An Extended Continuum Theory for Granular Media

Pasquale Giovine

Summary. In a dilatant granular material with rotating grains the kinetic energy in addition to the usual translational one consists of three terms owing to the microstructural motion; in particular, it includes the rotation of granules and the dilatational expansion and contraction of the individual (compressible) grains and of the grains relative to one another. Therefore the balance and constitutive equations of the medium are obtained by considering it as a continuum with a constrained affine microstructure. Moreover, the balance of granular energy is demonstrated to be a direct consequence of the balance of micromomentum, while the dilatational and the rotational microstresses are turned out to be of different physical nature. Finally, a kinetic energy theorem implies that, locally, the power of all inertial forces is the opposite of the time-rate of change of kinetic energy plus the divergence of a flux through the boundary. The peculiar case of a suspension of rotating rigid granules puts in evidence the possibility for granular materials of supporting shear stresses through the generation of microrotational gradients.

1 Introduction

In this study we extend the continuum theory of dilatant granular materials, as developed in [31], by the consideration of possible rotations of compressible granules (see also, [1] and [40]); that theory generalized the models of perfect fluids with microstructure of Capriz in Sect. 12 of [7] and of distributed bodies of Goodman and Cowin [33].

The theory of distributed continuum proposed in [33] was widely used to study the slow flows of granular materials and, in particular, the propagation of all sort of waves, the basic equations of the equilibrium theory obtained from variational principles, the multiphase granular mixtures, the shearing flows, etc. (see, e.g., [34], [38], [45]–[48] and [22]). The material was assumed to consist of dry cohesionless compressible spheres of uniform size and the flow behaviour has required a combination of suggestions from both fluid and solid mechanics owing to the fact that the material has an essentially fluid-like behaviour, but it can also be heaped and, moreover, its bulk compressibility depends on the initial voids distribution in the reference placement (see the experimental results in [41] and [2]). An additional equation of balance for the microinertia was needed for the new independent kinematical variable introduced in [33], the volume fraction of the grains which describes the local arrangements of the grains themselves: hence granular materials are a special case of continua with microstructure [7].

In [50] and [16] it was observed that the constitutive hypotheses made in [33] raised some uncertainties: these was partially rectified in Sect. 3 of [50] and in [20] and [28], at least in the case of incompressible grains. Instead, the compressible case was extensively analysed in [30], [27] and [31]. In particular, in [30] the dynamic equations of motion was obtained, in the conservative case, from a Hamiltonian variational principle of local type for a perfect fluid with microstructure, in accordance with the fluid-like behaviour of granular materials (the preference for a Eulerian variational principle, rather than Lagrangian, was not in contrast with the previous appeal to a reference placement because the difference between the former and the latter formulation is not so peremptory for such materials (see also, [3])). The choice of the expression of the total kinetic energy and of the independent constitutive variables was made in accordance with [4] (*"... the dilatational motion consists of expansion and contraction of the individual (compressible) grains ... and of the grains relative to one another ..."*) and [21] (*"... the gradient of solid's volume fraction is not, by itself, the appropriate second geometric measure of local structure ..."*), respectively.

An interesting application of the theory in [30] was investigated in [32] for the study of seismic waves propagating through a sediment filled basin in the case of rigid grains; one of the advantages of the model, with respect to purely propagative models, was the reproduction of a nonlinear effect experimentally observed for real seismic waves: site amplification decreases as the amplitude of the incident wave increases.

In this chapter we consider a suspension of elastic spheres in a compressible gas of negligible mass; we assume a volume concentration close to that of packed particles, so that the mean free path of the particles is very short in comparison to the size of the particles themselves (as it is the case of cohesionless soil or sand with rough surface grains).

In Sects. 2 and 3 we present the model for a dilatant granular material with rotating grains and make a proposal for the kinetic energy instead to mention momentum and inertia, because it appears easier to conceive an appropriate expression of the former quantity rather than of the latter. The total kinetic energy consists of four terms: in addition to the usual translational one, there are three types of microstructural motion that are modelled by our theory, the two dilatational motions previously mentioned and the rotation of granules.

In Sects. 4 and 5 we introduce the balance equations and the principal fields for continua with affine microstructure and analyse the meaning of objectivity for change in observer for these fields.

In Sect. 6 we study the kinematical constraint of spherical microstructure by imposing that the changes in the affine microstructure are conformal and then obtain the pure field equations that rule the time evolution of the macro- and micro-motion and of the temperature; moreover, we get an equation for reactions to the constraint.

In Sect. 7 we assume the validity of a kinetic energy theorem which implies that, locally, the power of all inertial forces be the opposite of the given time-rate of change of the kinetic energy plus the divergence of the flux through the boundary. Furthermore, we define the granular temperature in our theory and recover the balance of granular energy as a direct consequence of the balance of micromomentum.

In Sect. 8 we impose constitutive postulates for a thermoelastic granular medium, deduce that the Helmholtz free energy represents a sort of potential for stresses and microstresses, and compare the results with previous theories by using comments and remarks. In particular, we observe the different physical nature of the dilatational microstress with respect to the rotational one, the former expressing a sort of internal non-local action rather than the usual connection with boundary microtractions of the latter [12].

Finally, in Sect. 9 we consider the peculiar case of a suspension of rotating rigid granules in a fluid matrix and notice that the microstructure behaves as that of a microrigid Cosserat's continua. By considering possible rotations of grains during the motion, we also show that, even when the volume fraction of the grain distribution is constant, the model predicts the possibility of supporting shear stress through the generation of microrotational gradients.

2 A First Model

The continuum model for dilatant granular materials here considered is directly referred to the models proposed in [6] and [4]. The material elements of the body are a sort of quasi-particle, that will be called a 'chunk' of material, and are thought of as envelopes which fill the body without voids between them (see also, [13]): each one consists of a grain and its immediate neighbours as it is the case of a suspension of elastic particles in a compressible fluid, whose density is considered to be negligible compared with the proper density ρ_m of the suspended particles; so the chunk mass density ρ of the body equals ρ_m times the volume fraction ν of the grains

$$\rho = \rho_m \nu, \tag{1}$$

with $\nu \in [0, 1)$.

In [31] the motions allowed within the chunk were merely expansions (or contractions) of the inclusions and radial motions of the spherical crust due to the displacements of the grain relative to the centre of mass of the element itself; neither diffusion of the grains through the envelope, nor effects of relative rotations of the elements or of the granules themselves were considered: assumptions rather limiting for this type of media, but necessary to obtain suggestions for the choice of an appropriate expression of the density per unit mass of the additional kinetic coenergy $\chi\,(\rho_m,\,\dot{\rho}_m)$ due to the microstructure and related to the kinetic energy κ_d by the Legendre transformation

$$\frac{\partial \chi}{\partial \dot{\rho}_m} \cdot \dot{\rho}_m - \chi = \kappa_d \tag{2}$$

(see [10]). In (2) the dot denotes material time derivative, i.e.,

$$\dot{\rho_m} := \frac{\partial \rho_m}{\partial \tau} + \mathbf{v} \cdot \operatorname{grad} \rho_m. \tag{3}$$

In particular, if $\mathbf{v}$ denotes the velocity of the mass centre of the element, whose local position vector is $\mathbf{x}$ at the time τ ($\mathbf{x}_*$ being the reference one), thus the total kinetic coenergy of the material in [31] is homogeneous of second degree in the macro- and micro-velocities and so equal to the total kinetic energy κ_{tot} (see again, [10]); precisely, it is:

$$\kappa_{tot} = \kappa_t + \kappa_f + \kappa_d, \tag{4}$$

with

$$\kappa_t := \frac{1}{2}\mathbf{v} \cdot \mathbf{v}, \quad \kappa_f := \frac{1}{2}\gamma(\rho)\dot{\rho}^2, \quad \kappa_d := \frac{1}{2}\alpha(\rho_m)\dot{\rho}_m^2. \tag{5}$$

In (4) κ_t is the usual translational kinetic energy related to the velocity of the centre of mass of the macro-element; κ_f is the ‘fluctuation’ kinetic energy associated to the ‘dilatancy’, as defined by Reynolds [49], by means of the motion of individual grains relative to the centre of mass, i.e., the kinetic energy due to the variations of the volume of chunk interstitial voids and expressed in terms of the rate of change of the chunk mass density ρ, with $\gamma(\rho)$ a scalar constitutive coefficient; κ_d is the ‘dilatational’ kinetic energy related to local expansions (or contractions) of the inclusions in the chunk and written in terms of the rate of change of the proper mass density of the grains ρ_m, with $\alpha(\rho_m)$ another scalar constitutive function.

Explicit evaluations for the constitutive functions $\gamma(\rho)$ and $\alpha(\rho_m)$ can be obtained if one imagines simple microstructural motions and peculiar geometrical shapes for chunks and/or granules. In particular, if the grains and the chunks expand or contract homogeneously with independent motions and if the envelope of the chunk is imagined as a spherical surface of radius ς containing some spherical inclusions, the grains, of radius φ, which have the same radius ς_* and φ_*, respectively, in a reference placement $\mathcal{B}_*$ of the material, we calculate the following expressions (see the Appendix):

$$\gamma(\rho) = \gamma_* \rho^{-\frac{8}{3}}, \quad \alpha(\rho_m) = \alpha_* \rho_m^{-\frac{8}{3}}, \tag{6}$$

with

$$\gamma_* = \frac{16}{351}\rho_*^{\frac{2}{3}}\varsigma_*^2, \quad \alpha_* = \frac{1}{15}\rho_{m*}^{\frac{2}{3}}\varphi_*^2 \tag{7}$$

and where, now and in the course, the subscript $(\cdot)_*$ refers to the value of the quantity in the reference placement $\mathcal{B}_*$.

When other geometric configurations of the grains and of the elements are considered, it is possible to compute more general expressions for γ and α (see Sect. 2 of [4]).

3 Rotations

Hereafter we denote by Lin^+, Sym^+ and Orth^+ the collection of second-order tensors with positive determinant, symmetric and positive definite, and proper orthogonal, respectively. Moreover, sym $\mathbf{A}$ and skw $\mathbf{A}$ are the symmetric and skew parts of a second-order tensor $\mathbf{A}$, respectively, while the spherical and deviatoric parts of $\mathbf{A}$ are defined to be, respectively,

$$\mathrm{sph}\,\mathbf{A} := \frac{1}{3}(\mathrm{tr}\,\mathbf{A})\mathbf{I} \quad \text{and} \quad \mathrm{dev}\,\mathbf{A} := \mathrm{sym}\,\mathbf{A} - \mathrm{sph}\,\mathbf{A}, \tag{8}$$

where $\mathrm{tr}\,\mathbf{A} := \mathbf{A}\cdot\mathbf{I}$ is the trace of $\mathbf{A}$ and $\mathbf{I} := (\delta_{ik})$ the identity tensor with δ_{ik} the delta of Kronecker. Also, Skw is the collection of all skew second-order tensors and Sym that of all symmetric second-order tensors, direct sum of Sph and Dev, the subspaces of spherical and traceless elements of Sym, respectively.

Now we generalize the expression of the density of dilatational kinetic energy κ_d obtained in [31], and defined in $(5)_3$, in order to allow effects of relative rotations of the compressible granules.

