lattice cone can be represented as a as a cone of $\overline{\mathbb{R}}$-valued functions on some set X. The preceding considerations now allow us to identify the weak and strong order topologies as the topology of pointwise convergence in this representation.

Proposition 5.53. *Let $(\mathcal{P},\widehat{\mathcal{V}})$ be a complete lattice subcone of $\big(\mathcal{F}_{\widehat{\mathcal{V}}_b}(X,\overline{\mathbb{R}}),\widehat{\mathcal{V}}\big)$ for some set X and a neighborhood system $\widehat{\mathcal{V}}$ consisting of $\overline{\mathbb{R}}$-valued functions such that $\hat{v}(x)<+\infty$ for every $x\in X$ with some $\hat{v}\in\widehat{\mathcal{V}}$. Then the order topology, the weak order topology and the topology of pointwise convergence on X (with respect to the usual topology of $\overline{\mathbb{R}}$) all coincide on $\mathcal{P}$ and are Hausdorff.*

Proof. Under the assumptions of the Proposition, the order continuous lattice homomorphisms support the separation property for $\big(\mathcal{F}_{\widehat{\mathcal{V}}_b}(X,\overline{\mathbb{R}}),\widehat{\mathcal{V}}\big)$ (see 5.33(f)), hence also for the complete lattice subcone $(\mathcal{P},\widehat{\mathcal{V}})$. The coincidence of the order and the weak order topology was established in Proposition 5.51. Since for every $x\in X$ the point evaluation $f\mapsto f(x)$ is an order continuous lattice homomorphism on $\mathcal{P}$ (see 5.31(d)), week order convergence for a net in $\mathcal{P}$ implies pointwise convergence on X. A pointwise convergent net, on the other hand is seen to be almost order convergent and therefore convergent in the order topology. The three notions of convergence, hence the respective topologies therefore coincide. □

5.54 Extension of Linear Operators. A short inspection of the Hahn-Banach type extension results for linear functionals in [172] (see also Section 2) shows that they are still valid if the range $\overline{\mathbb{R}}$ for the functionals is replaced by some locally convex cone $(\mathcal{Q},\mathcal{W})$, provided that

(i) $(\mathcal{Q},\mathcal{W})$ is full and a complete lattice cone,
(ii) all elements of $\mathcal{Q}$, with the exception of the element $+\infty=\sup\mathcal{Q}$, are invertible,
(iii) the neighborhood system $\mathcal{W}$ consists of all (strictly) positive multiples of a single neighborhood $w\in\mathcal{W}$.

Requirement (ii) means of course that $\mathcal{Q}$ is a Dedekind complete Riesz space with an adjoint maximal element $+\infty$. Results about the extension of monotone linear operators between vector spaces and Dedekind complete Riesz spaces are due to Kantorovič ([96] and [98]) and can for example be found in Section 1.5 of [132]. Without furnishing the details of this, we reformulate Corollary 4.1 in [172] (see also Corollary 2.7 before).

Theorem 5.55. *Let $(\mathcal{N},\mathcal{V})$ be a subcone of the locally convex cone $(\mathcal{P},\mathcal{V})$. Suppose that $(\mathcal{Q},\mathcal{W})$ is a full locally convex complete lattice cone, that all elements of $\mathcal{Q}$ other than $+\infty$ are invertible, and that $\mathcal{W}=\{\alpha w\mid\alpha>0\}$ for some $w\in\mathcal{W}$. Then every continuous linear operator $T:\mathcal{N}\to\mathcal{Q}$ can be extended to a continuous linear operator $\overline{T}:\mathcal{P}\to\mathcal{Q}$.*

Unfortunately, a similar result is not generally available if the locally convex complete lattice cone $(\mathcal{Q},\mathcal{W})$ does not meet the stringent additional requirements of Theorem 5.55. However, we have the following:

Theorem 5.56. *Let $\mathcal{N}$ be a subcone of the locally convex cone $(\mathcal{P},\mathcal{V})$ and let $(\mathcal{Q},\mathcal{W})$ be a locally convex complete lattice cone. Every continuous linear operator $T:\mathcal{N}\to\mathcal{Q}$ can be uniquely extended to $\overline{\mathcal{N}}$, the closure of $\mathcal{N}$ in $\mathcal{P}$ with respect to the symmetric relative topology.*

Proof. Let $T:\mathcal{N}\to\mathcal{Q}$ be a continuous linear operator and let $a\in\overline{\mathcal{N}}$. There is a net $(a_i)_{i\in\mathcal{I}}$ in $\mathcal{N}$ converging to a in the symmetric relative topology. Given $w\in\mathcal{W}$ and $\varepsilon>0$ there is $v\in\mathcal{V}$ such that $T(b)\leq T(c)+w$ whenever $b\leq c+v$ for $b,c\in\mathcal{N}$. Because $(a_i)_{i\in\mathcal{I}}$ is a Cauchy net in $\mathcal{N}$, there is $i_0\in\mathcal{I}$ such that $a_i\in v_\varepsilon(a_k)$ for all $i,k\geq i_0$. This implies $T(a_i)\in w_\varepsilon\big(T(a_k)\big)$ for all $i,k\geq i_0$, hence $\big(T(a_i)\big)_{i\in\mathcal{I}}$ is a Cauchy net in $\mathcal{Q}$ as well. Proposition 5.41 shows that this net converges in $\mathcal{Q}$. Moreover, if $(b_j)_{j\in\mathcal{J}}$ is a second net in $\mathcal{N}$ converging toward the same element a, given $w\in\mathcal{W}$ and $\varepsilon>0$ we choose $v\in\mathcal{V}$ as above and find $i_0\in\mathcal{I}$ and $j_0\in\mathcal{J}$ such that both $a_i\in v_\varepsilon(b_j)$ and $b_j\in v_\varepsilon(a_i)$, hence $T(a_i)\in w_\varepsilon\big(T(b_j)\big)$ and $T(b_j)\in v_\varepsilon\big(T(a_i)\big)$, for all $i\geq i_0$ and $j\geq j_0$. This shows that both nets $\big(T(a_i)\big)_{i\in\mathcal{I}}$ and $\big(T(b_j)\big)_{j\in\mathcal{J}}$ have the same limit in $\mathcal{Q}$ which we denote $\overline{T}(a)$. It is now straightforward to verify that this procedure results in a bounded linear extension $\overline{T}:\overline{\mathcal{N}}\to\mathcal{Q}$ of the operator T. Uniqueness of this extension is obvious. □

5.57 The Standard Lattice Completion of a Locally Convex Cone. We proceed to establish that every locally convex cone $(\mathcal{P},\mathcal{V})$ can be canonically embedded into a locally convex complete lattice cone. For this, we use a representation for $(\mathcal{P},\mathcal{V})$ as a cone of $\overline{\mathbb{R}}$-valued functions on its dual cone $\mathcal{P}^*$, in analogy to the construction that we employed in the proof of Proposition 5.37: With the element $a\in\mathcal{P}$ we associate the $\overline{\mathbb{R}}$-valued function φ_a on $\mathcal{P}^*$ such that

$$\varphi_a(\mu)=\mu(a)\qquad\text{for all}\quad \mu\in\mathcal{P}^*.$$

The mapping $\Phi:\mathcal{P}\to\mathcal{F}(\mathcal{P}^*,\overline{\mathbb{R}})$ such that $\Phi(a)=\varphi_a$ for all $a\in\mathcal{P}$ is linear and monotone. Corresponding to the neighborhoods $v\in\mathcal{V}$ we consider the $\overline{\mathbb{R}}$-valued functions $\Phi(v)=\psi_v$ on $\mathcal{P}^*$ such that

$$\psi_v(\mu)=\inf\{\alpha>0\mid\mu\in\alpha v^\circ\}$$

for all $\mu\in\mathcal{P}^*$ and denote $\widehat{\mathcal{V}}=\{\hat{v}\mid v\in\mathcal{V}\}$. The neighborhoods $\hat{v}\in\widehat{\mathcal{V}}$ are defined for $\mathcal{F}(\mathcal{P}^*,\overline{\mathbb{R}})$ by

$$f\leq g+\hat{v}\qquad\text{if}\qquad f(\mu)\leq g(\mu)+\psi_v(\mu)\quad\text{for all}\quad\mu\in\mathcal{P}^*$$

(see 1.4(e)) for functions $f, g \in \mathcal{F}(\mathcal{P}^*, \overline{\mathbb{R}})$ and $\hat{v} \in \widehat{\mathcal{V}}$. We have $\psi_{(\alpha v)} = \alpha \psi_v$ and $\psi_{(u+v)} \geq \psi_u + \psi_v$ for all $u, v \in \mathcal{V}$ and $\alpha > 0$ (see the proof of 5.37 for details). Thus $\big(\mathcal{F}_{\widehat{\mathcal{V}}_b}(\mathcal{P}^*, \overline{\mathbb{R}}), \widehat{\mathcal{V}}\big)$ is a locally convex cone in the sense of 1.4(e), and a complete lattice since $\psi_v(\mu) < +\infty$ holds for all $\mu \in \mathcal{P}^*$ with some $\hat{v} \in \widehat{\mathcal{V}}$. We claim that $\Phi(\mathcal{P}) \subset \mathcal{F}_{\widehat{\mathcal{V}}_b}(\mathcal{P}^*, \overline{\mathbb{R}})$ and that

$$a \preccurlyeq b + v \qquad \text{if and only if} \qquad \varphi_a \leq \varphi_b + \psi_v.$$

holds for $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$. Indeed, suppose that $a \preccurlyeq b + v$. Then for every $\mu \in \mathcal{P}^*$ and $\alpha > 0$ such that $\mu \in \alpha v^\circ$ we have $\mu(a) \leq \mu(b) + \alpha$, that is $\varphi_a(\mu) \leq \varphi_b(\mu) + \psi_v(\mu)$, hence $\varphi_a \leq \varphi_b + \psi_v$. Conversely, if $a \not\preccurlyeq b + v$, then following Corollary 4.34 there is $\mu \in v^\circ \subset \mathcal{P}^*$ such that $\mu(a) > \mu(b) + 1$. The former implies $\psi_v(\mu) \leq 1$, hence $\varphi_a(\mu) > \varphi_b(\mu) + \psi_v(\mu)$ and therefore $\varphi_a \not\leq \varphi_b + \psi_v$. We infer in particular that the functions $\Phi(a) = \varphi_a$ are contained in $\mathcal{F}_{\widehat{\mathcal{V}}_b}(\mathcal{P}^*, \overline{\mathbb{R}})$ for all $a \in \mathcal{P}$. Indeed, given $a \in \mathcal{P}$ and $v \in \mathcal{V}$ there is $\lambda \geq 0$ such that $0 \preccurlyeq a + \lambda v$, hence $0 \leq \Phi(a) + \lambda \hat{v}$. Therefore the element $\Phi(a)$ is contained in $\mathcal{F}_{\widehat{\mathcal{V}}_b}(\mathcal{P}^*, \overline{\mathbb{R}})$ as claimed. Finally we establish that the linear operator $\Phi : \mathcal{P} \to \mathcal{F}_{\widehat{\mathcal{V}}_b}(\mathcal{P}^*, \overline{\mathbb{R}})$ is an embedding in the sense of 2.2 of the locally convex cone $(\mathcal{P}, \mathcal{V})$ into $\big(\mathcal{F}_{\widehat{\mathcal{V}}_b}(\mathcal{P}^*, \overline{\mathbb{R}}), \widehat{\mathcal{V}}\big)$, provided that we consider $(\mathcal{P}, \mathcal{V})$ in its weak preorder. Indeed, we set $\Phi(v) = \hat{v}$ for $v \in \mathcal{V}$ towards the extension

$$\Phi : (\mathcal{P} \cup \mathcal{V}) :\to \big(\mathcal{F}_{\widehat{\mathcal{V}}_b}(\Upsilon, \overline{\mathbb{R}}) \cup \widehat{\mathcal{V}}\big).$$

Then $\Phi(\mathcal{V}) = \widehat{\mathcal{V}}$, and by the above $a \preccurlyeq b + v$ holds for $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$ if and only if $\Phi(a) \leq \Phi(b) + \Phi(v)$, as required in 2.2.

