Chapter 6
Operations on Sheaves Via the Structure Data

Let I be a small category, S a scheme, and $\mathcal{P} := \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. To study sheaves on objects of $\mathcal{P}$, it is convenient to utilize the structure data of them, and then utilize the usual sheaf theory on schemes.

(6.1) Let $X_\bullet \in \mathcal{P}$. Let $\heartsuit$ be any of $\mathrm{PA}, \mathrm{AB}, \mathrm{PM}, \mathrm{Mod}$, and $\mathcal{M}, \mathcal{N} \in \heartsuit(X_\bullet)$. An element (φ_i) in $\prod \mathrm{Hom}_{\heartsuit(X_i)}(\mathcal{M}_i, \mathcal{N}_i)$ is given by some $\varphi \in \mathrm{Hom}_{\heartsuit(X_\bullet)}(\mathcal{M}, \mathcal{N})$ (by the canonical faithful functor $\heartsuit(X_\bullet) \to \prod \heartsuit(X_i)$), if and only if

$$\varphi_j \circ \alpha_\phi(\mathcal{M}) = \alpha_\phi(\mathcal{N}) \circ (X_\phi)^*_\heartsuit(\varphi_i) \tag{6.2}$$

holds (or equivalently, $\beta_\phi(\mathcal{N}) \circ \varphi_i = (X_\phi)_* \varphi_j \circ \beta_\phi(\mathcal{M})$ holds) for any $(\phi : i \to j) \in \mathrm{Mor}(I)$.

We say that a family of morphisms $(\varphi_i)_{i \in I}$ between structure data

$$\varphi_i : \mathcal{M}_i \to \mathcal{N}_i$$

is a morphism of structure data if φ_i is a morphism in $\heartsuit(X_i)$ for each i, and (6.2) is satisfied for any ϕ. Thus the categories of structure data of sheaves, presheaves, modules, and premodules on $X_\bullet$, denoted by $\mathfrak{D}_\heartsuit(X_\bullet)$ are defined, and the equivalence $\mathrm{Dat}_\heartsuit : \heartsuit(X_\bullet) \cong \mathfrak{D}_\heartsuit(X_\bullet)$ are given. This is the precise meaning of Lemma 4.8.

(6.3) Let $X_\bullet \in \mathcal{P}$ and $\mathcal{M}, \mathcal{N} \in \mathrm{Mod}(X_\bullet)$. As in Example 5.6, **5**, we have an isomorphism

$$m_i : \mathcal{M}_i \otimes_{\mathcal{O}_{X_i}} \mathcal{N}_i \cong (\mathcal{M} \otimes_{\mathcal{O}_{X_\bullet}} \mathcal{N})_i.$$

This is trivial for presheaves, and utilize the fact the sheafification is compatible with $(?)_i$ for sheaves. At the section level, for $\mathcal{M}, \mathcal{N} \in \mathrm{PM}(X_\bullet)$, $i \in I$, and $U \in \mathrm{Zar}(X_i)$,

$$m_i^p : \Gamma(U, \mathcal{M}_i \otimes^p_{\mathcal{O}_{X_i}} \mathcal{N}_i) \to \Gamma(U, (\mathcal{M} \otimes^p_{\mathcal{O}_{X_\bullet}} \mathcal{N})_i)$$

J. Lipman, M. Hashimoto, *Foundations of Grothendieck Duality for Diagrams of Schemes*, Lecture Notes in Mathematics 1960,

is nothing but the identification

$$\Gamma(U,\mathcal{M}_i)\otimes_{\Gamma(U,\mathcal{O}_{X_i})}\Gamma(U,\mathcal{N}_i)=\Gamma((i,U),\mathcal{M})\otimes_{\Gamma((i,U),\mathcal{O}_{X_\bullet})}\Gamma((i,U),\mathcal{N})$$
$$=\Gamma((i,U),\mathcal{M}\otimes^p_{\mathcal{O}_{X_\bullet}}\mathcal{N}).$$

For $\mathcal{M},\mathcal{N}\in\operatorname{Mod}(X_\bullet)$ and $i\in I$, m_i is given as the composite

$$\mathcal{M}_i\otimes_{\mathcal{O}_{X_i}}\mathcal{N}_i=a(q\mathcal{M}_i\otimes^p_{\mathcal{O}_{X_i}}q\mathcal{N}_i)\xrightarrow{c}a((q\mathcal{M})_i\otimes^p_{\mathcal{O}_{X_i}}(q\mathcal{N})_i)\xrightarrow{m_i^p}$$
$$a(q\mathcal{M}\otimes^p_{\mathcal{O}_{X_\bullet}}q\mathcal{N})_i\xrightarrow{\theta}(a(q\mathcal{M}\otimes^p_{\mathcal{O}_{X_\bullet}}q\mathcal{N}))_i=(\mathcal{M}\otimes_{\mathcal{O}_{X_\bullet}}\mathcal{N})_i,$$

see (2.52). Utilizing this identification, the structure map α_ϕ of $\mathcal{M}\otimes\mathcal{N}$ can be completely described via those of $\mathcal{M}$ and $\mathcal{N}$. Namely,

Lemma 6.4. *Let $X_\bullet\in\mathcal{P}(I,\underline{\mathrm{Sch}}/S)$, and $\mathcal{M},\mathcal{N}\in\heartsuit(X_\bullet)$, where $\heartsuit$ is* PM *or* Mod. *For $\phi\in I(i,j)$, $\alpha_\phi(\mathcal{M}\otimes\mathcal{N})$ agrees with the composite map*

$$X_\phi^*(\mathcal{M}\otimes\mathcal{N})_i\xrightarrow{m_i^{-1}}X_\phi^*(\mathcal{M}_i\otimes\mathcal{N}_i)\xrightarrow{\Delta}X_\phi^*\mathcal{M}_i\otimes X_\phi^*\mathcal{N}_i\xrightarrow{\alpha_\phi\otimes\alpha_\phi}\mathcal{M}_j\otimes\mathcal{N}_j\xrightarrow{m_j}(\mathcal{M}\otimes\mathcal{N})_j,$$

where $\otimes$ should be replaced by $\otimes^p$ when $\heartsuit=$ PM.

Proof (*sketch*). It is not so difficult to show that it suffices to show that $\beta_\phi(\mathcal{M}\otimes\mathcal{N})$ agrees with the composite

$$(\mathcal{M}\otimes\mathcal{N})_i\xrightarrow{m_i^{-1}}\mathcal{M}_i\otimes\mathcal{N}_i\xrightarrow{\beta\otimes\beta}(X_\phi)_*\mathcal{M}_j\otimes(X_\phi)_*\mathcal{N}_j\xrightarrow{m}$$
$$(X_\phi)_*(\mathcal{M}_j\otimes\mathcal{N}_j)\xrightarrow{m_j}(X_\phi)_*(\mathcal{M}\otimes\mathcal{N})_j. \quad (6.5)$$