We suppose that each grain of the continuum is capable of an affine deformation distinct from (and independent of) the local affine deformation ensuing from the macromotion (and so not adequately modelled by the classical gradient of deformation $\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{x}_*}(\mathbf{x}_*, \tau) \in \mathrm{Lin}^+$). In particular, we assume that the microstructure of the dilatant granular material is spherical, as defined in [16], i.e., the microstructural tensor field $\mathbf{G}$ of Lin^+, describing the changes in the affine structure, is conformal:

$$\mathbf{G}(\mathbf{x}_*, \tau) = \beta(\mathbf{x}_*, \tau)\,\mathbf{R}(\mathbf{x}_*, \tau), \tag{9}$$

with $\beta(\mathbf{x}_*, \tau) > 0$ and $\mathbf{R}(\mathbf{x}_*, \tau) \in \mathrm{Orth}^+$, and the reference microinertia tensor field $\mathbf{J}_* \in \mathrm{Sym}^+$ has spherical values:

$$\mathbf{J}_*(\mathbf{x}_*) = \mu_*^2(\mathbf{x}_*)\;\mathbf{I}, \quad \forall \mathbf{x}_* \in \mathcal{B}_*. \tag{10}$$

Remark: We observe that the reference microinertia tensor $\mathbf{J}_*$ is directly related to the Euler's microinertia tensor per unit mass $\mathbf{J}$ of the generic chunk with respect to its centre of mass $\mathbf{x}$ at time τ and to the corresponding kinetic energy density κ_s, two fields which have the following form, respectively:

$$\mathbf{J} = \mathbf{G}\mathbf{J}_*\mathbf{G}^T \in \mathrm{Sym}^+ \quad \text{and} \quad \kappa_s = \frac{1}{2}(\mathbf{V}\mathbf{J}_*)\cdot\mathbf{V}, \tag{11}$$

where $\mathbf{V}(\mathbf{x}_*, \tau) := \dot{\mathbf{G}}(\mathbf{x}_*, \tau)$ is the microvelocity over the current placement $\mathcal{B}_\tau = \mathbf{x}(\mathcal{B}_*, \tau)$ of the body $\mathcal{B}$ (see e.g., (2.10) and (2.35) of [16] and, more in general, (5) and (16) of [9]).

For dilatant granular materials with rotating grains, the microstructure is supposed spherical and relations (9) and (10) apply, hence the Euler's tensor $\mathbf{J}$ is always spherical and the inertia related to the admissible micromotions of grains is decomposed in two terms because the trace of the skew tensor product $\dot{\mathbf{R}}\mathbf{R}^T$ vanishes; they are expressed by

$$\mathbf{J} = \mu^2\,\mathbf{I} \quad \text{and} \quad \kappa_s = \frac{3}{2}\dot{\mu}^2 + \frac{1}{2}\mu^2\,\dot{\mathbf{R}}\cdot\dot{\mathbf{R}}, \tag{12}$$

respectively, with

$$\mu(\mathbf{x}_*, \tau) := \mu_*(\mathbf{x}_*)\beta(\mathbf{x}_*, \tau). \tag{13}$$

An explicit suggestion for the constitutive expression of μ is obtained by considering the previous model of Sect. 2 as a particular case of this one; thus, by restricting the rotation $\mathbf{R}$ to coincides with the identity tensor $\mathbf{I}$, the kinetic energy κ_s must reduce to the kinetic energy κ_d of $(5)_3$ with $\alpha(\rho_m)$ given by $(6)_2$. Thus the following relation is valid by identification (in the case $\mathbf{R} = \mathbf{I}$):

$$\frac{3}{2}\dot{\mu}^2 = \frac{1}{2}\alpha_*\,\rho_m^{-\frac{8}{3}}\,\dot{\rho}_m^2; \tag{14}$$

so that a straightforward integration of the latter equation yields the following requested constitutive term:

$$\mu(\rho_m) = \mu_* + \sqrt{3\alpha_*}(\rho_m^{-\frac{1}{3}} - \rho_{m*}^{-\frac{1}{3}}). \tag{15}$$

Therefore, by choosing $\mu_* = \rho_{m*}^{-\frac{1}{3}}\sqrt{3\alpha_*}$, we have that

$$\mu(\rho_m) = \rho_m^{-\frac{1}{3}}\sqrt{3\alpha_*} \quad \text{and} \quad \beta(\rho_m) = \left(\frac{\rho_{m*}}{\rho_m}\right)^{\frac{1}{3}}, \tag{16}$$

so the conformal coefficient β accounts for the homogeneous expansion or contraction of the grains.

At the end, in this chapter the total kinetic energy κ_{ext} of the extended model for dilatant granular media is:

$$\kappa_{ext} = \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + \frac{1}{2}\gamma(\rho)\dot{\rho}^2 + \frac{3}{2}\dot{\mu}^2(\rho_m) + \frac{1}{2}\mu^2(\rho_m)\,\dot{\mathbf{R}}\cdot\dot{\mathbf{R}}, \tag{17}$$

with $\gamma(\rho)$ and $\mu(\rho_m)$ given by $(6)_1$ and $(16)_1$, respectively.

4 Balance of Interactions for Material Bodies with Affine Microstructure

The local statements of the balance laws for granular materials will be obtained in Sect. 6 by the general ones for bodies with affine microstructure by imposing the internal constraint (9) on the tensor field $\mathbf{G}$ describing the changes in the affine structure. These equations of balance governing an admissible thermomechanical process are (see e.g., Sect. 21 of [7] and Sect. II.C of [43]):

$$\dot{\rho} + \rho \operatorname{tr} \mathbf{L} = 0, \tag{18}$$

$$\mathbf{c} + \operatorname{div} \mathbf{T} = \mathbf{0}, \tag{19}$$

$$\mathbf{C} - \mathbf{Z} + \operatorname{div} \mathbf{\Sigma} = \mathbf{0}, \tag{20}$$

$$\operatorname{skw} \mathbf{T} = \operatorname{skw} \left(\mathbf{G}\mathbf{Z}^T + \operatorname{grad} \mathbf{G} \odot \mathbf{\Sigma} \right), \tag{21}$$

$$\rho \dot{\epsilon} = \mathbf{T} \cdot \mathbf{L} + \mathbf{Z} \cdot \mathbf{V} + \mathbf{\Sigma} \cdot \operatorname{grad} \mathbf{V} + \rho \lambda - \operatorname{div} \mathbf{q}. \tag{22}$$

Equation (18) is the conservation law of mass and $\mathbf{L}$ is the usual velocity gradient: $\mathbf{L} := \operatorname{grad} \mathbf{v} \, (= \dot{\mathbf{F}}\mathbf{F}^{-1})$; equation (19) is the standard law of Cauchy's balance, where $\mathbf{c}$ is the vector density per unit volume of external bulk forces and $\mathbf{T}$ the stress tensor; equation (20) is the balance of microstructural interactions, in which $\mathbf{C}$ and $-\mathbf{Z}$ are the resultant tensor densities per unit volume of external bulk interactions on the microstructure and internal self-force, respectively, while $\mathbf{\Sigma}$ is the third-order microstress tensor that, in general, is not necessarily related to a sort of boundary microtractions, unless it is possible to define a physically significant connection on the manifold of values of the microstructure by which the gradient on it may be evaluated in covariant manner (see [11]); equation (21) is the balance law of angular momentum and the tensor product $\odot$ between third-order tensors is so defined:

$$(\operatorname{grad} \mathbf{G} \odot \mathbf{\Sigma})_{ij} := \mathbf{G}_{ih,k} \mathbf{\Sigma}_{jhk}; \tag{23}$$

equation (22) is the balance of mechanical energy in which ϵ is the specific internal energy per unit mass, λ the scalar rate of heat generation per unit mass due to irradiation and $\mathbf{q}$ the heat flux vector.

We accept here the principle of entropy as it applies in its classical form purely thermal: intrinsic production of entropy is always non-negative during every admissible thermodynamic process for the body. This production is given by the rate of variation of the specific entropy, whose density per unit mass is η, less the rate of heat exchange due to a flux of entropy through the boundary of vector density $-\theta^{-1}\mathbf{q}$, where θ is the (positive) absolute temperature, and a production owing to distributed entropy sources of specific density per unit mass $\lambda\theta^{-1}$. The local form of the principle is given by the Clausius–Duhem inequality

$$\rho \dot{\eta} + \operatorname{div} \left(\theta^{-1} \mathbf{q} \right) - \rho \lambda \theta^{-1} \geq 0; \tag{24}$$

moreover, if we introduce the Helmholtz free energy per unit mass $\psi := \epsilon - \theta\eta$ and use (22), we obtain a reduced version of this inequality, that is,

$$\rho\left(\dot{\psi} + \dot{\theta}\eta\right) + \theta^{-1}\mathbf{q}\cdot\mathbf{g} \leq \mathbf{T}\cdot\mathbf{L} + \mathbf{Z}\cdot\mathbf{V} + \boldsymbol{\Sigma}\cdot\operatorname{grad}\mathbf{V}. \tag{25}$$

where $\mathbf{g} := \operatorname{grad}\theta$.

Equations (20) and (21) are not immediately recognized to be the balance equations which are usually proposed for studying continua with affine microstructure (see e.g., [17]) or micromorphic media (see e.g., [26]), but, *modulo* some innocuous changes in notation and, after, by considering the effects of inertia of possible internal vibrations of the substructures, we can recover them.

Firstly, by transposing the balance equation of micromomentum (20) and multiplying both sides by the microstructural tensor variable $\mathbf{G}$, we have the following result:

$$\mathbf{G}\mathbf{C}^T - \mathbf{G}\mathbf{Z}^T - \operatorname{grad}\mathbf{G}\odot\boldsymbol{\Sigma} + \operatorname{div}\left(\mathbf{G}\oslash{}^t\boldsymbol{\Sigma}\right) = \mathbf{0}, \tag{26}$$

where the minor left transposition (of exponent t) on a tensor $\boldsymbol{\Omega}$ of the third order has the following meaning: $(({}^t\boldsymbol{\Omega}\,\mathbf{a})\mathbf{b})\mathbf{c} = ((\boldsymbol{\Omega}\,\mathbf{a})\mathbf{c})\mathbf{b}$, for each triple of vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, while the tensor product $\oslash$ between tensors of the second and the third order is so defined: $(\mathbf{A}\oslash\boldsymbol{\Omega})_{ijl} := \mathbf{A}_{ih}\boldsymbol{\Omega}_{hjl}$.

Then, by using the balance equation of moment of momentum (21) and by introducing the following second- and third-order tensors

$$\widetilde{\mathbf{C}} := \mathbf{G}\mathbf{C}^T,\ \ \widetilde{\mathbf{Z}} := \operatorname{sym}\left[\mathbf{G}\mathbf{Z}^T + \operatorname{grad}\mathbf{G}\odot\boldsymbol{\Sigma}\right]\ \text{ and }\ \widetilde{\boldsymbol{\Sigma}} := \mathbf{G}\oslash{}^t\boldsymbol{\Sigma} \tag{27}$$

into (26), it becomes

$$\widetilde{\mathbf{C}} - \widetilde{\mathbf{Z}} - \operatorname{skw}\mathbf{T} + \operatorname{div}\widetilde{\boldsymbol{\Sigma}} = \mathbf{0}. \tag{28}$$

Secondly, we decompose the volume forces $\mathbf{C}$ in their inertial $\mathbf{C}^{in}$ and noninertial $\rho\,\mathbf{B}$ parts as

$$\mathbf{C} = \mathbf{C}^{in} + \rho\mathbf{B} \tag{29}$$

and observe that $\mathbf{C}^{in}$ is the opposite of the Lagrangian derivative of the microstructural kinetic coenergy $\chi_s(\mathbf{V})$, homogeneous of second degree in the micro-velocity $\mathbf{V}$ and so equal to the kinetic energy κ_s defined in $(11)_2$, thus it is

$$\mathbf{C}^{in} = -\rho\left[\frac{d}{d\tau}\left(\frac{\partial\kappa_s}{\partial\mathbf{V}}\right) - \frac{\partial\kappa_s}{\partial\mathbf{G}}\right] = -\rho\dot{\mathbf{V}}\mathbf{J}_* \tag{30}$$

(see also, [10] or (60) of [43]).

