Chapter 13
Bökstedt–Neeman Resolutions and HyperExt Sheaves

(13.1) Let $\mathcal{T}$ be a triangulated category with small direct products. Note that a direct product of distinguished triangles is again a distinguished triangle (Lemma 3.1).

Let

$$\cdots \to t_3 \xrightarrow{s_3} t_2 \xrightarrow{s_2} t_1 \tag{13.2}$$

be a sequence of morphisms in $\mathcal{T}$. We define $d : \prod_{i\geq 1} t_i \to \prod_{i\geq 1} t_i$ by $p_i \circ d = p_i - s_{i+1} \circ p_{i+1}$, where $p_i : \prod_i t_i \to t_i$ is the projection. Consider a distinguished triangle of the form

$$M \xrightarrow{m} \prod_{i\geq 1} t_i \xrightarrow{d} \prod_{i\geq 1} t_i \xrightarrow{q} \Sigma M,$$

where Σ denotes the suspension.

We call M, which is determined uniquely up to isomorphisms, the *homotopy limit* of (13.2) and denote it by $\operatorname{holim} t_i$.

(13.3) Dually, *homotopy colimit* is defined and denoted by hocolim, if $\mathcal{T}$ has small coproducts.

(13.4) Let $\mathcal{A}$ be an abelian category which satisfies (AB3*). Let $(\mathbb{F}_\lambda)_{\lambda\in\Lambda}$ be a small family of objects in $K(\mathcal{A})$. Then for any $\mathbb{G} \in K(\mathcal{A})$, we have that

$$\begin{aligned}\operatorname{Hom}_{K(\mathcal{A})}(\mathbb{G}, \prod_\lambda \mathbb{F}_\lambda) &= H^0(\operatorname{Hom}^\bullet_{\mathcal{A}}(\mathbb{G}, \prod_\lambda \mathbb{F}_\lambda)) \cong H^0(\prod_\lambda \operatorname{Hom}^\bullet_{\mathcal{A}}(\mathbb{G}, \mathbb{F}_\lambda)) \\ &\cong \prod_\lambda H^0(\operatorname{Hom}^\bullet_{\mathcal{A}}(\mathbb{G}, \mathbb{F}_\lambda)) = \prod_\lambda \operatorname{Hom}_{K(\mathcal{A})}(\mathbb{G}, \mathbb{F}_\lambda).\end{aligned}$$

That is, the direct product $\prod_\lambda \mathbb{F}_\lambda$ in $C(\mathcal{A})$ is also a direct product in $K(\mathcal{A})$.

(13.5) Let $\mathcal{A}$ be a Grothendieck abelian category, and (t_λ) a small family of objects of $D(\mathcal{A})$. Let $(\mathbb{F}_\lambda)$ be a family of K-injective objects of $K(\mathcal{A})$ such that $\mathbb{F}_\lambda$ represents t_λ for each λ. Then $Q(\prod_\lambda \mathbb{F}_\lambda)$ is a direct product of t_λ in

J. Lipman, M. Hashimoto, *Foundations of Grothendieck Duality for Diagrams of Schemes*, Lecture Notes in Mathematics 1960,

$D(\mathcal{A})$ (note that the direct product $\prod_\lambda \mathbb{F}_\lambda$ exists, see [37, Corollary 7.10]). Hence $D(\mathcal{A})$ has small products.

Lemma 13.6. *Let I be a small category, S be a scheme, and let $X_\bullet \in \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. Let $\mathbb{F}$ be an object of $C(\mathrm{Mod}(X_\bullet))$. Assume that $\mathbb{F}$ has locally quasi-coherent cohomology groups. Then the following hold.*

1 *Let $\mathfrak{I}$ denote the full subcategory of $C(\mathrm{Mod}(X_\bullet))$ consisting of bounded below complexes of injective objects of $\mathrm{Mod}(X_\bullet)$ with locally quasi-coherent cohomology groups. There is an $\mathfrak{I}$-special inverse system $(I_n)_{n\in\mathbb{N}}$ with the index set $\mathbb{N}$ and an inverse system of chain maps $(f_n : \tau_{\geq -n}\mathbb{F} \to I_n)$ such that*

- **i** *f_n is a quasi-isomorphism for any $n \in \mathbb{N}$.*
- **ii** *$I_n^i = 0$ for $i < -n$.*

2 *If (I_n) and (f_n) are as in **1**, then the following hold.*

- **i** *For each $i \in \mathbb{Z}$, the canonical map $H^i(\varprojlim I_n) \to H^i(I_n)$ is an isomorphism for $n \geq \max(1, -i)$, where the projective limit is taken in the category $C(\mathrm{Mod}(X_\bullet))$, and $H^i(?)$ denotes the ith cohomology sheaf of a complex of sheaves.*
- **ii** *$\varprojlim f_n : \mathbb{F} \to \varprojlim I_n$ is a quasi-isomorphism.*
- **iii** *The projective limit $\varprojlim I_n$, viewed as an object of $K(\mathrm{Mod}(X))$, is the homotopy limit of (I_n).*
- **iv** *$\varprojlim I_n$ is K-injective.*

Proof. The assertion **1** is [39, (3.7)].

