Geometric interpretation of the cumulants for random matrices previously defined as convolutions on the symmetric group

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Summary. We show that, dealing with an appropriate basis, the cumulants for $N \times N$ random matrices $(A_1, \ldots, A_n)$, previously defined in [2] and [3], are the coordinates of $\mathbb{E}\{\Pi(A_1 \otimes \cdots \otimes A_n)\}$, where Π denotes the orthogonal projection of $A_1 \otimes \cdots \otimes A_n$ on the space of invariant vectors of $\mathcal{M}_N^{\otimes n}$ under the natural action of the unitary, respectively orthogonal, group. In this way we make the connection between [5] and [2], [3]. We also give a new proof in that context of the properties satisfied by these matricial cumulants.

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1 Introduction

For any $N \times N$ complex matrix X, we have constructed matricial cumulants $(C_n^U(X))_{n \leq N}$ in [2] (resp. $(C_n^O(X))_{n \leq N}$ in [3]) such that if X, Y are $N \times N$ independent complex matrices and U (resp. O) is a Haar distributed unitary (resp. orthogonal) $N \times N$ matrix independent of X, Y, then for any $n \leq N$,

$$C_n^U(X + UYU^*) = C_n^U(X) + C_n^U(Y),$$

$$C_n^O(X + OYO^t) = C_n^O(X) + C_n^O(Y).$$

We defined the $C_n^U(X)$ (resp. $C_n^O(X)$) as the value on the single cycle $(1 \ldots n)$ of a cumulant function $C^U(X)$ (resp. $C^O(X)$) on the symmetric group $\mathcal{S}_n$

(resp. $\mathcal{S}_{2n}$) of the permutations on $\{1,\ldots,n\}$ (resp. $\{1,\ldots,n,\bar{1},\ldots,\bar{n}\}$). Note that we defined more generally cumulant functions for a n-tuple $(X_1,\ldots,X_n)$ of $N\times N$ complex matrices. The aim of this paper is to give a geometrical interpretation of the values of the cumulant function $C^U(X_1,\ldots,X_n)$ (resp. $C^O(X_1,\ldots,X_n)$). It derives from the necessary confrontation of our results with the work of Collins and Sniady on the "Integration with respect to the Haar measure on unitary, orthogonal and symplectic group", see [5].

Let us roughly explain the key ideas of this interpretation and first introduce briefly some notations. Let π be a permutation in $\mathcal{S}_n$, denote by $\mathcal{C}(\pi)$ the set of all the disjoint cycles of π and by $\gamma_n(\pi)$ the number of these cycles. Let $\varepsilon=(\varepsilon_1,\ldots,\varepsilon_n)\in\{-1,1\}^n$. We set for any n-tuple $\mathbf{X}=(X_1,\ldots,X_n)$ of $N\times N$ complex matrices

$$r_\pi(\mathbf{X})=r_\pi(X_1,\ldots,X_n):=\prod_{C\in\mathcal{C}(\pi)}\mathrm{Tr}\left(\prod_{j\in C}X_j\right). \tag{1}$$

and

$$M^{\pm}_{\mathbf{X}}(g_{(\varepsilon,\pi)}):=r_\pi(X_1^{\varepsilon_1},\ldots,X_n^{\varepsilon_n}).$$

In this last expression we set X^{-1} for the transpose X^t of the matrix X and $g_{(\varepsilon,\pi)}$ denotes some particular permutation on the symmetric group $\mathcal{S}_{2n}$ which will be made precise in Section 3.1.