Hence, by transposing this relation, multiplying both sides by the microstructural tensor variable $\mathbf{G}$ and using relation $(11)_1$, we have that:

$$\mathbf{G}(\mathbf{C}^{in})^T = -\rho\mathbf{G}\mathbf{J}_*\dot{\mathbf{V}}^T = -\rho\mathbf{J}(\dot{\mathbf{V}}\mathbf{G}^{-1})^T, \tag{31}$$

Then, by introducing the second-order tensors $\widetilde{\mathbf{B}} := \mathbf{G}\mathbf{B}^T$ and using the relation (31) together with the (29) into (28), it becomes

$$\rho\mathbf{J}(\dot{\mathbf{V}}\mathbf{G}^{-1})^T = \rho\widetilde{\mathbf{B}} - \widetilde{\mathbf{Z}} - \operatorname{skw}\mathbf{T} + \operatorname{div}\widetilde{\mathbf{\Sigma}}. \tag{32}$$

Finally, let us insert the second-order kinematical tensor $\mathbf{W}$ for the micromotion corresponding to the velocity gradient $\mathbf{L}$ of the macromotion, i.e., the *wrenching* tensor

$$\mathbf{W}(\mathbf{x}_*, \tau) := \mathbf{V}(\mathbf{x}_*, \tau)\mathbf{G}^{-1}(\mathbf{x}_*, \tau); \tag{33}$$

for relation $(11)_1$ it satisfies the kinematical relation

$$\dot{\mathbf{J}} = \mathbf{J}\mathbf{W}^T + \mathbf{W}\mathbf{J} \tag{34}$$

that some Author calls the new fundamental conservation equation of microinertia, similar, in some sense, to the continuity equation (18) for macromotion (see, e.g., Theorem 5 in [26]): here, however, it is a simple consequence of definition $(11)_1$.

By using the wrenching (33) and relation (34) into (32), we are led to the requested classical form of equation for micromomentum (4.18) of [17]:

$$\rho\left[\overline{(\mathbf{J}\mathbf{W}^\mathbf{T})}^{\cdot} - \mathbf{W}\mathbf{J}\mathbf{W}^T\right] = \rho\widetilde{\mathbf{B}} - \widetilde{\mathbf{Z}} - \operatorname{skw}\mathbf{T} + \operatorname{div}\widetilde{\mathbf{\Sigma}}, \tag{35}$$

where $\widetilde{\mathbf{B}}$ is the generalized body moment, $\widetilde{\mathbf{\Sigma}}$ is the hyperstress and $\widetilde{\mathbf{Z}}$ represents the symmetric part of the generalized moment of interaction of the microstructure and the gross motion. By replacing these fields in (22), we obtain the related energy equation in presence of affine microstructure in the usual form (see also (2.5) of [16]):

$$\rho\dot{\epsilon} = \mathbf{T}\cdot\mathbf{L} + (\widetilde{\mathbf{Z}} + \operatorname{skw}\mathbf{T})\cdot\mathbf{W}^T + \widetilde{\mathbf{\Sigma}}\cdot\operatorname{grad}\left(\mathbf{W}^T\right) + \rho\lambda - \operatorname{div}\mathbf{q}. \tag{36}$$

5 Observers

Now, in order to give a suitable definition of a continuum with microstructure subject to internal kinematical constraints, as (9) and (10) are, and to study the consequences of them on the balance equations (18)–(22) and (25), we need an objective version of the total power density of mechanical internal actions $\mathcal{P}_{int}$ acting on the body $\mathcal{B}$, that is the quantity appearing, with the opposite sign, in the right-hand side of the reduced version of the imbalance of entropy (25) (see Sect. 3 of [31]):

$$\mathcal{P}_{int} = -\left(\mathbf{T}\cdot\mathbf{L} + \mathbf{Z}\cdot\mathbf{V} + \mathbf{\Sigma}\cdot\operatorname{grad}\mathbf{V}\right). \tag{37}$$

A change in observer of a body with affine microstructure $\mathcal{B}$ relates two processes $(\breve{\mathbf{x}}, \breve{\mathbf{G}}, \breve{\theta})(\tau)$ and $(\mathbf{x}, \mathbf{G}, \theta)(\tau)$ if, for any $(\mathbf{x}_*, \tau) \in \mathcal{B}_* \times \Re$,

$$\breve{\mathbf{x}}(\mathbf{x}_*, \tau) = \mathbf{c}(\tau) + \mathbf{Q}(\tau)\,\mathbf{x}(\mathbf{x}_*, \tau), \quad \breve{\mathbf{G}}(\mathbf{x}_*, \tau) = \mathbf{Q}(\tau)\,\mathbf{G}(\mathbf{x}_*, \tau) \tag{38}$$

and

$$\breve{\theta}(\mathbf{x}_*, \tau) = \theta(\mathbf{x}_*, \tau), \tag{39}$$

where $\mathbf{c}$ is a vector and $\mathbf{Q}$ a proper orthogonal tensor of Orth^+.

This means that $\mathbf{G}$ transforms like the deformation gradient $\mathbf{F}$ and can be considered as a double vector, while the velocity $\mathbf{v}$, the microvelocity $\mathbf{V}$ and the gradient of temperature $\mathbf{g}$ transform as follows:

$$\breve{\mathbf{v}} = \dot{\mathbf{c}} + \dot{\mathbf{Q}}\mathbf{d} + \mathbf{Q}\mathbf{v}, \quad \breve{\mathbf{V}} = \dot{\mathbf{Q}}\mathbf{G} + \mathbf{Q}\mathbf{V} \quad \text{and} \quad \breve{\mathbf{g}} = \mathbf{Q}\mathbf{g}, \tag{40}$$

where $\mathbf{d}$ is the position vector of $\mathbf{x}$ relative to a fixed origin in $\mathcal{E}$.

Now, let $\mathbf{D}\,(:= \mathrm{sym}\,\mathbf{L})$ and $\mathbf{Y}\,(:= -\mathrm{skw}\,\mathbf{L})$ be the stretching and the spin tensor, respectively, and $\widetilde{\mathbf{D}}\,(:= \mathrm{sym}\,\mathbf{W})$ and $\widetilde{\mathbf{Y}}\,(:= -\mathrm{skw}\,\mathbf{W})$ the micro-stretching and the micro-spin tensor, respectively, so that

$$\mathbf{L} = \mathbf{D} - \mathbf{Y} \quad \text{and} \quad \mathbf{W} = \widetilde{\mathbf{D}} - \widetilde{\mathbf{Y}}; \tag{41}$$

therefore one can compute from $(40)_{1,2}$ the transformation laws for $\breve{\mathbf{L}}$ and $\breve{\mathbf{W}}$:

$$\breve{\mathbf{L}} = \overline{\mathrm{gr\breve{a}d}}\,\breve{\mathbf{v}} = \frac{\partial \breve{\mathbf{L}}}{\partial \mathbf{x}_*}\breve{\mathbf{F}}^{-1} = \left(\dot{\mathbf{Q}}\mathbf{F} + \mathbf{Q}\frac{\partial \mathbf{v}}{\partial \mathbf{x}_*}\right)\mathbf{F}^{-1}\mathbf{Q}^T = \dot{\mathbf{Q}}\mathbf{Q}^T + \mathbf{Q}\mathbf{L}\mathbf{Q}^T \tag{42}$$

and

$$\breve{\mathbf{W}} = \breve{\mathbf{V}}\breve{\mathbf{G}}^{-1} = \left(\dot{\mathbf{Q}}\mathbf{G} + \mathbf{Q}\mathbf{V}\right)\mathbf{G}^{-1}\mathbf{Q}^T = \dot{\mathbf{Q}}\mathbf{Q}^T + \mathbf{Q}\mathbf{W}\mathbf{Q}^T; \tag{43}$$

consequently, $\breve{\mathbf{L}}$ can be split into the symmetric and skew part, respectively:

$$\breve{\mathbf{D}} = \mathbf{Q}\mathbf{D}\mathbf{Q}^T \quad \text{and} \quad \breve{\mathbf{Y}} = -\dot{\mathbf{Q}}\mathbf{Q}^T - \mathbf{Q}\mathbf{Y}\mathbf{Q}^T, \tag{44}$$

as well as $\breve{\mathbf{W}}$.

Owing to the transformation laws $(40)_2$ and $(42)_4$, the expression (37) for the power density $\mathcal{P}_{int}$ is not frame indifferent, apparently; instead, by using the balance of angular momentum (21) and relations (41), we have that

$$\begin{aligned} -\mathcal{P}_{int} &= \mathbf{D}\cdot\mathrm{sym}\,\mathbf{T} + \mathbf{Y}\cdot\mathrm{skw}\left(\mathbf{Z}\mathbf{G}^T + \mathbf{\Sigma}\odot\mathrm{grad}\,\mathbf{G}\right) + \\ &+ \mathbf{W}\cdot\left(\mathbf{Z}\mathbf{G}^T + \mathbf{\Sigma}\odot\mathrm{grad}\,\mathbf{G}\right) + (\mathbf{G}\oslash{}^t\mathbf{\Sigma})\cdot\mathrm{grad}\,(\mathbf{W}^T) = \\ &= \mathbf{D}\cdot\mathrm{sym}\,\mathbf{T} + (\mathbf{Y} - \widetilde{\mathbf{Y}})\cdot\mathrm{skw}\left(\mathbf{Z}\mathbf{G}^T + \mathbf{\Sigma}\odot\mathrm{grad}\,\mathbf{G}\right) + \\ &+ \widetilde{\mathbf{D}}\cdot\mathrm{sym}\left(\mathbf{Z}\mathbf{G}^T + \mathbf{\Sigma}\odot\mathrm{grad}\,\mathbf{G}\right) + (\mathbf{G}\oslash{}^t\mathbf{\Sigma})\cdot\mathrm{grad}\,(\widetilde{\mathbf{D}} + \widetilde{\mathbf{Y}}) \end{aligned} \tag{45}$$

and hence $\mathcal{P}_{int}$ is indifferent to changes in observer, as requested.

6 Dilatant Granular Materials with Rotating Grains

We now impose the perfect kinematical constraint of spherical microstructure, as described by formulas (9) and (10), in order to obtain the balance laws for granular materials which allow effects of microrotation of the compressible grains, other than the dilatancy of the chunks.