Finally, we denote by $\widehat{\mathcal{P}}$ the smallest locally convex complete lattice subcone of $\mathcal{F}_{\widehat{\mathcal{V}}_b}(\mathcal{P}^*, \overline{\mathbb{R}})$ that contains the embedding $\Phi(\mathcal{P})$ of $\mathcal{P}$. proposition 5.36 specifies that $\widehat{\mathcal{P}}$ consists of all functions in $\mathcal{F}_{\widehat{\mathcal{V}}_b}(\mathcal{P}^*, \overline{\mathbb{R}})$ that can be expressed in the following way:

$$\varphi_{\mathfrak{A}} = \inf \big\{ \sup \Phi(A) \mid A \in \mathfrak{A} \big\}.$$

where $\mathfrak{A}$ is family of subsets of $\mathcal{P}$ such that $\Phi(\mathfrak{A}) = \{\Phi(A) \mid A \in \mathfrak{A}\}$ is supbounded below in $\mathcal{F}_{\widehat{\mathcal{V}}_b}(\mathcal{P}^*, \overline{\mathbb{R}})$ (see 5.36). We call $(\widehat{\mathcal{P}}, \widehat{\mathcal{V}})$ the *standard lattice completion* of the locally convex cone $(\mathcal{P}, \mathcal{V})$. According to Proposition 5.52, the subcone $\widehat{\mathcal{P}}$ of $\mathcal{F}_{\widehat{\mathcal{V}}_b}(\mathcal{P}^*, \overline{\mathbb{R}})$ is closed in the order topology $\mathcal{O}\big(\mathcal{F}_{\widehat{\mathcal{V}}_b}(\mathcal{P}^*, \overline{\mathbb{R}})\big)$, and $\mathcal{O}(\widehat{\mathcal{P}})$ coincides with the restriction of $\mathcal{O}\big(\mathcal{F}_{\widehat{\mathcal{V}}_b}(\mathcal{P}^*, \overline{\mathbb{R}})\big)$ to $\widehat{\mathcal{P}}$. According to Proposition 5.53 the order topology, the weak order topology and the topology of pointwise convergence on X all coincide on $\widehat{\mathcal{P}}$ and are Hausdorff. Moreover, the restriction of this topology to the subcone $\Phi(\mathcal{P})$ of $\widehat{\mathcal{P}}$ is generally coarser than the image of the weak topology $\sigma(\mathcal{P}, \mathcal{P}^*)$ (see 4.6) on $\Phi(\mathcal{P})$. Indeed, while the domain of the functions $\varphi \in \widehat{\mathcal{P}}$ is the dual cone $\mathcal{P}^*$ of $\mathcal{P}$, pointwise convergence for a net $(\varphi_{a_i})_{i \in \mathcal{I}}$ of $\overline{\mathbb{R}}$-valued functions in

$\Phi(\mathcal{P})$ is treated differently from weak convergence for the corresponding net $(a_i)_{i\in\mathcal{I}}$ in $\mathcal{P}$ if the function value $+\infty \in \overline{\mathbb{R}}$ is involved. The order topology of $\overline{\mathbb{R}}$, which is used for pointwise convergence of the functions is coarser than the given (locally convex cone) topology of $\overline{\mathbb{R}}$ at this point (see 4.6 and 5.40). However, if all elements of $\mathcal{P}$ are bounded, that is for example in the case of a vector space, then continuous linear functionals take only finite values on $\mathcal{P}$, hence the elements of $\Phi(\mathcal{P})$ take only finite values as functions on $\mathcal{P}^*$. In this case the order topology, the weak order topology, the weak topology and the topology of pointwise convergence all coincide on $\Phi(\mathcal{P})$.

The embedding of a locally convex cone $(\mathcal{P},\mathcal{V})$ into some locally convex complete lattice cone is of course not unique. However, the standard lattice completion $(\widehat{\mathcal{P}},\widehat{\mathcal{V}})$ of $(\mathcal{P},\mathcal{V})$ is distinguished by the fact that every continuous linear operator from $\mathcal{P}$ into some locally convex complete lattice cone $(\mathcal{Q},\mathcal{W})$ can be extended to an order continuous lattice homomorphism from $(\widehat{\mathcal{P}},\widehat{\mathcal{V}})$ into $(\mathcal{Q},\mathcal{W})$. More precisely:

Proposition 5.58. *Let $(\mathcal{P},\mathcal{V})$ be a locally convex cone, and let Φ be the canonical embedding of $\mathcal{P}$ into its standard lattice completion $\widehat{\mathcal{P}}$. Let $(\mathcal{Q},\mathcal{W})$ be a locally convex complete lattice cone such that the order continuous lattice homomorphisms support the separation property for $\mathcal{Q}$. For every continuous linear operator $T:\mathcal{P}\to\mathcal{Q}$ there exists an order continuous lattice homomorphism $\widehat{T}:\widehat{\mathcal{P}}\to\mathcal{Q}$ such that $T=\widehat{T}\circ\Phi$. Moreover, if $v\in\mathcal{V}$ and $w\in\mathcal{W}$ are such that $a\le b+v$ implies $T(a)\le T(b)+w$ for $a,b\in\mathcal{P}$, then $\varphi\le\psi+\Phi(v)$ implies $\widehat{T}(\varphi)\le\widehat{T}(\psi)+w$ for $\varphi,\psi\in\widehat{\mathcal{P}}$.*

Proof. Let $(\mathcal{P},\mathcal{V})$, $(\mathcal{Q},\mathcal{W})$ and $T:\mathcal{P}\to\mathcal{Q}$ be as stated. The *adjoint operator* $T^*:\mathcal{Q}^*\to\mathcal{P}^*$ is defined as follows $\big($see II.2.15 in [100]$\big)$: For any $\nu\in\mathcal{Q}^*$ define the linear functional $T^*(\nu)$ on $\mathcal{P}$ by $T^*(\nu)\big(a\big)=\nu\big(T(a)\big)$ for all $a\in\mathcal{P}$. More precisely: If $\nu\in w^\circ$ for some $w\in\mathcal{W}$ and if $v\in\mathcal{V}$ is such that $T(a)\le T(b)+w$ whenever $a\le b+v$ for $a,b\in\mathcal{P}$, then $T^*(\nu)\in v^\circ$. Indeed, $a\le b+v$ for $a,b\in\mathcal{P}$ implies that

$$T^*\big(\nu\big)(a)=\nu\big(T(a)\big)\le\nu\big(T(b)\big)+1=T^*\big(\nu\big)(b)+1.$$

Now let $\mathfrak{A}$ be a family of subsets of $\mathcal{P}$ such that $\Phi(\mathfrak{A})=\{\Phi(A)\mid A\in\mathfrak{A}\}$ is sup-bounded below in $\mathcal{F}_{\widehat{\mathcal{V}}_b}(\mathcal{P}^*,\overline{\mathbb{R}})$. We claim that the family $T(\mathfrak{A})=\{T(A)\mid A\in\mathfrak{A}\}$ is sup-bounded below in $\mathcal{Q}$: Indeed, given $w\in W$ there is $v\in\mathcal{V}$ such that $T^*(w^\circ)\subset v^\circ$. There is $\lambda\ge0$ such that $0\le\sup\Phi(A)+\lambda v$ for all $A\in\mathfrak{A}$. This means $\mu\big(\sup\Phi(A)\big)\ge-\lambda$ for all $\mu\in v^\circ$. For an order continuous lattice homomorphism $\nu\in w^\circ$ set $\mu=T^*(\nu)\in v^\circ$. Then for $A\in\mathfrak{A}$ we have

$$\nu\big(\sup T(A)\big)=\sup\nu\big((T(A)\big)=\sup\mu(A)=\mu\big(\sup\Phi(A)\big)\ge-\lambda.$$

This shows $0\le\sup T(A)+\lambda v$, since the order continuous lattice homomorphisms are supposed to support the separation property for $\mathcal{Q}$. Our claim has therefore been verified. Now consider the elements of $\varphi_{\mathfrak{A}} \in \widehat{\mathcal{P}}$ and $\widetilde{\varphi}_{\mathfrak{A}} \in \mathcal{Q}$ defined as

$$\varphi_{\mathfrak{A}} = \inf\big\{\sup \Phi(A) \mid A \in \mathfrak{A}\big\} \qquad \text{and} \qquad \widetilde{\varphi}_{\mathfrak{A}} = \inf\big\{\sup T(A) \mid A \in \mathfrak{A}\big\},$$

where Φ denotes the canonical embedding of $\mathcal{P}$ into $\widehat{\mathcal{P}}$. For every $\mu \in \mathcal{P}^*$, that is the domain of the functions in $\widehat{\mathcal{P}}$, and for every order continuous lattice homomorphism $\nu \in \mathcal{Q}^*$ we calculate

$$\varphi_{\mathfrak{A}}(\mu) = \inf\big\{\sup \Phi(A) \mid A \in \mathfrak{A}\big\}(\mu) = \inf\big\{\sup \mu(A) \mid A \in \mathfrak{A}\big\}$$

and

$$\begin{aligned}\nu(\widetilde{\varphi}_{\mathfrak{A}}) &= \sup\big\{\inf \nu\big(T(A)\big) \mid A \in \mathfrak{A}\big\} = \sup\big\{\inf T^*(\nu)\big(A\big) \mid A \in \mathfrak{A}\big\} \\ &= \varphi_{\mathfrak{A}}\big((T^*(\nu)\big).\end{aligned}$$

Thus, if $w \in \mathcal{W}$, if $v \in \mathcal{V}$ is such that $T(a) \le T(b) + w$ whenever $a \le b + v$ for $a, b \in \mathcal{P}$, and if $\varphi_{\mathfrak{A}} \le \varphi_{\mathfrak{B}} + \Phi(v)$ for such families $\mathfrak{A}$ and $\mathfrak{B}$ of subsets of $\mathcal{P}$, then

$$\nu(\widetilde{\varphi}_{\mathfrak{A}}) = \varphi_{\mathfrak{A}}\big((T^*(\nu)\big) \le \varphi_{\mathfrak{B}}\big((T^*(\nu)\big) + \psi_v\big(T^*(\nu)\big) \le \nu(\widetilde{\varphi}_{\mathfrak{B}}) + 1$$

holds for all order continuous lattice homomorphisms $\nu \in w^\circ$, since $T^*(\nu) \in v^\circ$ implies that $\psi_v\big(T^*(\nu)\big) \le 1$. This shows

$$\widetilde{\varphi}_{\mathfrak{A}} \le \widetilde{\varphi}_{\mathfrak{B}} + w,$$

since these functionals are supposed to support the separation property for $\mathcal{Q}$. In particular, we infer that $\widetilde{\varphi}_{\mathfrak{A}} = \widetilde{\varphi}_{\mathfrak{B}}$ whenever $\varphi_{\mathfrak{A}} = \varphi_{\mathfrak{B}}$. This follows from the fact that both cones $\widehat{\mathcal{P}}$ and $\mathcal{Q}$ carry their respective weak preorders, which are supposed to be antisymmetric. We are now prepared to define the operator $\widehat{T} : \widehat{\mathcal{P}} \to \mathcal{Q}$. For a family $\mathfrak{A}$ of subsets of $\mathcal{P}$ such that $\Phi(\mathfrak{A}) = \{\Phi(A) \mid A \in \mathfrak{A}\}$ is sup-bounded below in $\mathcal{F}_{\widehat{\mathcal{V}}_b}(\mathcal{P}^*, \overline{\mathbb{R}})$ and the corresponding element $\varphi_{\mathfrak{A}} \in \widehat{\mathcal{P}}$ we set

$$\widehat{T}\big(\varphi_{\mathfrak{A}}\big) = \widetilde{\varphi}_{\mathfrak{A}}$$

and observe the following:

(i) The operator $\widehat{T}$ is linear. Indeed, we note that $\Phi(\alpha\mathfrak{A}) = \alpha\Phi(\mathfrak{A})$ and $T(\alpha\mathfrak{A}) = \alpha T(\mathfrak{A})$ as well as $\Phi(\mathfrak{A}+\mathfrak{B}) = \Phi(\mathfrak{A})+\Phi(\mathfrak{B})$ and $T(\mathfrak{A}+\mathfrak{B}) = T(\mathfrak{A})+T(\mathfrak{B})$ holds for any such families $\mathfrak{A}$ and $\mathfrak{B}$ of subsets of $\mathcal{P}$ and $\alpha \ge 0$. Then the arguments in Parts (i) and (ii) of the proof for Proposition 5.36 yield that

$$\varphi_{\mathfrak{A}} + \varphi_{\mathfrak{B}} = \varphi_{(\mathfrak{A}+\mathfrak{B})} \qquad \text{and} \qquad \widetilde{\varphi}_{\mathfrak{A}} + \widetilde{\varphi}_{\mathfrak{B}} = \widetilde{\varphi}_{(\mathfrak{A}+\mathfrak{B})}.$$

Thus

$$\widehat{T}\big(\varphi_{\mathfrak{A}} + \varphi_{\mathfrak{B}}\big) = \widehat{T}\big(\varphi_{(\mathfrak{A}+\mathfrak{B})}\big) = \widetilde{\varphi}_{(\mathfrak{A}+\mathfrak{B})} = \widetilde{\varphi}_{\mathfrak{A}} + \widetilde{\varphi}_{\mathfrak{B}} = \widehat{T}\big(\varphi_{\mathfrak{A}}\big) + \widehat{T}\big(\varphi_{\mathfrak{B}}\big).$$

Likewise, $\alpha\varphi_{\mathfrak{A}} = \varphi_{\alpha\mathfrak{A}}$ and $\alpha\widetilde{\varphi}_{\mathfrak{A}} = \widetilde{\varphi}_{\alpha\mathfrak{A}}$ yields $\widehat{T}\big(\alpha\varphi_{\mathfrak{A}}\big) = \alpha\widehat{T}\big(\varphi_{\mathfrak{A}}\big)$ for all $\alpha \geq 0$.

