9

p-Adic Grassmann Manifold

Summary. In Chap. 9 we give the analogous theory over the p-adic, giving the decomposition of the representation of $GL_d(\mathbb{Z}_p)$ afforded by the p-adic Grassmannian. The relative position of two planes $\mathfrak{p}, \mathfrak{q} \subseteq \mathbb{Z}_p^d$ is given by the type of the $\mathbb{Z}_p$-module $\mathfrak{p} \cap \mathfrak{q}$, i.e., by a partition. We calculate the measure on Ω_m^d, and describe the idempotents - the p-adic multivariable Jacobi polynomials.

9.1 Representation of $GL_d(\mathbb{Z}_p)$

9.1.1 Measures on $GL_d(\mathbb{Z}_p)$, V_m^d and X_m^d

Let p be a finite prime. First of all, we see that $GL_d(\mathbb{Z}_p)$ is expressed as the inverse limit;

$$GL_d(\mathbb{Z}_p) = \varprojlim G_{N^d},$$

where $G_{N^d} := GL_d(\mathbb{Z}/p^N)$. Then we obtain the following diagram by the determinant;

$$\begin{array}{ccc} \mathrm{Mat}_{d\times d}(\mathbb{Z}_p) & \xrightarrow{\det} & \mathbb{Z}_p \\ \cup & & \cup \\ GL_d(\mathbb{Z}_p) & \xrightarrow{\det} & \mathbb{Z}_p^* \end{array}$$

Note that $GL_d(\mathbb{Z}_p)$ is the maximal compact subgroup of $GL_d(\mathbb{Q}_p)$. This is similar to the real case. Namely, O_d is the maximal compact subgroup of $GL_d(\mathbb{R})$ and U_d is of $GL_d(\mathbb{C})$. But unlike the real case (where O_d and U_d are *closed* subset of $\mathrm{Mat}_{d\times d}$), the above diagram shows that $GL_d(\mathbb{Z}_p)$ is an *open* subset of $\mathrm{Mat}_{d\times d}(\mathbb{Z}_p)$. Now we have the measure on $\mathrm{Mat}_{d\times d}(\mathbb{Z}_p)$ defined by the additive Haar measure

$$dx := \bigotimes_{1\le i,j\le d} dx_{ij}.$$

This measure satisfies $d(gx) = |\det g|dx$ for $g \in \mathrm{Mat}_{d\times d}(\mathbb{Z}_p)$. In particular, dx is $GL_d(\mathbb{Z}_p)$-invariant measure on $\mathrm{Mat}_{d\times d}(\mathbb{Z}_p)$.

Let $A_1, \ldots, A_m \in \mathbb{Z}_p^{\oplus d} \subseteq \mathbb{Q}_p^{\oplus d}$. We call $A_1, \ldots, A_m$ orthonormal if

$$\mathbb{Z}_p^{\oplus d} \Big/ \sum_{1\leq i\leq m} \mathbb{Z}_p A_i \simeq \mathbb{Z}_p^{\oplus(d-m)}$$

as $\mathbb{Z}_p$-module. Let $\overline{A}_i$ be the image of A_i modulo p for $1 \leq i \leq m$. Then $A_1, \ldots, A_m$ are orthonormal if and only if $\overline{A}_1, \ldots, \overline{A}_m \in \mathbb{F}_p^{\oplus d}$ are linearly independent over $\mathbb{F}_p$. This is also equivalent to the existence of $B_1, \ldots, B_{d-m} \in \mathbb{Z}_p^{\oplus d}$ such that $(A_1, \ldots, A_m | B_1, \ldots, B_{d-m}) \in GL_d(\mathbb{Z}_p)$. Then we denote by

$$V_m^d := \big\{ A = (A_1, \ldots, A_m) \in \mathrm{Mat}_{d\times m}(\mathbb{Z}_p) \,\big|\, A_1, \ldots, A_m \text{ are orthonormal.} \big\}.$$

The group $GL_d(\mathbb{Z}_p)$ acts on V_m^d transitively and the stabilizer of the standard basis $1 = (E_1, \ldots, E_m)$ is given by $GL_{d-m}(\mathbb{Z}_p) \ltimes \mathrm{Mat}_{m\times(d-m)}(\mathbb{Z}_p)$. Hence it holds that

$$V_m^d \simeq GL_d(\mathbb{Z}_p)/GL_{d-m}(\mathbb{Z}_p) \ltimes \mathrm{Mat}_{m\times(d-m)}(\mathbb{Z}_p).$$

Note that the factor $\mathrm{Mat}_{m\times(d-m)}(\mathbb{Z}_p)$ does not appear in the real case. Let us first consider the case of $m = 1$. It is easy to see that

$$V_1^d = \big\{ A \in \mathbb{Z}_p^{\oplus d} \,\big|\, |A|_p = 1 \big\},$$

where $|A|_p = |{}^t(a_1, \ldots, a_d)|_p := \max_{1\leq i\leq d} |a_i|_p$. Then the condition $|A|_p = 1$ is equivalent to $\overline{A} \not\equiv 0$ modulo p. The measure of V_1^d can be calculated as follows;

$$\begin{aligned}
\int_{V_1^d} dx &= (1 - p^{-1}) + p^{-1} \int_{V_1^{d-1}} dx = \cdots \\
&= (1 - p^{-1}) + p^{-1}(1 - p^{-1}) + p^{-2}(1 - p^{-1}) + \cdots + p^{-(d-1)}(1 - p^{-1}) \\
&= 1 - p^{-d} \\
&= \frac{1}{\zeta_p(d)}.
\end{aligned}$$

Similarly, for general $m \geq 1$, we have

$$\int_{V_m^d} dx = \int_{V_1^d} dx \int_{V_{m-1}^{d-1}} dx = \cdots = \prod_{d-m<j\leq d} \frac{1}{\zeta_p(j)}.$$

In particular if we take $m = d$, we have $V_d^d = GL_d(\mathbb{Z}_p)$ and

$$\int_{GL_d(\mathbb{Z}_p)} dx = \prod_{1\leq j\leq d} \frac{1}{\zeta_p(j)}.$$

Normalizing the additive Haar measure dx by dividing by the above constant, one obtains the $GL_d(\mathbb{Z}_p)$ invariant probability measure on V_m^d. We denote by τ_m^d this measure on V_m^d, and $\tau^d := \tau_d^d$ the Haar measure on $GL_d(\mathbb{Z}_p)$.

Now we are interested in space

$$X_m^d := \mathrm{Grass}(m, d; \mathbb{Q}_p),$$

where $\mathrm{Grass}(m, d; \mathbb{Q}_p)$ is the Grassmann manifold of all m-dimensional space in d-dimensional plane over $\mathbb{Q}_p$. Note that $\mathrm{Grass}(m, d; \mathbb{Q}_p) = \mathrm{Grass}(m, d; \mathbb{Z}_p)$. Since $GL_d(\mathbb{Q}_p)$ (resp. $GL_d(\mathbb{Z}_p)$) acts transitively on X_m^d and the stabilizer of 1 is the Borel subgroup $B_{m,d-m}(\mathbb{Q}_p)$ (resp. $B_{m,d-m}(\mathbb{Z}_p)$) where

$$\begin{aligned} B_{m,d-m} :&= \left\{ \left(\begin{array}{c|c} A & B \\ \hline 0 & D \end{array} \right) \in \mathrm{Mat}_{d\times d} \,\middle|\, A \in GL_m,\ D \in GL_{d-m},\ B \in \mathrm{Mat}_{m\times(d-m)} \right\} \\ &= (GL_m \times GL_{d-m}) \ltimes \mathrm{Mat}_{m\times(d-m)}, \end{aligned}$$

we have

$$X_m^d = GL_d(\mathbb{Q}_p)/B_{m,d-m}(\mathbb{Q}_p) = GL_d(\mathbb{Z}_p)/B_{m,d-m}(\mathbb{Z}_p).$$