These n-linear forms $r_\pi,\pi\in\mathcal{S}_n$ or $M^{\pm}(g_{(\pi,\varepsilon)}),\pi\in\mathcal{S}_n,\varepsilon\in\{-1,1\}^n$, introduced on $\mathcal{M}_N^n$ for any integer $n\geq 1$, are respectively invariant under the action of the unitary group $\mathbb{U}_N$ for the first ones and the orthogonal group $\mathbb{O}_N$ for the second ones. From the point of view of [5], they canonically define linear forms on the tensor product $\mathcal{M}_N^{\otimes n}$ which also are invariant under the corresponding action of $\mathbb{U}_N$, respectively $\mathbb{O}_N$. As $\mathcal{M}_N^{\otimes n}$ is naturally endowed with a non degenerate quadratic form $(u,v)\mapsto\langle u,v\rangle$, these linear forms correspond in the first case to vectors u_π, $\pi\in\mathcal{S}_n$, of $\mathcal{M}_N^{\otimes n}$ which are $\mathbb{U}_N$-invariant, and in the second one to vectors $u_{\eta(g_{\varepsilon,\pi})}$, $\epsilon\in\{-1;1\}^n$, $\pi\in\mathcal{S}_n$, which are $\mathbb{O}_N$-invariant (η will be defined in Section 3.3). Thus they satisfy

$$r_\pi(X_1,\ldots,X_n)=\langle X_1\otimes\ldots\otimes X_n,u_\pi\rangle$$

respectively

$$M^{+}_{\mathbf{X}}(g_{(\varepsilon,\pi)})=\langle X_1\otimes\ldots\otimes X_n,u_{\eta(g_{\varepsilon,\pi})}\rangle.$$

Actually, for $n\leq N$, $\{u_\pi\ ;\ \pi\in\mathcal{S}_n\}$ forms a basis of the space $[\mathcal{M}_N^{\otimes n}]^{\mathbb{U}_N}$ of $\mathbb{U}_N$-invariant vectors, while a basis of the space $[\mathcal{M}_N^{\otimes n}]^{\mathbb{O}_N}$ of $\mathbb{O}_N$-invariant vectors can be extracted from $\{u_{\eta(g_{\varepsilon,\pi})};\epsilon\in\{-1;1\}^n,\ \pi\in\mathcal{S}_n\}$. Note that this last one needs the double parametrization by $\mathcal{S}_n$ and some ε in $\{-1,1\}^n$. This is the reason why, contrary to the unitary case where the adjoints are not involved, the transposes of matrices naturally occur in the orthogonal case.

We then prove that our matricial cumulants $C^U(X_1,\ldots,X_n)$ (respectively $C^O(X_1,\ldots,X_n)$) are the coordinates in this appropriate basis of $\mathbb{E}\left\{\int UX_1U^*\otimes\ldots\otimes UX_nU^*dU\right\}$ (respectively $\mathbb{E}\{\int OX_1O^t\otimes\ldots\otimes OX_nO^tdO\}$), where integration is taken with respect to the Haar measure on $\mathbb{U}_N$ (resp. $\mathbb{O}_N$).

The paper is split into two parts. The first one concerns the matricial $\mathbb{U}$-cumulants and the second one is devoted to the $\mathbb{O}$-cumulants. In each part we first recall the definition and fundamental properties satisfied by these cumulants (Sections 2.1, 2.2 and similarly 3.1, 3.2). Then we describe a basis of $[\mathcal{M}_N^{\otimes n}]^G$ in each case ($G=\mathbb{U}_N$ in Section 2.3 and $G=\mathbb{O}_N$ in Section 3.3) before giving the geometrical interpretation of our cumulants and ending with a new proof in that context of the properties they satisfy (Sections 2.4 and 3.4).

Note that the same development as for the orthogonal group can be carried out for the symplectic group $Sp(N)$. We just provide the corresponding basis of Sp-invariant vectors of $\mathcal{M}_N^{\otimes n}$ in the final section without giving more details.

Throughout the paper, we suppose $N\geq n$.

