
Bodies with Kinetic Substructure

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Summary. In some earlier papers an elementary approach was followed to suggest a set of balance laws governing, within a continuum theory, the evolution of bodies made up of countless molecules afflicted by chaotic agitation. The set is larger than usual to insure strict observer independence of consequent thermal entities. Here preliminary steps are taken to pursue the same goal but with an inception closely akin to that prefacing the kinetic theory of gases; the quest here, however, exacts divergence from the route followed in the latter theory. Thus some, possibly controversial, notions emerge and are proffered here for criticism.

1 Kinetics

Consider a body in its deportment at an instant τ , when it occupies a region \mathcal{B} ; a region which, in imagination, is envisaged as split into tiny spatial segments. Each segment ϵ contains many molecules and, although it is said to be located at a place x within \mathcal{B} , it must be imagined to have a microexpanse within which subplaces can be distinguished at a lower scale. Accordingly, and contrary to the bias mooted by the standard kinetic theory, of each molecule one presumes here to gauge not only the velocity w (which in principle can be any member of the vector space \mathcal{V}) but also the subplace z within ϵ (z being distinct, at our penetrating magnification, from x).

Consequently one seeks the distribution θ , valid for $\epsilon(x)$ at time τ , such that $\theta(\tau, x; z, w) dzdw$ gives the number of molecules passing in the vicinity of z and with velocity near w . θ is presumed to be such that all integrations involving it and mentioned below are convergent. In particular

$$\omega = \int_{\epsilon} \int_{\mathcal{V}} \theta(\tau, x; z, w), \quad [\theta] = L^{-6}T^3, \quad (1)$$

gives the (finite, though large) number of molecules in $\epsilon(x)$ at time τ . Take note that, at any instant τ , there may be many molecules passing through the immediate neighbourhood of z , possibly with widely different velocities.

If all molecules have the same mass μ (as always presumed below for simplicity) then

$$\mu\omega = (\text{meas } \epsilon) \rho, \quad (2)$$

where ρ is the gross mass density at x .

Some formulae below are shortened by use of the distribution $\tilde{\theta}$

$$\tilde{\theta}(\tau, x; z) = \int_{\mathcal{V}} \theta(\tau, x; z, w), \quad [\tilde{\theta}] = L^{-3}, \quad (3)$$

which counts the number of molecules near z whatever their velocity.

Vice versa, within the kinetic theory, as already mentioned, only the alternative reduced distribution $\hat{\theta}$ matters,

$$\hat{\theta}(\tau, x; w) = \int_{\epsilon} \theta(\tau, x; z, w), \quad [\hat{\theta}] = L^{-3}T^3, \quad (4)$$

which counts the number of molecules in the whole ϵ and velocity near w .

Using θ , or $\tilde{\theta}$, one determines the centre of gravity of all molecules in ϵ

$$x = \omega^{-1} \int_{\epsilon} \int_{\mathcal{V}} \theta z = \omega^{-1} \int_{\epsilon} \tilde{\theta} z; \quad (5)$$

it is after such x that the segment is labelled. Then z can be split into x and y , and, with a slight abuse of notation, one has

$$\int_{\epsilon} \int_{\mathcal{V}} \theta y = \int_{\epsilon} \tilde{\theta} y = 0. \quad (6)$$

As it is done, with success, in the standard theory of fluids, the velocity v assigned at x is, by fiat, the average velocity

$$v = \omega^{-1} \int_{\epsilon} \int_{\mathcal{V}} \theta w. \quad (7)$$

Similarly one attributes to the subelement at z the average velocity \tilde{w} of all molecules passing there

$$\tilde{w} = \tilde{\theta}^{-1} \int_{\mathcal{V}} \theta w \quad (8)$$

so that, in particular,

$$v = \omega^{-1} \int_{\epsilon} \tilde{\theta} \tilde{w}, \quad \omega = \int_{\epsilon} \tilde{\theta}. \quad (9)$$

Thus correct evaluation of total momentum for ϵ is assured summoning reduced quantities $\tilde{\theta}$ and \tilde{w} only. Actually v can be secured also by turning to $\hat{\theta}$ only

$$v = \omega^{-1} \int_{\mathcal{V}} \hat{\theta} w, \quad \omega = \int_{\mathcal{V}} \hat{\theta}, \quad (10)$$

as in the kinetic theory of gases, within which, though, no meaning can be attached to \tilde{w} .