It can also be expressed as

$$X_m^d = \left\{ \mathfrak{p} \subseteq \mathbb{Z}_p^{\oplus d} \,\middle|\, \mathbb{Z}_p^{\oplus d}/\mathfrak{p} \simeq \mathbb{Z}_p^{\oplus(d-m)} \right\}.$$

Note that in the real case the factor $\mathrm{Mat}_{m\times(d-m)}(\mathbb{Z}_\eta)$ disappear, and the real Grassmann manifold resembles more the space

$$\begin{aligned} \widetilde{X}_m^d &= \left\{ (\mathfrak{p}, \mathfrak{p}') \,\middle|\, \mathfrak{p} \simeq \mathbb{Z}_p^{\oplus m},\ \mathfrak{p}' \simeq \mathbb{Z}_p^{\oplus(d-m)},\ \mathfrak{p} \oplus \mathfrak{p}' \simeq \mathbb{Z}_p^{\oplus d} \right\} \\ &= GL_d(\mathbb{Z}_p)/GL_m(\mathbb{Z}_p) \times GL_{d-m}(\mathbb{Z}_p). \end{aligned}$$

The measure $\overline{\tau}_m^d$ on X_m^d is obtained as follows; Let pr be the projection

$$\mathrm{pr} : V_m^d \longrightarrow X_m^d = V_m^d/GL_m(\mathbb{Z}_p)\,; \quad \mathrm{pr}(A_1, \ldots, A_m) = \mathrm{Span}_{\mathbb{Z}_p}(A_1, \ldots, A_m).$$

Then we see that the image $\mathrm{pr}_*(\tau_m^d)$ of the probability measure τ_m^d is the unique $GL_d(\mathbb{Z}_p)$ invariant probability measure on X_m^d. Hence, by the uniqueness, we have $\overline{\tau}_m^d = \mathrm{pr}_*(\tau_m^d)$. On the other hand, notice that the set of matrices $X \in \mathrm{Mat}_{d\times m}(\mathbb{Z}_p)$ of rank $X = m$ is of full measure with respect to the additive Haar measure dx. Then we have the projection

$$\widetilde{\mathrm{pr}} : \mathrm{Mat}_{d\times m}(\mathbb{Z}_p) \longrightarrow X_m^d = V_m^d/GL_m(\mathbb{Z}_p)\,; \quad \widetilde{\mathrm{pr}}(X) = \mathrm{Span}_{\mathbb{Q}_p}(X_1, \ldots, X_m) \cap \mathbb{Z}_p^{\oplus d}$$

and also $\overline{\tau}_m^d = \widetilde{\mathrm{pr}}_*(dx)$. Note that $\widetilde{\mathrm{pr}}(X)$ is not the space spanned by X over $\mathbb{Z}_p$.

The space X_m^d can be also represented as the inverse limit;

$$X_m^d = GL_d(\mathbb{Z}_p)/B_{m,d-m}(\mathbb{Z}_p) = \varprojlim X_{N^m}^{N^d},$$

where $X_{N^m}^{N^d}$ is the finite set defined by $X_{N^m}^{N^d} := GL_d(\mathbb{Z}/p^N)/B_{m,d-m}(\mathbb{Z}/p^N) \simeq G_{N^d}/B_{N^m}$ and $B_{N^m} := B_{m,d-m}(\mathbb{Z}/p^N)$. One can also check that G_{N^d} acts on $X_{N^m}^{N^d}$ transitively and the stabilizer of 1 is given by B_{N^m}.

9.1.2 Unitary Representations of $GL_d(\mathbb{Z}_p)$ and G_{N^d}

We are interested in the unitary representation of $GL_d(\mathbb{Z}_p)$ defined by

$$\pi : GL_d(\mathbb{Z}_p) \longrightarrow U(H_m^d)\,; \quad \pi(g)f(x) := f(g^{-1}x),$$

where $H_m^d := L^2(X_m^d, \overline{\tau}_m^d)$. Now the Hilbert space H_m^d can be written as the direct limit of the finite dimensional spaces as follows;

$$H_m^d = \varinjlim H_{N^m}^{N^d},$$

where $H_{N^m}^{N^d} := L^2(X_{N^m}^{N^d})$. We have a unitary embedding from the finite dimensional space $H_{N^m}^{N^d}$ to H_m^d and $\bigcup_N H_{N^m}^{N^d}$ is dense in H_m^d. Moreover, each finite dimensional space is invariant under the group $GL_d(\mathbb{Z}_p)$ and the representation of $GL_d(\mathbb{Z}_p)$ on it factors through the projection $GL_d(\mathbb{Z}_p) \twoheadrightarrow G_{N^d} \to U(H_{N^m}^{N^d})$. The commutant of this representation are generated by the Hecke algebra

$$\mathcal{H}_m^d := C^\infty(\Omega_m^d).$$

Notice that, in the p-adic cases, smoothness means locally constant. Here

$$\Omega_m^d := B_{m,d-m}(\mathbb{Z}_p)\backslash GL_d(\mathbb{Z}_p)/B_{m,d-m}(\mathbb{Z}_p) = \varprojlim \Omega_{N^m}^{N^d},$$

where $\Omega_{N^m}^{N^d} := B_{N^m}\backslash G_{N^d}/B_{N^m}$. The commutant of the representation of the finite group G_{N^d} on the finite dimensional space $H_{N^m}^{N^d}$ is also generated by the Hecke algebra

$$\mathcal{H}_{N^m}^{N^d} = C^\infty(\Omega_{N^m}^{N^d}).$$

Again $\mathcal{H}_m^d$ is expressed as the direct limit of the space $\mathcal{H}_{N^m}^{N^d}$;

$$\mathcal{H}_m^d = \varinjlim \mathcal{H}_{N^m}^{N^d}.$$

More generally, if we want the intertwining operator of the various representation for different m, say $H_{N^m}^{N^d} \to H_{N^n}^{N^d}$, we have to consider the module

$$\mathcal{H}_{N^m,N^n}^{N^d} := C^\infty(B_{N^m}\backslash G_{N^d}/B_{N^n}).$$

Notice that we always assume $m \leq n \leq \frac{1}{2}d$.

Now remember the simple facts for finite $\mathbb{Z}_p$-modules. Let $\mathfrak{m}$ be a finite $\mathbb{Z}_p$-module (resp. $\mathbb{Z}/p^N$-module). Then it is of the form of

$$\mathfrak{m} \simeq \bigoplus_i \mathbb{Z}/p^{\lambda_i} =: \mathbb{Z}/p^\lambda,$$

where $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0)$ is a partition (resp. with $\lambda_1 \leq N$). In this case, we say the type of $\mathfrak{m}$ is λ and write $\mathrm{typ}(\mathfrak{m}) = \lambda$. This is a complete isomorphism invariant. Namely, two modules are isomorphic if and only if they have the same type. (Note that all partitions are *decreasing.* All the people working in real or q-special functions use increasing partition while Macdonald use decreasing partition ([Mac]). Hence we have to change the notation unfortunately if we treat both the real and the p-adic cases.) We also use the following notation

$$(1^{r_1}, 2^{r_2}, \ldots, N^{r_N}) := (\underbrace{N, \ldots, N}_{d}, \ldots, \underbrace{1, \ldots, 1}_{r_1})$$