First we prove this for the case that $\heartsuit=$ PM. For an open subset U of X_i, this composite map evaluated at U is

$$\Gamma((i,U),(\mathcal{M}\otimes\mathcal{N}))=\Gamma((i,U),\mathcal{M})\otimes_{\Gamma((i,U),\mathcal{O}_{X_\bullet})}\Gamma((i,U),\mathcal{N})\xrightarrow{\mathrm{res}\otimes\mathrm{res}}$$
$$\Gamma((j,X_\phi^{-1}(U)),\mathcal{M})\otimes_{\Gamma((i,U),\mathcal{O}_{X_\bullet})}\Gamma((j,X_\phi^{-1}(U)),\mathcal{N})\xrightarrow{p}$$
$$\Gamma((j,X_\phi^{-1}(U)),\mathcal{M})\otimes_{\Gamma((j,X_\phi^{-1}(U)),\mathcal{O}_{X_\bullet})}\Gamma((j,X_\phi^{-1}(U)),\mathcal{N})$$
$$=\Gamma((j,X_\phi^{-1}(U)),\mathcal{M}\otimes\mathcal{N}),$$

where $p(m\otimes n)=m\otimes n$. This composite map is nothing but the restriction map of $\mathcal{M}\otimes\mathcal{N}$. So by definition, it agrees with

$$\beta_\phi:\Gamma(U,(\mathcal{M}\otimes\mathcal{N})_i)\to\Gamma(U,(X_\phi)_*(\mathcal{M}\otimes\mathcal{N})_j).$$

Next we consider the case $\heartsuit=$ Mod. First note that the diagram

$$\begin{array}{ccc}(a(q\mathcal{M}\otimes^p q\mathcal{N}))_i & \xrightarrow{\quad\quad\quad\quad\beta\quad\quad\quad\quad} & (X_\phi)_*(a(q\mathcal{M}\otimes^p q\mathcal{N}))_j \\ \uparrow\theta & & \uparrow\theta \\ a(q\mathcal{M}\otimes^p q\mathcal{N})_i & \xrightarrow{\beta} a(X_\phi)_*(q\mathcal{M}\otimes^p q\mathcal{N})_j \xrightarrow{\theta} & (X_\phi)_*a(q\mathcal{M}\otimes^p q\mathcal{N})_j\end{array}$$

is commutative by Lemma 4.17. By the presheaf version of the lemma, which has been proved in the last paragraph, the diagram

$$\begin{tikzcd}
a(q\mathcal{M}\otimes^p q\mathcal{N})_i \arrow[dd, "\beta"] & a((q\mathcal{M})_i\otimes^p (q\mathcal{N})_i) \arrow[l, "m_i"'] \arrow[d, "\beta\otimes\beta"] \\
 & a((X_\phi)_*(q\mathcal{M})_j\otimes^p (X_\phi)_*(q\mathcal{N})_j) \arrow[d, "m"] \\
a(X_\phi)_*(q\mathcal{M}\otimes^p q\mathcal{N})_j & a(X_\phi)_*((q\mathcal{M})_j\otimes^p (q\mathcal{N})_j) \arrow[l, "m_j"']
\end{tikzcd}$$

is commutative. By the commutativity of the diagram (4.16), the diagram

$$\begin{tikzcd}
a((q\mathcal{M})_i\otimes^p (q\mathcal{N})_i) \arrow[r, "c\otimes c"] \arrow[dd, "\beta\otimes\beta"] & a(q\mathcal{M}_i\otimes^p q\mathcal{N}_i) \arrow[d, "\beta\otimes\beta"] \\
 & a(q(X_\phi)_*\mathcal{M}_j\otimes^p q(X_\phi)_*\mathcal{N}_j) \arrow[d, "c\otimes c"] \\
a((X_\phi)_*(q\mathcal{M})_j\otimes^p (X_\phi)_*(q\mathcal{N})_j) \arrow[r, "c\otimes c"] & a((X_\phi)_*q\mathcal{M}_j\otimes^p (X_\phi)_*q\mathcal{N}_j)
\end{tikzcd}$$

is commutative. Combining the commutativity of these three diagrams (and some other easy commutativity), it is not so difficult to show that the map

$$\beta : (\mathcal{M}\otimes\mathcal{N})_i = (a(q\mathcal{M}\otimes^p q\mathcal{N}))_i \to (X_\phi)_*(a(q\mathcal{M}\otimes^p q\mathcal{N}))_j = (X_\phi)_*(\mathcal{M}\otimes\mathcal{N})_j$$

agrees with the composite

$$\begin{aligned}(\mathcal{M}\otimes\mathcal{N})_i = (a(q\mathcal{M}\otimes^p q\mathcal{N}))_i &\xrightarrow{\theta^{-1}} a(q\mathcal{M}\otimes^p q\mathcal{N})_i \xrightarrow{m_i^{-1}} a((q\mathcal{M})_i\otimes^p (q\mathcal{N})_i) \\ \xrightarrow{c\otimes c} a(q\mathcal{M}_i\otimes^p q\mathcal{N}_i) &\xrightarrow{\beta\otimes\beta} a(q(X_\phi)_*\mathcal{M}_j\otimes^p q(X_\phi)_*\mathcal{N}_j) \xrightarrow{c\otimes c} \\ a((X_\phi)_*q\mathcal{M}_j\otimes^p (X_\phi)_*q\mathcal{N}_j) &\xrightarrow{m} a(X_\phi)_*(q\mathcal{M}_j\otimes^p q\mathcal{N}_j) \xrightarrow{\theta} (X_\phi)_*a(q\mathcal{M}_j\otimes^p q\mathcal{N}_j) \\ \xrightarrow{c\otimes c} (X_\phi)_*a((q\mathcal{M})_j\otimes^p (q\mathcal{N})_j) &\xrightarrow{m_j} (X_\phi)_*a((q\mathcal{M}\otimes^p q\mathcal{N})_j) \xrightarrow{\theta} \\ &(X_\phi)_*(a(q\mathcal{M}\otimes^p q\mathcal{N}))_j = (X_\phi)_*(\mathcal{M}\otimes\mathcal{N})_j.\end{aligned}$$

This composite map agrees with the composite map (6.5). This proves the lemma. □

(6.6) Let $X_\bullet \in \mathcal{P}$, and J a subcategory of I. The left adjoint functor $L_J^\heartsuit = Q(X_\bullet, J)_\#^\heartsuit$ of $(?)_J^\heartsuit$ is given by the structure data as follows explicitly. For $\mathcal{M} \in \heartsuit(X_\bullet|_J)$ and $i \in I$, we have

Lemma 6.7. *There is an isomorphism*

$$\lambda_{J,i} : (L_J^\heartsuit(\mathcal{M}))_i^\heartsuit \cong \varinjlim (X_\phi)_\heartsuit^*(\mathcal{M}_j),$$

where the colimit is taken over the subcategory $(I_i^{(J^{\mathrm{op}} \to I^{\mathrm{op}})})^{\mathrm{op}}$ *of* I/i *whose objects are* $(\phi : j \to i) \in I/i$ *with* $j \in \mathrm{ob}(J)$ *and morphisms are morphisms* φ *of* I/i *such that* $\varphi \in \mathrm{Mor}(J)$*. The translation map of the direct system is given as follows. For morphisms* $\phi : j \to i$ *and* $\psi : j' \to j$*, the translation map* $X_{\phi\psi}^* \mathcal{M}_{j'} \to X_\phi^* \mathcal{M}_j$ *is the composite*

$$X_{\phi\psi}^* \mathcal{M}_{j'} \xrightarrow{d} X_\phi^* X_\psi^* \mathcal{M}_{j'} \xrightarrow{\alpha_\psi} X_\phi^* \mathcal{M}_j.$$

Proof. We prove the lemma for the case that $\heartsuit = \mathrm{PM}, \mathrm{Mod}$. The case that $\heartsuit = \mathrm{PA}, \mathrm{AB}$ is similar and easier.