2 A Shadow Speck of Matter

Availability of the fields v and \tilde{w} grants us licence to invent a shadow speck of matter which, in imagination, simply translates with velocity v and within which, besides, the shadow subspeck at z flies with relative velocity \tilde{w} . We can then deal with the speck as it were a subbody (in the sense of standard theory of continua) rather than a collection of riotous molecules, a subbody occupying instantaneously the segment ϵ consisting of subplaces each identified by the variable z , where the material density is $\mu\hat{\theta}$.

One may now proceed to evaluate *Euler's inertia tensor* Y around x ,

$$Y = \omega^{-1} \int_{\epsilon} \int_{\mathcal{V}} \theta y \otimes y = \omega^{-1} \int_{\epsilon} \tilde{\theta} y \otimes y \quad (11)$$

and the tensor *moment of momentum* K

$$K = \omega^{-1} \int_{\epsilon} \int_{\mathcal{V}} \theta y \otimes (w - v) = \omega^{-1} \int_{\epsilon} \int_{\mathcal{V}} \theta y \otimes w = \omega^{-1} \int_{\epsilon} \tilde{\theta} y \otimes \tilde{w}. \quad (12)$$

Neither tensor could be defined with access to the distribution $\hat{\theta}$ only. Vice versa the kinetic energy tensor per unit mass W

$$W = \frac{1}{2} \omega^{-1} \int_{\epsilon} \int_{\mathcal{V}} \theta w \otimes w = \frac{1}{2} \omega^{-1} \int_{\mathcal{V}} \hat{\theta} w \otimes w \quad (13)$$

cannot be achieved with the distribution $\tilde{\theta}$ only. Thus, the 'reduced' tensor \tilde{W} acquires a decisive reserve rôle

$$\tilde{W} = \frac{1}{2} \omega^{-1} \int_{\epsilon} \int_{\mathcal{V}} \theta \tilde{w} \otimes \tilde{w} = \frac{1}{2} \omega^{-1} \int_{\epsilon} \tilde{\theta} \tilde{w} \otimes \tilde{w}. \quad (14)$$

The difference

$$W - \tilde{W} = \frac{1}{2} \omega^{-1} \int_{\epsilon} \int_{\mathcal{V}} \theta (w - \tilde{w}) \otimes (w - \tilde{w}) \quad (15)$$

will be relegated within some 'internal energy' tensor.

One can take now a further step and invent for the shadow speck a congruent affine kinetic field with a rate of deformation B , say; congruent in the sense that, for it, the tensor moment of momentum, now amounting to YB^T , is still equal to K : B need only be chosen to coincide with $K^T Y^{-1}$. There is a similarity here with the process that led to the selection of v : in that case it was the global momentum of the molecules pertaining to the segment which turned out to be equal to that which would have been experienced, within the segment, were all molecules to fly with the same velocity v . In the devised affine impetus, the tensor moment of momentum remains that occurring in the real molecular transit through ϵ .

Having also assembled the field $B(x, \tau)$, one can imagine it generated by a fictitious affine deformation G from an arbitrary constant reference. In principle one need only integrate the partial differential equation

$$\frac{\partial G}{\partial \tau} + (\text{grad}G)v = BG, \quad (16)$$

an integration which determines G a constant right factor apart. Basically the process is not different from that which leads to trajectories in ordinary fluid dynamics through an integration over v (a vector which we know to be an average over a population, not the property of a specific mass-point).

Abiding by the notation $\tilde{N} = GG^T$ and $R' = \tilde{N}^{-\frac{1}{2}}G$ used in an earlier paper [1] G can be split into the product

$$G = \tilde{N}^{\frac{1}{2}}R', \quad (17)$$

with the orthogonal tensor R' providing an intrinsic local reference \mathcal{R} . The inverse G^{-1} could be intended to express the retrogression of ϵ into a paragon segment ϵ_* and of the subplace y into a paragon subplace s : $y = Gs$ (in such a way, we recall, that K does not change if, in its definition, $w - v$ is substituted by $\dot{G}s$).