Before starting we would like to underline that the description of the subspace of invariant vectors relies on the following ideas. Note this first simple remark:

Lemma 1.1 *Let G and G' be two groups acting on a vector space V through the actions ρ and ρ' and let $[V]^G$ denote the subspace of G-invariant vectors of V. Then, when ρ and ρ' commute, for any vector $v\neq 0$ in $[V]^G$, $\{\rho'(g')\cdot v\ ;\ g'\in G'\}\subset [V]^G$.*

Hence $[V]^G$ is known as soon as we can find a suitable group G' and some vector v in $[V]^G$ for which we get $\{\rho'(g')\cdot v\ ;g'\in G'\}=[V]^G$. For the considered groups, the Schur-Weyl duality leads to the right G'. Thus for $G=GL(N,\mathbb{C})$ and $\mathbb{U}_N$, G' is chosen to be equal to $\mathcal{S}_n$. For $G=\mathbb{O}_N$ or $Sp(N)$, G' is $\mathcal{S}_{2n}$. This is well described in [8], see Theorem 4.3.1 for $GL(N,\mathbb{C})$ and Theorem 4.3.3 or Proposition 10.1.1 for $\mathbb{O}_N$ and $Sp(N)$. As for $\mathbb{U}_N$, note that any analytic function invariant by $\mathbb{U}_N$ is invariant by $GL(N,\mathbb{C})$ too (see Weyl's Theorem about analytic functions on $GL(N,\mathbb{C})$, [9]). For any $u\in[\mathcal{M}_N^{\otimes n}]^{\mathbb{U}_N}$, the analytic function on V, $A\mapsto\langle A,u\rangle$ is $\mathbb{U}_N$-invariant, hence is $GL(N,\mathbb{C})$-invariant. Thus, for any A in $\mathcal{M}_N^{\otimes n}$ and any $G\in GL(N,\mathbb{C})$, $\langle A,u\rangle=\langle A,G^{-1}uG\rangle$ and hence $u\in[\mathcal{M}_N^{\otimes n}]^{GL(N,\mathbb{C})}$. It readily comes that $[\mathcal{M}_N^{\otimes n}]^{GL(N,\mathbb{C})}=[\mathcal{M}_N^{\otimes n}]^{\mathbb{U}_N}$.

2 Matricial $\mathbb{U}$-cumulants

We refer the reader to [2] where the present section is developed and we just recall here the fundamental results.

2.1 Definition and first properties

Denote by $*$ the classical convolution operation on the space of complex functions on $\mathcal{S}_n$,

$$f * g(\pi) = \sum_{\sigma \in \mathcal{S}_n} f(\sigma) g(\sigma^{-1}\pi) = \sum_{\rho \in \mathcal{S}_n} f(\pi\rho^{-1}) g(\rho),$$

and by id the identity of $\mathcal{S}_n$. Recall that the $*$-unitary element is

$$\delta_{id} := \pi \to \begin{cases} 1 \text{ if } \ \pi = id \\ 0 \text{ else} \end{cases},$$

that is $f * \delta_{id} = \delta_{id} * f = f$ for all f. The inverse function of f for $*$, if it exists, is denoted by $f^{(-1)}$ and satisfies $f * f^{(-1)} = f^{(-1)} * f = \delta_{id}$. In particular the function $\pi \mapsto x^{\gamma_n(\pi)}$ is $*$-invertible for $n-1 < |x|$ (see [6]). Moreover, since γ_n is central (that is, constant on the conjugacy classes), x^{γ_n} and thus $(x^{\gamma_n})^{(-1)}$ commute with any function f defined on $\mathcal{S}_n$.
Recall the definition of the $\mathbb{U}$-cumulants introduced in [2].

Definition 2.1 *For $n \leq N$, for any n-tuple $\mathbf{X} = (X_1, \ldots, X_n)$ of random $N \times N$ complex matrices, the n-th $\mathbb{U}$-cumulant function $C^U(\mathbf{X}) : \mathcal{S}_n \to \mathbb{C}$, $\pi \mapsto C^U_\pi(\mathbf{X})$ is defined by the relation*

$$C^U(\mathbf{X}) := \mathbb{E}(r(\mathbf{X})) * (N^{\gamma_n})^{(-1)}.$$

The $\mathbb{U}$-cumulants of $\mathbf{X}$ are the $C^U_\pi(\mathbf{X})$ for single cycles π of $\mathcal{S}_n$.
For a single matrix X, $C^U(\mathbf{X})$ where $\mathbf{X} = (X, \cdots, X)$ will be simply denoted by $C^U(X)$.