In particular, $(N^d) = (\underbrace{N, \ldots, N}_{d})$ and hence

$$\mathbb{Z}/p^{(N^d)} \simeq \left(\mathbb{Z}/p^N\right)^{\oplus d}.$$

This is why we use the notation G_{N^d}, which is the automorphism group of $\left(\mathbb{Z}/p^N\right)^{\oplus d}$. These are the highly symmetric modules. If we take a module $\mathfrak{m} \subset \left(\mathbb{Z}/p^N\right)^{\oplus d}$ of $\mathrm{typ}(\mathfrak{m}) = \lambda$, there exist a basis $X_1, \ldots, X_d$ for the free module $\left(\mathbb{Z}/p^N\right)^{\oplus d}$ such that $p^{N-\lambda_1}X_1, \ldots, p^{N-\lambda_d}X_d$ is the basis for $\mathfrak{m}$. Here $y_1, \ldots, y_l$ is the basis for $\mathfrak{m}$ of type λ means that (note that $\mathfrak{m}$ is not *free*) y_i's generate $\mathfrak{m}$ and of order exactly λ_i. Equivalently, every $m \in \mathfrak{m}$ can be uniquely written as $m = a_1 y_1 + \cdots + a_l y_l$ for some $a_i \in \mathbb{Z}/p^{\lambda_i}$. For example, given such a module $\mathfrak{m} \subseteq \left(\mathbb{Z}/p^N\right)^{\oplus d}$ of $\mathrm{typ}(\mathfrak{m}) = \lambda$, we have

$$\mathrm{typ}\left(\left(\mathbb{Z}/p^N\right)^{\oplus d}/\mathfrak{m}\right) = (N - \lambda_d, \ldots, N - \lambda_1).$$

As a corollary of the elementary divisor, we have

Corollary 9.1.1. *Any isomorphism $g : \mathfrak{m} \to \mathfrak{m}'$ between two finite submodules $\mathfrak{m}, \mathfrak{m}' \subseteq \left(\mathbb{Z}/p^N\right)^{\oplus d}$ can be extended to $g \in \mathrm{Aut}\left(\left(\mathbb{Z}/p^N\right)^{\oplus d}\right) = G_{N^d}$.*

Therefore, the space of the relative positions $\Omega_{N^m}^{N^d}$ can be written as follows;

Corollary 9.1.2.

$$\Omega_{N^m}^{N^d} \simeq \left\{\lambda = (\lambda_1, \ldots, \lambda_l) \,\middle|\, \lambda_1 \leq N,\ \lambda'_1 \leq m\right\} =: \Lambda_{N^m},$$

where the isomorphism is given by

$$G_{N^d}(\mathfrak{m}_1, \mathfrak{m}_2) \longmapsto \mathrm{typ}(\mathfrak{m}_1 \cap \mathfrak{m}_2).$$

Here we denote by $\lambda' = (\lambda'_1 \ldots, \lambda'_n)$ the conjugate of λ defined by $\lambda'_j = \#\{i \,|\, \lambda_i \geq j\}$.

Indeed, if for some $g \in G_{N^d}$ with $g(\mathfrak{m}_i) = \mathfrak{m}'_i$, then we have $\mathrm{typ}(\mathfrak{m}_1 \cap \mathfrak{m}_2) = \mathrm{typ}(\mathfrak{m}'_1 \cap \mathfrak{m}'_2)$. Conversely, if $\mathrm{typ}(\mathfrak{m}_1 \cap \mathfrak{m}_2) = \mathrm{typ}(\mathfrak{m}'_1 \cap \mathfrak{m}'_2)$, we have an isomorphism $g : \mathfrak{m}_1 \cap \mathfrak{m}_2 \to \mathfrak{m}'_1 \cap \mathfrak{m}'_2$. By Corollary 9.1.1, this can be extended to isomorphisms $g_i : \mathfrak{m}_i \to \mathfrak{m}'_i$ for $i = 1, 2$. Hence we have an isomorphism $g : \mathfrak{m}_1 + \mathfrak{m}_2 \to \mathfrak{m}'_1 + \mathfrak{m}'_2$. By Corollary 9.1.1 again, g can be extended to $g \in G_{N^d}$. This shows that $g(\mathfrak{m}_1, \mathfrak{m}_2) = (\mathfrak{m}'_1, \mathfrak{m}'_2)$.

Since $\mathrm{typ}(\mathfrak{m}_1 \cap \mathfrak{m}_2) = \mathrm{typ}(\mathfrak{m}_2 \cap \mathfrak{m}_1)$, we have the following

Corollary 9.1.3. *The Hecke algebra $\mathcal{H}^{N^d}_{N^m}$ is commutative. The dimension of $\mathcal{H}^{N^d}_{N^m}$ is given by $\#\Lambda_{N^m} = \binom{N+m}{m}$. Hence their direct limit $\mathcal{H}^d_m = \varinjlim \mathcal{H}^{N^d}_{N^m}$ is also commutative.*

Therefore the representations of G_{N^d} and $GL_d(\mathbb{Z}_p)$ are multiplicity free, whence they decompose as follows

$$H^{N^d}_{N^m} = \bigoplus_{\lambda \in \Lambda_{N^m}} V_\lambda, \qquad H^d_m = \bigoplus_{\lambda'_1 \leq m} V_\lambda.$$

We have the following diagrams using the quotient maps from modulo p^N to modulo p^{N-1}. Here the projection $\Lambda_{N^m} \to \Lambda_{(N-1)^m}$ is given by "chopping the right-most column", that is, $(\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_N) \mapsto (\lambda'_1 \geq \lambda'_2 \geq \cdots \geq \lambda'_{N-1})$;

$$\begin{array}{ccc} G_{N^d} & \longrightarrow & X^{N^d}_{N^m} \\ \downarrow & & \downarrow \\ G_{(N-1)^d} & \longrightarrow & X^{(N-1)^d}_{(N-1)^m} \end{array} \qquad \begin{array}{ccc} \Omega^{N^d}_{N^m} & \xrightarrow{\sim} & \Lambda_{N^m} \\ \downarrow & & \downarrow \\ \Omega^{(N-1)^d}_{(N-1)^m} & \longrightarrow & \Lambda_{(N-1)^m} \end{array}$$

Taking the inverse limit, we have the following trees;

N-th layer	tree	boundary
$X^{N^d}_{N^m}$	$\bigsqcup_N X^{N^d}_{N^m}$	$X^d_m = \varprojlim X^{N^d}_{N^m}$
Λ_{N^m}	$\bigsqcup_N \Lambda_{N^m}$	$\Omega^d_m = \varprojlim \Omega^{N^d}_{N^m}$

Notice that, we have infinite partitions in $\varprojlim \Lambda_{N^m} \simeq \varprojlim \Omega^{N^d}_{N^m}$, that is,

$$\begin{aligned} \varprojlim \Lambda_{N^m} &= \Lambda_m \sqcup \Lambda_{m-1} \sqcup \cdots \sqcup \Lambda_1 \sqcup \Lambda_0 = \{\infty\}, \\ \Lambda_{m-j} &:= \bigl\{\lambda = (\underbrace{\infty, \ldots, \infty}_{j} > \lambda_{j+1} \geq \ldots \geq \lambda_m \geq 0)\bigr\}. \end{aligned}$$

We have the two types of embedding

$$\Omega^d_m = \varprojlim \Lambda_{N^m} \longrightarrow [0,1]^m$$

defined as follows;

$$\begin{aligned} \text{sin-embedding } &: \lambda \longmapsto (p^{-\lambda_1}, \ldots, p^{-\lambda_m}), \\ \text{cos-embedding } &: \lambda \longmapsto (1 - p^{-\lambda_1}, \ldots, 1 - p^{-\lambda_m}). \end{aligned}$$