The average molecular velocity with respect to \mathcal{R} , $\tilde{w} - v - \dot{G}s$, can be pulled back with the help of G to provide us with the ‘peculiar’ velocity c

$$c = G^{-1}(\tilde{w} - v - \dot{G}s). \quad (18)$$

Some additional remarks:

- (i) c is observer-independent; any rotation of the observer does not influence the reading of c .
- (ii) The choices of s and c are such that, not only $\int_{\epsilon} \tilde{\theta}s = 0$, $\int_{\epsilon} \tilde{\theta}c = 0$, but also

$$\int_{\epsilon} \tilde{\theta}s \otimes c = 0. \quad (19)$$

- (iii) Those choices make the integral

$$\int_{\epsilon} \tilde{\theta}(\tilde{w} - v - By)^2 \quad (20)$$

a minimum; thus, also in this sense, the option suggested for B is best fitting.

A crucial precondition for progress is to make it clear, even if repetitive, that the spatial segment ϵ is meant to be interpreted as the instantaneous placement of a fictitious material speck which translates with the velocity v and deforms affinely with a rate directed by B ; the placement ϵ derives from

another changeless fictional placement \mathbf{e}^* , with the central assumption that the transplacement from \mathbf{e}^* to \mathbf{e} preserves mass:

$$\overline{\mu\theta \det G} = 0. \quad (21)$$

A decisive corollary ensues: if \mathcal{G} is a sufficiently regular function of z and τ , then

$$\left(\int_{\mathbf{e}} \tilde{\theta} \mathcal{G} \right) \dot{} = \int_{\mathbf{e}} \tilde{\theta} \dot{\mathcal{G}}. \quad (22)$$

Hence, in particular, $\dot{\omega} = 0$.

The assumption (21) is compatible with macroscopic mass balance because molecules of one speck may protrude into and from neighbouring specks. Inside \mathbf{e} the agitation of the molecules is described only within the limits allowed by the assignment of the field \tilde{w} . A global estimate of the intensity of agitation at x (i.e., within \mathbf{e}) is offered by the tensor H :

$$H = \omega^{-1} \int_{\mathbf{e}} \tilde{\theta} (\tilde{w} - v - \dot{G}s) \otimes (\tilde{w} - v - \dot{G}s) = \omega^{-1} G \left(\int_{\mathbf{e}} \tilde{\theta} c \otimes c \right) G^T; \quad (23)$$

notice that

$$\tilde{W} = BYB^T + H. \quad (24)$$

Exploiting the shadow kinetics, one finds that

$$\begin{aligned} \dot{Y} &= \left(\omega^{-1} \int_{\mathbf{e}} \tilde{\theta} y \otimes y \right) \dot{} = \omega^{-1} \int_{\mathbf{e}} \overline{\dot{\theta}(y \otimes y)} = \\ &= \omega^{-1} \int_{\mathbf{e}} \tilde{\theta} (w \otimes y + y \otimes w) = K^T + K = BY + YB^T. \end{aligned} \quad (25)$$

On the other hand from the equation for G above one gets also

$$(GG^T) \dot{} = BGG^T + GG^TB^T. \quad (26)$$

Thus the ‘strain’ GG^T satisfies the same condition required of Y ; choosing the arbitrary factor so as to adjust also dimensions one is led to the identification

$$Y = (\text{meas } \mathbf{e})^{\frac{2}{3}} GG^T. \quad (27)$$

Y can be interpreted as an intrinsic metric at x and, ultimately at all points occupied by the body.

3 Straining and Allied Notions

It may be argued that our entire analysis balances precariously on the razor edge of ingrained ambiguities tied with the simultaneous concerns with two scales; misconceptions must be prevented already with regards to the notion of straining.