(ii) We observed before that, given $w \in \mathcal{W}$ and $v \in \mathcal{V}$ such that $a \leq b+v$ for $a, b \in \mathcal{P}$ implies $T(a) \leq T(b)+w$, then $\varphi_{\mathfrak{A}} \leq \varphi_{\mathfrak{B}} + \Phi(v)$ for $\varphi_{\mathfrak{A}}, \varphi_{\mathfrak{B}} \in \widehat{\mathcal{P}}$ implies

$$\widehat{T}(\varphi_{\mathfrak{A}}) = \widetilde{\varphi}_{\mathfrak{A}} \leq \widetilde{\varphi}_{\mathfrak{B}} + w = \widehat{T}(\varphi_{\mathfrak{B}}) + w.$$

This entails continuity for the operator $\widehat{T}$.

(iii) Let $a \in \mathcal{P}$ and set $\mathfrak{A} = \big\{\{a\}\big\}$. Then $\varphi_{\mathfrak{A}} = \Phi(a) \in \widehat{\mathcal{P}}$ and $\widetilde{\varphi}_{\mathfrak{A}} = T(a)$. Thus

$$\big(\widehat{T} \circ \Phi\big)(a) = \widehat{T}(\varphi_{\mathfrak{A}}) = \widetilde{\varphi}_{\mathfrak{A}} = T(a).$$

This shows $T = \widehat{T} \circ \Phi$.

(iv) Let $\{\mathfrak{A}_i\}_{i\in\mathcal{I}}$ be a collection of such families $\mathfrak{A}_i$ of subsets of $\mathcal{P}$. In a first instance, suppose that this collection is not empty, and let $\mathfrak{A} = \{\cup_{i\in\mathcal{I}} A_i \mid (A_i)_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} \mathfrak{A}_i\}$, that is the elements A of $\mathfrak{A}$ are all unions of the type $A = \cup_{i\in\mathcal{I}} A_i$, where $A_i \in \mathfrak{A}_i$. Then $\Phi(\mathfrak{A}) = \{\cup_{i\in\mathcal{I}} \Phi(A_i) \mid (A_i)_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} \mathfrak{A}_i\}$ and $T(\mathfrak{A}) = \{\cup_{i\in\mathcal{I}} T(A_i) \mid (A_i)_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} \mathfrak{A}_i\}$. Therefore Part (iii) in the proof for Proposition 5.36 yields

$$\sup_{i\in\mathcal{I}} \varphi_{\mathfrak{A}_i} = \varphi_{\mathfrak{A}} \qquad \text{and} \qquad \sup_{i\in\mathcal{I}} \widetilde{\varphi}_{\mathfrak{A}_i} = \widetilde{\varphi}_{\mathfrak{A}}.$$

This shows

$$\widehat{T}\big(\sup_{i\in\mathcal{I}} \varphi_{\mathfrak{A}_i}\big) = \sup_{i\in\mathcal{I}} \widehat{T}\big(\varphi_{\mathfrak{A}_i}\big).$$

In a second instance, suppose that the set $\{a_{\mathfrak{A}_i}\}_{i\in\mathcal{I}}$ is bounded below in $\mathcal{P}$, and let $\mathfrak{A} = \cup_{i\in\mathcal{I}} \mathfrak{A}_i$. Then $\Phi(\mathfrak{A}) = \{\cup_{i\in\mathcal{I}} \Phi(\mathfrak{A}_i) \mid i \in \mathcal{I}\}$ and $T(\mathfrak{A}) = \{\cup_{i\in\mathcal{I}} T(A_i) \mid i \in \mathcal{I}\}$. Part (iv) in the proof for Proposition 5.36 yields

$$\inf_{i\in\mathcal{I}} \varphi_{\mathfrak{A}_i} = \varphi_{\mathfrak{A}} \qquad \text{and} \qquad \inf_{i\in\mathcal{I}} \widetilde{\varphi}_{\mathfrak{A}_i} = \widetilde{\varphi}_{\mathfrak{A}}.$$

Thus

$$\widehat{T}\big(\inf_{i\in\mathcal{I}} \varphi_{\mathfrak{A}_i}\big) = \inf_{i\in\mathcal{I}} \widehat{T}\big(\varphi_{\mathfrak{A}_i}\big)$$

holds as well. The operator $\widehat{T} : \widehat{\mathcal{P}} \to \mathcal{Q}$ is therefore an order continuous lattice homomorphism. □

For $\mathcal{Q} = \overline{\mathbb{R}}$ in particular, Proposition 5.38 states that for $v \in \mathcal{V}$ and every linear functional $\mu \in v^\circ$ on $\mathcal{P}$ there is an order continuous lattice homomorphism $\hat{\mu} \in \big(\Phi(v)\big)^\circ$ on $\widehat{\mathcal{P}}$ such that $\mu = \hat{\mu} \circ \Phi$. We proceed to demonstrate that the standard lattice completion $(\widehat{\mathcal{P}}, \widehat{\mathcal{V}})$ of a locally convex cone $(\mathcal{P},\mathcal{V})$ is indeed the only (up to embedding) locally convex complete lattice cone which contains an embedding of $\mathcal{P}$ and satisfies this property.

Proposition 5.59. *Let $(\mathcal{P},\mathcal{V})$ be a locally convex cone, let $(\widetilde{\mathcal{P}},\widetilde{\mathcal{V}})$ be a locally convex complete lattice cone such that the order continuous lattice homomorphisms support the separation property for $\widetilde{\mathcal{P}}$. Suppose that there is an embedding $\Psi:\mathcal{P}\to\widetilde{\mathcal{P}}$ with respect to the weak preorder of $\mathcal{P}$ and that for every $v\in\mathcal{V}$ and every linear functional $\mu\in v^\circ$ on $\mathcal{P}$ there is an order continuous lattice homomorphism $\tilde{\mu}\in\big(\Psi(v)\big)^\circ$ on $\widetilde{\mathcal{P}}$ such that $\mu=\tilde{\mu}\circ\Psi$. Then there exists an embedding $\widehat{\Psi}:\widehat{\mathcal{P}}\to\widetilde{\mathcal{P}}$, where $(\widehat{\mathcal{P}},\widehat{\mathcal{V}})$ denotes the standard lattice completion of $(\mathcal{P},\mathcal{V})$. This embedding preserves the lattice operations for $\widehat{\mathcal{P}}$ and $\widetilde{\mathcal{P}}$.*

Proof. We shall use the notations from the proof of the preceding proposition, in particular we denote by $\Phi:\mathcal{P}\to\widehat{\mathcal{P}}$ the canonical embedding of $(\mathcal{P},\mathcal{V})$ into its standard lattice completion $(\widehat{\mathcal{P}},\widehat{\mathcal{V}})$. Now suppose that the linear operator $\Psi:\mathcal{P}\to\widetilde{\mathcal{P}}$ is also an embedding in the sense of 2.2, that is there is an extension

$$\Psi:(\mathcal{P}\cup\mathcal{V})\to(\widetilde{\mathcal{P}}\cup\widetilde{\mathcal{V}})$$

with the required properties. According to Proposition 5.58 there exists an order continuous lattice homomorphism $\widehat{\Psi}:\widehat{\mathcal{P}}\to\widetilde{\mathcal{P}}$ such that $\Psi=\widehat{\Psi}\circ\Phi$ and

$$\varphi_{\mathfrak{A}}\le\varphi_{\mathfrak{B}}+\Phi(v)\qquad\text{implies that}\qquad\widehat{\Psi}(\varphi_{\mathfrak{A}})\le\widehat{\Psi}(\varphi_{\mathfrak{B}})+\Psi(v)$$

for $\varphi_{\mathfrak{A}},\varphi_{\mathfrak{B}}\in\widehat{\mathcal{P}}$ and $v\in\mathcal{V}$. As before we abbreviate $\hat{v}=\Phi(v)\in\widehat{\mathcal{V}}$ for $v\in\mathcal{V}$ and use this notation for the extension

$$\widehat{\Psi}:(\widehat{\mathcal{P}}\cup\widehat{\mathcal{V}})\to(\widetilde{\mathcal{P}}\cup\widetilde{\mathcal{V}})$$

setting $\widehat{\Psi}(\hat{v})=\Psi(v)$ for all $v\in\mathcal{V}$. Clearly $\widehat{\Psi}(\widehat{\mathcal{V}})=\Psi(\mathcal{V})=\widetilde{\mathcal{V}}$, and rewriting the above yields that

$$\varphi_{\mathfrak{A}}\le\varphi_{\mathfrak{B}}+\hat{v}\qquad\text{implies that}\qquad\widehat{\Psi}(\varphi_{\mathfrak{A}})\le\widehat{\Psi}(\varphi_{\mathfrak{B}})+\widehat{\Psi}(\hat{v})$$

for $\varphi_{\mathfrak{A}},\varphi_{\mathfrak{B}}\in\widehat{\mathcal{P}}$ and $\hat{v}\in\widehat{\mathcal{V}}$. All left to verify for $\widehat{\Psi}:\widehat{\mathcal{P}}\to\widetilde{\mathcal{P}}$ to be an embedding is the reverse implication in the above. For this, suppose that $\widehat{\Psi}(\varphi_{\mathfrak{A}})\le\widehat{\Psi}(\varphi_{\mathfrak{B}})+\widehat{\Psi}(\hat{v})$ and let $\mu\in v^\circ$. Following our assumption there is an order continuous lattice homomorphism $\tilde{\mu}\in\big(\Psi(v)\big)^\circ$ on $\widetilde{\mathcal{P}}$ such that $\mu=\tilde{\mu}\circ\Psi$. Then

$$\tilde{\mu}\big(\widehat{\Psi}(\varphi_{\mathfrak{A}})\big)\le\tilde{\mu}\big(\widehat{\Psi}(\varphi_{\mathfrak{B}})\big)+1.$$

On the other hand, we have

$$\begin{aligned}\varphi_{\mathfrak{A}}(\mu) &= \inf\big\{\sup \mu(A) \mid A \in \mathfrak{A}\big\}\\ &= \inf\big\{\sup \tilde{\mu}\big(\Psi(A)\big) \mid A \in \mathfrak{A}\big\}\\ &= \tilde{\mu}\left(\inf\big\{\sup \Psi(A) \mid A \in \mathfrak{A}\big\}\right)\end{aligned}$$

and

$$\begin{aligned}\inf\big\{\sup \Psi(A) \mid A \in \mathfrak{A}\big\} &= \inf\big\{\sup \widehat{\Psi}\big(\Phi(A)\big) \mid A \in \mathfrak{A}\big\}\\ &= \widehat{\Psi}\big(\inf\big\{\sup \Phi(A) \mid A \in \mathfrak{A}\big\}\big)\\ &= \widehat{\Psi}(\varphi_{\mathfrak{A}})\end{aligned}$$

since $\Psi = \widehat{\Psi} \circ \Phi$ by 5.58 and since $\widehat{\Psi} : \widehat{\mathcal{P}} \to \widetilde{\mathcal{P}}$ is an order continuous lattice homomorphism. Combining the above yields $\varphi_{\mathfrak{A}}(\mu) = \tilde{\mu}\big(\widehat{\Psi}(\varphi_{\mathfrak{A}})\big)$ and, likewise $\varphi_{\mathfrak{B}}(\mu) = \tilde{\mu}\big(\widehat{\Psi}(\varphi_{\mathfrak{B}})\big)$. Therefore

$$\varphi_{\mathfrak{A}}(\mu) \leq \varphi_{\mathfrak{B}}(\mu) + 1$$

holds for all $\mu \in v^{\circ}$. Form this we infer that $\varphi_{\mathfrak{A}}(\mu) \leq \varphi_{\mathfrak{B}}(\mu) + \psi_v(\mu)$ holds for all $\mu \in \mathcal{P}^*$, and conclude that $\varphi_{\mathfrak{A}} \leq \varphi_{\mathfrak{B}} + \hat{v}$, as claimed. □