Consider the case $\heartsuit = \mathrm{PM}$ first. For any object $(\phi, h) : (i, U) \to (j, V)$ of $I_{(i,U)}^{\mathrm{Zar}(X_\bullet|_J) \hookrightarrow \mathrm{Zar}(X_\bullet)}$, consider the obvious map

$$\begin{aligned}
\Gamma((i,U), \mathcal{O}_{X_\bullet}) \otimes_{\Gamma((j,V), \mathcal{O}_{X_\bullet|_J})} \Gamma((j,V), \mathcal{M}) &= \Gamma(U, \mathcal{O}_{X_i}) \otimes_{\Gamma(V, \mathcal{O}_{X_j})} \Gamma(V, \mathcal{M}_j) \\
&\to \varinjlim_{X_\phi^{-1}(V') \supset U} \Gamma(U, \mathcal{O}_{X_i}) \otimes_{\Gamma(V', \mathcal{O}_{X_j})} \Gamma(V', \mathcal{M}_j) \\
&= \Gamma(U, X_\phi^* \mathcal{M}_j) \to \varinjlim \Gamma(U, X_{\phi'}^* \mathcal{M}_{j'}),
\end{aligned}$$

where the last $\varinjlim$ is taken over $(\phi' : j' \to i) \in (I_i^{(J^{\mathrm{op}} \to I^{\mathrm{op}})})^{\mathrm{op}}$. This map induces a unique map

$$\begin{aligned}
&\Gamma(U, (L_J \mathcal{M})_i) = \Gamma((i,U), L_J \mathcal{M}) = \\
&\quad \varinjlim \Gamma((i,U), \mathcal{O}_{X_\bullet}) \otimes_{\Gamma((j,V), \mathcal{O}_{X_\bullet|_J})} \Gamma((j,V), \mathcal{M}) \to \varinjlim \Gamma(U, X_{\phi'}^* \mathcal{M}_{j'}).
\end{aligned}$$

It is easy to see that this defines $\lambda_{J,i}$.

We define the inverse of $\lambda_{J,i}$ explicitly. Let $(\phi : j \to i) \in (I_i^{(J^{\mathrm{op}} \to I^{\mathrm{op}})})^{\mathrm{op}}$. Let $U \in \mathrm{Zar}(X_i)$ and $V \in \mathrm{Zar}(X_j)$ such that $U \subset X_\phi^{-1}(V)$. We have an obvious map

$$\begin{aligned}
\Gamma(U, \mathcal{O}_{X_i}) \otimes_{\Gamma(V, \mathcal{O}_{X_j})} \Gamma(V, \mathcal{M}_j) &= \Gamma((i,U), \mathcal{O}_{X_\bullet}) \otimes_{\Gamma((j,V), \mathcal{O}_{X_\bullet|_J})} \Gamma((j,V), \mathcal{M}) \\
&\to \varinjlim \Gamma((i,U), \mathcal{O}_{X_\bullet}) \otimes_{\Gamma((j,V), \mathcal{O}_{X_\bullet|_J})} \Gamma((j,V), \mathcal{M}) \\
&= \Gamma((i,U), L_J \mathcal{M}) = \Gamma(U, (L_J \mathcal{M})_i),
\end{aligned}$$

which induces

$$\Gamma(U, X_\phi^* \mathcal{M}) = \varinjlim \Gamma(U, \mathcal{O}_{X_i}) \otimes_{\Gamma(V, \mathcal{O}_{X_j})} \Gamma(V, \mathcal{M}_j) \to \Gamma(U, (L_J \mathcal{M})_i).$$

This gives a morphism $X_\phi^*\mathcal{M} \to (L_J\mathcal{M})_i$. It is easy to see that this defines $\varinjlim X_\phi^*\mathcal{M} \to (L_J\mathcal{M})_i$, which is the inverse of $\lambda_{J,i}$. This completes the proof for the case that $\heartsuit = \mathrm{PM}$.

Now consider the case $\heartsuit = \mathrm{Mod}$. Define $\lambda_{J,i}^{\mathrm{Mod}}$ to be the composite

$$\begin{aligned}(L_J^{\mathrm{Mod}}\mathcal{M})_i = (?)_i a L_J^{\mathrm{PM}} q\mathcal{M} \xrightarrow{\theta^{-1}} a(?)_i L_J^{\mathrm{PM}} q\mathcal{M} \xrightarrow{\lambda_{J,i}^{\mathrm{PM}}} a \varinjlim X_\phi^*(q\mathcal{M})_j \\ \cong \varinjlim aX_\phi^*(q\mathcal{M})_j \xrightarrow{c} \varinjlim aX_\phi^* q\mathcal{M}_j = \varinjlim X_\phi^*\mathcal{M}_j.\end{aligned}$$

As the morphisms appearing in the composition are all isomorphisms, $\lambda_{J,i}^{\mathrm{Mod}}$ is an isomorphism. $\square$

In particular, we have an isomorphism

$$\lambda_{j,i} : (L_j^\heartsuit(\mathcal{M}))_i^\heartsuit \cong \bigoplus_{\phi \in I(j,i)} (X_\phi)_\heartsuit^*(\mathcal{M}). \tag{6.8}$$

(6.9) As announced in (2.61), we show that the monoidal adjoint pair $((?)_\#^{\mathrm{Mod}}, (?)^\#_{\mathrm{Mod}})$ in Lemma 2.55 is not Lipman.

We define a finite category $\mathcal{K}$ by $\mathrm{ob}(\mathcal{K}) = \{s,t\}$, and $\mathcal{K}(s,t) = \{u,v\}$, $\mathcal{K}(s,s) = \{\mathrm{id}_s\}$, and $\mathcal{K}(t,t) = \{\mathrm{id}_t\}$. Pictorially, $\mathcal{K}$ looks like $t \underset{v}{\overset{u}{\leftleftarrows}} s$. Let k be a field, and define $X_\bullet \in \mathcal{P}(\mathcal{K}, \underline{\mathrm{Sch}})$ by $X_s = X_t = \operatorname{Spec} k$, and $X_u = X_v = \mathrm{id}$. Then $\Gamma(X_t, (L_s\mathcal{O}_{X_s})_t)$ is two-dimensional by (6.8). So $L_s\mathcal{O}_{X_s}$ and $\mathcal{O}_{X_\bullet}$ are not isomorphic by the dimension reason. Similarly, $L_s(\mathcal{O}_{X_s} \otimes_{\mathcal{O}_{X_s}} \mathcal{O}_{X_s})$ cannot be isomorphic to $L_s\mathcal{O}_{X_s} \otimes_{\mathcal{O}_{X_\bullet}} L_s\mathcal{O}_{X_s}$.

Similarly, $((?)_\#^{\mathrm{PM}}, (?)^\#_{\mathrm{PM}})$ in Lemma 2.55 is not Lipman.