The body $\mathcal{B}$ is said to be *internally constrained* if the allowed velocity, microvelocity and temperature gradient distributions are such that not all values of the objective factors $\mathbf{D}$, $\widetilde{\mathbf{D}}$, $(\mathbf{Y}-\widetilde{\mathbf{Y}})$, $\operatorname{grad}\widetilde{\mathbf{D}}$, $\operatorname{grad}\widetilde{\mathbf{Y}}$ and $\mathbf{g}$ are accessible. In our case the wrenching $\mathbf{W}$, the micro-stretching $\widetilde{\mathbf{D}}$ and the micro-spin $\widetilde{\mathbf{Y}}$ are given by

$$\mathbf{W}=\dot{\beta}\beta^{-1}\mathbf{I}+\dot{\mathbf{R}}\mathbf{R}^T, \quad \widetilde{\mathbf{D}}=\dot{\beta}\beta^{-1}\mathbf{I}, \quad \widetilde{\mathbf{Y}}=-\dot{\mathbf{R}}\mathbf{R}^T, \tag{46}$$

respectively, and so the macromotion is not constrained at all, while

$$\operatorname{grad}(\mathbf{W}^T)=\mathbf{I}\otimes\operatorname{grad}(\dot{\beta}\beta^{-1})+\operatorname{grad}(\dot{\mathbf{R}}\mathbf{R}^T). \tag{47}$$

Furthermore, we follow classical theories (see [36] and [18]) and suppose that each quantity, which, in absence of the constraint, is ruled by a constitutive prescription (that is $\mathbf{T}$, $\mathbf{Z}$, $\boldsymbol{\Sigma}$, $\mathbf{q}$, ϵ, η, ψ) is now the direct sum of one *active* and one *reactive* component

$$\mathbf{T}=\mathbf{T}_a+\mathbf{T}_r,\ \mathbf{Z}=\mathbf{Z}_a+\mathbf{Z}_r,\ \text{etc.} \tag{48}$$

and only the active component is bound through suitable constitutive relations to the independent thermokinetic variables.

The additional request that the constraint is *perfect*, i.e., internally frictionless, is specified, in this wider thermomechanic rather than purely mechanical context, by the property that the entropy production due to the reaction is null, that is the contribution of the reactions to the inequality (25) are identically zero for every process allowed by the constraint (see also, Sect. 27 of [7]):

$$\rho\left(\dot{\psi}_r+\eta_r\dot{\theta}\right)+\theta^{-1}\mathbf{q}_r\cdot\mathbf{g}=\mathbf{T}_r\cdot\mathbf{L}+\mathbf{Z}_r\cdot\mathbf{V}+\boldsymbol{\Sigma}_r\cdot\operatorname{grad}\mathbf{V}. \tag{49}$$

By using the representation $(45)_1$ of $\mathcal{P}_{int}$, the constraint relation (9), (46) and (47) into (49), we have

$$\begin{aligned}\rho\left(\dot{\psi}_r+\eta_r\dot{\theta}\right)+\theta^{-1}\mathbf{q}_r\cdot\mathbf{g}&=\operatorname{sym}\mathbf{T}_r\cdot\mathbf{D}-\operatorname{skw}\mathbf{T}_r\cdot\mathbf{Y}+\\ +(\dot{\beta}\beta^{-1})\left[\beta\mathbf{Z}_r\cdot\mathbf{R}+\boldsymbol{\Sigma}_r\cdot\operatorname{grad}(\beta\mathbf{R})\right]&+\left(\beta\boldsymbol{\Sigma}_r^T\,\mathbf{R}^T\right)\cdot\operatorname{grad}\left(\dot{\beta}\beta^{-1}\right)-\\ -\operatorname{skw}\left[\beta\mathbf{Z}_r\mathbf{R}^T+\boldsymbol{\Sigma}_r\odot\operatorname{grad}(\beta\mathbf{R})\right]&\cdot\widetilde{\mathbf{Y}}+(\beta\mathbf{R}\oslash{}^t\boldsymbol{\Sigma}_r)\cdot\operatorname{grad}\widetilde{\mathbf{Y}},\end{aligned} \tag{50}$$

for every totally free choice of $\dot{\theta}$ and $(\dot{\beta}\beta^{-1})$ among the scalars, $\mathbf{g}$ among the vectors, $\mathbf{Y}$ and $\widetilde{\mathbf{Y}}$ among the skew tensors and $\mathbf{D}$ among the symmetric tensors; in (50) the transposition of exponent T on a tensor $\mathbf{\Omega}$ of the third order has the following meaning: $((\mathbf{\Omega}^T\mathbf{a})\mathbf{b})\mathbf{c} = ((\mathbf{\Omega}\,\mathbf{c})\mathbf{b})\mathbf{a}$, for each triple of vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

The reactions are then characterized by the following requirements:

$$\psi_r = \text{const.}, \quad \eta_r = 0, \quad \mathbf{q}_r = \mathbf{0}, \tag{51}$$

$$\begin{gathered} \mathbf{T}_r = \mathbf{0}, \quad \mathbf{Z}_r \cdot \beta\mathbf{R} + \mathbf{\Sigma}_r \cdot \operatorname{grad}(\beta\mathbf{R}) = 0, \\ \operatorname{skw}\left[\beta\mathbf{Z}_r\mathbf{R}^T + \mathbf{\Sigma}_r \odot \operatorname{grad}(\beta\mathbf{R})\right] = \mathbf{0}, \\ \mathbf{\Sigma}_r^T\,\mathbf{R}^T = \mathbf{0} \quad \text{and} \quad \operatorname{skw}\left[\beta\mathbf{R}\,(\mathbf{\Sigma}_r\mathbf{w})^T\right] = \mathbf{0},\ \forall\,\text{vector}\,\mathbf{w}; \end{gathered} \tag{52}$$

hence, from definitions $(27)_{2,3}$, we have that reactions must be such that

$$\begin{gathered} \widetilde{\mathbf{Z}}_r = \left[\beta\,\mathbf{Z}_r\mathbf{R}^T + \mathbf{\Sigma}_r \odot \operatorname{grad}(\beta\mathbf{R})\right] \in \text{Dev}, \\ \widetilde{\mathbf{\Sigma}}_r\,\mathbf{w} = \beta\,\mathbf{R}\,(\mathbf{\Sigma}_r\,\mathbf{w})^T \in \text{Dev},\ \forall\,\text{vector}\,\mathbf{w}, \end{gathered} \tag{53}$$

and, accordingly,

$$\widetilde{\mathbf{Z}}_a \in \text{Sph} \quad \text{and} \quad \left(\widetilde{\mathbf{\Sigma}}_a\,\mathbf{w}\right) \in \text{Sph} \oplus \text{Skw},\ \forall\,\text{vector}\,\mathbf{w}, \tag{54}$$

while, for $(52)_1$, $\mathbf{T}_a$ is a free tensor field, not necessarily symmetric-valued.

Now we are able to obtain a set of *pure* equations which rules the thermomechanical evolution of our model of dilatant granular material $\mathcal{B}$; in fact, by splitting the stress tensor $\mathbf{T}$ into its symmetric and skew parts and by using the condition $(52)_1$ into $(48)_1$, together with the balance of moment of momentum (21) and condition $(52)_3$, the following reaction-free expression for the stress $\mathbf{T}$ follows:

$$\mathbf{T} = \operatorname{sym}\mathbf{T}_a + \operatorname{skw}\left[\beta\mathbf{R}\mathbf{Z}_a^T + \operatorname{grad}(\beta\mathbf{R}) \odot \mathbf{\Sigma}_a\right], \tag{55}$$

which will be the object of a constitutive prescription and it is clearly not symmetric, in general.

Moreover, by using relations (53) and (54), the balance for micromomentum in the shape (28), broken up into spherical, skew and deviatoric part, delivers

$$\operatorname{sph}\left(\widetilde{\mathbf{C}} - \widetilde{\mathbf{Z}}_a + \operatorname{div}\widetilde{\mathbf{\Sigma}}_a\right) = \mathbf{0}, \quad \operatorname{skw}\left(\widetilde{\mathbf{C}} - \mathbf{T} + \operatorname{div}\widetilde{\mathbf{\Sigma}}_a\right) = \mathbf{0} \tag{56}$$

$$\text{and} \quad \widetilde{\mathbf{Z}}_r - \operatorname{div}\widetilde{\mathbf{\Sigma}}_r = \operatorname{dev}\widetilde{\mathbf{C}}, \tag{57}$$

respectively; therefore, the constraint (9), definitions (27) and the skew part of the stress tensor furnished by (55) permit us to write the following equations:

$$\beta\,(\mathbf{C} - \mathbf{Z}_a + \operatorname{div}\mathbf{\Sigma}_a) \cdot \mathbf{R} = 0,\ \beta\operatorname{skw}\left[(\mathbf{C} - \mathbf{Z}_a + \operatorname{div}\mathbf{\Sigma}_a)\,\mathbf{R}^T\right] = \mathbf{0} \tag{58}$$

$$\text{and} \quad \mathbf{Z}_r - \operatorname{div}\mathbf{\Sigma}_r = \left[\operatorname{dev}\left(\mathbf{C}\,\mathbf{R}^T\right)\right]\mathbf{R}\,. \tag{59}$$

In conclusion, only the active constitutive components of the fields of stress, internal actions and microstress appear in the Cauchy equation (19), with $\mathbf{T}$ given by (55), and in the spherical and skew parts of equation for micromomentum (58): these are the pure equations which rule the mechanical evolution of the body.

Once a motion is ensued from them, the corresponding reactions to the constraint are obtained by the condition (59) (other than by (51)) within the intrinsic indeterminacy generated from equation itself for $\mathbf{Z}_r$ and $\boldsymbol{\Sigma}_r$, as pointed out in Sects. 205 and 227 of [51] or in Remark 1, Sect. 3 of [15].

Now let us use the definition of the Helmholtz free energy ψ and the results (51), (52) and (55) in the balance equation for energy (22); on repeating the same procedure leading to (50), we immediately get

$$\rho\overline{(\psi_a + \theta\eta_a)}^{\cdot} = \mathbf{D} \cdot \operatorname{sym} \mathbf{T}_a + \left(\dot{\beta}\beta^{-1}\right) \left[\beta \mathbf{Z}_a \cdot \mathbf{R} + \boldsymbol{\Sigma}_a \cdot \operatorname{grad}(\beta \mathbf{R})\right] + \\ + \left(\mathbf{Y} + \dot{\mathbf{R}}\mathbf{R}^T\right) \cdot \operatorname{skw} \left[\beta \mathbf{Z}_a \mathbf{R}^T + \boldsymbol{\Sigma}_a \odot \operatorname{grad}(\beta \mathbf{R})\right] + \\ + \left(\beta \boldsymbol{\Sigma}_a^T \mathbf{R}^T\right) \cdot \operatorname{grad}\left(\dot{\beta}\beta^{-1}\right) - \left(\beta \mathbf{R} \oslash {}^t\boldsymbol{\Sigma}_a\right) \cdot \operatorname{grad}\left(\dot{\mathbf{R}}\mathbf{R}^T\right) + \rho\lambda - \operatorname{div} \mathbf{q}_a, \tag{60}$$

where there is no trace of effects due to the constraint: we have obtained the pure equation of evolution for the temperature of the body.

We observe that (60) will be greatly simplified when the constitutive prescriptions for the active fields will be given and the consequences of the Clausius–Duhem inequality (25) will be taken into account.

7 Inertia Forces and Balance of Granular Energy

The fundamental pure equations of balance (19) (with the stress tensor $\mathbf{T}$ given by (55)), (58) and (60) presented in the previous sections apply to the general class of materials with spherical microstructure.

The material properties of granular media are assigned through constitutive hypotheses of thermomechanic and kinematical character: the former will be rendered explicit in the next section with the choice of constitutive postulates for a thermoelastic continuum; the latter involve the delicate argument of the connection between an appropriate choice of the densities of macro- and micro-structural inertia forces and the chosen expression (17) for the total kinetic energy density κ_{ext}.