Remarks 5.60. (a) We shall make extensive use of the standard lattice completion $(\widehat{\mathcal{P}}, \widehat{\mathcal{V}})$ of a locally convex cone $(\mathcal{P}, \mathcal{V})$ in the integration theory for cone-valued functions in Chapters II and III. However, many of the results will refer only to the order closure of the embedding of $\mathcal{P}$ into $\widehat{\mathcal{P}}$. It is therefore useful to observe that the elements of this order closure can be interpreted as elements of some second dual of $\mathcal{P}$. Indeed, let $\varphi \in \widehat{\mathcal{P}}$ be an element of this closure. Since convergence in the order topology of $\widehat{\mathcal{P}}$ coincides with pointwise convergence on $\mathcal{P}^*$, there is a net $(a_i)_{i \in \mathcal{I}}$ in $\mathcal{P}$ such that the functions $\varphi_{a_i} \in \widehat{\mathcal{P}}$ from above converge pointwise to φ. Thus for $\mu, \nu \in \mathcal{P}^*$ we have

$$\varphi(\mu + \nu) = \lim_{i \in \mathcal{I}} \varphi_{a_i}(\mu + \nu) = \lim_{i \in \mathcal{I}} \varphi_{a_i}(\mu) + \lim_{i \in \mathcal{I}} \varphi_{a_i}(\nu) = \varphi(\mu) + \varphi(\nu)$$

by (OT2). Since $\varphi(\alpha\mu) = \alpha\varphi(\mu)$ holds for all $\varphi \in \widehat{\mathcal{P}}$ and $\mu \in \mathcal{P}^*$ and $\alpha \geq 0$, the function φ is an $\overline{\mathbb{R}}$-valued linear functional on $\mathcal{P}^*$, that is an element of $\mathcal{P}^{**}$, the dual cone of $\mathcal{P}^*$ under its finest locally convex topology which renders all linear functionals on $\mathcal{P}^*$ continuous (see 7.3(i) below). Moreover, as an element of $\widehat{\mathcal{P}}$, the functional φ is bounded below on all polars of neighborhoods in $\mathcal{V}$.

(b) If $(\mathcal{P}, \mathcal{V})$ is a locally convex vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ in its symmetric (modular) topology (see Example 1.4(d)), that is a locally convex topological vector space, then the dual cone $\mathcal{P}^*$ of $\mathcal{P}$ consist of the real parts μ of all continuous $\mathbb{K}$-linear functionals $\mu_{\mathbb{K}}$ in the vector space dual $\mathcal{P}^*_{\mathbb{K}}$ of $\mathcal{P}$ (see 2.1(d)). Similarly, in 2.1(d) we established a correspondence between real-valued linear (with respect to multiplication by non-negative scalars) functionals φ on $\mathcal{P}^*$ and $\mathbb{K}$-valued $\mathbb{K}$-linear functionals $\varphi_{\mathbb{K}}$ on $\mathcal{P}^*_{\mathbb{K}}$. This correspondence is given by

$$\varphi(\mu) = \mathfrak{Re}\big(\varphi_{\mathbb{K}}(\mu_{\mathbb{K}})\big)$$

for all $\mu \in \mathcal{P}^*$, and

$$\varphi_{\mathbb{K}}(\mu_{\mathbb{K}}) = \varphi(\mu) \qquad \text{or} \qquad \varphi_{\mathbb{K}}(\mu_{\mathbb{K}}) = \big(\varphi(\mu) - i\,\varphi(i\mu)\big)$$

for all $\mu_{\mathbb{B}} \in \mathcal{P}^*_{\mathbb{K}}$ in the real or complex case, respectively. If the real-linear functional φ on $\mathcal{P}^*$ is contained in $\widehat{\mathcal{P}}$, that is for example if φ is contained in the order closure of the embedding of $\mathcal{P}$ $\big($see Part (a) before$\big)$ into $\widehat{\mathcal{P}}$, then φ is bounded below on the polars $v^\circ \subset \mathcal{P}^*$ of all neighborhoods $v \in \mathcal{V}$. Therefore the corresponding $\mathbb{K}$-linear functional $\varphi_{\mathbb{K}}$ in the second vector space dual $\mathcal{P}^{**}_{\mathbb{K}}$ of $\mathcal{P}$ is also bounded on all polars of neighborhoods in $\mathcal{V}$. In the case of a normed space $(\mathcal{P}, \|\ \|)$, for example, the latter implies that $\varphi_{\mathbb{K}}$ is an element of the (strong) second vector space dual of $\mathcal{P}$.

(c) For a concrete example to (b) let $\mathcal{P} = \mathbb{K}$ endowed with the Euclidean topology, that is the neighborhood system $\mathcal{V} = \{\varepsilon\mathbb{B} \mid \varepsilon > 0\}$, where $\mathbb{B}$ is the unit ball in $\mathbb{K}$. The vector space dual $\mathcal{P}_{\mathbb{K}}$ of $\mathbb{K}$ then is of course $\mathbb{K}$ itself, which corresponds to the dual cone $\mathcal{P}^*$ of $\mathbb{K}$ as a locally convex cone as elaborated in 2.1(d), that is every $z \in \mathbb{K}$ defines a real-linear functional in $\mathcal{P}^*$ via

$$a \mapsto \mathfrak{Re}(za) : \mathbb{K} \to \mathbb{R}.$$

On the other hand, every real-valued linear functional φ on $\mathcal{P}^* = \mathbb{K}$ corresponds to an element $z \in \mathbb{K}$, that is the second vector space dual of $\mathbb{K}$, by

$$z = \varphi(1) \qquad \text{or} \qquad z = \varphi(1) - i\varphi(i)$$

in the real or complex case, respectively.

(d) If under the assumptions of (b), $(\mathcal{Q}, \mathcal{W})$ is a second locally convex vector space over $\mathbb{K}$, then we shall say that a linear operator $T : \mathcal{Q} \to \widehat{\mathcal{P}}$ is $\mathbb{K}$-*linear* if

(i) $T\big(f\big)(\mu + \nu) = T\big(f\big)(\mu) + T\big(f\big)(\nu)$ and
(ii) $T\big(\alpha f\big)(\mu) = T\big(f\big)(\alpha\mu)$

holds for all $f \in \mathcal{Q}$, $\mu, \nu \in \mathcal{P}^*$ and $\alpha \in \mathbb{K}$. In this case T corresponds to a $\mathbb{K}$-linear operator $\widetilde{T} : \mathcal{Q} \to \mathcal{P}^{**}_{\mathbb{K}}$.

5.61 Simplified Standard Lattice Completion. It is often preferable to realize a lattice completion of a locally convex cone $(\mathcal{P}, \mathcal{V})$ as a cone of $\overline{\mathbb{R}}$-valued functions on a suitable subset of $\mathcal{P}^*$ rather than on the whole of $\mathcal{P}^*$. For this we use a subset Υ of $\mathcal{P}^*$ which supports the separation property for $\mathcal{P}$ in the sense of 5.32. (Following Corollary 4.34 this holds true for $\Upsilon = \mathcal{P}^*$). Let us denote by $\widehat{\mathcal{P}}_\Upsilon$ and $\widehat{\mathcal{V}}_\Upsilon$ the restrictions to Υ of the functions in $\widehat{\mathcal{P}}$ and of the associated neighborhood functions in $\widehat{\mathcal{V}}$. Then

$(\widehat{\mathcal{P}}_\Upsilon, \widehat{\mathcal{V}}_\Upsilon)$ is again a full locally convex complete lattice cone. Consider the restriction map $\widehat{\Psi} : \widehat{\mathcal{P}} \to \widehat{\mathcal{P}}_\Upsilon$ and its composition $\Psi = \widehat{\Psi} \circ \Phi$ with the canonical embedding Φ of $\mathcal{P}$ into $\widehat{\mathcal{P}}$. We claim that $\Psi : \mathcal{P} \to \widehat{\mathcal{P}}_\Upsilon$ is an embedding of $\mathcal{P}$ into $\widehat{\mathcal{P}}_\Upsilon$ if we consider $\mathcal{P}$ in its weak preorder. Indeed, if $a \preccurlyeq b + \psi_v$ for $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$, then $\widehat{\Psi}(\varphi_a) \leq \widehat{\Psi}(\varphi_b) + \psi_v$ holds as well in $\widehat{\mathcal{P}}_\Upsilon$, that is $\Psi(a) \leq \Psi(b) + \Psi(v)$. (We use the earlier notations.) Conversely, if $a \not\preccurlyeq b + v$, then by our assumption there is $\alpha \geq 0$ and $\mu \in \Upsilon \cap \alpha v^\circ$ such that $\mu(a) > \mu(b) + \alpha$. Then $\psi_v(\mu) \leq \alpha$, hence $\varphi_a(\mu) > \varphi_b(\mu) + \psi_v(\mu)$. This shows $\widehat{\Psi}(\varphi_a) \not\leq \widehat{\Psi}(\varphi_b) + \psi_v$, that is $\Psi(a) \not\leq \Psi(b) + \Psi(v)$. Thus $\Phi : \mathcal{P} \to \widehat{\mathcal{P}}_\Upsilon$ is an embedding in the sense of 2.2 as claimed. Since the lattice operations are performed pointwise, we have $\widehat{\Psi}(\sup A) = \sup\big(\widehat{\Psi}(A)\big)$ for every non-empty subset A of $\widehat{\mathcal{P}}$ and $\widehat{\Psi}(\inf A) = \inf\big(\widehat{\Psi}(A)\big)$ for every non-empty bounded below subset A of $\widehat{\mathcal{P}}$. The operator

$$\widehat{\Psi} : \widehat{\mathcal{P}} \to \widehat{\mathcal{P}}_\Upsilon$$

is therefore a surjective order continuous lattice homomorphism, but not necessarily an embedding. Because the operator $\widehat{\Psi}$ is also continuous with respect to the order topologies on $\widehat{\mathcal{P}}$ and $\widehat{\mathcal{P}}_\Upsilon$ (Proposition 5.46), the image under $\widehat{\Psi}$ of the order closure of $\Phi(\mathcal{P})$ in $\widehat{\mathcal{P}}$ is contained in the order closure of $\Psi(\mathcal{P})$ in $\widehat{\mathcal{P}}_\Upsilon$. According to the preceding Proposition 5.59, $\widehat{\Psi}$ is an embedding and indeed an isomorphism if and only if for every $v \in \mathcal{V}$ and every linear functional $\mu \in v^\circ$ on $\mathcal{P}$ there is an order continuous lattice homomorphism $\tilde{\mu} \in \big(\Psi(v)\big)^\circ$ on $\widehat{\mathcal{P}}_\Upsilon$ such that $\mu = \tilde{\mu} \circ \Psi$. This condition is satisfied if for every $\mu \in \mathcal{P}^*$ there is $\nu \in \Upsilon$ and $\alpha \geq 0$ such that $\mu = \alpha\nu$. In this case the conclusion of Proposition 5.58 applies to $\widehat{\mathcal{P}}_\Upsilon$ as it does to $\widehat{\mathcal{P}}$.

We shall at times us such a simplified standard lattice completion $(\widehat{\mathcal{P}}_\Upsilon, \widehat{\mathcal{V}}_\Upsilon)$ and the order continuous lattice homomorphism $\Psi : \widehat{\mathcal{P}} \to \widehat{\mathcal{P}}_\Upsilon$ in order to represent and visualize results that were obtained in the standard lattice completion $(\widehat{\mathcal{P}}, \widehat{\mathcal{V}})$ of a locally convex cone $(\mathcal{P}, \mathcal{V})$.

Examples 5.62. In the preceding Examples 5.33 we investigated a range of locally convex cones $(\mathcal{P}, \mathcal{V})$ and identified subsets Υ of the dual cone which support the separation property. All of these choices are suitable for the construction of a simplified standard lattice completion $(\widehat{\mathcal{P}}_\Upsilon, \widehat{\mathcal{V}}_\Upsilon)$. Let us elaborate on the most important of these situations.

(a) For $\mathcal{P} = \mathbb{R}$ with the usual order and neighborhood system $\mathcal{V} = \{\varepsilon \in \mathbb{R} \mid \varepsilon > 0\}$ the dual cone is $\mathcal{P}^* = \{\alpha \in \mathbb{R} \mid \alpha \geq 0\}$. Therefore the standard lattice completion $\widehat{\mathbb{R}}$ of $\mathbb{R}$ is the cone of all linear $\overline{\mathbb{R}}$-valued functions on $\mathcal{P}^*$. This can be visualized more easily if we choose the subset $\Upsilon = \{1\}$ of $\mathcal{P}^*$ for the above simplified construction, since $\widehat{\mathcal{P}}_\Upsilon$ then coincides with $\overline{\mathbb{R}}$. Following the remark in 5.61, we realize that the standard lattice completion of $\mathbb{R}$ is isomorphic to $\overline{\mathbb{R}}$.