(6.10) Let $\psi : i \to i'$ be a morphism. The structure map

$$\alpha_\psi : (X_\psi)_\heartsuit^*((L_J^\heartsuit(\mathcal{M}))_i^\heartsuit) \to (L_J^\heartsuit(\mathcal{M}))_{i'}^\heartsuit$$

is induced by

$$(X_\psi)_\heartsuit^*((X_\phi)_\heartsuit^*(\mathcal{M}_j)) \cong (X_{\psi\phi})_\heartsuit^*(\mathcal{M}_j).$$

More precisely, for $\psi : i \to i'$, the diagram

$$\begin{array}{ccc} X_\psi^*((L_J\mathcal{M})_i) \xrightarrow{\lambda_{J,i}} X_\psi^* \varinjlim X_\phi^*\mathcal{M}_j & \cong & \varinjlim X_\psi^* X_\phi^*\mathcal{M}_j \\ \downarrow \alpha_\psi & & \downarrow h \\ (L_J\mathcal{M})_{i'} \qquad \xrightarrow{\qquad\lambda_{J,i'}\qquad} & & \varinjlim X_{\phi'}^*\mathcal{M}_{j'} \end{array}$$

is commutative, where $\phi : i \to j$ runs through $(I_i^f)^{\mathrm{op}}$, and $\phi' : i' \to j'$ runs through $(I_{i'}^f)^{\mathrm{op}}$, where $f : J^{\mathrm{op}} \to I^{\mathrm{op}}$ is the inclusion. The map h is induced by $d : X_\psi^* X_\phi^* \to (X_\phi X_\psi)^* = X_{\psi\phi}^*$. This is checked at the section level directly when $\heartsuit = \mathrm{PM}$.

We consider the case that $\heartsuit = \mathrm{Mod}$. Then the composite

$$X_\psi^*(?)_i L_J \xrightarrow{\lambda_{J,i}} X_\psi^* \varinjlim X_\phi^*(?)_j \cong \varinjlim X_\psi^* X_\phi^*(?)_j \xrightarrow{h} \varinjlim X_{\phi'}^*(?)_{j'}$$

agrees with the composite

$$\begin{aligned} X_\psi^*(?)_i L_J = aX_\psi^* q(?)_i a L_J q \xrightarrow{\theta^{-1}} aX_\psi^* qa(?)_i L_J q \xrightarrow{\lambda_{J,i}^{\mathrm{PM}}} aX_\psi^* qa \varinjlim X_\phi^*(?)_j q \\ \xrightarrow{\cong} aX_\psi^* q \varinjlim aX_\phi^*(?)_j q \xrightarrow{c} aX_\psi^* q \varinjlim aX_\phi^* q(?)_j \xrightarrow{\cong} \varinjlim aX_\psi^* qaX_\phi^* q(?)_j \xrightarrow{u^{-1}} \\ \varinjlim aX_\psi^* X_\phi^* q(?)_j \xrightarrow{d} \varinjlim aX_{\psi\phi}^* q(?)_j \to \varinjlim aX_{\phi'}^* q(?)_{j'} = \varinjlim X_{\phi'}^*(?)_{j'}. \end{aligned}$$

Using Lemma 2.60, it is straightforward to show that this map agrees with

$$\begin{aligned} X_\psi^*(?)_i L_J = aX_\psi^* q(?)_i a L_J q \xrightarrow{c} aX_\psi^*(?)_i qa L_J q \xrightarrow{\alpha_\psi} a(?)_{i'} qa L_J q \xrightarrow{c} \\ aq(?)_{i'} a L_J q \xrightarrow{\varepsilon} (?)_{i'} a L_J q \xrightarrow{\theta^{-1}} a(?)_{i'} L_J q \xrightarrow{\lambda_{J,i'}} a \varinjlim X_{\phi'}^*(?)_{j'} q \xrightarrow{\cong} \\ \varinjlim aX_{\phi'}^*(?)_{j'} q \xrightarrow{c} \varinjlim aX_{\phi'}^* q(?)_{j'} = \varinjlim X_{\phi'}^*(?)_{j'}. \end{aligned}$$

This composite map agrees with

$$X_\psi^*(?)_i L_J \xrightarrow{\alpha_\psi} (?)_{i'} L_J \xrightarrow{\lambda_{J,i'}} \varinjlim X_{\phi'}^*(?)_{j'}$$

by (4.20) and the definition of $\lambda_{J,i'}$ for sheaves (see the proof of Lemma 6.7). This is what we wanted to prove.

The case that $\heartsuit = \mathrm{PA}, \mathrm{AB}$ is proved similarly.

(6.11) In the remainder of this chapter, we do not give detailed proofs, since the strategy is similar to the above (just check the commutativity at the section level for presheaves, and sheafify it).

(6.12) The counit map $\varepsilon : L_J(?)_J \to \mathrm{Id}$ is given as a morphism of structure data as follows.

$$\varepsilon_i : (?)_i L_J(?)_J \to (?)_i$$

agrees with

$$(?)_i L_J(?)_J \xrightarrow{\lambda_{J,i}} \varinjlim X_\phi^*(?)_j(?)_J \xrightarrow{c} \varinjlim X_\phi^*(?)_j \xrightarrow{\alpha} (?)_i,$$

where α is induced by $\alpha_\phi : X_\phi^*(?)_j \to (?)_i$.

(6.13) The unit map $u : \mathrm{Id} \to (?)_J L_J$ is also described, as follows.

$$u_j : (?)_j \to (?)_j (?)_J L_J$$

agrees with

$$(?)_j \xrightarrow{\mathfrak{f}^{-1}} X^*_{\mathrm{id}_j}(?)_j \to \varinjlim X^*_\phi(?)_k \xrightarrow{\lambda^{-1}_{J,j}} (?)_j L_J \cong (?)_j(?)_J L_J,$$

where the colimit is taken over $(\phi : k \to j) \in (I_j^{(J^{\mathrm{op}} \subset I^{\mathrm{op}})})^{\mathrm{op}}$.

(6.14) Let $X_\bullet \in \mathcal{P}$, and J a subcategory of I. The right adjoint functor $R_J^\heartsuit$ of $(?)_J^\heartsuit$ is given as follows explicitly. For $\mathcal{M} \in \heartsuit(X_\bullet|_J)$ and $i \in I$, we have

$$\rho^{J,i} : (R_J^\heartsuit(\mathcal{M}))_i^\heartsuit \cong \varprojlim (X_\phi)_*^\heartsuit(\mathcal{M}_j),$$

where the limit is taken over $I_i^{(J \to I)}$, see (2.6) for the notation. The descriptions of α, u, and ε for the right induction are left to the reader.

Lemma 6.15. *Let $X_\bullet \in \mathcal{P}$, and J a full subcategory of I. Then we have the following.*

1 The counit of adjunction $\varepsilon : (?)_J^\heartsuit \circ R_J^\heartsuit \to \mathrm{Id}$ is an isomorphism. In particular, $R_J^\heartsuit$ is full and faithful.
2 The unit of adjunction $u : \mathrm{Id} \to (?)_J^\heartsuit \circ L_J^\heartsuit$ is an isomorphism. In particular, $L_J^\heartsuit$ is full and faithful.

Proof. **1** For $i \in J$, the restriction

$$\varepsilon_i : (?)_i^\heartsuit (?)_J^\heartsuit R_J^\heartsuit \mathcal{M} = \varprojlim (X_\phi)_*^\heartsuit(\mathcal{M}_j) \to (X_{\mathrm{id}_i})_* \mathcal{M}_i = \mathcal{M}_i = (?)_i \mathcal{M}$$

is nothing but the canonical map from the projective limit, where the limit is taken over $(\phi : i \to j) \in I_i^{(J \to I)}$. As J is a full subcategory, we have $I_i^{(J \to I)}$ equals i/J, and hence id_i is its initial object. So the limit is equal to $\mathcal{M}_i$, and ε_i is the identity map. Since ε_i is an isomorphism for each $i \in J$, we have that ε is an isomorphism.