Above the tensor G was sought from knowledge of B ; likewise the formal construction of the placement gradient F can be effected. However, whereas the former rendition is conceived strictly within ϵ , the latter demands knowledge of v over all elements in the immediate gross neighbourhood of x , so that $L = \text{grad}_x v$ be available. Then F can be sought as a solution of $\dot{F} = FL$ and there is no geometric reason for F to be conditioned by G , nor, of course, L by B . Furthermore, within ϵ , one can evaluate $\text{grad}_y \tilde{w}$:

$$\text{grad}_y \tilde{w} = B + G(\text{grad}_y c), \quad (28)$$

yet another distinct tensor, which averaged over ϵ

$$\omega^{-1} \int_{\epsilon} \tilde{\theta} \text{grad}_y \tilde{w} = B + \omega^{-1} G \left(\int_{\epsilon} \tilde{\theta} \text{grad}_y c \right) \quad (29)$$

leads to a new field over \mathcal{B} , say $J(x, \tau)$, which assesses a sort of micro-stretching and spin evoked from the molecular maelstrom and to be, possibly, attributed to ϵ .

Thus B , chosen to estimate most fittingly the relative kinetic energy is not quite as successful in matching average micro straining. Of course, the additional term might still vanish or, at least, amount to little and thus be negligible; only the scrutiny of many special instances will offer evidence one way or another. Rewriting the correction to B in the form

$$\omega^{-1} G \left(\int_{\epsilon} \tilde{\theta} \text{grad}_y c \right) = \omega^{-1} G \left(\int_{\partial \epsilon} \tilde{\theta} c \otimes n - \int_{\epsilon} c \otimes \text{grad}_y \tilde{\theta} \right) \quad (30)$$

(n the normal to $\partial \epsilon$) evidences a contribution due to a flux through $\partial \epsilon$ and one due to a rearrangement within ϵ .

Below attention is focused on the requited rôle of F versus G or of L versus B . In a sense, G may be envisioned to account for:

- (i) The influence within the element of the macrostretch F , plus
- (ii) The rearrangement of molecules within the macrostretched element insofar as a crowding near the centre implies a smaller moment of inertia than a crowding at the periphery, and
- (iii) The protrusion of molecules beyond the element bounds after they are expanded by the macrostretch and insofar as they can be accounted for affinely.

Above the concepts of stretch are, of course, virtual as quantities derived from an irregularly evolving reality. They might, nevertheless, take up direct capacity within some ensuing developments; then the formal splitting of G into the product of GF^{-1} by F (or, rather, of F by $F^{-1}G$) separates nominally the outcome of action (i) from the other two; to the combined effect of the latter the contribution of (ii) could be measured by GG^T though such choice includes consequences of protrusion proper (though excluding, however, the effects mentioned in the previous paragraph).

In an essay on perfect pseudofluids [2], the following strain characteristics were invoked

$$C = F^T F, \quad N = G^T G, \quad X = G^{-1} F, \quad (31)$$

leading to the rates

$$\dot{C} = 2F^T (\text{sym}L) F, \quad \dot{N} = 2G^T (\text{sym}B) G, \quad \dot{X} = G^{-1} (L - B) F. \quad (32)$$

Notice that \dot{X} is not independent of \dot{C} , \dot{N} . Thus, strictly, X is not the appropriate characteristic to pool with C and N ; rather that rôle could be properly taken by

$$Q = R'^T R, \quad (33)$$

where R and R' are the orthogonal tensors associated with F and G respectively, with

$$F = RC^{\frac{1}{2}}, \quad G = R'N^{\frac{1}{2}}. \quad (34)$$

In fact,

$$\dot{Q} = R'^T \left(\dot{R}R^T - \dot{R}'R'^T \right) R = R'^T (\text{skw}L - \text{skw}B) R \quad (35)$$

is evidently independent of \dot{C} , \dot{N} .

Protrusion does not necessarily mean loss or gain of molecules: in an element number density may easily be balanced by intrusion from neighbouring elements. Thus a discrepancy between F and G by itself is insufficient to imply mass variation, it might simply give a hint as to the extent of interpenetration. A scalar measure of the latter could be the different change of volume attributed by F and G : $\alpha = \det (FG^{-1})$, leading to the rate

$$\dot{\alpha} = \alpha \text{tr} (L - B). \quad (36)$$

Rather, it is only in the presence of a relatively steep gradient of α or, more generally, of X that protrusion implies deviant features. Thus that gradient enters necessarily among descriptive variables, perhaps through associated quantities, such as wryness, torsion, Burgers' vector, but also 'extra matter'.