(b) If $(\mathcal{P}, \| \|)$ is a normed vector space $\big($see 5.33(b)$\big)$, then we may choose the dual unit sphere for $\Upsilon \subset \mathcal{P}^*$. According to our preceding remark, the lattice completion $(\widehat{\mathcal{P}}_\Upsilon, \widehat{\mathcal{V}}_\Upsilon)$ then is isomorphic to the standard lattice completion $(\widehat{\mathcal{P}}, \widehat{\mathcal{V}})$. Alternatively, we may choose $\Upsilon = Ex(\mathbb{B})$, that is the set of all extreme points of the dual unit ball $\mathbb{B}$ in $\mathcal{P}^*$. However, the conclusion of Proposition 5.58 does not generally apply to $\widehat{\mathcal{P}}_\Upsilon$ for the latter choice of Υ. In both cases the lattice completion $\widehat{\mathcal{P}}_\Upsilon$ of $\mathcal{P}$ consists of a cone of $\overline{\mathbb{R}}$-valued bounded below functions on Υ, endowed with the topology of uniform convergence.

(c) For a special case of 5.33(f) consider the locally convex cone $\big(\mathcal{F}_{b_\mathcal{Y}}(X, \overline{\mathbb{R}}), \mathcal{V}_\mathcal{Y}\big)$ of $\overline{\mathbb{R}}$-valued functions on a set X endowed with the topology of uniform convergence on the sets in a family $\mathcal{Y}$ of subsets of X $\big($see 1.4(e)$\big)$. Then

$$\Upsilon = \left\{ \varepsilon_x \;\middle|\; x \in \bigcup_{Y \in \mathcal{Y}} Y \right\} \subset \mathcal{F}_{b_\mathcal{Y}}(X, \overline{\mathbb{R}})^*,$$

where ε_x denotes the point evaluation at $x \in X$, supports the separation property for $\mathcal{F}_{b_\mathcal{Y}}(X, \mathcal{P})$ $\big($see 5.33(f)$\big)$. The corresponding lattice completion $\widehat{\mathcal{F}_{b_\mathcal{Y}}(X, \overline{\mathbb{R}})}_\Upsilon$ of $\mathcal{F}_{b_\mathcal{Y}}(X, \overline{\mathbb{R}})$ then consists of $\overline{\mathbb{R}}$-valued bounded below functions on Υ, endowed with the topology of uniform convergence.

(d) For a special case of (c) let X be a compact set and let $\mathcal{P} = \mathcal{C}(X)$ be the space of all continuous real-valued functions on X, endowed with the pointwise operations and order. The neighborhood system $\mathcal{V}$ consisting of all positive constants generates the topology of uniform convergence. The set Υ of all point evaluations ε_x for $x \in X$ supports the separation property, and the lattice completion $(\widehat{\mathcal{P}}, \widehat{\mathcal{V}})$ of $(\mathcal{P}, \mathcal{V})$ can be realized as a cone of $\overline{\mathbb{R}}$-valued functions on X.

(e) In Section 7 below we shall provide another example, that is cones $\mathfrak{H}(\mathcal{N}, \mathcal{M})$ of linear operators from a cone $\mathcal{N}$ into a second cone $\mathcal{M}$, endowed with suitable locally convex cone topologies, where the canonical choice for Υ for a lattice completion $\widehat{\mathfrak{H}}(\mathcal{N}, \mathcal{M})$ is a proper subset rather than the whole dual cone of $\mathfrak{H}(\mathcal{N}, \mathcal{M})$.

(f) Let $\mathcal{P} = \mathbb{K}$, endowed with the Euclidean topology, that is the neighborhood system $\mathcal{V} = \{\varepsilon\mathbb{B} \mid \varepsilon > 0\}$, where $\mathbb{B}$ is the unit ball in $\mathbb{K}$ (see the preceding Remark 5.60(c)). The vector space dual $\mathcal{P}_\mathbb{K}$ of $\mathbb{K}$ then is $\mathbb{K}$ itself which corresponds to the dual cone $\mathcal{P}^*$ of $\mathbb{K}$ as a locally convex cone as elaborated in 2.1(d) and in 5.60(c).

For the construction of a simplified standard lattice completion $\widehat{\mathbb{K}}_\Upsilon$ of $\mathbb{K}$ we choose $\Upsilon = \Gamma$, the unit circle in $\mathbb{K}$. It is straightforward to verify that $\widehat{\mathbb{K}}_\Upsilon$ consists of all bounded below $\overline{\mathbb{R}}$-valued functions on Γ, endowed with the (strictly) positive constants as neighborhoods. A function $\varphi \in \widehat{\mathbb{K}}_\Upsilon$ can be canonically extended to a real-linear functional on all of $\mathcal{P}^* = \mathbb{K}$ if and only if it takes only finite values in $\mathbb{R}$ and if

$$\sum_{i=1}^{n} \alpha_i \varphi(\gamma_i) = 0$$

holds whenever $\sum_{i=1}^{n} \alpha_i \gamma_i = 0$ for $\alpha_i \in \mathbb{R}$ and $\gamma_i \in \Gamma$. In the real case, that is for $\mathbb{K} = \mathbb{R}$, the latter requires just that

$$\varphi(-1) = -\varphi(1).$$

If the above condition holds, then the corresponding $\mathbb{K}$-linear functional $\varphi_{\mathbb{K}}$ in the second vector space dual of $\mathbb{K}$, that is $\mathbb{K}$ itself, is represented by the number

$$\varphi_{\mathbb{K}} = \varphi(1) \in \mathbb{R} \qquad \text{or} \qquad \varphi_{\mathbb{K}} = \varphi(1) - i\varphi(i) \in \mathbb{C}$$

in the real or in the complex case, respectively.

6. Quasi-Full Locally Convex Cones

In Section 1, a locally convex cone $(\mathcal{P}, \mathcal{V})$ was defined to be a subcone of a full locally convex cone, inheriting both the order and the algebraic structure from the latter. Using only the convex quasiuniform structure of $\mathcal{P}$ (see I.3), a procedure described in Chapter I.5 of [100] allows to recover such a full locally convex cone containing $\mathcal{P}$. However this construction is rather unwieldy and far from unique. In situations like in our upcoming measure and integration theory we shall require more immediate access to a canonically constructed full locally convex cone, containing the given cone of interest. This will be possible for a restricted class of locally convex cones which we shall define and describe in the following.

6.1 Quasi-Full Locally Convex Cones. In a locally convex cone $(\mathcal{P}, \mathcal{V})$ the scalar multiples and sums for neighborhoods in $\mathcal{V}$ are not necessarily reflected in the corresponding operations for their upper, lower or symmetric neighborhoods as subsets of $\mathcal{P}$. In general we only have

$$\alpha\, v(a) = \big(\alpha v\big)(\alpha a) \qquad \text{and} \qquad u(a) + v(b) \subset \big(u + v\big)(a + b)$$

for $u, v \in \mathcal{V}$, $a, b \in \mathcal{P}$ and $\alpha > 0$, as well as similar relations for the lower and symmetric neighborhoods. Stronger links for the addition are however desirable in some cases. In this vein, we shall say that a locally convex cone $(\mathcal{P}, \mathcal{V})$ is *quasi-full* if for $a, b \in \mathcal{P}$ and $u, v \in \mathcal{V}$

(QF1) *$a \le b + v$ for $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$ if and only if $a \le b + s$ for some $s \in \mathcal{P}$ such that $s \le v$,* and

(QF2) *$a \le u + v$ for $a \in \mathcal{P}$ and $u, v \in \mathcal{V}$ if and only if $a \le s + t$ for some $s, t \in \mathcal{P}$ such that $s \le u$ and $t \le v$.*

These conditions can be reformulated as

$$v(a) = v(0) + a \qquad \text{and} \qquad \big(u+v\big)(0) = \downarrow\!\big(u(0)+v(0)\big),$$

where $\downarrow\! A = \{b \in \mathcal{P} \mid b \leq a \text{ for some } a \in A\}$ denotes the decreasing hull of a subset A of $\mathcal{P}$. Indeed, the first statement is clearly equivalent to (QF1), whereas $\downarrow\!\big(u(0)+v(0)\big) \subset \big(u+v\big)(0)$ always holds as the latter set is decreasing. The reverse inclusion is equivalent to (QF2).

Obviously, every full cone, that is every locally convex cone that contains its neighborhoods as elements, is quasi-full. Most importantly, every ordered locally convex topological vector space $(\mathcal{P}, \mathcal{V})$, where $\mathcal{V}$ denotes a basis of balanced convex neighborhoods of the origin, is seen to be a quasi-full locally convex cone in this sense. Recall from Example 1.4(c) that the cone topologies on $\mathcal{P}$ are defined for elements $a, b \in \mathcal{P}$ and $V \in \mathcal{V}$ by

$$a \leq b + V \qquad \text{if} \qquad a - b \leq s \quad \text{for some} \quad s \in V.$$

(QF1) is evident, since $s \in V$ implies $s \leq V$. For (QF2), let $a \leq (U+V)$ for $a \in \mathcal{P}$ and $U, V \in \mathcal{V}$. Then $a \leq s + t$ for some $s \in U$ and $t \in V$, since the addition in $\mathcal{V}$ is the usual addition for subsets of $\mathcal{P}$. As $s \leq U$ and $t \leq V$, this yields (QF2). Recall that equality is a possible choice for the order on $\mathcal{P}$.

In fact, quasi-full locally convex cones are close to locally convex topological vector spaces in the sense that the neighborhoods of every element $a \in \mathcal{P}$ are already determined by the neighborhoods of the element $0 \in \mathcal{P}$. The sum of two neighborhoods in $\mathcal{V}$ coincides with the usual sum of the corresponding subsets of $\mathcal{P}$, that is $u(a) + v(b) = \big(u+v\big)(a+b)$ and $(a)u + (b)v = (a+b)\big(u+v\big)$ for $u, v \in \mathcal{V}$ and $a, b \in \mathcal{P}$.

Another advantage of quasi-full locally convex cones is that for complete lattice structures in the sense of Section 5.4, Condition $\big(\bigvee 2\big)$ transfers from zero-neighborhoods to general ones and from individual neighborhoods to their sums, thus needs to be checked only for a subsystem of zero-neighborhoods that span the entire neighborhood system. Indeed, suppose that $(\mathcal{P}, \mathcal{V})$ is a quasi-full locally convex cone that contains suprema of non-empty sets, and that Condition $\big(\bigvee 2\big)$ holds with $b = 0$ and the neighborhoods u and v in $\mathcal{V}$, that is for a non-empty subset $A \subset \mathcal{P}$, $a \leq v$ for all $a \in A$ implies $\sup A \leq v$, and $a \leq u$ for all $a \in A$ implies $\sup A \leq u$. Now if there is $b \in \mathcal{P}$ such that $a \leq b + (u+v)$ for all $a \in A$, then $a \leq b + s_a + t_a$ for some $s_a \leq u$ and $t_a \leq v$. Then $s = \sup_{a \in A} s_a \leq u$ and $t = \sup_{a \in A} t_a \leq v$ by our assumption on u and v. This shows that $a \leq b + (s+t)$ for all $a \in A$, hence $\sup A \leq b + (s+t) \leq b + (u+v)$, as claimed. If $(\mathcal{P}, \mathcal{V})$ contains both suprema of non-empty and infima of bounded below subsets and satisfies $\big(\bigwedge 1\big)$, then Condition $\big(\bigvee 2\big)$ for some neighborhood $v \in \mathcal{V}$ implies $\big(\bigwedge 2\big)$ for the same v. Indeed, suppose that there is $b \in \mathcal{P}$ such that $b \leq a + v$ holds for all elements a of some bounded below subset

$A \subset \mathcal{P}$. Then $b \le a + t_a$ for some $t_a \le v$, hence $b \le a + t$ for all $a \in A$, where $t = \sup_{a \in A} t_a \le v$. This yields $b \le \inf(A + t) = \inf A + t \le \inf A + v$ by $(\bigwedge 1)$, hence our claim.