For example, if $\mathrm{tr}_N = \frac{1}{N}\mathrm{Tr}$,

$$C^U_{(1)}(X) = \mathbb{E}(\mathrm{tr}_N(X))$$

$$C^U_{(1)(2)}(X_1, X_2) = \frac{N\mathbb{E}\{\mathrm{Tr}(X_1)\mathrm{Tr}(X_2)\} - \mathbb{E}\{\mathrm{Tr}(X_1X_2)\}}{N(N^2-1)}$$

$$C^U_{(1\,2)}(X_1, X_2) = \frac{-\mathbb{E}\{\mathrm{Tr}(X_1)\mathrm{Tr}(X_2)\} + N\mathbb{E}\{\mathrm{Tr}(X_1X_2)\}}{N(N^2-1)}.$$

Here are some basic properties remarked in [2]. First, for each π in $\mathcal{S}_n$, $(X_1, \ldots, X_n) \mapsto C^U_\pi((X_1, \ldots, X_n))$ is obviously n-linear. Moreover it is clear that for any unitary matrix U,

$$C^U_\pi(U^*X_1U, \ldots, U^*X_nU) = C^U_\pi(X_1, \ldots, X_n).$$

Now,

1. For any π and σ in $\mathcal{S}_n$,

$$C_\pi^U((X_{\sigma(1)},\ldots,X_{\sigma(n)})) = C_{\sigma\pi\sigma^{-1}}^U((X_1,\ldots,X_n)). \tag{2}$$

2. $C_\pi^U(X)$ depends only of the conjugacy class of π.

Thus the cumulants $C_\pi^U(X)$ of a matrix X for single cycles π of $\mathcal{S}_n$ are all equal so that we denote by $C_n^U(X)$ this common value. We call it *cumulant of order n* of the matrix X. In particular, $C_1^U(X) = \mathbb{E}(\mathrm{tr}_N X)$ and $C_2^U(X) = \frac{N}{N^2-1}\left[\mathbb{E}\{\mathrm{tr}_N(X^2)\} - \mathbb{E}\{(\mathrm{tr}_N X)^2\}\right]$. We also proved the following

Proposition 2.1 *For any $k < n \leq N$, any π in $\mathcal{S}_n$, then*

$$C_\pi^U(X_1,\ldots,X_k,I_N,\ldots,I_N) = \begin{cases} C_\rho^U(X_1,\ldots,X_k) & \text{if } \pi = (n)\ldots(k+1)\rho \text{ for some } \rho\in\mathcal{S}_k, \\ 0 & \text{else.}\end{cases}$$

Now recall the fundamental properties we proved in [2] and which motivated the terminology of cumulants.

2.2 Fundamental properties

2.2.1 Mixed moments of two independent tuples

In [2] we have proved the following theorem with great analogy with the results of [10] about the multiplication of free n-tuples.

Theorem 2.1 *Let $\mathbf{X} = (X_1,\ldots,X_n)$ and $\mathbf{B} = (B_1,\ldots,B_n)$ two independent n-tuple of $N\times N$ random complex matrices such that the distribution of $\mathbf{X}$ is invariant under unitary conjugations, namely $\forall U \in \mathbb{U}_N$, $\mathcal{L}(UX_1U^*,\ldots,UX_nU^*) = \mathcal{L}(X_1,\ldots,X_n)$. Then we have for any π in $\mathcal{S}_n$:*

$$\mathbb{E}\left(r_\pi(B_1X_1,\ldots,B_nX_n)\right) = \{\mathbb{E}(r(\mathbf{B})) * C^U(\mathbf{X})\}(\pi) = \{C^U(\mathbf{B}) * \mathbb{E}(r(\mathbf{X}))\}(\pi)$$

From Theorem 2.1 we readily get the following convolution relation which has to be related to Theorem 1.4 in [10].