The proof of **2** is similar, and we omit it. □

Let $\mathcal{C}$ be a small category. A connected component of $\mathcal{C}$ is a full subcategory of $\mathcal{C}$ whose object set is one of the equivalence classes of $\mathrm{ob}(\mathcal{C})$ with respect to the transitive symmetric closure of the relation $\sim$ given by

$$c \sim c' \iff \mathcal{C}(c, c') \neq \emptyset.$$

Definition 6.16. We say that a subcategory J of I is *admissible* if

1 For $i \in I$, the category $(I_i^{(J^{\mathrm{op}} \subset I^{\mathrm{op}})})^{\mathrm{op}}$ is pseudofiltered.
2 For $j \in J$, we have id_j is the initial object of one of the connected components of $I_j^{(J^{\mathrm{op}} \subset I^{\mathrm{op}})}$ (i.e., id_j is the terminal object of one of the connected components of $(I_j^{(J^{\mathrm{op}} \subset I^{\mathrm{op}})})^{\mathrm{op}}$).

Note that for $j \in I$, the subcategory $j = (\{j\}, \{\mathrm{id}_j\})$ of I is admissible.

In Lemma 6.7, the colimit in the right hand side is pseudo-filtered and hence it preserves exactness, if **1** is satisfied. In particular, if **1** is satisfied, then $Q(X_\bullet, J) : \mathrm{Zar}(X_\bullet|_J) \to \mathrm{Zar}(X_\bullet)$ is an admissible functor. As in the proof of Lemma 6.15, $(?)_j$ is a direct summand of $(?)_j \circ L_J$ for $j \in J$ so that L_J is faithful, if **2** is satisfied. We have the following.

Lemma 6.17. *Let $X_\bullet \in \mathcal{P}(I, \underline{\mathrm{Sch}}/S)$, and $K \subset J \subset I$ be admissible subcategories of I. Then $L^{\mathrm{PA}}_{J,K}$ is faithful and exact. The morphism of sites $Q(X_\bullet|_J, K)$ is admissible. If, moreover, X_ϕ is flat for any $\phi \in I(k, j)$ with $j \in J$ and $k \in K$, then $L^{\heartsuit}_{J,K}$ is faithful and exact for $\heartsuit = \mathrm{Mod}$.*

Proof. Assume that $\mathcal{M} \in \heartsuit(X_\bullet|_K)$, $\mathcal{M} \neq 0$, and $L_{J,K}\mathcal{M} = 0$. There exists some $k \in K$ such that $M_k \neq 0$. Since $L_{J,K}\mathcal{M} = 0$, we have that $0 \cong (?)_k L_{I,J} L_{J,K}\mathcal{M} \cong (?)_k L_{I,K}\mathcal{M}$. This contradicts the fact that $\mathcal{M}_k$ is a direct summand of $(L_{I,K}\mathcal{M})_k$. Hence $L_{J,K}$ is faithful.

We prove that $L^{\heartsuit}_{J,K}$ is exact. It suffices to show that for any $j \in J$, $(?)_j L_{J,K}$ is exact. As J is admissible, $(?)_j$ is a direct summand of $(?)_j L_{I,J}$. Hence it suffices to show that $(?)_j L_{I,K} \cong (?)_j L_{I,J} L_{J,K}$ is exact. By Lemma 6.7, $(?)_j L_{I,K} \cong \varinjlim (X_\phi)^*_\heartsuit (?)_k$, where the colimit is taken over $(\phi\colon k \to j) \in (I_j^{K^{\mathrm{op}} \subset I^{\mathrm{op}}})^{\mathrm{op}}$. By assumption, $(X_\phi)^*_\heartsuit$ is exact for any ϕ in the colimit. As $(I_j^{K^{\mathrm{op}} \subset I^{\mathrm{op}}})^{\mathrm{op}}$ is pseudo-filtered by assumption, $(?)_j L_{I,K}$ is exact, as desired. □

(6.18) As in Example 5.6, **2**, we have an isomorphism

$$c_{i,f_\bullet} : (?)_i \circ (f_\bullet)_* \cong (f_i)_* \circ (?)_i. \tag{6.19}$$

The translation α_ϕ is described as follows.

Lemma 6.20. *Let $f_\bullet\colon X_\bullet \to Y_\bullet$ be a morphism in $\mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. For $\phi \in I(i, j)$,*

$$\alpha_\phi (f_\bullet)_* : Y_\phi^* (?)_i (f_\bullet)_* \to (?)_j (f_\bullet)_*$$

agrees with

$$\begin{aligned} &Y_\phi^*(?)_i(f_\bullet)_* \xrightarrow{c_{i,f_\bullet}} Y_\phi^*(f_i)_*(?)_i \xrightarrow{\text{via } \theta} (f_j)_* X_\phi^*(?)_i \\ &\qquad \xrightarrow{(f_j)_*\alpha_\phi} (f_j)_*(?)_j \xrightarrow{c_{j,f_\bullet}^{-1}} (?)_j(f_\bullet)_*, \end{aligned} \tag{6.21}$$

where θ is Lipman's theta [26, (3.7.2)].

One of the definitions of θ is the composite

$$\theta : Y_\phi^*(f_i)_* \xrightarrow{\text{via } u} Y_\phi^*(f_i)_*(X_\phi)_* X_\phi^* \xrightarrow{c} Y_\phi^*(Y_\phi)_*(f_j)_* X_\phi^* \xrightarrow{\text{via } \varepsilon} (f_j)_* X_\phi^*.$$

Proof. Note that the diagram

$$\begin{tikzcd}
(?)_i(f_\bullet)_* \arrow[r, "\beta"] \arrow[d, "c"] & (Y_\phi)_*(?)_j(f_\bullet)_* \arrow[r, "c"] & (Y_\phi)_*(f_j)_*(?)_j \arrow[d, "c"] \\
(f_i)_*(?)_i \arrow[rr, "(f_i)_*\beta"] & & (f_i)_*(X_\phi)_*(?)_j
\end{tikzcd} \tag{6.22}$$

is commutative. Indeed, when we apply the functor $\Gamma(U, ?)$ for an open subset U of Y_i, then we get an obvious commutative diagram

$$\begin{tikzcd}
\Gamma((i, f_i^{-1}(U)), ?) \arrow[r, "\mathrm{res}"] \arrow[d, "\mathrm{id}"] & \Gamma((j, f_j^{-1}(Y_\phi^{-1}(U))), ?) \arrow[r, "\mathrm{id}"] & \Gamma((j, f_j^{-1}(Y_\phi^{-1}(U))), ?) \arrow[d, "\mathrm{id}"] \\
\Gamma((i, f_i^{-1}(U)), ?) \arrow[rr, "\mathrm{res}"] & & \Gamma((j, X_\phi^{-1}(f_i^{-1}(U))), ?).
\end{tikzcd}$$