We prove **2**, **i**. Let $j \in \mathrm{ob}(I)$ and U an affine open subset of X_j. Then for any $n \geq 1$, I_n^i and $H^i(I_n)$ are $\Gamma((j,U), ?)$-acyclic for each $i \in \mathbb{Z}$. As I_n is bounded below, each $Z^i(I_n)$ and $B^i(I_n)$ are also $\Gamma((j,U), ?)$-acyclic, and the sequence

$$0 \to \Gamma((j,U), Z^i(I_n)) \to \Gamma((j,U), I_n^i) \to \Gamma((j,U), B^{i+1}(I_n)) \to 0 \quad (13.7)$$

and

$$0 \to \Gamma((j,U), B^i(I_n)) \to \Gamma((j,U), Z^i(I_n)) \to \Gamma((j,U), H^i(I_n)) \to 0 \quad (13.8)$$

are exact for each i, as can be seen easily, where B^i and Z^i respectively denote the ith coboundary and the cocycle sheaves.

In particular, the inverse system $(\Gamma((j,U), B^i(I_n)))$ is a Mittag-Leffler inverse system of abelian groups by (13.7), since $(\Gamma((j,U), I_n^i))$ is. On the other hand, as we have $H^i(I_n) \cong H^i(\mathbb{F})$ for $n \geq \max(1, -i)$, the inverse system $(\Gamma((j,U), H^i(I_n)))$ stabilizes, and hence we have $(\Gamma((j,U), Z^i(I_n)))$ is also Mittag-Leffler.

Passing through the projective limit,

$$0 \to \Gamma((j,U), Z^i(\varprojlim I_n)) \to \Gamma((j,U), \varprojlim I_n) \to \Gamma((j,U), \varprojlim B^{i+1}(I_n)) \to 0$$

is exact. Hence, the canonical map $B^i(\varprojlim I_n) \to \varprojlim B^i(I_n)$ is an isomorphism, since (j, U) with U an affine open subset of X_j generates the topology of $\mathrm{Zar}(X_\bullet)$.

Taking the projective limit of (13.8), we have

$$0 \to \Gamma((j,U), B^i(\varprojlim I_n)) \to \Gamma((j,U), Z^i(\varprojlim I_n)) \to \Gamma((j,U), \varprojlim H^i(I_n)) \to 0$$

is an exact sequence for any j and any affine open subset U of X_j.

Hence, the canonical maps

$$\Gamma((j,U), H^i(I_n)) \cong \Gamma((j,U), \varprojlim H^i(I_n)) \leftarrow \Gamma((j,U), H^i(\varprojlim I_n))$$

are all isomorphisms for $n \geq \max(1, -i)$, and we have $H^i(I_n) \cong H^i(\varprojlim I_n)$ for $n \geq \max(1, -i)$.

The assertion **ii** is now trivial.

The assertion **iii** is now a consequence of [7, Remark 2.3] (one can work at the presheaf level where we have the (AB4*) property). The assertion **iv** is now obvious. □

Let I be a small category, S a scheme, and $X_\bullet \in \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$.

Lemma 13.9. *Assume that $X_\bullet$ has flat arrows. Let J be a subcategory of I, and let $\mathbb{F} \in D_{\mathrm{EM}}(X_\bullet)$ and $\mathbb{G} \in D(X_\bullet)$. Assume one of the following.*

a $\mathbb{G} \in D^+(X_\bullet)$.
b $\mathbb{F} \in D^+_{\mathrm{EM}}(X_\bullet)$.
c $\mathbb{G} \in D_{\mathrm{Lqc}}(X_\bullet)$.

Then the canonical map

$$H_J : (?)_J R\,\underline{\mathrm{Hom}}^\bullet_{\mathrm{Mod}(X_\bullet)}(\mathbb{F}, \mathbb{G}) \to R\,\underline{\mathrm{Hom}}^\bullet_{\mathrm{Mod}(X_\bullet|_J)}(\mathbb{F}_J, \mathbb{G}_J)$$

is an isomorphism of functors to $D(\mathrm{PM}(X_\bullet|_J))$ (here $\underline{\mathrm{Hom}}^\bullet_{\mathrm{Mod}(X_\bullet)}(?, *)$ is viewed as a functor to $\mathrm{PM}(X_\bullet)$, and similarly for $\underline{\mathrm{Hom}}_{\mathrm{Mod}(X_\bullet|_J)}(?, *)$). In particular, it is an isomorphism of functors to $D(X_\bullet|_J)$.

Proof. By Lemma 1.39, we may assume that $J = i$ for an object i of I.