6.2 The Standard Full Extension of a Quasi-Full Cone. We shall construct a canonical embedding of a quasi-full locally convex cone $(\mathcal{P}, \mathcal{V})$ into a full locally convex cone in the following manner. Let

$$\mathcal{P}_{\mathcal{V}} = \big\{ a \oplus v \mid a \in \mathcal{P},\ \ v \in \mathcal{V} \cup \{0\} \big\}.$$

We use the obvious algebraic operations on $\mathcal{P}_{\mathcal{V}}$, that is

$$(a \oplus v) + (b \oplus u) = (a + b) \oplus (v + u) \qquad \text{and} \qquad \alpha(a \oplus v) = (\alpha a \oplus \alpha v)$$

for $a, b \in \mathcal{P}$, $u, v \in \mathcal{V} \cup \{0\}$ and $\alpha \ge 0$. The order on $\mathcal{P}_{\mathcal{V}}$ is defined as

$$a \oplus v \ \le\ b \oplus u$$

if $c \le a + v$ implies that $c \le b + u$ for all $c \in \mathcal{P}$. This order relation is reflexive, and transitive, as for $a, b, c \in \mathcal{P}$ and $u, v, w \in \mathcal{V} \cup \{0\}$ such that $a \oplus v \le b \oplus u$ and $b \oplus u \le c \oplus w$, for every $d \in \mathcal{P}$ such that $d \le a + v$, we have $d \le b + u$, hence $d \le c + w$. Thus $a \oplus v \le c \oplus w$. Similarly, one verifies compatibility with the algebraic operations: Compatibility with the multiplication by positive scalars is obvious; for compatibility with the addition, let $(a \oplus v), (b \oplus u), (c \oplus w) \in \mathcal{P}_{\mathcal{V}}$ such that $a \oplus v \le b \oplus u$. If $d \le (a + c) + (v + w)$, then $d \le (a + c) + s$ for some $s \le v + w$ by (QF1), and $s \le s' + s''$ for some $s' \le v$ and $s'' \le w$ by (QF2.) Hence $d \le (a + s') + (c + s'')$. Because $a + s' \le a + v$ implies that $a + s' \le b + u$, we infer that $d \le (b + c) + (u + w)$. This shows $(a + c) \oplus (v + w) \le (b + c) \oplus (u + w)$. The embedding

$$a \mapsto a \oplus 0 \ :\ \mathcal{P} \to \mathcal{P}_{\mathcal{V}}$$

therefore preserves the algebraic operations and the order of $\mathcal{P}$, since $a \le b$ holds for elements $a, b \in \mathcal{P}$ if and only $a \oplus 0 \le b \oplus 0$ holds in $\mathcal{P}_{\mathcal{V}}$. Moreover, for a neighborhood $v \in \mathcal{V}$ and $a, b \in \mathcal{P}$ we have $a \le b + v$ in $\mathcal{P}$ if and only if $a \oplus 0 \le (b \oplus 0) + (0 \oplus v) = b \oplus v$ holds in $\mathcal{P}_{\mathcal{V}}$. We may therefore identify the neighborhoods $v \in \mathcal{V}$ with the elements $0 \oplus v$ in $\mathcal{P}_{\mathcal{V}}$. In this way $\mathcal{V}$ is embedded into $\mathcal{P}_{\mathcal{V}}$ as well, and $(\mathcal{P}_{\mathcal{V}}, \mathcal{V})$ becomes a full locally convex cone, containing $(\mathcal{P}, \mathcal{V})$ as a subcone. If a certain neighborhood $v \in \mathcal{V}$ is already contained in the given cone $\mathcal{P}$, then the above definition of the order in $\mathcal{P}_{\mathcal{V}}$ yields that both $v \oplus 0 \le 0 \oplus v$ and $0 \oplus v \le v \oplus 0$. The elements $v \oplus 0$ and $0 \oplus v$ are therefore equivalent with respect to the canonical equivalence relation defined by the order on $\mathcal{P}_{\mathcal{V}}$. Thus for a full cone $\mathcal{P}$, this extension $\mathcal{P}_{\mathcal{V}}$ yields only elements that in terms of the order relation are equivalent to existing ones in $\mathcal{P}$. We shall call $(\mathcal{P}_{\mathcal{V}}, \mathcal{V})$ the *standard full extension* of the locally convex cone $(\mathcal{P}, \mathcal{V})$.

Theorem 6.3. *Let $(\mathcal{P},\mathcal{V})$ be a quasi-full locally convex cone, and let $(\mathcal{Q},\mathcal{W})$ be a locally convex complete lattice cone. Every continuous linear operator $T:\mathcal{P}\to\mathcal{Q}$ can be extended to a continuous linear operator $\overline{T}:\mathcal{P}_{\mathcal{V}}\to\mathcal{Q}$.*

Proof. Let $(\mathcal{P},\mathcal{V})$ be quasi-full, $(\mathcal{Q},\mathcal{W})$ a complete lattice cone, and let $T:\mathcal{P}\to\mathcal{Q}$ be a continuous linear operator. Recall from Section 3 that a continuous linear operator between locally convex cones is monotone with respect to the respective weak preorders. Because $\mathcal{Q}$ carries its weak preorder, this implies monotonicity with respect to the given orders of $\mathcal{P}$ and $\mathcal{Q}$ as well. For an element $a\oplus v\in\mathcal{P}_{\mathcal{V}}$ we define

$$\overline{T}(a\oplus v)=\sup\big\{T(b)\mid b\in\mathcal{P},\ b\le a+v\big\}\in\mathcal{Q}.$$

Let us first check linearity: Clearly $\overline{T}(\alpha a\oplus\alpha v)=\alpha\overline{T}(a\oplus v)$ for $\alpha\ge 0$. For additivity, let $(a\oplus v),(b\oplus u)\in\mathcal{P}_{\mathcal{V}}$. Using Lemma 5.5(a), we infer

$$\begin{aligned}\overline{T}(a\oplus v)+\overline{T}(b\oplus u)&=\sup\big\{T(c)\mid c\in\mathcal{P},\ c\le a+v\big\}\\&\qquad+\sup\big\{T(d)\mid d\in\mathcal{P},\ d\le b+u\big\}\\&=\sup\big\{T(c+d)\mid c,d\in\mathcal{P},\ \ c\le a+v,\ \ d\le b+u\big\}\\&\le\sup\big\{T(e)\mid e\in\mathcal{P},\ \ e\le(a+b)+(v+u)\big\}\\&=\overline{T}\big((a\oplus v)+(b\oplus u)\big).\end{aligned}$$

If on the other hand $c\le(a+b)+(v+u)$ for $c\in\mathcal{P}$, then $c\le c'+c''$ for some $c',c''\in\mathcal{P}$ such that $c'\le a+v$ and $c''\le b+v$. by (QF1) and (QF2). Thus

$$T(c)\le T(c')+T(c'')\le\overline{T}(a\oplus v)+\overline{T}(b\oplus u).$$

Taking the supremum over all such elements $c\le(a+b)+(v+u)$ on the left-hand side yields

$$\overline{T}\big((a\oplus v)+(b\oplus u)\big)\le\overline{T}(a\oplus v)+\overline{T}(b\oplus u).$$

Next we observe that the operator $\overline{T}$ is monotone. Indeed, let $a\oplus v\le b\oplus u$ for $(a\oplus v),(b\oplus u)\in\mathcal{P}_{\mathcal{V}}$, and let $c\in\mathcal{P}$ such that $c\le a+v$. Then $c\le b+u$ by our definition of the order in $\mathcal{P}_{\mathcal{V}}$. This shows $T(c)\le\overline{T}(b\oplus u)$. Taking the supremum over all such elements $c\le a+v$ on the left-hand side yields $\overline{T}(a\oplus v)\le\overline{T}(b\oplus u)$. Finally, for every $w\in\mathcal{W}$ there is $v\in\mathcal{V}$ such that $a\le b+v$ implies that $T(a)\le T(b)+w$ for all $a,b\in\mathcal{P}$. Thus

$$\overline{T}(0\oplus v)=\sup\big\{T(s)\mid s\in\mathcal{P},\ s\le v\big\}\le w.$$

As $(\mathcal{P}_{\mathcal{V}},\mathcal{V})$ is a full locally convex cone, this demonstrates the continuity of the monotone linear operator $\overline{T}:\mathcal{P}_{\mathcal{V}}\to\mathcal{Q}$. All left to verify is that $\overline{T}$ is indeed an extension of T if we consider $\mathcal{P}$ as a subcone of $\mathcal{P}_{\mathcal{V}}$ via its canonical embedding $a \mapsto (a \oplus 0)$. But this is obvious, as for $a \in \mathcal{P}$ we have

$$\overline{T}(a \oplus 0) = \sup\left\{T(b) \mid b \in \mathcal{P},\ b \le a\right\} = T(a).$$

□

Remarks 6.4. Let $(\mathcal{P}, \mathcal{V})$ be a (not necessarily quasi-full) locally convex cone satisfying the following condition:

(QF*) *For every $a \in \mathcal{P}$ and $v \in \mathcal{V}$ there is $s \in \mathcal{P}$ such that $s \le v$ and $\lambda \ge 0$ such that $0 \le a + \lambda s$.*

In this case we may remodel $\mathcal{P}$ into a quasi-full locally convex cone if we define an alternative neighborhood system $\mathfrak{V}$ consisting of all families $(r_v)_{v\in\mathcal{V}}$, where r_v is a non-negative real and $r_v > 0$ for at least one $v \in \mathcal{V}$ and $r_v = 0$ else. Endowed with componentwise defined algebraic operations and order $\mathfrak{V}_0 = \mathfrak{V} \cup 0$ is an ordered cone. Let $\widetilde{\mathcal{P}} = \mathcal{P} \oplus \mathfrak{V}_0$ be the direct sum of $\mathcal{P}$ and $\mathcal{V}_0$. We define the order on $\widetilde{\mathcal{P}}$ in the following way: We set $a \oplus r \le b \oplus s$ for elements $a, b \in \mathcal{P}$ and $r, s \in \mathcal{V}_0$ if $r \le s$ and if there are elements $c_1, \ldots, c_n \in \mathcal{P}$ such that $a \le b + (c_1 + \ldots + c_n)$ and $c_i \le (s_{v_i} - r_{v_i})v_i$ for distinct elements $v_1, \ldots, v_n \in \mathcal{V}$. In this way, $(\widetilde{\mathcal{P}}, \mathfrak{V})$ becomes a full locally convex cone. Condition (QF*) in particular guarantees that its elements are bounded below. The neighborhoods $u \in \mathcal{V}$ may be identified with the elements $r(u) \in \mathfrak{V}$ such that $r(u)_u = 1$ and $r(u)_v = 0$ else. As a subcone of $(\widetilde{\mathcal{P}}, \mathfrak{V})$, the locally convex cone $(\mathcal{P}, \mathfrak{V})$ is seen to be quasi-full. (Conditions (QF1) and (QF2) from 6.1 are implied by our definition of the neighborhoods in $\mathfrak{V}$.) Because $a \le b \oplus r(v)$ implies $a \le b + v$ for $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$, the (upper, lower, symmetric) topologies induced on $\mathcal{P}$ by $\mathfrak{V}$ are generally finer than the given ones. The dual cone $\mathcal{P}^*_{\mathfrak{V}}$ of $\mathcal{P}$ under this new locally convex topology is therefore larger than the given dual cone $\mathcal{P}^*$. The polar r^* of a neighborhood $r \in \mathfrak{V}$ consists of all monotone $\overline{\mathbb{R}}$-valued linear functionals μ on $\mathcal{P}$ satisfying $\mu(c) \le 1$ for all $c \in \mathcal{P}$ such that $c \le r$.