We follow Mariano [43] and Capriz [7] and, firstly, decompose the volume force density $\mathbf{c}$ in its inertial $\mathbf{c}^{in}$ and noninertial $\rho\,\mathbf{f}$ part, as made in (29) for $\mathbf{C}$:

$$\mathbf{c} = \mathbf{c}^{in} + \rho\mathbf{f}; \tag{61}$$

after we assume the validity of a kinetic energy theorem, which implies that, locally, the power for unit volume of inertial forces be the opposite of the time-rate of change of the kinetic energy density per unit mass κ_{ext}, times ρ, plus the divergence of the flux of kinetic energy density $\mathbf{k}$ through the boundary, that is

$$\mathbf{c}^{in} \cdot \mathbf{v} + \mathbf{C}^{in} \cdot \mathbf{V} = -\rho\, \dot{\kappa}_{ext} + \operatorname{div} \mathbf{k}. \tag{62}$$

It is easy to check that

$$\begin{aligned} \rho\, \dot{\kappa}_{ext} = \rho \left[\mathbf{v} \cdot \dot{\mathbf{v}} + \dot{\rho} \left(\gamma(\rho) \ddot{\rho} + \tfrac{1}{2} \gamma'(\rho) \dot{\rho}^2 \right) + 3\dot{\mu}\ddot{\mu} + \mu\dot{\mu}\, \dot{\mathbf{R}} \cdot \dot{\mathbf{R}} + \mu^2\, \dot{\mathbf{R}} \cdot \ddot{\mathbf{R}} \right] = \\ = \left\{ \rho \dot{\mathbf{v}} + \operatorname{grad} \left[\rho^2 \left(\gamma(\rho) \ddot{\rho} + \tfrac{1}{2} \gamma'(\rho) \dot{\rho}^2 \right) \right] \right\} \cdot \mathbf{v} + \\ + \mu_* \rho \left(\ddot{\mu} \mathbf{R} + 2\dot{\mu} \dot{\mathbf{R}} + \mu \ddot{\mathbf{R}} \right) \cdot \left(\dot{\beta} \mathbf{R} + \beta \dot{\mathbf{R}} \right) - \operatorname{div} \left[\rho^2 \left(\gamma(\rho) \ddot{\rho} + \tfrac{1}{2} \gamma'(\rho) \dot{\rho}^2 \right) \mathbf{v} \right], \end{aligned} \tag{63}$$

where the continuity equation (18), the relation (13) and the properties $\mathbf{R} \cdot \mathbf{R} = 3$ and $\mathbf{R} \cdot \dot{\mathbf{R}} = 0$ of the orthogonal tensor $\mathbf{R}$ are used; the prime $(\cdot)'$ denotes differentiation with respect to the argument. Therefore, it must be:

$$\begin{aligned} \mathbf{c}^{in} = -\rho\, \dot{\mathbf{v}} - \operatorname{grad} \left[\rho^2 \left(\gamma(\rho) \ddot{\rho} + \tfrac{1}{2} \gamma'(\rho) \dot{\rho}^2 \right) \right], \\ \mathbf{C}^{in} = -\mu_* \rho \left(\ddot{\mu} \mathbf{R} + 2\dot{\mu} \dot{\mathbf{R}} + \mu \ddot{\mathbf{R}} \right) \text{ and } \mathbf{k} = -\rho^2 \left(\gamma(\rho) \ddot{\rho} + \tfrac{1}{2} \gamma'(\rho) \dot{\rho}^2 \right) \mathbf{v}. \end{aligned} \tag{64}$$

We observe that, for the constraints (9) and (10), $\mathbf{C}^{in}$ satisfies again relation $(30)_2$, while the expression $(64)_1$ for $\mathbf{c}^{in}$ was already obtained in Appendix B of [28] and Sect. 3 of [30] with variational procedures.

Furthermore, we also note from $(64)_1$ that there is a contribution to the total Cauchy stress tensor $\widetilde{\mathbf{T}}$ in addition to the classical surface actions exerted through the boundary and coming from an influx of linear momentum described by a tensor of inertia flux $\mathbf{M}$ which is the Lagrangian derivative, times $\rho\, \mathbf{I}$, of the fluctuation energy κ_f and measures the agitation within a chunk of material (see [29]): hence, it is

$$\widetilde{\mathbf{T}} = \mathbf{T} + \mathbf{M} \quad \text{with} \quad \mathbf{M} := -\rho^2 \left[\gamma(\rho) \ddot{\rho} + \frac{1}{2} \gamma'(\rho) \dot{\rho}^2 \right] \mathbf{I} \tag{65}$$

with $\mathbf{T}$ given by (55) (see also, the collisional–translational contribution to the total stress tensor in (2.6) of [39] or, for (18), the spherical part of a type of Reynolds stress tensor of the turbulence theory in (3.14) of the review paper [37], in which many other granular theories that split the stress tensor are examined); therefore, by using (61), the Cauchy equation in this context is so written:

$$\rho \dot{\mathbf{v}} = \rho \mathbf{f} + \operatorname{div} \widetilde{\mathbf{T}}. \tag{66}$$

From relations (29), $(64)_2$ and (13) and the properties of $\mathbf{R}$, it follows that the frictionless micromomentum balances (58) are now:

$$\begin{aligned} \rho \left(3\mu\ddot{\mu} - \mu^2 \widetilde{\mathbf{Y}} \cdot \widetilde{\mathbf{Y}} \right) = \beta \mathbf{R} \cdot \left(\rho \mathbf{B} - \mathbf{Z}_a + \operatorname{div} \boldsymbol{\Sigma}_a \right) \quad \text{and} \\ \rho \left(\mu^2 \widetilde{\mathbf{Y}} \right)^{\cdot} = \operatorname{skw} \left[\beta \mathbf{R} \left(\rho \mathbf{B} - \mathbf{Z}_a + \operatorname{div} \boldsymbol{\Sigma}_a \right)^T \right] \end{aligned} \tag{67}$$

or, by inserting the constitutive expressions (16) and $(6)_2$,

$$3\rho\rho_m^{\frac{4}{3}}\Big[\alpha(\rho_m)\left(\ddot{\rho}_m - \rho_m\widetilde{\mathbf{Y}}\cdot\widetilde{\mathbf{Y}}\right) + \tfrac{1}{2}\alpha'(\rho_m)\dot{\rho}_m^2\Big] = \rho_{m*}^{\frac{1}{3}}\mathbf{R}\cdot(\rho\mathbf{B} - \mathbf{Z}_a + \operatorname{div}\boldsymbol{\Sigma}_a)$$
$$\text{and}\quad 3\rho\rho_m^{\frac{1}{3}}\left(\rho_m^2\alpha(\rho_m)\widetilde{\mathbf{Y}}\right)^{\cdot} = \rho_{m*}^{\frac{1}{3}}\text{skw}\left[\mathbf{R}\,(\rho\mathbf{B} - \mathbf{Z}_a + \operatorname{div}\boldsymbol{\Sigma}_a)^T\right]. \quad (68)$$

Equation (59) for the reactions is now

$$\mathbf{Z}_r - \operatorname{div}\boldsymbol{\Sigma}_r = \rho\left[\operatorname{dev}\left(\mathbf{B}\,\mathbf{R}^T + \mu_*\mu^2\widetilde{\mathbf{Y}}^2\right)\right]\mathbf{R}\,, \quad \text{or}$$
$$\mathbf{Z}_r - \operatorname{div}\boldsymbol{\Sigma}_r = \rho\left[\operatorname{dev}\left(\mathbf{B}\,\mathbf{R}^T + 3\rho_{m*}^{-\frac{1}{3}}\,\rho_m^{\frac{7}{3}}\,\alpha(\rho_m)\,\widetilde{\mathbf{Y}}^2\right)\right]\mathbf{R}\,. \quad (69)$$

In the sequel of this section, we recover the relation of evolution for the granular temperature of the body (the granular heat transfer equation (4.6) of [13] or the balance of pseudo-thermal energy (2.7) of [39]) as a direct consequence of our equations for micromomentum balance (67).

The quantity that is usually introduced as granular temperature ϑ represents a fraction of the extra energy due to grains agitation and to chunks dilatancy (and is the trace of the so-called Reynolds tensor which measures the momentum flux in fluid dynamics); in our theory it corresponds to the fluctuation energy κ_f plus the roto-dilatational kinetic energy κ_s (multiplied by $\frac{2}{3}$):

$$\vartheta := \frac{2}{3}(\kappa_f + \kappa_s) = \frac{1}{3}\gamma(\rho)\dot{\rho}^2 + \dot{\mu}^2(\rho_m) + \frac{1}{3}\mu^2(\rho_m)\,\widetilde{\mathbf{Y}}\cdot\widetilde{\mathbf{Y}}, \quad (70)$$

where relation $(46)_3$ was used.

By differentiating with respect to the time and by using (17), (63) and (18), the representation $(65)_2$ of the inertia flux tensor $\mathbf{M}$ and the antisymmetry of $\widetilde{\mathbf{Y}}$, we obtain

$$\frac{3}{2}\rho\dot{\vartheta} = \mathbf{M}\cdot\mathbf{L} + \rho\dot{\mu}\mu^{-1}\left(3\mu\ddot{\mu} - \mu^2\widetilde{\mathbf{Y}}\cdot\widetilde{\mathbf{Y}}\right) + \rho\widetilde{\mathbf{Y}}\cdot\left(\mu^2\widetilde{\mathbf{Y}}\right)^{\cdot}; \quad (71)$$

at the end, the equations for micromomentum balance (67) give

$$\frac{3}{2}\rho\dot{\vartheta} \;= \operatorname{div}\mathbf{u} + \mathbf{M}\cdot\mathbf{L} + \iota + \rho\mathbf{B}\cdot(\beta\mathbf{R})^{\cdot}\,. \quad (72)$$

Equation (72) is the so-called balance of granular energy in which we can easily recognize, with appropriate identifications, usual terms introduced in granular theories: $\mathbf{u} := \left[\boldsymbol{\Sigma}_a^T\left(\beta\mathbf{R}^T\right)^{\cdot}\right]$ is the granular heat flux vector, an interstitial work flux of mechanical nature, in excess of the usual flux due to surface tractions, owing to interactions between chunks and due to grains–boundary collisions or to exchange of granules through the chunk boundary itself as well as to weakly nonlocal spatial effects (see [31], [24], [25] and [23]); $(\mathbf{M}\cdot\mathbf{L})$ is the rate of working of the inertia component of the stress tensor; $\left[\rho\mathbf{B}\cdot(\beta\mathbf{R})^{\cdot}\right]$ is a granular heat source, that some author call the 'stir' due to external actions; $\iota := -\left[\mathbf{Z}_a\cdot(\beta\mathbf{R})^{\cdot}+\boldsymbol{\Sigma}_a\cdot\mathrm{grad}\,(\beta\mathbf{R})^{\cdot}\right]$ is the local rate of dissipation due to the inelastic nature of collisions between particles, dissipation which also appears, when (9) and $(52)_{2,3,4,5}$ are taken into account, on the right-hand side of the balance of thermic internal energy (22) with the opposite sign (see also, (2.4) of [39]).

8 Constitutive Restrictions in the Thermoelastic Case

The peculiar flow behaviour of granular materials can be considered similar to fluid one, except that its bulk compressibility and temperature distribution depend on the initial porosity (see e.g., [2] and the experimental results in [4]) and thus the medium has a preferred reference placement with respect to volume distribution.