So what we want to prove is for any complex in $\mathrm{Mod}(X_\bullet)$ with equivariant cohomology groups $\mathbb{F}$ and any K-injective complex $\mathbb{G}$ in $\mathrm{Mod}(X_\bullet)$,

$$H_i : \underline{\mathrm{Hom}}_{\mathrm{Mod}(X_\bullet)}(\mathbb{F}, \mathbb{G})_i \to \underline{\mathrm{Hom}}_{\mathrm{Mod}(X_i)}(\mathbb{F}_i, \mathbb{G}_i)$$

is a quasi-isomorphism of complexes in $\mathrm{PM}(X_i)$ (in particular, it is a quasi-isomorphism of complexes in $\mathrm{Mod}(X_i)$), under the additional assumptions corresponding to **a**, **b**, or **c**. Indeed, if so, $\mathbb{G}_i$ is K-injective by Lemma 8.4.

First consider the case that $\mathbb{F}$ is a single equivariant object. Then the assertion is true by Lemma 6.36. By the way-out lemma [17, Proposition I.7.1], the case that $\mathbb{F}$ is bounded holds. Under the assumption of **a**, the case that $\mathbb{F}$ is bounded above holds.

Now consider the general case for **a**. As the functors in question on $\mathbb{F}$ changes coproducts to products, the map in question is a quasi-isomorphism if $\mathbb{F}$ is a direct sum of complexes bounded above with equivariant cohomology groups. Indeed, a direct product of quasi-isomorphisms of complexes of $\mathrm{PM}(X_i)$ is again quasi-isomorphic. In particular, the lemma holds if $\mathbb{F}$ is a homotopy colimit of objects of $D^-_{\mathrm{EM}}(X_\bullet)$. As any object $\mathbb{F}$ of $D_{\mathrm{EM}}(X_\bullet)$ is the homotopy colimit of $(\tau_{\leq n}\mathbb{F})$, we are done.

The proof for the case **b** is similar. As $\mathbb{F}$ has bounded below cohomology groups, $\tau_{\leq n}\mathbb{F}$ has bounded cohomology groups for each n.

We prove the case **c**. By Lemma 13.6, we may assume that $\mathbb{G}$ is a homotopy limit of K-injective complexes with locally quasi-coherent bounded below cohomology groups. As the functors on $\mathbb{G}$ in consideration commute with homotopy limits, the problem is reduced to the case **a**. □

Lemma 13.10. *Let I be a small category, S a scheme, and $X_\bullet \in \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. Assume that $X_\bullet$ has flat arrows and is locally noetherian. Let $\mathbb{F} \in D^-_{\mathrm{Coh}}(X_\bullet)$ and $\mathbb{G} \in D^+_{\mathrm{Lqc}}(X_\bullet)$ (resp. $D^+_{\mathrm{Lch}}(X_\bullet)$), where* Lch *denotes the plump subcategory of* Mod *consisting of locally coherent sheaves. Then $\underline{\mathrm{Ext}}^i_{\mathcal{O}_{X_\bullet}}(\mathbb{F}, \mathbb{G})$ is locally quasi-coherent (resp. locally coherent) for $i \in \mathbb{Z}$. If, moreover, $\mathbb{G}$ has quasi-coherent (resp. coherent) cohomology groups, then $\underline{\mathrm{Ext}}^i_{\mathcal{O}_{X_\bullet}}(\mathbb{F}, \mathbb{G})$ is quasi-coherent (resp. coherent) for $i \in \mathbb{Z}$.*

Proof. We prove the assertion for the local quasi-coherence and the local coherence. By Lemma 13.9, we may assume that $X_\bullet$ is a single scheme. This case is [17, Proposition II.3.3].

We prove the assertion for the quasi-coherence (resp. coherence), assuming that $\mathbb{G}$ has quasi-coherent (resp. coherent) cohomology groups. By [17, Proposition I.7.3], we may assume that $\mathbb{F}$ is a single coherent sheaf, and $\mathbb{G}$ is an injective resolution of a single quasi-coherent (resp. coherent) sheaf.

As $X_\bullet$ has flat arrows and the restrictions are exact, it suffices to show that

$$\alpha_\phi : X_\phi^*(?)_i \,\underline{\mathrm{Hom}}^\bullet_{\mathrm{Mod}(X_\bullet)}(\mathbb{F}, \mathbb{G}) \to (?)_j \,\underline{\mathrm{Hom}}^\bullet_{\mathrm{Mod}(X_\bullet)}(\mathbb{F}, \mathbb{G})$$

is a quasi-isomorphism for any morphism $\phi : i \to j$ in I.

As X_ϕ is flat, $\alpha_\phi : X_\phi^*\mathbb{F}_i \to \mathbb{F}_j$ and $\alpha_\phi : X_\phi^*\mathbb{G}_i \to \mathbb{G}_j$ are quasi-isomorphisms. In particular, the latter is a K-injective resolution.

By the derived version of (6.37), it suffices to show that

$$P : X_\phi^* R \,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{X_i}}(\mathbb{F}_i, \mathbb{G}_i) \to R \,\underline{\mathrm{Hom}}^\bullet_{\mathcal{O}_{X_j}}(X_\phi^*\mathbb{F}_i, X_\phi^*\mathbb{G}_i)$$

is an isomorphism. This is [17, Proposition II.5.8]. □