Here we understand $p^{-\infty} = 0$. It is important to note that we have two types of topologies in Ω_m^d, that is, the inverse limit topology, and the topology induced from $[0,1]^m$, and these are the same topology. This shows that the set of all finite partitions Λ_m (these do not have the 0 coordinate in $[0,1]^m$ by the embedding above) is an open and dense subspace of Ω_m^d; it is also of full measure with respect to the probability measure $\overline{\tau}_{m,n}^d$ on Ω_m^d. Here the measure $\overline{\tau}_{m,n}^d$ is obtained as follows; Let us write

$$\Omega_m^d = B_{m,d-m}(\mathbb{Z}_p) \backslash GL_d(\mathbb{Z}_p) / B_{n,d-n}(\mathbb{Z}_p)$$

Then $\overline{\tau}_{m,n}^d$ is the measure induced from the Haar measure τ^d on $GL_d(\mathbb{Z}_p)$. As in the case of the reals, $\overline{\tau}_{m,n}^d$ can be obtained by $t_*(dx \otimes dy)$. Here $dx \otimes dy$ is the additive measure on $\mathrm{Mat}_{d\times(m+n)}(\mathbb{Z}_p)$ and t is the map

$$t : \mathrm{Mat}_{d\times(m+n)}(\mathbb{Z}_p) \xrightarrow{\widetilde{\mathrm{pr}}} X_m^d \times X_n^d \xrightarrow{\mathrm{typ}} \Omega_m^d$$

It can be also expressed as

$$\overline{\tau}_{m,n}^d = t_*(dx \otimes dy) = t_*(dx \otimes \delta_{y_0}) = t_*(\delta_{x_0} \otimes dy)$$

for some $x_0 \in \mathrm{Mat}_{d\times m}(\mathbb{Z}_p)$, or some $y_0 \in \mathrm{Mat}_{d\times n}(\mathbb{Z}_p)$. We get the Markov chain on $\bigsqcup_N \Lambda_{N^m}$ with harmonic measure $\overline{\tau}_{m,n}^d$ (remember that we have the Markov chain if we have a tree and a measure on the boundary).

Now we try to see the relative position more like in the real case. Let $A, B \in \mathbb{P}^{d-1}(\mathbb{Z}_p^{\oplus d})$. Define

$$|(A,B)| = 1 - \rho(A,B) := \sup\bigl\{1 - p^{-n} \bigm| A \equiv B \pmod{p^n},\ n \geq 0\bigr\}.$$

For example, we have

$$\begin{aligned} A \not\equiv B \pmod{p} &\iff |(A,B)| = 1 - p^0 = 0 \\ &\iff A, B \text{ are orthonormal,} \end{aligned}$$

and

$$\begin{aligned} A = B &\iff A \equiv B \pmod{p^n} \text{ for all } n \geq 0 \\ &\iff |(A,B)| = 1. \end{aligned}$$

Hence we have for $\mathfrak{p} \in X_m^d$ and $\mathfrak{q} \in X_n^d$, $\mathrm{typ}(\mathfrak{p}, \mathfrak{q}) = \lambda \in \varprojlim \Lambda_{N^m}$ if and only if there exists orthonormal basis $A_1, \ldots, A_m$ for $\mathfrak{p}$ and $B_1, \ldots, B_n$ for $\mathfrak{q}$ such that $|(A_i, B_j)| = \delta_{i,j}(1 - p^{-\lambda_i})$.

9.2 Harmonic Measure

9.2.1 Notations

Let $\lambda, \mu, \overline{\mu}$ be partitions. We put $G_\lambda := \mathrm{Aut}(\mathbb{Z}/p^\lambda)$ and fixing $\mathfrak{m}_0 = \mathbb{Z}/p^\lambda$ we define

$$X^\lambda_\mu : = \mathrm{Grass}\big(\mathfrak{m} \subseteq \mathfrak{m}_0 \,\big|\, \mathrm{typ}(\mathfrak{m}) = \mu\big),$$
$$X^\lambda_{\mu,\overline{\mu}} : = \mathrm{Grass}\big(\mathfrak{m} \subseteq \mathfrak{m}_0 \,\big|\, \mathrm{typ}(\mathfrak{m}) = \mu,\ \mathrm{typ}(\mathfrak{m}_0/\mathfrak{m}) = \overline{\mu}\big).$$

More generally, for the modules $\mathfrak{m}_0$ of $\mathrm{typ}(\mathfrak{m}_0) = \lambda$ and $\mathfrak{m} \subseteq \mathfrak{m}_0$ of $\mathrm{typ}(\mathfrak{m}) = \mu$, we get the sequence of the partitions $\{\mathrm{typ}(\mathfrak{m}_0/\mathfrak{m} \cap p^i\mathfrak{m}_0)\}_{i=0,1,\ldots}$ from $\overline{\mu}$ to λ. Hence

$$T := \big\{\mathrm{typ}(\mathfrak{m}_0/\mathfrak{m} \cap p^i\mathfrak{m}_0)\big\}_{i\geq 0}$$

is a tableau of shape $\mathrm{sh}(T) = \lambda\backslash\overline{\mu}$ and weight $\mathrm{wt}(T) = \mu$. We define for a given tableau T

$$X^\lambda_T : = \mathrm{Grass}\big(\mathfrak{m} \subseteq \mathfrak{m}_0 \,\big|\, \big\{\mathrm{typ}(\mathfrak{m}_0/\mathfrak{m} \cap p^i\mathfrak{m}_0)\big\}_{i\geq 0} = T\big).$$

Then the group G_{N^d} acts on the spaces X^λ_μ, $X^\lambda_{\mu,\overline{\mu}}$ and X^λ_T. Note that $X^\lambda_\mu = \bigcup_{\overline{\mu}} X^\lambda_{\mu,\overline{\mu}}$ and $X^\lambda_{\mu,\overline{\mu}} = \bigcup_T X^\lambda_T$ the union taken over T with $\mathrm{sh}(T) = \lambda\backslash\overline{\mu}$ and $\mathrm{wt}(T) = \mu$. G_{N^d} is not transitive on X^λ_T (It is very difficult combinatorial problem to describe all the equivalence classes of embedding $\mathbb{Z}/p^\mu \hookrightarrow \mathbb{Z}/p^\lambda$). We denote respectively by

$$\binom{\lambda}{T}_p := \#X^\lambda_T,\quad \binom{\lambda}{\mu,\overline{\mu}}_p := \#X^\lambda_{\mu,\overline{\mu}} = \sum_T \binom{\lambda}{T}_p,\quad \binom{\lambda}{\mu}_p := \#X^\lambda_\mu = \sum_{\overline{\mu}} \binom{\lambda}{\mu,\overline{\mu}}_p$$

$\binom{\lambda}{T}_p$ are monic polynomials in p, and $\binom{\lambda}{\mu,\overline{\mu}}_p$ are the Hall polynomial (see [Mac]). One can see that the leading term of $\binom{\lambda}{\mu,\overline{\mu}}_p$ is the number $c^\lambda_{\mu,\overline{\mu}}$ of tableau T with $\mathrm{sh}(T) = \lambda\backslash\overline{\mu}$ and $\mathrm{wt}(T) = \mu$; $c^\lambda_{\mu,\overline{\mu}}$ are the Littlewood-Richardson coefficients.