Strain measures like C , N and Q appear inappropriate when addressing phenomena in fluids; in fact one may deem the bare pull-back linked to F , or G , as artificial; although a reference state could still be imagined: e.g., one where molecules are distributed homogenously within the element at some standard number density. Also, one must not disregard the opportunity offered apparently by those strain measures, to compare and contrast models of semisolids subject to 'configurational' changes, i.e. to mutations of background.

Strictly, when seeking theories for fluids, one should rather evidence measures bearing only on the current state such as the metrics

$$\tilde{C} = FF^T \quad \text{and} \quad \tilde{N} = (\text{meas } \epsilon)^{-\frac{2}{3}} Y = GG^T, \quad (37)$$

the rotation

$$\tilde{Q} = R'R^T, \quad (38)$$

a *wryness* w defined as the gradient of GF^{-1} , the consequent *torsion* h , dislocation density and Burgers' vector b (relative to any plane of normal n) given by

$$w = \text{grad}(GF^{-1}), \quad h = \frac{1}{2}(w - w^t), \quad b = (eh^T)n, \quad (39)$$

where the exponents t and T mean minor right transposition and major transposition respectively in the third-order tensors w and h , e is Ricci's permutation tensor. The common invariants of all those tensors have then a crucial rôle to play.

4 Balance Laws

The scenario promoted in the previous sections evidences within the region \mathcal{B} , once totals over each ϵ are affected, the substratum provided by the fields of gross density ρ and moment of inertia Y

$$\rho = \frac{\mu}{(\text{meas } \epsilon)} \int_{\epsilon} \tilde{\theta}, \quad Y = \omega^{-1} \int_{\epsilon} \tilde{\theta}y \otimes y \quad (40)$$

and, later, the kinematic fields v , B and H

$$v = \omega^{-1} \int_{\epsilon} \tilde{\theta}\tilde{w}, \quad B = \left(\int_{\epsilon} \tilde{\theta}\tilde{w} \otimes y \right) \left(\int_{\epsilon} \tilde{\theta}y \otimes y \right)^{-1}, \quad (41)$$

$$H = \omega^{-1} \int_{\epsilon} \tilde{\theta}(\tilde{w} - v - By) \otimes (\tilde{w} - v - By). \quad (42)$$

Thus, the intention is not to press the depth of description of events in the body down to the details of the distribution θ (or, yet less deeply, $\tilde{\theta}$) but to stop at the stage set by those fields. Further, one expects that the evolution of the latter be ruled by balance laws also lingering at their level, hence involving, on the one hand, the time derivatives of v , B (or, better, K), H and, on the other hand, totals over ϵ of impact and/or bonding effects be those intimate (or close, i.e. among subspecks within ϵ), internal (among distinct specks), external to the body. Such totals per unit mass are formally expressed by the integrals

$$\omega^{-1} \int_{\epsilon} \tilde{\theta}g^c, \quad \omega^{-1} \int_{\epsilon} \tilde{\theta}g^i, \quad \omega^{-1} \int_{\epsilon} \tilde{\theta}g^e \quad (43)$$

and could, in principle, be given substance once a collision/coherence operator (as occurs in Boltzmann equation) were known. Be that as it may, the presumption below is that \tilde{w} equals the sum $g^c + g^i + g^e$ as per Newton law.

Disregarding, as said above, possible deeper inhomogeneities (which would be gauged by $\tilde{\theta}(\tau, z)$ and would be related to *div* \tilde{w}), *conservation of mass* is invoked by the standard law

$$\dot{\rho} + \rho \operatorname{div} v = 0. \tag{44}$$

What could be called *law of conservation of moment of inertia* was already written (see (27)) and follows immediately from the definition (40)₂

$$\dot{Y} = 2 \operatorname{sym} K. \tag{45}$$

Because totals of intimate interactions vanish, *conservation of momentum* embodied by (see (41)₁ and (43))

$$\rho \dot{v} = \rho \omega^{-1} \int_{\epsilon} \tilde{\theta} (g^i + g^e), \tag{46}$$

might take the usual form

$$\rho \dot{v} = \rho b + \operatorname{div} T, \tag{47}$$

though here one should justify anew the presumption that external actions sum up into a functional absolutely continuous with gross volume, whereas internal actions obey Cauchy’s assertions.