Corollary 2.1 *With the hypothesis of Theorem 2.1,*

$$C^U(X_1B_1,\ldots,X_nB_n) = C^U(\mathbf{X}) * C^U(\mathbf{B}).$$

If $\mathbf{X} = (X_1,\ldots,X_n)$ and $\mathbf{B} = (B_1,\ldots,B_n)$ are two independent n-tuple of $N\times N$ random complex matrices such that the distribution of $\mathbf{X}$ is invariant under orthogonally conjugations, namely $\forall O \in \mathbb{O}_N$, $\mathcal{L}(OX_1O^t,\ldots,OX_nO^t) = \mathcal{L}(X_1,\ldots,X_n)$, the mixed moments $\mathbb{E}\left(r_\pi(B_1X_1,\ldots,B_nX_n)\right)$ can still be expressed by a convolution relation but on $\mathcal{S}_{2n}$; consequently we were led to introduce in [3] another cumulant function $C^O : \mathcal{S}_{2n} \to \mathbb{C}$, recalled in Section 3.

2.2.2 Linearizing property

Proposition 2.1 together with Corollary 2.1 imply that the cumulants $C_n^U(X_1, \ldots, X_n)$ vanish as soon as the involved matrices $(X_1, \ldots, X_n)$ are taken in two independent sets, one having distribution invariant under unitary conjugation; therefore they do linearize the convolution, namely if X_1, X_2 are two independent matrices such that $\mathcal{L}(UX_1U^*) = \mathcal{L}(X_1)$, $\forall U \in \mathbb{U}_N$, then

$$C_n^U(X_1 + X_2) = C_n^U(X_1) + C_n^U(X_2).$$

2.2.3 Asymptotic behavior

We refer the reader to [12] for noncommutative probability space and freeness and to [11] and [10] for free cumulants. Let $(\mathcal{A}, \Phi)$ be a noncommutative probability space. For any noncommutative random variables $(a_1, \ldots, a_n)$ in $(\mathcal{A}, \Phi)$ and for any $\pi = \prod_{i=1}^r \pi_i$ in $\mathcal{S}_n$ with $\pi_i = (l_{i,1}, l_{i,2}, \ldots, l_{i,n_i})$, we write

$$\phi_\pi(a_1, \ldots, a_n) := \prod_{i=1}^r \phi(a_{l_{i,1}} a_{l_{i,2}} \cdots a_{l_{i,n_i}}),$$

$$k_\pi(a_1, \ldots, a_n) := \prod_{i=1}^r k_{n_i}(a_{l_{i,1}}, a_{l_{i,2}}, \ldots, a_{l_{i,n_i}}),$$

where $(k_n)_{n\in\mathbb{N}}$ stand for the free cumulants. For any n-tuple $(X_1, \ldots, X_n)$ of $N \times N$ matrices, we define the *normalized generalized moments* $\mathbb{E}(r_\pi^{(N)}(X_1, \ldots, X_n))$ where π is in $\mathcal{S}_n$ by setting

$$\mathbb{E}(r_\pi^{(N)}(X_1, \ldots, X_n)) = \frac{1}{N^{\gamma_n(\pi)}} \mathbb{E}(r_\pi(X_1, \ldots, X_n)) = \mathbb{E}\left(\prod_{C \in \mathcal{C}(\pi)} \frac{1}{N} \mathrm{Tr}\big(\prod_{j \in C} X_j \big) \right).$$

We also define the normalized cumulants by

$$(C_\pi^U)^{(N)}(X_1, \ldots, X_n) := N^{n-\gamma_n(\pi)} C_\pi^U(X_1, \ldots, X_n).$$

In [2] we prove the following equivalence.