Let

$$[n]_p := \frac{1}{\zeta_p(n)} = 1 - p^{-n},\quad [n]_p! := [n]_p \cdots [1]_p,\quad \begin{bmatrix} n \\ m \end{bmatrix}_p := \frac{[n]_p!}{[m]_p![n-m]_p!}.$$

Then it is easy to see that

$$\#\mathrm{Hom}(\mathbb{Z}/p^\lambda, \mathbb{Z}/p^\mu) = p^{\langle\lambda',\mu'\rangle},$$
$$\#\mathrm{Hom}^{1:1}(\mathbb{Z}/p^\lambda, \mathbb{Z}/p^\mu) = p^{\langle\lambda',\mu'\rangle} \prod_i \frac{[\mu'_i - \lambda'_{i+1}]_p!}{[\mu'_i - \lambda'_i]_p!},$$

where $\langle \lambda', \mu' \rangle := \sum_i \lambda'_i \mu'_i$. In particular, taking $\mu = \lambda$, we have

$$\# G_\lambda = p^{\langle \lambda', \lambda' \rangle} \prod_i [\lambda'_i - \lambda'_{i+1}]_p!.$$

Hence we have

$$\binom{\lambda}{\mu}_p = \# X^\lambda_\mu = \sum_{\overline{\mu}} \binom{\lambda}{\mu, \overline{\mu}}_p = \frac{\# \mathrm{Hom}^{1:1}(\mathbb{Z}/p^\mu, \mathbb{Z}/p^\lambda)}{\# G_\mu}$$
$$= p^{\langle \mu', \lambda' - \mu' \rangle} \prod_i \begin{bmatrix} \lambda'_i - \mu'_{i+1} \\ \lambda'_i - \mu'_i \end{bmatrix}_p.$$

Also we set

$$\{n\}_p := \frac{-1}{\zeta_p(-n)} = p^n - 1, \quad \{n\}_p! := \{n\}_p \cdots \{1\}_p, \quad \begin{Bmatrix} n \\ m \end{Bmatrix}_p := \frac{\{n\}_p!}{\{m\}_p!\{n-m\}_p!}.$$

These are useful notations when we count things. On the other hand we use the notation $[n]_p$ when we are working with the probability measure. Notice that

$$\begin{Bmatrix} n \\ m \end{Bmatrix}_p = \begin{bmatrix} n \\ m \end{bmatrix}_p p^{m(n-m)}.$$

Then it can be calculated as

$$\binom{N^d}{N^m} = \# X^{N^d}_{N^m} = p^{Nm(d-m)} \begin{bmatrix} d \\ d-m \end{bmatrix}_p.$$

Similarly, for a general partition λ, it is useful to calculate

$$\frac{\begin{bmatrix} N^d \\ \lambda \end{bmatrix}_p}{\begin{bmatrix} (N-1)^d \\ \overline{\lambda} \end{bmatrix}_p} = \frac{\# X^{N^d}_\lambda}{\# X^{(N-1)^d}_{\overline{\lambda}}} = \frac{p^{\sum_{i=1}^N \lambda'_i (d - \lambda'_i)} \prod_{i=1}^N \begin{bmatrix} d-\lambda'_{i+1} \\ d-\lambda'_i \end{bmatrix}_p}{p^{\sum_{i=1}^{N-1} \lambda'_i (d - \lambda'_i)} \prod_{i=1}^{N-1} \begin{bmatrix} d-\overline{\lambda}'_{i+1} \\ d-\overline{\lambda}'_i \end{bmatrix}_p}$$
$$= p^{\lambda'_N (d - \lambda'_N)} \frac{\begin{bmatrix} d \\ d-\lambda'_N \end{bmatrix}_p \begin{bmatrix} d-\lambda'_N \\ d-\lambda'_{N-1} \end{bmatrix}_p}{\begin{bmatrix} d \\ d-\lambda'_{N-1} \end{bmatrix}_p}.$$

Here $\overline{\lambda}$ is the projection of λ; $\overline{\lambda'} = (\lambda'_1, \ldots, \lambda'_{N-1})$.

9.2.2 Harmonic Measure on Ω^d_m

Now we determine the harmonic measure $\tau := \overline{\tau}^d_{m,n}$ on the boundary space $\Omega^d_m = \varprojlim \Lambda_{N^m}$ of the relative positions of m-plane and n-plane from the transition probability of the Markov chain (see Sect. 9.2). We here work with the conjugate coordinate, that is, $\lambda = (\lambda'_1, \ldots, \lambda'_N)$ and $\overline{\lambda} = (\lambda'_1, \ldots, \lambda'_{N-1})$.

Let τ_N be the probability measure on Λ_{N^m}. First of all, let us calculate the measure in the finite layer $\tau_1(\lambda_1')$. Fix a subspace $\mathfrak{q}_1 = \mathbb{F}_p^{\lambda_1'}$ with $\mathfrak{q}_1 \subseteq \mathfrak{q}_0 = \mathbb{F}_p^n \subseteq \mathbb{F}_p^d$. Note that

$$\#\{\mathfrak{p} \subseteq \mathbb{F}_p^d \,|\, \dim\mathfrak{p} = m,\ \mathfrak{p} \cap \mathfrak{q}_0 = \mathfrak{q}_1\} = \left\{ \begin{matrix} d-n \\ m-\lambda_1' \end{matrix} \right\}_p p^{(m-\lambda_1')(n-\lambda_1')},$$

$$\#\{\mathfrak{q}_1 \subseteq \mathbb{F}_p^n \,|\, \dim\mathfrak{q}_1 = \lambda_1'\} = \left\{ \begin{matrix} n \\ \lambda_1' \end{matrix} \right\}_p,$$

$$\#\{\mathfrak{p} \subseteq \mathbb{F}_p^d \,|\, \dim\mathfrak{p} = m\} = \left\{ \begin{matrix} d \\ m \end{matrix} \right\}_p.$$

Hence the first transition probability of the Markov chain is calculated as

$$\begin{aligned}
\tau_1(\lambda_1') &= \frac{\#\{\mathfrak{p} \subseteq \mathbb{F}_p^d \,|\, \dim\mathfrak{p} = m,\ \dim\mathfrak{p} \cap \mathfrak{q}_0 = \lambda_1'\}}{\#\{\mathfrak{p} \subseteq \mathbb{F}_p^d \,|\, \dim\mathfrak{p} = m\}} \\
&= \frac{\#\{\mathfrak{p} \subseteq \mathbb{F}_p^d \,|\, \dim\mathfrak{p} = m,\ \mathfrak{p} \cap \mathfrak{q}_0 = \mathfrak{q}_1\} \cdot \#\{\mathfrak{q}_1 \subseteq \mathbb{F}_p^n \,|\, \dim\mathfrak{q}_1 = \lambda_1'\}}{\#\{\mathfrak{p} \subseteq \mathbb{F}_p^d \,|\, \dim\mathfrak{p} = m\}} \\
&= \frac{\left\{ \begin{smallmatrix} d-n \\ m-\lambda_1' \end{smallmatrix} \right\}_p p^{(m-\lambda_1')(n-\lambda_1')} \cdot \left\{ \begin{smallmatrix} n \\ \lambda_1' \end{smallmatrix} \right\}_p}{\left\{ \begin{smallmatrix} d \\ m \end{smallmatrix} \right\}_p} \\
&= \frac{\left[\begin{smallmatrix} d-n \\ m-\lambda_1' \end{smallmatrix} \right]_p \left[\begin{smallmatrix} n \\ \lambda_1' \end{smallmatrix} \right]_p}{\left[\begin{smallmatrix} d \\ m \end{smallmatrix} \right]_p} p^{-\lambda_1'(d-n-m+\lambda_1')}.
\end{aligned} \tag{9.1}$$