Conservation of moment of momentum follows from the definition of K (see (12) and, again, (43))

$$\dot{K} = \omega^{-1} \int_{\epsilon} \tilde{\theta} \tilde{w} \otimes \tilde{w} + \omega^{-1} \int_{\epsilon} \tilde{\theta} y \otimes (g^c + g^i + g^e), \tag{48}$$

from the link (24) and the property (25)

$$\omega^{-1} \int_{\epsilon} \tilde{\theta} \tilde{w} \otimes \tilde{w} = \omega^{-1} \int_{\epsilon} \tilde{\theta} (\dot{G}s + Gc) \otimes (\dot{G}s + Gc) = BK + H, \tag{49}$$

with the conclusion

$$\dot{K} - BK - H = \omega^{-1} \int_{\epsilon} \tilde{\theta} y \otimes (g^c + g^i + g^e). \tag{50}$$

Notation introduced in earlier papers could be called upon

$$M = \omega^{-1} \int_{\epsilon} \tilde{\theta} y \otimes g^e, \quad A = -\frac{1}{\rho} \int_{\epsilon} \tilde{\theta} y \otimes g^c. \tag{51}$$

No impelling case, but analogy and convenience, is yet available to declare that the third addendum in the right-hand side of (50) be expressible as the

divergence of a third-order tensor \mathbf{m} , the factor ρ^{-1} apart, again as used in earlier papers. But, when that is the case the next balance law reads

$$\rho \left(\dot{K} - BK - H \right) = \rho M - A + \operatorname{div} \mathbf{m}. \quad (52)$$

Finally one finds, again with reference to (43), (24), (25)

$$\dot{H} = 2 \operatorname{sym} \left[\omega^{-1} \int_{\epsilon} \tilde{\theta} \left(g^c + g^i + g^e - \dot{G}c \right) \otimes Gc \right] \quad (53)$$

or

$$\dot{H} + 2 \operatorname{sym} BH = 2 \omega^{-1} \operatorname{sym} \int_{\epsilon} \tilde{\theta} \left(g^c + g^i + g^e \right) \otimes Gc. \quad (54)$$

Again, using notation of earlier papers for tensor virials

$$S = 2 \omega^{-1} \operatorname{sym} \int_{\epsilon} \tilde{\theta} g^e \otimes Gc, \quad (55)$$

$$Z = -\frac{1}{\rho} \operatorname{sym} \int_{\epsilon} \tilde{\theta} g^c \otimes Gc, \quad (56)$$

and presuming again that also the virial of internal actions have contact character so that they be expressed as the divergence of a third-order tensor \mathbf{s} , the last balance equation takes the disguise

$$\rho \left(\dot{H} + 2 \operatorname{sym} BH \right) = \rho S - Z + \operatorname{div} \mathbf{s}. \quad (57)$$

5 Balance of Kinetic Energy

Energy has the leading rôle in the continuum discussed here. Thus it seems appropriate to assemble a few results below, even if largely mentioned elsewhere.

Within our model the kinetic energy tensor per unit mass W can be split thus

$$W = \tilde{W} + U \quad (58)$$

with a thermal contribution

$$U = \frac{1}{2} \omega^{-1} \int_{\epsilon} \int_{\mathcal{V}} \theta \left(w - \tilde{w} \right) \otimes \left(w - \tilde{w} \right) \quad (59)$$

and a properly kinetic one

$$\tilde{W} = \frac{1}{2\omega} \int_{\epsilon} \tilde{\theta} \tilde{w} \otimes \tilde{w} = \frac{1}{2\omega} \int_{\epsilon} \tilde{\theta} \left(v + \dot{G}s + Gc \right) \otimes \left(v + \dot{G}s + Gc \right), \quad (60)$$

or, see remarks at the end of Sect. 2,

$$\tilde{W} = v \otimes v + BYB^T + H. \quad (61)$$

The tensorial kinetic energy theorem follows from the balance equations (47), (52), (57) multiplying tensorially the first by v , the second by B , summing the two with the third one (divided by 2) term by term, taking the symmetric part of both sides and integrating, by parts where appropriate, over the region occupied by the body