Therefore, we assume that the overall response of a thermoelastic dilatant granular materials with rotating grains depends on the set $\mathcal{S}\equiv\{\rho_*,\rho,\mathbf{s}:=\mathrm{grad}\,\rho,\mathbf{S}:=\mathrm{grad}^{\,2}\rho,\beta,\mathbf{p}:=\mathrm{grad}\,\beta,\mathbf{P}:=\mathrm{grad}^{\,2}\beta,\mathbf{R},\boldsymbol{\Pi}:=\mathrm{grad}\,\mathbf{R},\theta_*,\theta,\mathbf{g}\}$. The symmetric tensors $\mathbf{S}$ and $\mathbf{P}$ are inserted among variables not only for consistency with the results of the conservative case in absence of rotations [30], but also because they seem the appropriate second geometric measures of local structure, namely, a sort of rough measurements of anisotropy of grains and chunks distributions, respectively (see, also, [21] and [44]).

The equipresence principle requires that each dependent constitutive field is given by a smooth function of the set $\mathcal{S}$, i.e.,

$$\{\psi_a,\,\eta_a,\,\mathrm{sym}\,\mathbf{T}_a,\,\mathbf{Z}_a,\,\boldsymbol{\Sigma}_a,\,\mathbf{q}_a\}=\left\{\hat{\psi},\,\hat{\eta},\,\hat{\mathbf{T}},\,\hat{\mathbf{Z}},\,\hat{\boldsymbol{\Sigma}},\,\hat{\mathbf{q}}\right\}(\mathcal{S});\qquad(73)$$

now let us check the compatibility of these prescriptions with the Clausius–Duhem inequality, in its reduced version (25), by incorporating the condition (49) of perfect constraint and the functional dependence of the free energy ψ_a and by using the chain rule, the conservation of mass (18) and the identities

$$\overline{\mathrm{grad}\,\mathbf{R}}^{\cdot}=\mathrm{grad}\,\dot{\mathbf{R}}-(\mathrm{grad}\,\mathbf{R})\mathbf{L}\quad\text{and}\quad\overline{\mathrm{grad}\,\omega}^{\cdot}=\mathrm{grad}\,\dot{\omega}-\mathbf{L}^T\mathrm{grad}\,\omega,\qquad(74)$$

for each scalar function ω.

We require that the entropy imbalance (25) be valid for any choice of the fields in the set $\mathcal{S}$ and their derivatives, consequently, when the terms are appropriately ordered, the inequality reads

$$\left\{\mathrm{sym}\left[\mathbf{T}_a+\rho\left(\mathbf{s}\otimes\hat{\psi}_{\mathbf{s}}+\mathbf{p}\otimes\hat{\psi}_{\mathbf{p}}+\boldsymbol{\Pi}^T\odot\hat{\psi}^T_{\boldsymbol{\Pi}}\right)\right]+\rho^2\left(\hat{\psi}_\rho+\hat{\psi}_{\mathbf{s}}\cdot\mathbf{s}\right)\mathbf{I}\right\}\cdot\mathbf{D}-$$
$$-\,\mathrm{skw}\left[\beta\mathbf{R}\hat{\mathbf{Z}}^T+\mathrm{grad}\,(\beta\mathbf{R})\odot\hat{\boldsymbol{\Sigma}}+\rho\left(\mathbf{s}\otimes\hat{\psi}_{\mathbf{s}}+\mathbf{p}\otimes\hat{\psi}_{\mathbf{p}}+\boldsymbol{\Pi}^T\odot\hat{\psi}^T_{\boldsymbol{\Pi}}\right)\right]\cdot\mathbf{Y}-$$
$$-\,\rho\left(\hat{\eta}+\hat{\psi}_\theta\right)\dot{\theta}+\left[\beta\hat{\mathbf{Z}}\cdot\mathbf{R}+\hat{\boldsymbol{\Sigma}}\cdot\mathrm{grad}\,(\beta\mathbf{R})-\rho\beta\hat{\psi}_\beta-\rho\hat{\psi}_{\mathbf{p}}\cdot\mathbf{p}\right]\left(\dot{\beta}\beta^{-1}\right)+$$

$$+\operatorname{skw}\left[\beta\mathbf{R}\hat{\mathbf{Z}}^T + \operatorname{grad}(\beta\mathbf{R})\odot\hat{\mathbf{\Sigma}} + \rho\left(\hat{\psi}_{\mathbf{R}}\mathbf{R}^T + \hat{\psi}_{\mathbf{\Pi}}\odot\mathbf{\Pi}\right)\right]\cdot\widetilde{\mathbf{Y}} -$$
$$-\rho\left(\hat{\psi}_{\mathbf{S}}\cdot\dot{\mathbf{S}} + \hat{\psi}_{\mathbf{P}}\cdot\dot{\mathbf{P}} + \hat{\psi}_{\mathbf{g}}\cdot\dot{\mathbf{g}}\right) + \beta\left(\hat{\mathbf{\Sigma}}^T\mathbf{R}^T - \rho\hat{\psi}_{\mathbf{p}}\right)\cdot\operatorname{grad}\left(\dot{\beta}\beta^{-1}\right) +$$
$$+\left[\mathbf{R}\oslash\left(\beta\,{}^t\hat{\mathbf{\Sigma}} - \rho\,{}^t\hat{\psi}_{\mathbf{\Pi}}\right)\right]\cdot\operatorname{grad}\widetilde{\mathbf{Y}}\cdot + \rho^2\left(\mathbf{I}\otimes\hat{\psi}_{\mathbf{s}}\right)\cdot\operatorname{grad}\mathbf{D} - \theta^{-1}\mathbf{q}\cdot\mathbf{g}\geq 0, \tag{75}$$

where subscripts denote partial differentiation with respect to the shown field, e.g., $\hat{\psi}_{\mathbf{p}} := \frac{\partial\hat{\psi}}{\partial\mathbf{p}}$.

The left-hand member of inequality (75) is linear in $\mathbf{D}$, $\mathbf{Y}$, $\dot{\theta}$, $(\dot{\beta}\beta^{-1})$, $\widetilde{\mathbf{Y}}$, $\dot{\mathbf{S}}$, $\dot{\mathbf{P}}$, $\dot{\mathbf{g}}$, grad $\left(\dot{\beta}\beta^{-1}\right)$, grad $\widetilde{\mathbf{Y}}$ and grad $\mathbf{D}$ and hence, because one can imagine, for each material element, thermomechanical processes along which these quantities take up arbitrary values at a given instant, its fulfillment implies that the coefficients in the linear expression must all vanish:

$$\hat{\psi} = \hat{\psi}(\rho_*, \rho, \beta, \mathbf{p}, \mathbf{R}, \mathbf{\Pi}, \theta_*, \theta), \quad \hat{\eta} = -\hat{\psi}_\theta,$$
$$\hat{\mathbf{T}} = -\rho\left[\rho\hat{\psi}_\rho\mathbf{I} + \operatorname{sym}\left(\mathbf{p}\otimes\hat{\psi}_{\mathbf{p}} + \mathbf{\Pi}^T\odot\hat{\psi}_{\mathbf{\Pi}}^T\right)\right],$$
$$\operatorname{skw}\left[\beta\mathbf{R}\hat{\mathbf{Z}}^T + \operatorname{grad}(\beta\mathbf{R})\odot\hat{\mathbf{\Sigma}} + \rho\left(\mathbf{p}\otimes\hat{\psi}_{\mathbf{p}} + \mathbf{\Pi}^T\odot\hat{\psi}_{\mathbf{\Pi}}^T\right)\right] = \mathbf{0},$$
$$\beta\hat{\mathbf{Z}}\cdot\mathbf{R} + \hat{\mathbf{\Sigma}}\cdot\operatorname{grad}(\beta\mathbf{R}) = \rho\left(\beta\hat{\psi}_\beta + \hat{\psi}_{\mathbf{p}}\cdot\mathbf{p}\right), \tag{76}$$
$$\operatorname{skw}\left[\beta\mathbf{R}\hat{\mathbf{Z}}^T + \operatorname{grad}(\beta\mathbf{R})\odot\hat{\mathbf{\Sigma}} + \rho\left(\hat{\psi}_{\mathbf{R}}\mathbf{R}^T + \hat{\psi}_{\mathbf{\Pi}}\odot\mathbf{\Pi}\right)\right] = \mathbf{0},$$
$$\hat{\mathbf{\Sigma}}^T\mathbf{R}^T = \rho\hat{\psi}_{\mathbf{p}},\ \operatorname{skw}\left\{\mathbf{R}\left[\left(\beta\,\hat{\mathbf{\Sigma}} - \rho\,\hat{\psi}_{\mathbf{\Pi}}\right)\mathbf{w}\right]^T\right\} = \mathbf{0},\ \forall\,\text{vector}\,\mathbf{w},$$

while the heat flux $\hat{\mathbf{q}}$ must satisfy identically the Fourier inequality

$$\hat{\mathbf{q}}\cdot\mathbf{g}\leq 0. \tag{77}$$

The following compatibility condition on the free energy $\hat{\psi}$, which comes out from $(76)_4$ and $(76)_6$:

$$\operatorname{skw}\left(\mathbf{p}\otimes\hat{\psi}_{\mathbf{p}} + \mathbf{R}\,\hat{\psi}_{\mathbf{R}}^T + \mathbf{\Pi}^T\odot\hat{\psi}_{\mathbf{\Pi}}^T + \mathbf{\Pi}\odot\hat{\psi}_{\mathbf{\Pi}}\right) = \mathbf{0}, \tag{78}$$

expresses simply the condition of frame-indifference for $\hat{\psi}$, namely,

$$\hat{\psi}\left(\rho_*, \rho, \beta, \mathbf{Q}\mathbf{p}, \mathbf{Q}\mathbf{R}, (\mathbf{Q}\oslash\mathbf{\Pi})\,\mathbf{Q}^T, \theta_*, \theta\right) = \hat{\psi}(\rho_*, \rho, \beta, \mathbf{p}, \mathbf{R}, \mathbf{\Pi}, \theta_*, \theta), \tag{79}$$

for each $\mathbf{Q}\in\mathrm{Orth}^+$.

Moreover, the total Cauchy stress tensor $\widetilde{\mathbf{T}}$ for a thermoelastic medium is given by (65), (55) and $(76)_{3,4}$:

$$\widetilde{\mathbf{T}} = -\rho^2 \left[\gamma(\rho)\ddot{\rho} + \frac{1}{2}\gamma'(\rho)\dot{\rho}^2 + \hat{\psi}_\rho\right]\mathbf{I} - \rho\left(\mathbf{p}\otimes\hat{\psi}_{\mathbf{p}} + \boldsymbol{\Pi}^T \odot \hat{\psi}^T_{\boldsymbol{\Pi}}\right), \tag{80}$$

where we recognize the usual thermodynamic pressure for fluids $\pi := \rho^2\hat{\psi}_\rho$, related to the compressibility of granules, a stress of Ericksen's type $(-\rho\,\mathbf{p}\otimes\hat{\psi}_{\mathbf{p}})$ that justifies the ability of granular continua to support shear in equilibrium also in absence of microrotation, as evidenced by the characteristic angle of repose of these materials, and a further stress term $\left(-\rho\,\boldsymbol{\Pi}^T\odot\hat{\psi}^T_{\boldsymbol{\Pi}}\right)$, which shows that they could still sustain shear stresses when the grains are rigid, giving rise to the generation of microrotation gradients.