Next we work on the N-th layer. For details see [On1]. Fix also a subspace $\mathfrak{q}_0 = \left(\mathbb{Z}/p^N\right)^{\oplus n} \subseteq \left(\mathbb{Z}/p^N\right)^{\oplus d}$. Then we have

$$\begin{aligned}
\frac{\tau_N(\lambda)}{\tau_{N-1}(\overline{\lambda})} &= \frac{\binom{N^d}{N^m}_p^{-1} \#\{\mathfrak{p} \subseteq (\mathbb{Z}/p^N)^{\oplus d} \,|\, \mathrm{typ}(\mathfrak{p}) = N^m,\ \mathrm{typ}(\mathfrak{p} \cap \mathfrak{q}_0) = \lambda\}}{\binom{(N-1)^d}{(N-1)^m}_p^{-1} \#\{\overline{\mathfrak{p}} \subseteq (\mathbb{Z}/p^{N-1})^{\oplus d}, \,|\, \mathrm{typ}(\overline{\mathfrak{p}}) = (N-1)^m,\ \mathrm{typ}(\overline{\mathfrak{p}} \cap \overline{\mathfrak{q}_0}) = \overline{\lambda}\}} \\
&= \frac{\binom{N^d}{N^m}_p^{-1} \binom{N^n}{\lambda}_p \#\{\mathfrak{p} \subseteq (\mathbb{Z}/p^N)^{\oplus d} \,|\, \mathrm{typ}(\mathfrak{p}) = N^m,\ \mathfrak{p} \cap \mathfrak{q}_0 = \mathfrak{q}_1\}}{\binom{(N-1)^d}{(N-1)^m}_p^{-1} \binom{(N-1)^n}{\overline{\lambda}}_p \#\{\overline{\mathfrak{p}} \subseteq (\mathbb{Z}/p^{N-1})^{\oplus d} \,|\, \mathrm{typ}(\overline{\mathfrak{p}}) = (N-1)^m,\ \overline{\mathfrak{p}} \cap \overline{\mathfrak{q}_0} = \overline{\mathfrak{q}_1}\}}
\end{aligned}$$

Here we fix $\mathfrak{q}_1$ of $\mathrm{typ}(\mathfrak{q}_1) = \lambda$. The independence of this choice of $\mathfrak{q}_1$ is justified by the symmetry of $\mathfrak{q}_0$. Hence we have

$$\begin{aligned}
\frac{\tau_N(\lambda)}{\tau_{N-1}(\overline{\lambda})} &= p^{-m(d-m)} \begin{bmatrix} \lambda_{N-1}' \\ \lambda_N' \end{bmatrix}_p p^{\lambda_N'(n-\lambda_N')} \\
&\times \#\{\mathfrak{p} \subseteq \left(\mathbb{Z}/p^N\right)^{\oplus d} \,|\, \mathrm{typ}(\mathfrak{p}) = N^m,\ \mathfrak{p} \cap \mathfrak{q}_0 = \mathfrak{q}_1,\ \overline{\mathfrak{p}} = \overline{\mathfrak{p}_0}\}.
\end{aligned} \tag{9.2}$$

Here we again fix the submodule $\overline{\mathfrak{p}_0} \subseteq \left(\mathbb{Z}/p^{N-1}\right)^{\oplus d}$ of $\mathrm{typ}(\overline{\mathfrak{p}_0}) = (N-1)^m$ such that $\overline{\mathfrak{p}_0} \cap \overline{\mathfrak{q}_0} = \overline{\mathfrak{q}_1}$. To calculate (9.2), without loss of generality, we assume that

$$\lambda'_N = 0 \tag{9.3}$$

since we can factor out $\left(\mathbb{Z}/p^N\right)^{\oplus\lambda'_N} \subseteq \mathfrak{p}\cap\mathfrak{q}$. Hence, to calculate (9.2), it is sufficient to count

$$\#\{\mathfrak{p}\subseteq \left(\mathbb{Z}/p^N\right)^{\oplus(d-\lambda'_N)} \,\big|\, \mathrm{typ}(\mathfrak{p}) = N^{m-\lambda'_N},\ \mathfrak{p}\cap\mathfrak{q}_0 = \mathfrak{q}_1,\ \overline{\mathfrak{p}} = \overline{\mathfrak{p}_0}\}. \tag{9.4}$$

Fix $\mathfrak{q}_0 \subseteq \left(\mathbb{Z}/p^N\right)^{\oplus n-\lambda'_N}$, $\mathfrak{q}_1$ of $\mathrm{typ}(\mathfrak{q}_1) = \overline{\lambda}$ and $\overline{\mathfrak{p}_0}$ of $\mathrm{typ}(\overline{\mathfrak{p}_0}) = (N-1)^{m-\lambda'_N}$. Also fix a lifting $\mathfrak{p}_0$ of $\overline{\mathfrak{p}_0}$. Let $\mathfrak{A} := \{A_1,\dots,A_m\}$ be a basis for $\mathfrak{p}_0$ and $\mathfrak{B} := \{B_1,\dots,B_m\}$ a completion to a basis for $\left(\mathbb{Z}/p^N\right)^{\oplus d}$. Then any other lifting $\mathfrak{p}$ of $\overline{\mathfrak{p}_0}$ has basis of the form

$$A_i + p^{N-1}\{\sum_{1\le j\le m} a_{ij}A_j + \sum_{1\le k\le m} b_{ik}B_k\},$$

where $a_{ij}, b_{ik} \in \mathbb{F}_p$. Now note that a_{ij}'s do not change $\mathfrak{p}$. Hence we ignore a_{ij}'s and consider only b_{ik}'s. Note also that, for two liftings $\mathfrak{p}$ and $\mathfrak{p}'$ of $\overline{\mathfrak{p}_0}$ given by $\{b_{ik}\}$ and $\{b'_{ik}\}$ respectively, it holds that $\mathfrak{p} = \mathfrak{p}'$ if and only if $b_{ik} = b'_{ik}$. Therefore the choice of the $\{b_{ik}\}$ determines the space uniquely. Now let $\mathfrak{C} = \{C_1,\dots,C_n\}$ be the basis for $\mathfrak{q}_0$ such that $p^{N-\lambda_i}C_i = p^{N-\lambda_i}A_i$ is a basis for $\mathfrak{q}_0\cap\mathfrak{p}_0 = \mathfrak{q}_1 = \mathbb{Z}/p^{\lambda}$. Write

$$\mathfrak{A} = \bigsqcup_{0\le k\le N} \mathfrak{A}^{(k)}, \qquad \mathfrak{C} = \bigsqcup_{0\le k\le N} \mathfrak{C}^{(k)}$$

with $p^{N-k}\mathfrak{C}^{(k)} = p^{N-k}\mathfrak{A}^{(k)}$. Hence we have $\#\mathfrak{C}^{(k)} = \#\mathfrak{A}^{(k)} = \#\{i \,|\, \lambda_i = k\}$. By the assumption (9.3), $\#\mathfrak{C}^{(N)} = \mathfrak{A}^{(N)} = 0$. Note that the element of $\bigsqcup_{0\le k<N-1}\mathfrak{A}^{(k)}$ can be changed arbitrary and the number of such choices (i.e., the choices of $\{b_{ij}\}$'s) are equal to $p^{(d-m)(n-\lambda'_{N-1})}$. On the other hand the element of $\mathfrak{A}^{(N-1)} = \{A_{m-\lambda'_{N-1}+1},\dots,A_m\}$ cannot be changed arbitrary, only by b_{ij}'s, which avoid $p^{N-1}(\mathfrak{C}\backslash\mathfrak{C}^{(N-1)})$ (notice that $\dim_{\mathbb{F}_p}(\mathfrak{C}\backslash\mathfrak{C}^{(N-1)}) = n-\lambda'_{N-1}$). Hence when we chose $\{b_{ij}\}$'s, we have to avoid not only the space $\mathfrak{C}\backslash\mathfrak{C}^{(N-1)}$ but also the space spanned by $(i-1)$ elements chosen previously. Because we fix $\mathfrak{p}_0\cap\mathfrak{q}_1 = \mathfrak{q}_0$, the number of choices of such elements is $p^{d-m} - p^{n-\lambda'_{N-1}+i-1}$. Therefore all the number (9.4) is given by

$$\begin{aligned}
&p^{(d-m)(m-\lambda'_{N-1})}(p^{d-m} - p^{n-\lambda'_{N-1}})(p^{d-m} - p^{n-\lambda'_{N-1}+1})\cdots\\
&\qquad (p^{d-m} - p^{n-\lambda'_{N-1}+\lambda'_{N-1}-1})\\
&= p^{(d-m)m}(1-p^{-(d-m-n+\lambda'_{N-1})})\cdots(1-p^{-(d-m-n+1)})\\
&= p^{(d-m)m}\frac{[d-m-n+\lambda'_{N-1}]_p!}{[d-m-n]_p!}.
\end{aligned}$$