$$\begin{aligned} \int_{\mathcal{B}} \rho \dot{\tilde{W}} &= \int_{\mathcal{B}} \rho \text{sym} \left(v \otimes f + BM + \frac{1}{2}S \right) - \\ &\quad - \int_{\mathcal{B}} \left(\text{sym} \left(\frac{1}{2}Z + LT^T \right) + BA + \mathbf{bm}^t \right) + \\ &\quad + \int_{\partial\mathcal{B}} \text{sym} \left(v \otimes Tn + B(mn) + \frac{1}{2}sn \right), \end{aligned} \quad (62)$$

where n is the unit normal vector to $\partial\mathcal{B}$, \mathbf{b} is the gradient of B , and the exponent t to \mathbf{m} indicates minor right transposition: $(\mathbf{bm}^t)_{ij} = B_{ia,b}m_{ajb}$. The central term in the right-hand side must be interpreted as the tensor power of intimate and internal actions, with densities respectively

$$-\text{sym} \left(\frac{1}{2}Z + BA \right) \quad \text{and} \quad -\text{sym} (LT^T + \mathbf{bm}^t). \quad (63)$$

Hence the density of scalar power is given by

$$-\left(\frac{1}{2}\text{tr}Z + L \cdot T + B \cdot A^T + \mathbf{b} \cdot (\mathbf{m}^t)^T \right). \quad (64)$$

The equation of balance of moment of momentum (52) does not secure here observer independence of (64), as occurs in the classical case for the vectorial version. Two observers on frames in relative motion read different values of L and B : the change in both is the addition of the same skew tensor. Hence observer independence is assured if and only if

$$\text{skw}T = \text{skw}A. \quad (65)$$

If one were to demand observer independence of the tensor power then the stronger condition

$$T = -A^T \quad (66)$$

would be required, when the tensor power would reduce to

$$-\text{sym} \left(\frac{1}{2}Z + (L - B)T^T + \mathbf{bm}^t \right). \quad (67)$$

It was already remarked in Sect. 3 that $symL$ and $symB$ can be expressed in terms of the strain rates \dot{C} and \dot{N} respectively and $skw(L - B)$ in terms of \dot{Q}

$$\begin{aligned} symL &= \frac{1}{2}F^T\dot{C}F^{-1}, & symB &= \frac{1}{2}G^T\dot{N}G^{-1}, \\ skw(L - B) &= R'\dot{Q}R^T. \end{aligned} \quad (68)$$

Longer algebra shows that

$$\mathbf{b}_{ijk} + \mathbf{b}_{jik} = G_{Bi}^{-1}\dot{\mathbf{n}}_{ABK}G_{Aj}^{-1}F_{Kk}^{-1} - \dot{N}_{AB}G_{Ba}^{-1}G_{aC,k} \left(G_{Ai}^{-1}G_{Cj}^{-1} + G_{Aj}^{-1}G_{Ci}^{-1} \right) \quad (69)$$

where $\mathbf{n} = (gradN)F$.

Thus the scalar power density can be written as an affine function of \dot{C} , \dot{N} , \dot{Q} , $\dot{\mathbf{n}}$

$$\begin{aligned} -trZ - ({}^t\mathbf{b} - \mathbf{b}) \cdot \mathbf{m} - (F^{-1}(symT)F^{-T}) \cdot \dot{C} - (G^{-1}(symA)G^{-T}) \cdot \dot{N} + \\ + \left(G_{Ai}^{-1}G_{Cj}^{-1} + G_{Aj}^{-1}G_{Ci}^{-1} \right) G_{Ba}^{-1}G_{aC,k} \mathbf{m}_{ijk} \dot{N}_{AB} - \\ - 2 [R'^T(skwT)R] \cdot \dot{Q} - G_{Ai}^{-1}G_{Bj}^{-1} \mathbf{m}_{ijk} F_{Ck}^{-1} \dot{\mathbf{n}}_{ABC}. \end{aligned} \quad (70)$$