Proposition 2.2 *Let $(X_1, \ldots, X_n)$ be a n-tuple of $N \times N$ matrices. Let $(x_1, \ldots, x_n)$ be non commutative variables in $(\mathcal{A}, \phi)$. The following equivalence holds,*

$$\mathbb{E}(r_\pi^{(N)}(X_1, \ldots, X_n)) \underset{N \to +\infty}{\longrightarrow} \phi_\pi(x_1, \ldots, x_n), \quad \forall\, \pi \in \mathcal{S}_n$$

$$\Leftrightarrow (C_\pi^U)^{(N)}(X_1, \ldots, X_n) \underset{N \to +\infty}{\longrightarrow} k_\pi(x_1, \ldots, x_n), \quad \forall\, \pi \in \mathcal{S}_n.$$

2.3 Action of the unitary group on the space of complex matrices

We first need to precisely state some basic generalities and notations. Let $(e_1, \dots, e_N)$ be the canonical basis of $\mathbb{C}^N$. Endow $\mathbb{C}^N$ with the usual Hermitian product $\langle \sum_i u_i e_i, \sum_i v_i e_i \rangle_{\mathbb{C}^N} = \sum_i u_i \overline{v_i}$. Thus the dual space $(\mathbb{C}^N)^*$ is composed by the linear forms $v^* : \mathbb{C}^N \to \mathbb{C}, u \mapsto \langle u, v \rangle_{\mathbb{C}^N}$ with $v \in \mathbb{C}^N$. Let $(e_1^*, \dots, e_N^*)$ be the dual basis. First consider the tensor product $\mathbb{C}^N \otimes (\mathbb{C}^N)^*$ with orthonormal basis $e_i \otimes e_j^*$, $i, j = 1, \dots, N$ with respect to the Hermitian product

$$\langle u_1 \otimes v_1^*, u_2 \otimes v_2^* \rangle_{\mathbb{C}^N \otimes (\mathbb{C}^N)^*} = \langle u_1, u_2 \rangle_{\mathbb{C}^N} \langle v_2, v_1 \rangle_{\mathbb{C}^N}.$$

The unitary group $\mathbb{U}_N$ acts on $\mathbb{C}^N \otimes (\mathbb{C}^N)^*$ as follows:

$$\widetilde{\rho}(U)(e_i \otimes e_j^*) = Ue_i \otimes (Ue_j)^*.$$

Now consider $\mathcal{M}_N$ with canonical basis $(E_{a,b})_{a,b=1,\dots,N}$ defined by $(E_{a,b})_{ij} = \delta_{a,i}\delta_{b,j}$, and with Hermitian product $\langle A, B \rangle_{\mathcal{M}_N} = \mathrm{Tr}(AB^*)$. It is well-known that $\mathcal{M}_N$ and $\mathbb{C}^N \otimes (\mathbb{C}^N)^*$ are isomorphic Hermitian vector spaces when we identify any $M = (M_{ij})_{1 \le i,j \le N} \in \mathcal{M}_N$ with $\tilde{M} = \sum_{1 \le i,j \le N} M_{ij} e_i \otimes e_j^*$ (and hence $\tilde{E}_{a,b} = e_a \otimes e_b^*$). Besides the action $\widetilde{\rho}$ corresponds on $\mathcal{M}_N$ to

$$\rho(U)(M) = UMU^*.$$

Note also that the inner product AB in $\mathcal{M}_N$ corresponds to the product defined by

$$(u_1 \otimes v_1^*).(u_2 \otimes v_2^*) = \langle u_2, v_1 \rangle_{\mathbb{C}^N} \; u_1 \otimes v_2^*,$$

and the adjoint A^* to the following rule: $(u \otimes v^*)^* = v \otimes u^*$.

More generally, for any n, the tensor products $\mathcal{M}_N^{\otimes n}$ and $(\mathbb{C}^N \otimes (\mathbb{C}^N)^*)^{\otimes n}$ are isomorphic through the map: $A = A_1 \otimes \cdots \otimes A_n \mapsto \tilde{A} = \tilde{A}_1 \otimes \cdots \otimes \tilde{A}_n$ and with Hermitian product

$$\begin{aligned} &\langle A_1 \otimes \cdots \otimes A_n, B_1 \otimes \cdots \otimes B_n \rangle_{\mathcal{M}_N^{\otimes n}} \\ &= \prod_{i=1}^n \mathrm{Tr}(A_i B_i^*) = \prod_{i=1}^n \langle \widetilde{A}_i, \widetilde{B}_i \rangle_{\mathbb{C}^N \otimes (\mathbb{C}^N)^*} \\ &= \langle \widetilde{A}_1 \otimes \cdots \otimes \widetilde{A}_n, \widetilde{B}_1 \otimes \cdots \otimes \widetilde{B}_n \rangle_{(\mathbb{C}^N \otimes (\mathbb{C}^N)^*)^{\otimes n}}. \end{aligned}$$