7. Cones of Linear Operators

Endowed with the canonical (pointwise) algebraic operations, the linear operators between two cones $\mathcal{N}$ and $\mathcal{M}$ form again a cone $L(\mathcal{N}, \mathcal{M})$. We may introduce neighborhoods for $L(\mathcal{N}, \mathcal{M})$ in the following way (for a similar construction in the case of vector spaces see III.3 in [185]): Let $\mathcal{W}$ be a neighborhood system and let $\le$ be an order for $\mathcal{M}$ such that $(\mathcal{M}, \mathcal{W})$ is a locally convex cone. Let $\mathfrak{Z}$ be a family of subsets of $\mathcal{N}$, directed upward by set inclusion. For every $Z \in \mathfrak{Z}$ and $w \in W$ we define a neighborhood $V_{(Z,w)}$, setting $S \le T + V_{(Z,w)}$ for linear operators $S, T \in L(\mathcal{N}, \mathcal{M})$ if

$$S(a) \le T(a) + w \quad \text{for all} \quad a \in Z.$$

The collection $\mathfrak{V}_{(\mathfrak{Z},\mathcal{W})} = \{V_{(Z,w)} \mid Z \in \mathfrak{Z},\ w \in \mathcal{W}\}$ of these neighborhoods defines a convex quasiuniform structure on a subcone $\mathfrak{H}(\mathcal{N},\mathcal{M})$ of $L(\mathcal{N},\mathcal{M})$ in the sense of I.5.3 in [100] provided that its elements are bounded below, that is if for each $T \in \mathfrak{H}(\mathcal{N},\mathcal{M})$ and $Z \in \mathfrak{Z}$ and $w \in \mathcal{W}$ there is $\lambda \geq 0$ such that

$$0 \leq T(a) + \lambda w \quad \text{for all} \quad a \in Z.$$

It is elaborated in I.5.3 [100] how such a convex quasiuniform structure can be used to construct an abstract neighborhood system $\mathfrak{V}$ for the cone $\mathfrak{H}(\mathcal{N},\mathcal{M})$, turning $(\mathfrak{H}(\mathcal{N},\mathcal{M}),\mathfrak{V})$ into a locally convex cone in such a way that the neighborhoods in $\mathfrak{V}_{(\mathfrak{Z},\mathcal{W})}$ form a basis for the neighborhood system $\mathfrak{V}$. In fact, all that needs to be done is to define suitable sums of the elements of $\mathfrak{V}_{(\mathfrak{Z},\mathcal{W})}$ and thus create a cone that can be adjoined to $\mathfrak{H}(\mathcal{N},\mathcal{M})$. The induced order for $\mathfrak{H}(\mathcal{N},\mathcal{M})$ is given by $S \leq T$ for operators $S, T \in \mathfrak{H}(\mathcal{N},\mathcal{M})$ if

$$S(a) \leq T(a) + w \qquad \text{for all} \qquad a \in \bigcup_{Z\in\mathfrak{Z}} Z \quad \text{and} \quad w \in \mathcal{W}.$$

Alternatively, if Condition (QF*) from 6.4 holds for the neighborhoods in $\mathfrak{V}_{\mathfrak{Z}}$ (with the order from above), then we may use the procedure from 6.4 in order to turn $\mathfrak{H}(\mathcal{N},\mathcal{M})$ into a quasi-full locally convex cone $(\mathfrak{H}(\mathcal{N},\mathcal{M}),\widetilde{\mathfrak{V}})$. As elaborated in 6.4, the topologies induced by $\widetilde{\mathfrak{V}}$ are generally finer than those resulting from $\mathfrak{V}$.

Remark 7.1. The standard lattice completion $\widehat{\mathfrak{H}}(\mathcal{N},\mathcal{M})$ of $\mathfrak{H}(\mathcal{N},\mathcal{M})$ (see 5.57) leads to a rather unwieldy setting in this case. It consists of $\overline{\mathbb{R}}$-valued functions defined on the dual cone $\mathfrak{H}(\mathcal{N},\mathcal{M})^*$ which is difficult to approach and depends on the particular topology of $\mathfrak{H}(\mathcal{N},\mathcal{M})$, that is the choice for the family $\mathfrak{Z}$ of subsets of $\mathcal{N}$. It is therefore preferable to employ a simplified lattice completion $\widehat{\mathfrak{H}}(\mathcal{N},\mathcal{M})_{\Upsilon}$ in the sense of 5.61 for which we shall use the subset $\Upsilon = \left(\bigcup_{Z\in\mathfrak{Z}} Z\right) \times M^*$ of $\mathfrak{H}(\mathcal{N},\mathcal{M})^*$, consisting of all continuous linear functionals whose elements (a,μ) act as linear functionals on $\mathfrak{H}(\mathcal{N},\mathcal{M})$ as

$$(a,\mu)(T) = \mu\big(T(a)\big) \qquad \text{for all} \qquad T \in \mathfrak{H}(\mathcal{N},\mathcal{M}).$$

By our definition of the neighborhoods in $\mathfrak{H}(\mathcal{N},\mathcal{M})$, this set Υ supports the separation property. The locally convex cone $\mathfrak{H}(\mathcal{N},\mathcal{M})$ is therefore embedded into $\widehat{\mathfrak{H}}(\mathcal{N},\mathcal{M})_{\Upsilon}$, which in turn permits a more easily accessible realization of the lattice completion of $\mathfrak{H}(\mathcal{N},\mathcal{M})$. In case that the subcone spanned by the sets $Z \in \mathfrak{Z}$ is all of $\mathcal{N}$, we may interpret the elements of the order closure of $\mathfrak{H}(\mathcal{N},\mathcal{M})$ in its lattice completion $\widehat{\mathfrak{H}}(\mathcal{N},\mathcal{M})$ as linear operators from $\mathcal{N}$ into $\mathcal{M}^{**}$, the second dual of $\mathcal{M}$. Indeed, we observed in 5.60(a) that every element $\varphi \in \widehat{\mathfrak{H}}(\mathcal{N},\mathcal{M})$ in the order closure of $\mathfrak{H}(\mathcal{N},\mathcal{M})$ is a linear functional on $\mathfrak{H}(\mathcal{N},\mathcal{M})^*$. Since $\Upsilon = \mathcal{N} \times \mathcal{M}^* \subset \mathfrak{H}(\mathcal{N},\mathcal{M})^*$ as elaborated above, the function $\varphi : \mathcal{N} \times \mathcal{M}^* \to \overline{\mathbb{R}}$ is linear in both arguments from $\mathcal{N}$ and from $\mathcal{M}^*$. Thus the mapping

$$a \mapsto \varphi_a : \mathcal{N} \to \mathcal{M}^{**},$$

where $\varphi_a(\mu) = \varphi(a, \mu)$ for $\mu \in \mathcal{M}^*$ is indeed a linear operator from $\mathcal{N}$ into $\mathcal{M}^{**}$. Moreover, if both $\mathcal{N}$ and $\mathcal{M}$ are in fact vector spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ and if all operators in $\mathfrak{H}(\mathcal{N}, \mathcal{M})$ are $\mathbb{K}$-linear, then a similar argument shows the every function φ in the order closure of $\mathfrak{H}(\mathcal{N}, \mathcal{M})$ in $\widehat{\mathfrak{H}}(\mathcal{N}, \mathcal{M})$ can be interpreted as a $\mathbb{K}$-linear operator from $\mathcal{N}$ into $\mathcal{M}^{**}$.

Examples 7.2. (a) If both $(\mathcal{N}, \mathcal{U})$ and $(\mathcal{M}, \mathcal{W})$ are locally convex cones, and if all the sets $Z \in \mathfrak{Z}$ are bounded below in $\mathcal{N}$, then every continuous linear operator from $\mathcal{N}$ in to $\mathcal{M}$ is bounded below with respect to the neighborhoods in $\mathfrak{V}_{(\mathfrak{Z}, \mathcal{W})}$. Thus, if $\mathfrak{H}(\mathcal{N}, \mathcal{M})$ is a cone of continuous linear operators from $\mathcal{N}$ into $\mathcal{M}$, we may consider either of the following:

(i) If $\mathfrak{Z}$ is the family of all bounded below subset of $\mathcal{N}$, we obtain the *uniform operator topology* for $\mathfrak{H}(\mathcal{N}, \mathcal{M})$. We may alternatively choose the families $\mathfrak{Z}$ of all bounded or of all relatively bounded subsets of $\mathcal{P}$ (see 4.24) in this case.

(ii) If $\mathfrak{Z}$ is the family of all finite subsets of $\mathcal{N}$, we obtain the *strong operator topology* for $\mathfrak{H}(\mathcal{N}, \mathcal{M})$.

We shall also consider topologies on $\mathfrak{H}(\mathcal{N}, \mathcal{M})$ that arise if $\mathcal{M}$ is endowed with an alternative weak topology $\sigma(\mathcal{M}, \mathcal{L})$ generated by a third cone $\mathcal{L}$ and a bilinear form on $\mathcal{M} \times \mathcal{L}$ (see II.3 in [100]). In particular:

(iii) If $\mathfrak{Z}$ is the family of all finite subsets of $\mathcal{N}$, and if $\mathcal{M}$ is endowed with the topology $\sigma(\mathcal{M}, \mathcal{M}^*)$, we obtain the *weak operator topology* for $\mathfrak{H}(\mathcal{N}, \mathcal{M})$.

(iv) If $\mathfrak{Z}$ is the family of all finite subsets of $\mathcal{N}$, and if $\mathcal{M} = \mathcal{L}^*$ is the dual cone of some locally convex cone $(\mathcal{L}, \mathcal{V})$, endowed with the topology $\sigma(\mathcal{L}^*, \mathcal{L})$, we obtain the *weak* operator topology* for $\mathfrak{H}(\mathcal{N}, \mathcal{L}^*)$.

(b) If $\mathfrak{Z}$ consists of the set $Z = \mathcal{N}$, then $V = 0$ is the only resulting neighborhood for $L(\mathcal{N}, \mathcal{M})$, and boundedness from below requires that we consider linear operators that take only positive values on $\mathcal{N}$ for the cone $\mathfrak{H}(\mathcal{N}, \mathcal{M})$. The resulting order for operators $S, T \in \mathfrak{H}(\mathcal{N}, \mathcal{M})$ is $S \leq T$ if $S(a) \leq T(a)$ for all $a \in \mathcal{N}$. If on the other hand, $\mathfrak{Z}$ consists of the set $Z = \{0\}$, then $V = \infty$ is the only resulting neighborhood and the indiscrete topology arises for any subcone $\mathfrak{H}(\mathcal{N}, \mathcal{M})$ of $L(\mathcal{N}, \mathcal{M})$.

(c) If $\mathcal{N} = \mathcal{M}$, then $\mathfrak{H}(\mathcal{M}, \mathcal{M}) = \mathbb{R}_+ = \{a \in \mathbb{R} \mid a \geq 0\}$ is an example of a cone of linear operators on $\mathcal{M}$, with the scalar multiplication as its operation. If $(\mathcal{M}, \mathcal{W})$ is a locally convex cone and if $\mathfrak{Z}$ is an upward directed family of bounded below subsets of subsets of $\mathcal{M}$, then the neighborhood $V_{(Z,w)}$ in $\mathbb{R}_+$ corresponding to some $Z \in \mathfrak{Z}$ and $w \in \mathcal{W}$ according to the above is given by $\alpha \leq \beta + V_{(Z,w)}$ for $\alpha, \beta \in \mathbb{R}_+$ if

$$\alpha a \leq \beta a + w \qquad \text{for all} \qquad a \in Z.$$

If all elements of the set Z are bounded in $\mathcal{M}$, then this condition can be interpreted as follows: Let $\delta = \inf\{\lambda \geq 0 \mid 0 \leq a + \lambda w \text{ for all } a \in Z\}$ and $\gamma = \inf\{\lambda \geq 0 \mid a \leq \lambda w \text{ for all } a \in Z\}$. A simple argument using the cancellation rule I.4.2 in [100] then yields that the above is equivalent to

$$\beta \leq \alpha + \frac{1}{\delta} \qquad \text{and} \qquad \alpha \leq \beta + \frac{1}{\gamma}.$$

(We set of course $\frac{1}{0} = +\infty$ and $\frac{1}{+\infty} = 0$ is these expressions.) Thus depending on our choice for $\mathfrak{Z}$, one of the following can emerge as the upper neighborhoods $V_{(Z,w)}(\alpha)$ for an element $\alpha \in \mathbb{R}_+$: The intervals (for $\varepsilon > 0$) (i) $[\alpha - \varepsilon, \alpha + \varepsilon]$, yielding the Euclidean topology with equality as order; (ii) $[0, \alpha + \varepsilon]$, yielding the upper Euclidean topology with the natural order; (iii) $[\alpha - \varepsilon, +\infty)$, yielding the lower Euclidean topology with reverse natural order; (iv) $[\alpha - \varepsilon, \alpha]$, yielding the equality as order; (v) $[0, \alpha]$, yielding the natural order. Note that only in cases (ii) and (v) the resulting locally convex cone $(\mathbb{R}_+, \mathfrak{V})$ is quasi-full.

If $(\mathcal{N}, \mathcal{U})$ is indeed a locally convex topological vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, endowed with its (modular) symmetric topology, then we may also consider $\mathfrak{H}(\mathcal{N}, \mathcal{N}) = \mathbb{K}$. Most useful choices for $\mathfrak{Z}$ will yield the Euclidean neighborhoods $\mathbb{B}_\varepsilon(\alpha) = \{\beta \in \mathbb{K} \mid |\beta - \alpha| \leq \varepsilon\}$ for elements α of $\mathbb{K}$ and the equality as order. Alternatively, there may be a subcone C of negative elements in $\mathbb{K}$ in this case, and the upper neighborhoods are the sets $\mathbb{B}_\varepsilon(\alpha) + C$.