Now the assertion of the lemma follows from the commutativity of the diagram

$$\begin{tikzcd}
Y_\phi^*(?)_i(f_\bullet)_* \arrow[r, "\alpha"] \arrow[d, "\mathrm{id}"] \arrow[dr, phantom, "\text{(a)}"] & (?)_j(f_\bullet)_* \arrow[r, "c"] \arrow[dr, phantom, "\text{(b)}"] & (f_j)_*(?)_j \arrow[r, "\mathrm{id}"] & (f_j)_*(?)_j \arrow[dddd, "\mathrm{id}"] \\
Y_\phi^*(?)_i(f_\bullet)_* \arrow[r, "\beta"] \arrow[d, "c"] & Y_\phi^*(Y_\phi)_*(?)_j(f_\bullet)_* \arrow[r, "c"] \arrow[u, "\varepsilon"] \arrow[d, phantom, "\text{(c)}"] & Y_\phi^*(Y_\phi)_*(f_j)_*(?)_j \arrow[u, "\varepsilon"] \arrow[d, "c"] & \\
Y_\phi^*(f_i)_*(?)_i \arrow[rr, "\beta"] \arrow[d, "\theta"] & {} \arrow[d, phantom, "\text{(d)}"] & Y_\phi^*(f_i)_*(X_\phi)_*(?)_j \arrow[d, "\theta"] \arrow[r, phantom, "\text{(f)}"] & {} \\
(f_j)_*X_\phi^*(?)_i \arrow[rr, "\beta"] \arrow[d, "\mathrm{id}"] & {} \arrow[d, phantom, "\text{(e)}"] & (f_j)_*X_\phi^*(X_\phi)_*(?)_j \arrow[d, "\varepsilon"] & \\
(f_j)_*X_\phi^*(?)_i \arrow[rr, "\alpha"] & {} & (f_j)_*(?)_j & (f_j)_*(?)_j. \arrow[l, "\mathrm{id}"']
\end{tikzcd}$$

Indeed, the commutativity of (a) and (e) is the definition of α. The commutativity of (b) follows from the naturality of ε. The commutativity of (c) follows from the commutativity of (6.22). The commutativity of (d) is the naturality of θ. The commutativity of (f) follows from the definition of θ and the fact that the composite

$$(X_\phi)_* \xrightarrow{u} (X_\phi)_* X_\phi^* (X_\phi)_* \xrightarrow{\varepsilon} (X_\phi)_*$$

is the identity. $\square$

Proposition 6.23. *Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism in $\mathcal{P}$, J a subcategory of I, and $i \in I$. Then the composite map*

$$(?)_i L_J (f_\bullet|_J)_* \xrightarrow{\text{via } \theta} (?)_i (f_\bullet)_* L_J \xrightarrow{\text{via } c_{i,f_\bullet}} (f_i)_* (?)_i L_J$$

agrees with the composite map

$$(?)_i L_J (f_\bullet|_J)_* \xrightarrow{\text{via } \lambda_{J,i}} \varinjlim Y^*_\phi(?)_j (f_\bullet|_J)_* \xrightarrow{\text{via } c_{j,f_\bullet|_J}} \varinjlim Y^*_\phi(f_j)_*(?)_j$$
$$\xrightarrow{\text{via } \theta} \varinjlim (f_i)_* X^*_\phi(?)_j \to (f_i)_* \varinjlim X^*_\phi(?)_j \xrightarrow{\text{via } \lambda^{-1}_{J,i}} (f_i)_*(?)_i L_J.$$

Proof. Note that θ in the first composite map is the composite

$$\theta = \theta(J, f_\bullet) : L_J(f_\bullet|_J)_* \xrightarrow{\text{via } u} L_J(f_\bullet|_J)_*(?)_J L_J \xrightarrow{c} L_J(?)_J(f_\bullet)_* L_J \xrightarrow{\varepsilon} (f_\bullet)_* L_J.$$

The description of u and ε are already given, and the proof is reduced to the iterative use of (6.10), (6.12), (6.13), and Lemma 6.20. The detailed argument is left to a patient reader. The reason why the second map involves θ is Lemma 6.20. □

Similarly, we have the following.

Proposition 6.24. *Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism in $\mathcal{P}$, J a subcategory of I, and $i \in I$. Then the composite map*

$$(f_i)^*(?)_i L_J \xrightarrow{\text{via } \theta(f_\bullet,i)} (?)_i (f_\bullet)^* L_J \xrightarrow{\text{via } d_{f_\bullet,J}} (?)_i L_J (f_\bullet|_J)^*$$

agrees with the composite map

$$(f_i)^*(?)_i L_J \xrightarrow{\text{via } \lambda_{J,i}} (f_i)^* \varinjlim Y^*_\phi(?)_j \cong \varinjlim (f_i)^* Y^*_\phi(?)_j$$
$$\xrightarrow{d} \varinjlim X^*_\phi(f_j)^*(?)_j \xrightarrow{\text{via } \theta(f_\bullet|_J,j)} \varinjlim X^*_\phi(?)_j (f_\bullet|_J)^* \xrightarrow{\text{via } \lambda^{-1}_{J,i}} (?)_i L_J (f_\bullet|_J)^*.$$

The proof is left to the reader. The proof of Proposition 6.23 and Proposition 6.24 are formal, and the propositions are valid for $\heartsuit = \mathrm{PM}, \mathrm{Mod}, \mathrm{PA}$, and AB.

Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism in $\mathcal{P}$, and $J \subset I$ a subcategory. The inverse image $(f_\bullet)^*_\heartsuit$ is compatible with the restriction $(?)_J$.

Lemma 6.25. *The natural map*

$$\theta_\heartsuit = \theta_\heartsuit(f_\bullet, J) : ((f_\bullet)|_J)^*_\heartsuit \circ (?)_J \to (?)_J \circ (f_\bullet)^*_\heartsuit$$

is an isomorphism for $\heartsuit = \mathrm{PA}, \mathrm{AB}, \mathrm{PM}, \mathrm{Mod}$. *In particular,* $f_\bullet^{-1} : \mathrm{Zar}(Y_\bullet) \to \mathrm{Zar}(X_\bullet)$ *is an admissible continuous functor.*

Proof. We consider the case where $\heartsuit = \mathrm{PM}$.

Let $\mathcal{M} \in \mathrm{PM}(Y_\bullet)$, and $(j, U) \in \mathrm{Zar}(X_\bullet|_J)$. We have

$$\Gamma((j,U), (f_\bullet|_J)^* \mathcal{M}_J) = \varinjlim \Gamma((j,U), \mathcal{O}_{X_\bullet}) \otimes_{\Gamma((j',V),\mathcal{O}_{Y_\bullet})} \Gamma((j',V), \mathcal{M}),$$

where the colimit is taken over $(j', V) \in (I^{(f_\bullet|_J)^{-1}}_{(j,U)})^{\mathrm{op}}$. On the other hand, we have

$$\Gamma((j,U), (?)_J f^*_\bullet \mathcal{M}) = \varinjlim \Gamma((j,U), \mathcal{O}_{X_\bullet}) \otimes_{\Gamma((i,V),\mathcal{O}_{Y_\bullet})} \Gamma((i,V), \mathcal{M}),$$

where the colimit is taken over $(i,V) \in (I^{f_\bullet^{-1}}_{(j,U)})^{\mathrm{op}}$. There is an obvious map from the first to the second. This obvious map is θ, see (2.57).

To verify that this is an isomorphism, it suffices to show that the category $(I^{(f_\bullet|_J)^{-1}}_{(j,U)})^{\mathrm{op}}$ is final in the category $(I^{f_\bullet^{-1}}_{(j,U)})^{\mathrm{op}}$. In fact, any $(\phi,h):(j,U)\to(i,f_i^{-1}(V))$ with $(i,V)\in\mathrm{Zar}(Y_\bullet)$ factors through

$$(\mathrm{id}_j,h):(j,U)\to(j,f_j^{-1}Y_\phi^{-1}(V)).$$

Hence, $\theta_\heartsuit$ is an isomorphism for $\heartsuit=\mathrm{PM}$. The construction for the case where $\heartsuit=\mathrm{PA}$ is similar.