To remove the assumption (9.3), we substitute $d - \lambda'_N$ for d, and so on, i.e., subtract λ'_N from d, m, n and λ'_{N-1}. Then we have

$$\begin{aligned}\#\{\mathfrak{p} \subseteq \left(\mathbb{Z}/p^N\right)^{\oplus d} \,|\, \mathrm{typ}(\mathfrak{p}) = N^m,\, \mathfrak{p} \cap \mathfrak{q}_0 = \mathfrak{q}_1,\, \overline{\mathfrak{p}} = \overline{\mathfrak{p}_0}\} \\ = \frac{[d-m-n+\lambda'_{N-1}]_p!}{[d-m-n+\lambda'_N]_p!} p^{(d-m)(m-\lambda'_N)}. \end{aligned} \tag{9.5}$$

Hence, from (9.2) and (9.5), the transition probability is given by

$$\begin{aligned}\frac{\tau_N(\lambda)}{\tau_{N-1}(\overline{\lambda})} &= p^{-m(d-m)} \begin{bmatrix} \lambda'_{N-1} \\ \lambda'_N \end{bmatrix}_p p^{\lambda'_N(n-\lambda'_N)} \frac{[d-m-n+\lambda'_{N-1}]_p!}{[d-m-n+\lambda'_N]_p!} p^{(d-m)(m-\lambda'_N)} \\ &= \begin{bmatrix} \lambda'_{N-1} \\ \lambda'_N \end{bmatrix}_p \frac{[d-m-n+\lambda'_{N-1}]_p!}{[d-m-n+\lambda'_N]_p!} p^{-\lambda'_N(d-m-n+\lambda'_N)}.\end{aligned}$$

Therefore, taking the product of all $0 \le j \le N$ of the transition probability, the measure τ_N on the N-th layer is calculated as follows;

$$\begin{aligned}\tau_N(\lambda) &= \frac{\begin{bmatrix} n \\ \lambda'_1 \end{bmatrix}_p \begin{bmatrix} d-n \\ m-\lambda'_1 \end{bmatrix}_p}{\begin{bmatrix} d \\ m \end{bmatrix}_p} p^{-\lambda'_1(d-n-m+\lambda'_1)} \\ &\quad \prod_{1\le j\le N} \begin{bmatrix} \lambda'_{j-1} \\ \lambda'_j \end{bmatrix}_p \frac{[d-m-n+\lambda'_{j-1}]_p!}{[d-m-n+\lambda'_j]_p!} p^{-\lambda'_j(d-m-n+\lambda'_j)} \\ &= \begin{bmatrix} n \\ n-\lambda'_1, \lambda'_1-\lambda'_2, \ldots, \lambda'_N \end{bmatrix}_p \\ &\quad \frac{\frac{[d-n]_p!}{[m-\lambda'_1]_p![d-m-n+\lambda_N]_p!} p^{-\sum_{i=1}^N \lambda'_i(d-m-n+\lambda'_i)}}{\begin{bmatrix} d \\ m \end{bmatrix}_p},\end{aligned}$$

where

$$\begin{bmatrix} n \\ m_1, \ldots, m_N \end{bmatrix}_p := \frac{[n]_p!}{[m_1]_p! \cdots [m_N]_p!} \qquad (m_1 + \cdots + m_N = n)$$

is the multinomial coefficient. Then the harmonic measure τ is obtained by taking the limit $N \to \infty$ of the measure τ_N on the N-th layer;

$$\tau(\lambda) := \begin{bmatrix} n \\ n-\lambda'_1, \lambda'_1-\lambda'_2, \ldots \end{bmatrix}_p \frac{\frac{[d-n]_p!}{[m-\lambda'_1]_p![d-m-n]_p!} p^{-\sum_{i\ge1} \lambda'_i(d-m-n+\lambda'_i)}}{\begin{bmatrix} d \\ m \end{bmatrix}_p}.$$

It can be written as follows

$$\tau(\lambda) = \frac{\begin{bmatrix} d \\ m+n \end{bmatrix}_p}{\begin{bmatrix} d \\ m \end{bmatrix}_p \begin{bmatrix} d \\ n \end{bmatrix}_p} \frac{[m+n]!}{[m-\lambda'_1]_p![n-\lambda'_1]_p!} \prod_{j\ge1} \frac{1}{[\lambda'_j - \lambda'_{j+1}]_p!} p^{-\sum_{i\ge1} \lambda_i(d-m-n+2i-1)}. \tag{9.6}$$

This expression shows that $\tau(\lambda)$ is symmetric in m and n. We call this measure the harmonic Selberg measure, which is a p-adic analogue of the Selberg measure.

9.3 Basis for the Hecke Algebra

In the last section, we see the unitary representation of $G_{N^d} = GL_d(\mathbb{Z}/p^N)$; $\pi : G_{N^d} \to U(H_{N^m}^{N^d})$ where $H_{N^m}^{N^d} := L^2(X_{N^m}^{N^d}, \tau)$. The commutant is generated by the Hecke algebra $\mathcal{H}_{N^m}^{N^d} = L^2(\Lambda_{N^m})$. We have the geometric basis $\{\delta_\lambda\}_{\lambda \subseteq N^m}$ for $\mathcal{H}_{N^m}^{N^d}$, which act on the function in $H_{N^m}^{N^d}$ as

$$\delta_\lambda \varphi(y) := \int_{\mathrm{typ}(x \cap y) = \lambda} \varphi(x)\tau(x) \qquad (\varphi \in H_{N^m}^{N^d}).$$

On the other hand, we denote by $\ell^2(X_{N^m}^{N^d})$ the Hilbert space with the counting measure (not normalized to be a probability measure). In this case, we denote by g_λ the geometric basis, acting via

$$g_\lambda \varphi(y) := \sum_{\mathrm{typ}(x \cap y) = \lambda} \varphi(x) \qquad (\varphi \in H_{N^m}^{N^d}).$$

Note that g_λ is up to constant identical with δ_λ, that is,

$$g_\lambda = \binom{N^d}{N^m}_p \delta_\lambda.$$

Let $\lambda' \subseteq \lambda \subseteq N^d$. We define "gradient" and "divergent" operators

$$\ell^2(X_\lambda^{N^d}) \underset{T_{\lambda \supseteq \lambda'}}{\overset{T_{\lambda' \subseteq \lambda}}{\rightleftarrows}} \ell^2(X_{\lambda'}^{N^d})$$

by

$$T_{\lambda' \subseteq \lambda} \varphi(x') := \sum_{x' \subseteq x} \varphi(x), \qquad T_{\lambda \supseteq \lambda'} \varphi(x) := \sum_{x \supseteq x'} \varphi(x').$$

It is clear that these operators are adjoint to each other and commute with the action of G_{N^d} on the Grassmann manifolds. Let $\lambda_1, \lambda_2 \subseteq \lambda \subseteq N^d$. Then we also define

$$T_{\lambda_1,\lambda_2}^{\lambda} : \ell^2(X_{\lambda_2}^{N^d}) \longrightarrow \ell^2(X_{\lambda_1}^{N^d})$$

by

$$T_{\lambda_1,\lambda_2}^{\lambda} \varphi(x_1) = \sum_{\mathrm{typ}(x_1 + x_2) = \lambda} \varphi(x_2).$$

Then we have $(T_{\lambda_1,\lambda_2}^{\lambda})^* = T_{\lambda_2,\lambda_1}^{\lambda}$. This also commutes with G_{N^d}-action.