This result suggests the possible existence of continua for which a potential $\varphi(C, N, Q, \mathbf{m})$ exists and is such that

$$symT = 2\rho F \frac{\partial \varphi}{\partial C} F^T, \quad skwT = skwA = \rho R' \frac{\partial \varphi}{\partial Q} R^T, \quad (71)$$

$$\mathbf{m}_{ijk} = 2\rho G_{iA} G_{jB} F_{kC} \frac{\partial \varphi}{\partial \mathbf{n}_{ABC}}; \quad (72)$$

thus \mathbf{m} is symmetric in the first two indices and, as a consequence, the second term in the sum (70) vanishes. The factor multiplying \dot{N}_{AB} is equal to

$$G_{Ai}^{-1}(symA)_{ij}G_{Bj}^{-1} + 2\rho G_{iR}G_{jS}F_{kT} \frac{\partial \varphi}{\partial \mathbf{n}_{RST}} \left(G_{Ai}^{-1}G_{Cj}^{-1} + G_{Aj}^{-1}G_{Ci}^{-1} \right) G_{Ba}^{-1}G_{aC,k} \quad (73)$$

and hence

$$(symA)_{ij} = 2\rho G_{iA} \frac{\partial \varphi}{\partial N_{AB}} G_{jB} - 2\rho (G_{iR}G_{jS})_k F_{kT} \frac{\partial \varphi}{\partial \mathbf{n}_{RST}}. \quad (74)$$

Finally

$$trZ = 2\rho\dot{\varphi}. \quad (75)$$

When the constitutive laws above apply, the balance equations of momentum and tensor moment of momentum acquire the rôle of evolution equations for v and B (or x and G). The rule of progress for H needs additional physical insight.

6 The First Principle

A deeper kinetic energy theorem ensues if molecular events are graded more finely inside the distribution θ rather than $\tilde{\theta}$. Then, some intriguing corollaries ensue; their deduction is barely sketched below omitting adscititious qualifications to display the essence.

Choose $\theta(z, w)h \, dw dz$ to represent the resultant of the forces acting on the molecules belonging to the immediate neighbourhood of z, w , molecules numbering $\theta dw dz$ and h to be eventually split into the sum $h^c + h^i + h^e$, as g was earlier.

Then $\dot{w} = h$ and

$$\dot{W} = \left(\frac{1}{2} \omega^{-1} \int_{\epsilon} \int_{\mathcal{V}} \theta w \otimes w \right) \dot{} = \omega^{-1} \int_{\epsilon} \int_{\mathcal{V}} \theta \text{sym}(w \otimes h). \quad (76)$$

Recall notation introduced at the beginning of Sect. 5

$$W = \tilde{W} + U. \quad (77)$$

Hence

$$\dot{\tilde{W}} + \dot{U} = \omega^{-1} \int_{\epsilon} \int_{\mathcal{V}} \theta \text{sym}(w \otimes (h - g)) + \omega^{-1} \int_{\epsilon} \int_{\mathcal{V}} \theta \text{sym}(w \otimes g); \quad (78)$$

but, from the restricted kinetic energy theorem and the appropriate interpretation of terms

$$\begin{aligned} \int_{\mathcal{B}} \rho \dot{\tilde{W}} &= \int_{\mathcal{B}} \mu (\text{meas } \epsilon)^{-1} \int_{\epsilon} \int_{\mathcal{V}} \theta \text{sym}(w \otimes g) - \\ &\quad - \int_{\mathcal{B}} \text{sym} \left(\frac{1}{2} Z + LT^T + BA + \mathbf{b}m^t \right) \end{aligned} \quad (79)$$

so that

$$\begin{aligned} \int_{\mathcal{B}} \rho \dot{U} &= \int_{\mathcal{B}} \mu (\text{meas } \epsilon)^{-1} \int_{\epsilon} \int_{\mathcal{V}} \theta \text{sym}(w \otimes (h - g)) + \\ &\quad + \int_{\mathcal{B}} \text{sym} \left(\frac{1}{2} Z + LT^T + BA + \mathbf{b}m^t \right). \end{aligned} \quad (80)$$

Finally, through the standard criterion of localization justified by the fact that the law above would equally apply when the integrals were extended to any subbody of \mathcal{B} ,

$$\begin{aligned} \rho \dot{U} &= \mu (\text{meas } \epsilon)^{-1} \int_{\epsilon} \int_{\mathcal{V}} \theta \text{sym}(w \otimes (h - g)) + \\ &\quad + \text{sym} \left(\frac{1}{2} Z + LT^T + BA + \mathbf{b}m^t \right). \end{aligned} \quad (81)$$

Such is the local equation which expresses, under the circumstances, the first principle of thermodynamics.

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