As $(?)_J$ is compatible with the sheafification by Lemma 2.31, we have that θ is an isomorphism for $\heartsuit=\mathrm{Mod},\mathrm{AB}$ by Lemma 2.59. □

Corollary 6.26. *The conjugate*

$$\xi_\heartsuit=\xi_\heartsuit(f_\bullet,J):(f_\bullet)_*^\heartsuit R_J\to R_J(f_\bullet|_J)_*^\heartsuit$$

of $\theta_\heartsuit(f_\bullet,J)$ *is an isomorphism for* $\heartsuit=\mathrm{PA},\mathrm{AB},\mathrm{PM},\mathrm{Mod}$.

Proof. Obvious by Lemma 6.25. □

(6.27) By Corollary 6.26, we may define the composite

$$\begin{aligned}\mu_\heartsuit=\mu_\heartsuit(f_\bullet,J):f_\bullet^*R_J&\xrightarrow{u}f_\bullet^*R_J(f_\bullet|_J)_*(f_\bullet|_J)^*\\&\xrightarrow{\xi^{-1}}f_\bullet^*(f_\bullet)_*R_J(f_\bullet|_J)^*\xrightarrow{\varepsilon}R_J(f_\bullet|_J)^*.\end{aligned}$$

Observe that the diagram

$$\begin{tikzcd}
(?)_if_\bullet^*R_J \arrow[r, "\theta^{-1}"] \arrow[d, "\mu"] & f_i^*(?)_iR_J \arrow[r, "\rho"] & f_i^*\varinjlim(Y_\phi)_*(?)_j \arrow[r] & \varinjlim f_i^*(Y_\phi)_*(?)_j \arrow[dl, "\theta"] \\
(?)_iR_Jf_\bullet|_J^* \arrow[r, "\rho"] & \varinjlim(X_\phi)_*(?)_jf_\bullet|_J^* \arrow[r, "\theta^{-1}"] & \varinjlim(X_\phi)_*f_j^*(?)_j &
\end{tikzcd}$$

is commutative.

Lemma 6.28. *Let the notation be as above, and* $\mathcal{M},\mathcal{N}\in\heartsuit(Y_\bullet)$*. Then the diagram*

$$\begin{array}{ccccc}(f_\bullet|_J)^*_\heartsuit(\mathcal{M}_J\otimes\mathcal{N}_J) & \xrightarrow{m} & (f_\bullet|_J)^*_\heartsuit((\mathcal{M}\otimes\mathcal{N})_J) & \xrightarrow{\theta} & ((f_\bullet)^*_\heartsuit(\mathcal{M}\otimes\mathcal{N}))_J\\ \downarrow\Delta & & & & \downarrow(?)_J\Delta\\ (f_\bullet|_J)^*_\heartsuit\mathcal{M}_J\otimes(f_\bullet|_J)^*_\heartsuit\mathcal{N}_J & \xrightarrow{\theta\otimes\theta} & ((f_\bullet)^*_\heartsuit\mathcal{M})_J\otimes((f_\bullet)^*_\heartsuit\mathcal{N})_J & \xrightarrow{m} & ((f_\bullet)^*_\heartsuit\mathcal{M}\otimes(f_\bullet)^*_\heartsuit\mathcal{N})_J\end{array}\tag{6.29}$$

is commutative.

Proof. This is an immediate consequence of Lemma 1.44. □

Corollary 6.30. *The adjoint pair* $((?)^*_{\mathrm{Mod}}, (?)^{\mathrm{Mod}}_*)$ *over the category* $\mathcal{P}(I, \underline{\mathrm{Sch}}/S)$ *is Lipman.*

Proof. Let $f_\bullet : X_\bullet \to Y_\bullet$ be a morphism of $\mathcal{P}(I, \underline{\mathrm{Sch}}/S)$. It is easy to see that the diagram

$$\begin{tikzcd}
(?)_i \mathcal{O}_{Y_\bullet} \arrow[r, "="] \arrow[d, "(?)_i \eta"] & \mathcal{O}_{Y_i} \arrow[dr, "\eta"] & \\
(?)_i (f_\bullet)_* \mathcal{O}_{X_\bullet} \arrow[r, "c"] & (f_i)_* (?)_i \mathcal{O}_{X_\bullet} \arrow[r, "="] & (f_i)_* \mathcal{O}_{X_i}
\end{tikzcd}$$

is commutative. So utilizing Lemma 1.25, it is easy to see that

$$\begin{tikzcd}
(?)_i f_\bullet^* \mathcal{O}_{Y_\bullet} \arrow[r, "\theta^{-1}"] \arrow[d, "(?)_i C"] & f_i^* (?)_i \mathcal{O}_{Y_\bullet} \arrow[r, "="] & f_i^* \mathcal{O}_{Y_i} \arrow[dl, "C"] \\
(?)_i \mathcal{O}_{X_\bullet} \arrow[r, "="] & \mathcal{O}_{X_i} &
\end{tikzcd}$$

is also commutative. Since $C\colon f_i^* \mathcal{O}_{Y_i} \to \mathcal{O}_{X_i}$ is an isomorphism by Corollary 2.65, $(?)_i C$ is an isomorphism for any $i \in I$. Hence $C\colon f_\bullet^* \mathcal{O}_{Y_\bullet} \to \mathcal{O}_{X_\bullet}$ is also an isomorphism.

Let us consider $\mathcal{M}, \mathcal{N} \in \heartsuit(Y_\bullet)$. To verify that Δ is an isomorphism, it suffices to show that

$$(?)_i \Delta : (f_\bullet^*(\mathcal{M} \otimes \mathcal{N}))_i \to (f_\bullet^* \mathcal{M} \otimes f_\bullet^* \mathcal{N})_i$$

is an isomorphism for any $i \in \mathrm{ob}(I)$. Now consider the diagram (6.29) for $J = i$. Horizontal maps in the diagram are isomorphisms by (6.3) and Lemma 6.25. The left Δ is an isomorphism, since f_i is a morphism of single schemes. By Lemma 6.28, $(?)_i \Delta$ is also an isomorphism. $\square$

(6.31) The description of the translation map α_ϕ for $f_\bullet^*$ is as follows. For $\phi \in I(i, j)$,

$$\alpha_\phi : X_\phi^* (?)_i f_\bullet^* \to (?)_j f_\bullet^*$$

is the composite

$$X_\phi^* (?)_i f_\bullet^* \xrightarrow{X_\phi^* \theta^{-1}} X_\phi^* f_i^* (?)_i \xrightarrow{d} f_j^* Y_\phi^* (?)_i \xrightarrow{f_j^* \alpha_\phi} f_j^* (?)_j \xrightarrow{\theta} (?)_j f_\bullet^*.$$

(6.32) Let $X_\bullet \in \mathcal{P}$, and $\mathcal{M}, \mathcal{N} \in \heartsuit(X_\bullet)$. Although there is a canonical map

$$H_i : \underline{\mathrm{Hom}}_{\heartsuit(X_\bullet)}(\mathcal{M}, \mathcal{N})_i \to \underline{\mathrm{Hom}}_{\heartsuit(X_i)}(\mathcal{M}_i, \mathcal{N}_i)$$

arising from the closed structure for $i \in I$, this may not be an isomorphism. However, we have the following.

Lemma 6.33. *Let $i \in I$. If $\mathcal{M}$ is equivariant, then the canonical map*

$$H_i : \underline{\mathrm{Hom}}_{\heartsuit(X_\bullet)}(\mathcal{M},\mathcal{N})_i \to \underline{\mathrm{Hom}}_{\heartsuit(X_i)}(\mathcal{M}_i,\mathcal{N}_i)$$

is an isomorphism of presheaves. In particular, it is an isomorphism in $\heartsuit(X_i)$.