The "Laplacian" $c_\lambda : \ell^2(X_{N^m}^{N^d}) \to \ell^2(X_{N^m}^{N^d})$ is expressed in terms of the geometric basis:

$$c_\lambda = T_{N^m \supseteq \lambda} \circ T_{\lambda \subseteq N^m} = \sum_{\lambda \subseteq \lambda' \subseteq N^m} \binom{\lambda'}{\lambda}_p g_{\lambda'}.$$

The collection $\{c_\lambda\}_{\lambda \subseteq N^m}$ is called the cellular basis for $\mathcal{H}_{N^m}^{N^d}$. Note that the matrix $\{\binom{\lambda'}{\lambda}_p\}_{\lambda,\lambda'}$, which transforms the geometric basis $\{g_\lambda\}$ to the cellular basis $\{c_\lambda\}$, is upper triangular with $\binom{\lambda}{\lambda}_p = 1$. Let $\{\binom{\lambda'}{\lambda}_p^*\}_{\lambda,\lambda'} := \{\binom{\lambda'}{\lambda}_p\}_{\lambda,\lambda'}^{-1}$ denote the coefficients of inverse matrix. Then we have

$$g_\lambda = \sum_{p^{\lambda'} \subseteq \lambda \subseteq \lambda' \subseteq N^m} \binom{\lambda'}{\lambda}_p^* c_{\lambda'}$$

Moreover, we have explicit expression of $\binom{\beta}{\lambda}_p^*$;

$$\binom{\beta}{\lambda}_p^* := (-1)^{|\beta|-|\lambda|} p^{n(\beta)-n(\lambda)} \prod_{1 \le i \le m} \begin{bmatrix} \beta_i' - \beta_{i+1}' \\ \beta_i' - \lambda_i' \end{bmatrix}_p,$$

where $|\lambda| := \sum_i \lambda_i$ and $n(\lambda) := \sum_i \lambda_i (i-1)$ (these are the standard notations for partitions). Remark that

$$\mathcal{H}_{N^m}^{N^d}(\lambda) := \mathrm{Span}\{c_\alpha \mid \alpha \subseteq \lambda\}, \qquad \mathcal{H}_{N^m}^{N^d}(\lambda^-) := \mathrm{Span}\{c_\alpha \mid \alpha \subsetneq \lambda\}$$

are ideals of $\mathcal{H}_{N^m}^{N^d}$. Let us consider the quotient

$$\mathcal{W}_\lambda := \mathcal{H}_{N^m}^{N^d}(\lambda)/\mathcal{H}_{N^m}^{N^d}(\lambda^-) \qquad (\lambda \in \Lambda_m).$$

Then $\{\mathcal{W}_\lambda\}_{\lambda \in \Lambda_m}$ gives the complete list of the irreducible representations of the Hecke algebra $\mathcal{H}_{N^m}^{N^d}$. Note that $\dim \mathcal{W}_\lambda = 1$. Hence we have idempotents $\{\varphi_\lambda\}_{\lambda \in \Lambda_{N^m}}$ for $\mathcal{H}_{N^m}^{N^d}$:

$$\mathcal{W}_\lambda = \mathbb{C} \cdot \varphi_\lambda = \mathcal{H}_{N^m}^{N^d} * \varphi_\lambda.$$

The function φ_λ is characterized as

$$\begin{aligned} c_\alpha \cdot \varphi_\lambda &= 0 \quad \text{for } \alpha \subsetneq \lambda, \\ c_\lambda \cdot \varphi_\lambda &\neq 0. \end{aligned}$$

Then, the irreducible decomposition of $\ell^2(X_{N^m}^{N^d})$ is given by

$$\ell^2(X_{N^m}^{N^d}) = \bigoplus_{\lambda \in \Lambda_m} \mathcal{V}_\lambda,$$

we have also $\mathcal{V}_\lambda = \ell^2(X_{N^m}^{N^d}) * \varphi_\lambda$. Notice that $\mathcal{V}_\lambda$ is the unique irreducible representation of G_{N^d} which occurs in the Grassmann manifold $X_\lambda^{N^d}$ but does not occur in $X_\alpha^{N^d}$ for any $\alpha \subsetneq \lambda$.

Let us write

$$c_\lambda = \sum_{\alpha \subseteq \lambda} A_{\lambda,\alpha} \varphi_\alpha, \qquad \varphi_\lambda = \sum_{\alpha \subseteq \lambda} A^*_{\lambda,\alpha} c_\alpha$$

with some coefficients $A_{\lambda,\alpha}$ and $A^*_{\lambda,\alpha}$. Then the matrix $\{A_{\lambda,\alpha}\}_{\lambda,\alpha}$ is lower triangular and $\{A^*_{\lambda,\alpha}\}_{\lambda,\alpha} = \{A_{\lambda,\alpha}\}^{-1}_{\lambda,\alpha}$. Moreover, let

$$c_{\lambda_1} * c_{\lambda_2} = \sum_{\alpha \subseteq \lambda_1, \lambda_2} C_\alpha^{\lambda_1,\lambda_2} \cdot c_\alpha.$$

Then we have $A_{\lambda,\alpha} = C_\alpha^{\lambda,\alpha}$ for $\alpha \subseteq \lambda$. One can explicitly calculate the number $A_{\lambda,\alpha}$ as follows: Fixed submodules $\mathbb{Z}/p^\alpha \subseteq \big(\mathbb{Z}/p^N\big)^{\oplus m}$ and $\mathbb{Z}/p^\lambda, \mathbb{Z}/p^\alpha \subseteq \big(\mathbb{Z}/p^N\big)^{\oplus d}$ such that $\mathbb{Z}/p^\lambda \cap \mathbb{Z}/p^\alpha = 0$. Then we have

$$\begin{aligned} A_{\lambda,\alpha} &= \#\{\mathfrak{m} \subseteq \big(\mathbb{Z}/p^N\big)^{\oplus m} \mid \mathbb{Z}/p^\alpha \subseteq \mathfrak{m},\ \mathrm{typ}(\mathfrak{m}) = \lambda\} \\ &\quad \times \#\{\mathfrak{m} \subseteq \big(\mathbb{Z}/p^N\big)^{\oplus d} \mid \mathbb{Z}/p^\lambda \subseteq \mathfrak{m},\ \mathfrak{m} \cap \mathbb{Z}/p^\alpha = 0,\ \mathrm{typ}(\mathfrak{m}) = N^m\} \binom{N^d}{N^m}_p^{-1} \\ &= p^{-(d-2m)|\lambda| - m|\alpha| - \langle \lambda', \lambda' - \alpha' \rangle} \begin{bmatrix} m - \alpha'_1 \\ m - \lambda'_1 \end{bmatrix}_p \prod_{i \ge 1} \begin{bmatrix} \lambda'_i - \alpha'_{i+1} \\ \lambda'_i - \lambda'_{i+1} \end{bmatrix}_p \begin{bmatrix} d - \lambda'_1 - \alpha'_1 \\ m - \lambda'_1 \end{bmatrix}_p \begin{bmatrix} d \\ m \end{bmatrix}_p^{-1}. \end{aligned}$$

It seems that the inverse matrix $\{A^*_{\lambda,\alpha}\}_{\lambda,\alpha}$ should be also calculated explicitly, however, unfortunately, we can not obtain this (it should be possible). For the reference of this section, see [BO1].