(d) Every locally convex cone $(\mathcal{P}, \mathcal{V})$ can be represented as a locally convex cone of linear operators. Indeed, algebraically, $\mathcal{P}$ coincides with the cone $\mathfrak{H}(\mathbb{R}_+, \mathcal{P})$ of all linear operators from $\mathbb{R}_+ = \{a \in \mathbb{R} \mid a \geq 0\}$ into $\mathcal{P}$ if we identify an element $a \in \mathcal{P}$ with the operator $\alpha \mapsto \alpha a$ in $\mathfrak{H}(\mathbb{R}_+, \mathcal{P})$. The neighborhoods of $\mathcal{P}$ may be recovered for $\mathfrak{H}(\mathbb{R}_+, \mathcal{P})$ if we use the above procedure with $\mathfrak{Z}$ containing only the singleton set $\{1\} \subset \mathbb{R}_+$. We obtain a copy of the locally convex cone $(\mathcal{P}, \mathcal{V})$.

7.3 Cones of Linear Functionals. The Second Dual. Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone. In the general settings of this section we choose $(\mathcal{N}, \mathcal{U}) = (\mathcal{P}, \mathcal{V})$ and $\mathcal{M} = \overline{\mathbb{R}}$ with its usual neighborhood system $\mathcal{W} = \{\varepsilon \in \mathbb{R} \mid \varepsilon > 0\}$ (see 1.4(a)). For the subcone $\mathfrak{H}(\mathcal{P}, \overline{\mathbb{R}})$ of $L(\mathcal{P}, \overline{\mathbb{R}})$ we choose the dual $\mathcal{P}^*$ of $\mathcal{P}$. As in 7.2(a) let $\mathfrak{Z}$ be a family of bounded below subsets of $\mathcal{P}$. In this way, $(\mathcal{P}^*, \mathfrak{V})$ becomes a locally convex cone. Its own dual cone, that is the second dual of $\mathcal{P}$, then is well-defined and depends on the choice for the topology of $\mathcal{P}^*$, that is on the choice for the family $\mathfrak{Z}$ of subsets of $\mathcal{P}$. Considering the particular choices for $\mathfrak{Z}$ as elaborated in 7.2(a) we shall use the following notations for the second dual of a locally convex cone $(\mathcal{P}, \mathcal{V})$:

(i) $\mathcal{P}^{**}$ denotes the cone of all $\overline{\mathbb{R}}$-valued linear functionals on $\mathcal{P}^*$.

(ii) $\mathcal{P}^{**}_{sl}$, $\mathcal{P}^{**}_{sr}$ and $\mathcal{P}^{**}_{sl}$ denote the dual of $(\mathcal{P}^*, \mathfrak{V})$ if $\mathfrak{Z}$ consists of all (bounded below, relatively bounded) or bounded subsets of $\mathcal{P}$. These are referred to as the *(lower strong, relative strong)* or *strong second dual* of $\mathcal{P}$.

(iii) $\mathcal{P}_w^{**}$ denotes the *weak second dual* of $\mathcal{P}$, that is the dual of $(\mathcal{P}^*, \mathfrak{V})$ if $\mathfrak{Z}$ consists of all finite subsets of $\mathcal{P}$.

Since every element $a \in \mathcal{P}$ acts as an $\overline{\mathbb{R}}$-valued linear functional φ_a on $\mathcal{P}^*$, and since this linear functional is obviously contained in the polar of the neighborhood $V_{(Z,1)}$, where $Z = \{a\} \subset \mathcal{P}$, the given cone $\mathcal{P}$ can be envisioned as a subcone of its second dual $\mathcal{P}_w^{**}$. Indeed, we have

$$\mathcal{P} \subset \mathcal{P}_w^{**} \subset \mathcal{P}_s^{**} \subset \mathcal{P}_{sr}^{**} \subset \mathcal{P}_{sl}^{**} \subset \mathcal{P}^{**}$$

in general. Now let us recall the construction of the standard lattice completion $\widehat{\mathcal{P}}$ of $\mathcal{P}$ from Section 5.57. The elements of $\widehat{\mathcal{P}}$ were realized as $\overline{\mathbb{R}}$-valued functions on $\mathcal{P}^*$, and in Remark 5.60(a) we observed that the elements of the order closure of $\mathcal{P}$ in $\widehat{\mathcal{P}}$ are linear on $\mathcal{P}^*$. This order closure can therefore also be considered as a subcone of $\mathcal{P}^{**}$.

Furthermore, we observe that for every choice of the family $\mathfrak{Z}$ of bounded below subsets of $\mathcal{P}$ the linear functionals in the thus generated second dual $\mathcal{P}_{\mathfrak{Z}}^{**}$ of $\mathcal{P}$, if considered as $\overline{\mathbb{R}}$-valued functions on $\mathcal{P}^*$, are bounded below on the polars of all neighborhoods in $\mathcal{V}$. Indeed, every functional φ in the second dual $\mathcal{P}_{\mathfrak{Z}}^{**}$ of $\mathcal{P}$ is contained in the polar of some neighborhood $V_{(Z,\varepsilon)}$ for $Z \in \mathfrak{Z}$ and $\varepsilon > 0$. Given $v \in \mathcal{V}$ there is $\lambda \geq 0$ such that $0 \leq z + \lambda v$ for all $z \in Z$. Then for every $\mu \in v^\circ$ we have $0 \leq \mu(z) + \lambda$ for all $z \in Z$, hence $0 \leq \mu + (\lambda/\varepsilon) V_{(Z,\varepsilon)}$ by the definition of the neighborhood $V_{(Z,\varepsilon)}$. Since $\varphi \in V_{(Z,\varepsilon)}^\circ$, this yields $\varphi(\mu) \geq -(\lambda/\varepsilon)$ for all $\mu \in v^\circ$. Consequently, for every such choice of the family $\mathfrak{Z}$, the resulting second dual $\mathcal{P}_{\mathfrak{Z}}^{**}$ of $\mathcal{P}$ may be considered as a subcone of the locally convex complete lattice cone $\mathcal{F}_{\widehat{\mathcal{V}}_b}(\mathcal{P}^*, \overline{\mathbb{R}})$ from 5.57. Recall that the standard lattice completion $\widehat{\mathcal{P}}$ of $\mathcal{P}$ had been introduced as the smallest locally convex complete lattice subcone of $\mathcal{F}_{\widehat{\mathcal{V}}_b}(\mathcal{P}^*, \overline{\mathbb{R}})$ that contains $\mathcal{P}$ (see 5.57). We have $\mathcal{P} \subset \mathcal{P}_{\mathfrak{Z}}^{**} \subset \mathcal{F}_{\widehat{\mathcal{V}}_b}(\mathcal{P}^*, \overline{\mathbb{R}})$ for any such choice of the family $\mathfrak{Z}$ by the above.

Therefore both the order closure of $\mathcal{P}$ in $\widehat{\mathcal{P}}$ and the second dual $\mathcal{P}_{\mathfrak{Z}}^{**}$ are contained in the intersection of $\mathcal{P}^{**}$ and $F_{\widehat{\mathcal{V}}_b}(\mathcal{P}^*, \overline{\mathbb{R}})$, but it is in general not possible to identify one of these as a subcone of the other.

We can, however, add the following often helpful observation: Let Z be a bounded below subset of $\mathcal{P}$, and suppose that the element $\varphi \in \widehat{\mathcal{P}}$ is in the closure with respect to the order topology of (the embedding of) Z in $\widehat{\mathcal{P}}$. This means that there is a net $(a_i)_{i \in \mathcal{I}}$ in Z converging pointwise as functions on $\mathcal{P}^*$ towards φ, that is

$$\varphi(\mu) = \lim_{i \in \mathcal{I}} \mu(a_i)$$

for all $\mu \in \mathcal{P}^*$. (Convergence is meant in the usual (order) topology of $\overline{\mathbb{R}}$.) Then the function φ is linear on $\mathcal{P}^*$ and $\mu \leq \nu + V_{(Z,1)}$ for elements $\mu, \nu \in \mathcal{P}^*$ implies that $\mu(a_i) \leq \nu(a_i) + 1$ holds for all $i \in \mathcal{I}$, and therefore

$\varphi(\mu) \le \varphi(\nu)+1$ as well. This shows that $\varphi \in V^{\circ}_{(Z,1)}$. Hence the element $\varphi \in \widehat{\mathcal{P}}$ is contained in the dual cone $\mathcal{P}^{**}_{\mathfrak{Z}}$ of $(\mathcal{P}^*, \mathfrak{V})$ whenever the neighborhood generating family $\mathfrak{Z}$ contains the bounded below set Z.

For a locally convex vector space $(\mathcal{P}, \mathcal{V})$ over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ the different notions in (ii) for the strong second dual coincide, and according to 2.1(d) every real-valued (real) linear functional $\varphi \in \mathcal{P}^{**}$ corresponds canonically to a $\mathbb{K}$-valued $\mathbb{K}$-linear functional $\varphi_{\mathbb{K}}$ on $\mathcal{P}^*_{\mathbb{K}}$, that is an element of $\mathcal{P}^{**}_{\mathbb{K}}$, the (algebraic) second vector space dual of $\mathcal{P}$.

These final observations will prove particularly useful in the subsequent chapters when we shall investigate integrals of cone-valued functions.

8. Notes and Remarks

The theory of locally convex cones originated in a joint work [100] by the author and K. Keimel in 1992. We were then looking for a suitable setting for the formulation of Korovkin-type approximation theory which deals with certain restricted classes of continuous linear operators on locally convex vector spaces. These may be positive operators on ordered spaces, contractions on normed spaces, multiplicative operators on Banach algebras, etc. Approximation processes are modeled by sequences or nets of operators in such a class. The given restrictions then guarantee convergence towards the identity operator on a large subset of their domain if this property can be checked for a relatively small test set. The use of locally convex cones instead of locally convex vector spaces turns out to be very advantageous in this context, since it allows to formulate all those different restriction properties for the operators in terms of the order structure alone, thus yielding a unifying approach. Subsequently the theory of locally convex cones has been expanded, mostly by the author of this book. Readers interested in further aspects of the subject should in particular familiarize themselves with the Hahn-Banach type extension and separation results that were laid out in [172] and form the foundations for the duality theory of locally convex cones. Ordered cones were earlier studied by various authors, in particular Fuchssteiner and Lusky [63] whose book contains a Hahn-Banach type sandwich theorem for $\big(\mathbb{R}\cup\{-\infty\}\big)$-valued linear functionals on an ordered cone, a non-topological predecessor to the results from [172]. An in-depth investigation for the relationship between order and topology can be found in the seminal work [135] by Nachbin. The compendia of continuous lattices [68] and [69] by Gierz, Hofmann, Keimel, Lawson, Mislove and Scott contain a detailed analysis of various ways to introduce topologies on lattices.

The weak (global) preorder $\preccurlyeq$ as defined in Section 3 has an earlier analogue in the (global) preorder $\preceq$ which was defined in Section I.3 of [100] for elements $a, b \in \mathcal{P}$ as follows:

$$a \preceq b \qquad \text{if} \qquad a \leq b + v$$

for all $v \in \mathcal{V}$. Clearly $a \leq b$ implies $a \preceq b$ which in turn implies that $a \preccurlyeq b$. In some sense the preorder $\preceq$ can be considered as a topological closure of the given order $\leq$ whereas the weak preorder $\preccurlyeq$ signifies a closure with respect to both topology and the linear structure. Like for the weak preorder there is also a local version $\preceq_v$ of the preorder (see I.3 in [100]) referring to a particular neighborhood $v \in \mathcal{V}$ rather than the whole neighborhood system. Relationships between the different orders of a locally convex cone are investigated in detail in [175]. Since it provides the separation properties from Section 4, the weak preorder turns out to be the most suitable one for our purposes.

An excellent historical account of the extensive literature on ordered topological vector spaces can be found in the classical book by Day [39]. The notions of order convergence and of order topology for complete locally convex lattice cones from Section 5 are also used in ordered vector spaces, but introduced in a slightly different way which does not require a given topological or lattice structure (see Chapter V.6 in [185]). However, on topological vector lattices this notion coincides with ours form Section 5. Topological vector lattices had first been introduced as Banach lattices, and comprehensive treatments can for example be found in the books by Schäfer [184] and [185] and by Meyer-Nieberg [132]. In locally convex vector spaces there are compatibility requirements between the algebraic and the lattice operations as well as the topology. These are reflected in the corresponding requirements of Section 5 for locally convex cones. The strong conditions for locally convex lattice cones mirror those for M-topologies in topological vector lattices. Since under circumstances the latter permit representations as function spaces (see [94]), the result of Proposition 5.37 is not unexpected. Proposition 5.37 gives also the reason for using $\overline{\mathbb{R}}$-valued functions in the standard lattice completion of a locally convex cone. General lattices carrying different orders leading to notions of order convergence and of approximation of elements are thoroughly investigated in [68] and [69].