Proof. It suffices to prove that

$$H_i : \mathrm{Hom}_{\heartsuit(\mathrm{Zar}(X_\bullet)/(i,U))}(\mathcal{M}|_{(i,U)},\mathcal{N}|_{(i,U)}) \to \mathrm{Hom}_{\heartsuit(U)}(\mathcal{M}_i|_U,\mathcal{N}_i|_U)$$

is an isomorphism for any Zariski open set U in X_i.

To give an element of $\varphi \in \mathrm{Hom}_{\heartsuit(\mathrm{Zar}(X_\bullet)/(i,U))}(\mathcal{M}|_{(i,U)},\mathcal{N}|_{(i,U)})$ is the same as to give a family $(\varphi_\phi)_{\phi:i\to j}$ with

$$\varphi_\phi \in \mathrm{Hom}_{\heartsuit(X_\phi^{-1}(U))}(\mathcal{M}_j|_{X_\phi^{-1}(U)},\mathcal{N}_j|_{X_\phi^{-1}(U)})$$

such that for any $\phi\colon i \to j$ and $\psi\colon j \to j'$,

$$\varphi_{\psi\phi} \circ (\alpha_\psi(\mathcal{M}))|_{X_{\psi\phi}^{-1}(U)} = (\alpha_\psi(\mathcal{N}))|_{X_{\psi\phi}^{-1}(U)} \circ ((X_\psi)|_{X_{\psi\phi}^{-1}(U)})^*_\heartsuit(\varphi_\phi). \quad (6.34)$$

As $\alpha_\phi(\mathcal{M})$ is an isomorphism for any $\phi : i \to j$, we have that such a (φ_ϕ) is uniquely determined by φ_{id_i} by the formula

$$\varphi_\phi = (\alpha_\phi(\mathcal{N}))|_{X_\phi^{-1}(U)} \circ ((X_\phi)|_{X_\phi^{-1}(U)})^*_\heartsuit(\varphi_{\mathrm{id}_i}) \circ (\alpha_\phi(\mathcal{M}))|^{-1}_{X_\phi^{-1}(U)}. \quad (6.35)$$

Conversely, fix φ_{id_i}, and define φ_ϕ by (6.35). Consider the diagram

$$\begin{tikzcd}
X^*_{\psi\phi}\mathcal{M}_i \arrow[r, "d^{-1}"] \arrow[d, "\varphi_{\mathrm{id}_i}"'] \arrow[dr, phantom, "(a)"] & X^*_\psi X^*_\phi \mathcal{M}_i \arrow[r, "\alpha_\phi"] \arrow[d, "\varphi_{\mathrm{id}_i}"] \arrow[dr, phantom, "(b)"] & X^*_\psi \mathcal{M}_j \arrow[r, "\alpha_\psi"] \arrow[d, "\varphi_\phi"] \arrow[dr, phantom, "(c)"] & \mathcal{M}_{j'} \arrow[d, "\varphi_{\psi\varphi}"] \\
X^*_{\psi\varphi}\mathcal{N}_i \arrow[r, "d^{-1}"] & X^*_\psi X^*_\phi \mathcal{N}_i \arrow[r, "\alpha_\phi"] & X^*_\psi \mathcal{N}_j \arrow[r, "\alpha_\psi"] & \mathcal{N}_{j'} .
\end{tikzcd}$$

The diagram (a) is commutative by the naturality of d^{-1}. The diagram (b) and (a)+(b)+(c) are commutative, by the definition of φ_ϕ and $\varphi_{\psi\phi}$ (6.35), respectively. Since d^{-1} and $\alpha_\phi(\mathcal{M})$ are isomorphisms, the diagram (c) is commutative, and hence (6.34) holds. Hence H_i is bijective, as desired. □

Lemma 6.36. *Let J be a subcategory of I. If $\mathcal{M}$ is equivariant, then the canonical map*

$$H_J : \underline{\mathrm{Hom}}_{\heartsuit(X_\bullet)}(\mathcal{M},\mathcal{N})_J \to \underline{\mathrm{Hom}}_{\heartsuit(X_\bullet|_J)}(\mathcal{M}_J,\mathcal{N}_J)$$

is an isomorphism of presheaves. In particular, it is an isomorphism in $\heartsuit(X_\bullet|_J)$.

Proof. It suffices to show that

$$(H_J)_i : (\underline{\mathrm{Hom}}_{\heartsuit(X_\bullet)}(\mathcal{M},\mathcal{N})_J)_i \to \underline{\mathrm{Hom}}_{\heartsuit(X_J)}(\mathcal{M}_J,\mathcal{N}_J)_i$$

is an isomorphism for each $i \in J$. By Lemma 1.39, the composite map

$$\underline{\mathrm{Hom}}_{\heartsuit(X_\bullet)}(\mathcal{M},\mathcal{N})_i \cong (\underline{\mathrm{Hom}}_{\heartsuit(X_\bullet)}(\mathcal{M},\mathcal{N})_J)_i \xrightarrow{(H_J)_i} \underline{\mathrm{Hom}}_{\heartsuit(X_J)}(\mathcal{M}_J,\mathcal{N}_J)_i \xrightarrow{H_i} \underline{\mathrm{Hom}}_{\heartsuit(X_i)}(\mathcal{M}_i,\mathcal{N}_i)$$

agrees with H_i. As $\mathcal{M}_J$ is also equivariant, we have that the two H_i are isomorphisms by Lemma 6.33, and hence $(H_J)_i$ is an isomorphism for any $i \in J$. □

(6.37) By the lemma, the sheaf $\underline{\mathrm{Hom}}_{\heartsuit(X_\bullet)}(\mathcal{M},\mathcal{N})$ is given by the collection

$$(\underline{\mathrm{Hom}}_{\heartsuit(X_i)}(\mathcal{M}_i,\mathcal{N}_i))_{i\in I}$$

provided $\mathcal{M}$ is equivariant. The structure map is the canonical composite map

$$\alpha_\phi : (X_\phi)^*_\heartsuit \underline{\mathrm{Hom}}_{\heartsuit(X_i)}(\mathcal{M}_i,\mathcal{N}_i) \xrightarrow{P} \underline{\mathrm{Hom}}_{\heartsuit(X_j)}((X_\phi)^*_\heartsuit\mathcal{M}_i, (X_\phi)^*_\heartsuit\mathcal{N}_i) \xrightarrow{\underline{\mathrm{Hom}}_{\heartsuit(X_j)}(\alpha_\phi^{-1},\alpha_\phi)} \underline{\mathrm{Hom}}_{\heartsuit(X_j)}(\mathcal{M}_j,\mathcal{N}_j).$$

Similarly, the following is also easy to prove.

Lemma 6.38. *Let $i \in I$ be an initial object of I. Then the following hold:*

1 *If $\mathcal{M} \in \heartsuit(X_\bullet)$ is equivariant, then*

$$(?)_i : \mathrm{Hom}_{\heartsuit(X_\bullet)}(\mathcal{M},\mathcal{N}) \to \mathrm{Hom}_{\heartsuit(X_i)}(\mathcal{M}_i,\mathcal{N}_i)$$

is an isomorphism.

2 $(?)_i : \mathrm{EM}(X_\bullet) \to \mathrm{Mod}(X_i)$ *is an equivalence, whose quasi-inverse is L_i.*

The fact that $L_i(\mathcal{M})$ is equivariant for $\mathcal{M} \in \mathrm{Mod}(X_i)$ is checked directly from the definition.