As observed at the end of Sect. 6, the evolution equation for the temperature of granular materials (60) simplifies considerably and reduces to the classical one, that is,

$$\rho\theta\dot{\hat{\eta}} = \rho\lambda - \operatorname{div}\hat{\mathbf{q}}. \tag{81}$$

Furthermore, with the use of constitutive relations (76) in (73), we are able to express the dependent fields on the right-hand side of pure equations of micromotion (68) in function of the Helmholtz free energy $\hat{\psi}$; precisely, by using $(76)_{5,7}$ in the former and $(76)_{6,8}$ in the latter, we have

$$\left(\operatorname{div}\hat{\boldsymbol{\Sigma}} - \hat{\mathbf{Z}}\right)\cdot\mathbf{R} = \operatorname{div}\left(\rho\hat{\psi}_{\mathbf{p}}\right) - \rho\hat{\psi}_\beta \quad \text{and} \tag{82}$$

$$\operatorname{skw}\left[\beta\,\mathbf{R}\left(\operatorname{div}\hat{\boldsymbol{\Sigma}} - \hat{\mathbf{Z}}\right)^T\right] = \operatorname{skw}\left\{\mathbf{R}\left[\operatorname{div}\left(\rho\hat{\psi}_{\boldsymbol{\Pi}}\right) - \rho\hat{\psi}_{\mathbf{R}}\right]^T\right\},$$

where $\hat{\psi}$ represents a sort of potential for stresses and microstresses.

Now, if we introduce in (82) internal forces of dilatancy δ and of rotation $\mathbf{N}$, the dilating microstress vector $\mathbf{h}_{dil}$ and the third order spinning hyperstress tensor $\boldsymbol{\Sigma}_{spi}$, defined by

$$\delta := \tfrac{1}{3}\rho\left(\beta\hat{\psi}_\beta + \mathbf{p}\cdot\hat{\psi}_{\mathbf{p}}\right), \quad \mathbf{N} := \tfrac{1}{3}\rho\operatorname{skw}\left(\mathbf{R}\,\hat{\psi}^T_{\mathbf{R}} + \boldsymbol{\Pi}\otimes\hat{\psi}_{\boldsymbol{\Pi}}\right), \tag{83}$$

$$\mathbf{h}_{dil} := \tfrac{1}{3}\rho\beta\hat{\psi}_{\mathbf{p}} \quad \text{and} \quad \boldsymbol{\Sigma}_{spi}\mathbf{w} := \tfrac{1}{3}\operatorname{skw}\left[\rho\mathbf{R}\left(\hat{\psi}_{\boldsymbol{\Pi}}\mathbf{w}\right)^T\right], \ \forall\,\text{vector}\,\mathbf{w}, \tag{84}$$

as a consequence the balances of dilatational and rotational micromomentum (68) are, respectively:

$$\rho\rho_m\left[\alpha(\rho_m)\left(\ddot{\rho}_m - \rho_m\widetilde{\mathbf{Y}}\cdot\widetilde{\mathbf{Y}}\right) + \tfrac{1}{2}\alpha'(\rho_m)\dot{\rho}^2_m\right] = \rho\,\phi - \delta + \operatorname{div}\mathbf{h}_{dil}$$

$$\text{and} \quad \rho\left(\rho^2_m\alpha(\rho_m)\widetilde{\mathbf{Y}}\right)^{\cdot} = \rho\,\mathbf{O} - \mathbf{N} + \operatorname{div}\boldsymbol{\Sigma}_{spi}, \tag{85}$$

where $\phi := \frac{1}{3}\beta\,\mathbf{B}\cdot\mathbf{R}$ and $\mathbf{O} := \frac{1}{3}\operatorname{skw}\left(\beta\mathbf{R}\mathbf{B}^T\right)$ are the external dilatational force and the external tensor moment per unit mass, respectively.

These equations for micromomentum together with the balance of mass (18) and the Cauchy's balance of linear momentum (66) ($\widetilde{\mathbf{T}}$ given by (80)) are the pure field equations of motion for thermoelastic dilatant granular materials with rotating grains, the evolution of the temperature being ruled by (81).

Remark 1: We observe now that $\mathbf{h}_{dil}$ and $\mathbf{\Sigma}_{spi}$, defined in (84), are particular examples of the stirring and the twisting hyperstress tensor defined in [14] and [8], but, unlike those papers, we think that it is not possible the assignment of prescribed boundary conditions to both of them, the stirrer $\mathbf{h}_{dil}\hat{\mathbf{n}}$ and the twister $\mathbf{\Sigma}_{spi}\hat{\mathbf{n}}$ ($\hat{\mathbf{n}}$ is the exterior unit normal to the boundary surface).

In fact they are of different physical nature: while for the twister the boundary distribution of the external couples could be assigned in analogy to the microrigid Cosserat brother's continua [19], on the contrary, for the stirrer, it appears difficult to imagine a direct way to act on the proper grain compressibility through the boundary itself; rather, only the sum $(-\delta + \operatorname{div}\mathbf{h}_{dil})$ has sense, has the right properties of covariance and could express weakly non-local effects (see [7], pages 26–27, [42], page 21, and [5]).

In [11] a wide discussion about the manifold of values of the microstructures with, or without, physically significant connection and the consequent presence, or absence, of the related microstress is presented.

Remark 2: In this context, the mechanical interstitial work flux $\mathbf{u}$, introduced at the end of Sect. 7 in the balance of granular energy (72), is now written as

$$\mathbf{u} = 3\left[\left(\dot{\beta}\beta^{-1}\right)\mathbf{h}_{dil} + \mathbf{\Sigma}_{spi}^{T}\widetilde{\mathbf{Y}}\right]; \tag{86}$$

thus terms related to contractions or dilatations of grains and to rotations appear clearly put in evidence.

9 Suspension of Rigid Granules in a Fluid Matrix

In the analysis of flows of a large number of discrete inelastic particles at relatively high concentrations and with interstices filled with a fluid or a gas of negligible mass (as it is the case of cohesionless soil, such as sand with rough surface grains, or of fluidized particulate beds), we must assume that the granules are incompressible; therefore, the proper mass density ρ_m is constant and, for relation (1), the chunk mass density ρ comes down to be proportional to the volume fraction ν of grains ($\rho = \rho_{m*}\nu$) and so the conservation of mass (18) gives

$$\dot{\nu} + \nu\,\mathrm{tr}\,\mathbf{L} = 0. \tag{87}$$

Furthermore, for condition $(16)_2$, the coefficient β in the constraint relation (9) disappears ($\beta \equiv 1$) and $\mathbf{G} = \mathbf{R}$, so that

$$\kappa_s = \frac{1}{2}\mu_*^2\,\dot{\mathbf{R}}\cdot\dot{\mathbf{R}},\;\; \mathbf{W} = \dot{\mathbf{R}}\mathbf{R}^T = \widetilde{\mathbf{Y}}^T,\;\; \widetilde{\mathbf{D}} = \mathbf{0} \text{ and } \operatorname{grad}(\mathbf{W}^T) = \operatorname{grad}\widetilde{\mathbf{Y}}. \tag{88}$$

Remark: When the effects of relative rotations of the chunks and of the grains are also negligible ($\mathbf{R} = \mathbf{I}$), we recover the essence of the theory in [28] and in Sect. 6 of [31]: in particular, in both of them the Coulomb's model for the stress at equilibrium in a granular material with incompressible grains:

$$\mathbf{T}^e = \left(\beta_0 - \beta_1 \nu^2 + \beta_2 \mathrm{grad}\,\nu \cdot \mathrm{grad}\,\nu + 2\,\beta_3\,\nu\,\Delta\nu\right)\mathbf{I} - 2\,\beta_4\,\mathrm{grad}\,\nu \otimes \mathrm{grad}\,\nu$$

with β_i material constants for $i = 0, 1, 2, 3, 4$, is obtained as a peculiar example (see also, (9.1) of [33]). Alternatively, the complementary case in which $\mathbf{R} = \mathbf{I}$, but the grains are elastic, is studied in [30] and again in [31].

We focus here on the simple inelastic case for which relations (88) apply and we develop calculations of Sects. 6–8 with few adjustments.

Firstly, we obtain the following prescriptions for reactions:

$$\begin{gathered}\left(\mathbf{Z}_r \mathbf{R}^T + \mathbf{\Sigma}_r \odot \mathrm{grad}\,\mathbf{R}\right) \in \mathrm{Sym}, \quad \mathbf{R}\,(\mathbf{\Sigma}_r\,\mathbf{w})^T \in \mathrm{Sym},\ \forall\,\mathrm{vector}\,\mathbf{w}, \\ \mathbf{T}_r = \mathbf{0}, \quad \psi_r = \mathrm{const.}, \quad \eta_r = 0, \quad \mathbf{q}_r = \mathbf{0},\end{gathered} \tag{89}$$

and, correspondingly, for actions

$$\begin{gathered}\left(\mathbf{Z}_a \mathbf{R}^T + \mathbf{\Sigma}_a \odot \mathrm{grad}\,\mathbf{R}\right) \in \mathrm{Skw}, \quad \mathbf{R}\ (\mathbf{\Sigma}_a\,\mathbf{w})^T \in \mathrm{Skw},\ \forall\,\mathrm{vector}\,\mathbf{w}, \\ \mathbf{T} = \mathrm{sym}\,\mathbf{T}_a - \mathrm{skw}\,\left(\mathbf{Z}_a \mathbf{R}^T + \mathbf{\Sigma}_a \odot \mathrm{grad}\,\mathbf{R}\right).\end{gathered} \tag{90}$$

Secondly, the reaction-free equation of micromomentum balance for our suspension of rigid granules is now

$$\rho \mu_*^2 \dot{\widetilde{\mathbf{Y}}} = \mathrm{skw}\,\left[\mathbf{R}\,(\rho\mathbf{B} - \mathbf{Z}_a + \mathrm{div}\,\mathbf{\Sigma}_a)^T\right], \tag{91}$$

while the equation for the reactions to the constraint is

$$\mathbf{Z}_r - \mathrm{div}\,\mathbf{\Sigma}_r = \rho\left[\mathrm{sym}\,\left(\mathbf{B}\,\mathbf{R}^T + \mu_*^2 \widetilde{\mathbf{Y}}^2\right)\right]\mathbf{R}. \tag{92}$$

We observe that the (91) for the microstructural actions is the same that rules the micromotion for the microrigid Cosserat's continua (see (23.1) of [7] or (63) of [35]).

Thirdly, the set of constitutive variables for a thermoelastic materials with rotating rigid grains is now

$$\mathcal{S}_{rigid} \equiv \{\nu_*, \nu, \mathrm{grad}\,\nu, \mathrm{grad}^2\nu, \mathbf{R}, \mathbf{\Pi}, \theta_*, \theta, \mathbf{g}\},$$

and so the entropy imbalance (25) and relations (65) and (87) give the following constitutive prescriptions for dependent fields:

$$\begin{gathered}\rho_{m*}\psi_a = \bar{\psi}(\nu_*, \nu, \mathbf{R}, \mathbf{\Pi}, \theta_*, \theta) = \\ = \bar{\psi}\left(\nu_*, \nu, \mathbf{Q}\mathbf{R}, (\mathbf{Q} \oslash \mathbf{\Pi})\,\mathbf{Q}^T, \theta_*, \theta\right), \quad \forall \mathbf{Q} \in \mathrm{Orth}^+, \\ \widetilde{\mathbf{T}} = -\nu^2\left[\bar{\gamma}(\nu)\ddot{\nu} + \tfrac{1}{2}\bar{\gamma}'(\nu)\dot{\nu}^2 + \bar{\psi}_\nu\right]\mathbf{I} - \nu\mathbf{\Pi}^T \odot \bar{\psi}_{\mathbf{\Pi}}^T, \quad \eta_a = -\bar{\psi}_\theta, \\ \mathrm{skw}\,\left\{\mathbf{R}\left[\left(\mathbf{\Sigma}_a - \nu\,\bar{\psi}_{\mathbf{\Pi}}\right)\mathbf{w}\right]^T\right\} = \mathbf{0}, \quad \forall\,\mathrm{vector}\,\mathbf{w}, \quad \mathbf{q}_a \cdot \mathbf{g} \leq 0,\end{gathered} \tag{93}$$

with

$$\bar{\gamma}(\nu) = \bar{\gamma}_* \nu^{-\frac{8}{3}} \quad \text{and} \quad \bar{\gamma}_* = \frac{16}{351} \rho_{m*} \nu_*^{\frac{2}{3}} \varsigma_*^2. \tag{94}$$

In $(93)_3$ the thermodynamic pressure is now $\bar{\pi} := \nu^2 \bar{\psi}_\nu$ and is related to the compressibility of chunks, while the stresses of Reynolds' and of Ericksen's type measure, respectively, the agitation within a chunk of material and the ability of rigid granular continua to support shear stresses in equilibrium, by inducing the generation of microrotation gradients, even when the proper mass density and the volume fraction of the grain distribution is constant.

Finally, the balance of rotational micromomentum (68) is given by

$$\rho_{m*} \nu \mu_*^2 \dot{\bar{\mathbf{Y}}} = \rho_{m*} \nu \, \bar{\mathbf{O}} - \bar{\mathbf{N}} + \operatorname{div} \bar{\mathbf{\Sigma}}_{spi}, \tag{95}$$

where

$$\bar{\mathbf{O}} := \operatorname{skw} \left(\mathbf{R} \mathbf{B}^T \right), \quad \bar{\mathbf{N}} := \nu \operatorname{skw} \left(\mathbf{R} \, \bar{\psi}_{\mathbf{R}}^T + \mathbf{\Pi} \otimes \bar{\psi}_{\mathbf{\Pi}} \right), \tag{96}$$

$$\bar{\mathbf{\Sigma}}_{spi} \mathbf{w} := \operatorname{skw} \left[\nu \mathbf{R} \left(\bar{\psi}_{\mathbf{\Pi}} \mathbf{w} \right)^T \right], \; \forall \, \text{vector} \, \mathbf{w}, \tag{97}$$

are the new external and internal rotational tensor moment per unit mass and the new third order spinning hyperstress tensor, respectively.

The pure field equations of mass, macro- and micromotion and of temperature for granular materials with rigid rotating grains are then (87), (66) with $\widetilde{\mathbf{T}}$ given by $(93)_3$, (95) and (81).

Appendix: Kinetic Energy Coefficients

To compute explicitly the constitutive functions $\gamma(\rho)$ and $\alpha(\rho_m)$ we imagine the chunk consisting, in a mental magnification, of a spherical grain and its immediate spherical neighbours (see [6]), and the envelope of the chunk as a spherical surface of variable radius ς containing all these spherical compressible inclusions of variable radius φ with interstices filled with a fluid or a gas of negligible mass; the envelopes and the grains have the same radius ς_* and φ_*, respectively, in a reference placement $\mathcal{B}_*$ of the material.

Moreover, we assume that the chunks and the grains expand and/or contract homogeneously with independent radial motions; therefore, if we indicate with $\widetilde{\varsigma}$ the distance from the centre of mass of the chunk to the centre of mass of a grain in the chunk itself, and with $\widetilde{\varphi}$ the distance from the centre of mass of a grain to the element of volume dv_m, they are related to ς and φ by

$$\tilde{\varsigma} = \frac{\tilde{\varsigma}_*}{\varsigma_*} \varsigma \quad \text{and} \quad \tilde{\varphi} = \frac{\tilde{\varphi}_*}{\varphi_*} \varphi, \tag{98}$$

respectively.

Furthermore, the average density of kinetic energy $(\kappa_f^{ch} + \kappa_d^{ch})$ per unit volume associated to each chunk, as effect of the homogeneous expansions or contractions of a typical chunk itself and of the inclusions (in addition to the classical kinetic energy of translation κ_t^{ch}), will be written

$$\kappa_f^{ch} + \kappa_d^{ch} = \frac{1}{2}\frac{1}{\frac{4}{3}\pi\varsigma^3}\sum^{n}\left(m\dot{\tilde{\varsigma}}^2 + \int_{\mathcal{V}_m}\rho_m\dot{\tilde{\varphi}}^2\, d\mathcal{V}_m\right), \tag{99}$$

where $\sum$ denotes summation over all of the grains of the chunk, n is the number of the grains in a chunk, $\mathcal{V}_m$ and m are the volume and the mass of a typical grain, respectively, i.e.,

$$\mathcal{V}_m = \frac{4}{3}\pi\varphi^3 \quad \text{and} \quad m = \frac{4}{3}\rho_m\pi\varphi^3 = \frac{4}{3}\rho_{m*}\pi\varphi_*^3 \tag{100}$$

(see also, (2.3) of [4]); hence, from relations $(98)_2$ and $(100)_3$, the time rate of change of $\tilde{\varphi}$ can be expressed in terms of the rate of change of ρ_m:

$$\dot{\tilde{\varphi}} = -\frac{1}{3}\tilde{\varphi}_*\,\rho_{m*}^{\frac{1}{3}}\,\rho_m^{-\frac{4}{3}}\,\dot{\rho}_m. \tag{101}$$

We observe that the quasi-particles are assumed to fill the space of the granular material, without voids between them, and so the volume fraction ν of the chunk is

$$\nu = \frac{1}{\frac{4}{3}\pi\varsigma^3}\sum^{n}\frac{4}{3}\pi\varphi^3 = \sum^{n}\left(\frac{\varphi}{\varsigma}\right)^3, \tag{102}$$

while (being ς_* and φ_* constants in the same chunk)

$$\nu_* = \sum^{n_*}\left(\frac{\varphi_*}{\varsigma_*}\right)^3 = n_*\left(\frac{\varphi_*}{\varsigma_*}\right)^3. \tag{103}$$

Moreover, the granules are supposed homogeneous, strictly packed and such that they do not diffuse throughout the envelope of the chunk; therefore, ρ_{m*} = const., the immediate neighbours of a grain are twelve, with $n = n_* = 13$, and, finally, the volume and the mass density of a macroelement are, respectively,

$$\mathcal{V} = \frac{4}{3}\pi\varsigma^3 \quad \text{and} \quad \rho = \frac{\rho_*\mathcal{V}_*}{\mathcal{V}} = \rho_*\left(\frac{\varsigma_*}{\varsigma}\right)^3. \tag{104}$$

Thus, it follows from $(98)_1$ and $(104)_3$ that the time rate of change of $\tilde{\varsigma}$ can be expressed in terms of the rate of change of ρ

$$\dot{\tilde{\varsigma}} = -\frac{1}{3}\tilde{\varsigma}_*\,\rho_*^{\frac{1}{3}}\,\rho^{-\frac{4}{3}}\,\dot{\rho} \tag{105}$$

and, from $(100)_3$, $(104)_3$ and (105), that the 'fluctuation' kinetic energy κ_f^{ch} of the chunk is

$$\kappa_f^{ch} = \frac{1}{2}\frac{1}{\frac{4}{3}\pi\varsigma^3}\sum^{n} m\dot{\tilde{\varsigma}}^2 = \frac{\rho_*^{\frac{2}{3}}\dot{\rho}^2}{18\rho^{\frac{8}{3}}\varsigma^3}\sum^{n_*}\rho_{m*}\varphi_*^3\tilde{\varsigma}_*^2 = \frac{\rho_{m*}\varphi_*^3\dot{\rho}^2}{18\varsigma_*^3\rho_*^{\frac{1}{3}}\rho^{\frac{5}{3}}}\sum^{n_*}\tilde{\varsigma}_*^2, \qquad (106)$$

where we used the fact that the single grains of a chunk are homogeneous and of the same radius φ_* in the reference placement $\mathcal{B}_*$ of the material.

Nevertheless, we supposed the granules strictly packed in $\mathcal{B}_*$, therefore, the centre of mass of the chunk coincides with the centre of the main grain (hence the related $\tilde{\varsigma}_*$ vanishes), while the centre of mass of its twelve immediate spherical neighbours are distant two time the constant radius φ_* of a grain from the centre of the main grain (hence, $\tilde{\varsigma}_* = 2\varphi_*$); moreover, the radius of the chunk envelope ς_* is three time the radius φ_*, i.e., $\varsigma_* = 3\varphi_*$ and $\tilde{\varsigma}_* = \frac{2}{3}\varsigma_*$.

Thus, by using also relations (1) and (103), we have

$$\rho_{m*}\left(\frac{\varphi_*}{\varsigma_*}\right)^3 = \frac{\rho_*}{n_*} \quad \text{and} \quad \sum^{n_*}\tilde{\varsigma}_*^2 = \frac{4}{9}(n_* - 1)\varsigma_*^2; \qquad (107)$$

at the end, by inserting (107) and $n_* = 13$ in $(106)_3$, we obtain the density of kinetic energy κ_f^{ch} per unit volume associated to each chunk

$$\kappa_f^{ch} = \frac{1}{2}\rho\left(\frac{16}{351}\rho_*^{\frac{2}{3}}\varsigma_*^2\right)\rho^{-\frac{8}{3}}\dot{\rho}^2. \qquad (108)$$

Now, if we consider the 'dilatational' kinetic energy κ_d^{ch} of the chunk and use relations (100), (101), $(104)_3$ and, after, (1) and $(103)_1$, we obtain

$$\kappa_d^{ch} = \frac{1}{2}\frac{1}{\frac{4}{3}\pi\varsigma^3}\sum^{n}\int_{\mathcal{V}_m}\rho_m\dot{\tilde{\varphi}}^2\,d\mathcal{V}_m = \frac{\rho\rho_{m*}^{\frac{5}{3}}\dot{\rho}_m^2}{24\pi\rho_*\varsigma_*^3\rho_m^{\frac{8}{3}}}\sum^{n_*}\int_0^{\varphi_*}4\pi\tilde{\varphi}_*^4\,d\tilde{\varphi}_* =$$

$$= \frac{\rho\rho_{m*}^{\frac{2}{3}}\varphi_*^2\dot{\rho}_m^2}{30\rho\varsigma_*^3\rho_m^{\frac{8}{3}}}\left[\frac{\rho_{m*}}{\rho_*}\sum^{n_*}\left(\frac{\varphi_*}{\varsigma_*}\right)^3\right] = \frac{1}{2}\rho\left(\frac{1}{15}\rho_{m*}^{\frac{2}{3}}\varphi_*^2\right)\rho_m^{-\frac{8}{3}}\dot{\rho}_m^2. \qquad (109)$$

The constitutive expressions (6) and (7) for the coefficients $\gamma(\rho)$ and $\alpha(\rho_m)$, which appear in formula (5) are then easily recognized in (108) and $(109)_4$.

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