Here again the following actions of $\mathbb{U}_N$ are equivalent:

$$\begin{aligned} &\text{on } (\mathbb{C}^N \otimes (\mathbb{C}^N)^*)^{\otimes n} \quad \widetilde{\rho}_n(U)(e_{i_1} \otimes e_{i_{\bar{1}}}^* \otimes \cdots \otimes e_{i_n} \otimes e_{i_{\bar{n}}}^*) \\ &\qquad\qquad = Ue_{i_1} \otimes (Ue_{i_{\bar{1}}})^* \otimes \cdots \otimes Ue_{i_n} \otimes (Ue_{i_{\bar{n}}})^*, \\ &\text{on } \mathcal{M}_N^{\otimes n} \qquad\qquad \rho_n(U)(A_1 \otimes \cdots \otimes A_n) = UA_1U^* \otimes \cdots \otimes UA_nU^*. \end{aligned}$$

Denote by $[V]^{\mathbb{U}_N}$ the subspace of $\mathbb{U}_N$-invariant vectors of V with $V = \mathcal{M}_N^{\otimes n}$ or $(\mathbb{C}^N \otimes (\mathbb{C}^N)^*)^{\otimes n}$. Clearly $[\mathcal{M}_N^{\otimes n}]^{\mathbb{U}_N}$ and $[(\mathbb{C}^N \otimes (\mathbb{C}^N)^*)^{\otimes n}]^{\mathbb{U}_N}$ are isomorphic too. Consequently from now on we identify $\mathcal{M}_N^{\otimes n}$ and $(\mathbb{C}^N \otimes (\mathbb{C}^N)^*)^{\otimes n}$. We also simply denote the Hermitian product by $\langle .,. \rangle$ from now on throughout Section 2. Note lastly that the inner product in $\mathcal{M}_N^{\otimes n}$ is defined by

$$(A_1 \otimes \cdots \otimes A_n).(B_1 \otimes \cdots \otimes B_n) = A_1 B_1 \otimes \cdots \otimes A_n B_n,$$

and the adjunction by $(A_1 \otimes \cdots \otimes A_n)^* = A_1^* \otimes \cdots \otimes A_n^*$. They satisfy for any $u, v, w \in \mathcal{M}_N^{\otimes n}$:

$$\langle u.v, w \rangle = \langle v, u^*.w \rangle = \langle u, w.v^* \rangle. \tag{3}$$

In the following proposition we determine a basis of $[\mathcal{M}_N^{\otimes n}]^{\mathbb{U}_N}$. We use the previous identification in the proof.

Proposition 2.3 *For any permutation σ in $\mathcal{S}_n$, define*

$$u_\sigma := \sum_{i_1,\ldots,i_n} E_{i_{\sigma^{-1}(1)} i_1} \otimes \cdots \otimes E_{i_{\sigma^{-1}(n)} i_n}.$$

Then $\{u_\sigma \ ; \ \sigma \in \mathcal{S}_n\}$ generates $[\mathcal{M}_N^{\otimes n}]^{\mathbb{U}_N}$. Moreover when $N \geq n$, it is a basis of $[\mathcal{M}_N^{\otimes n}]^{\mathbb{U}_N}$.

Proof: The first part of Proposition 2.3 derives from Theorem 4.3.1 in [8]. We briefly recall how this set is introduced before showing that it forms a basis of $[\mathcal{M}_N^{\otimes n}]^{\mathbb{U}_N}$. We work on $(\mathbb{C}^N \otimes (\mathbb{C}^N)^*)^{\otimes n}$ where we consider another group action and a specific invariant vector in order to apply lemma 1.1. Define

$$\Theta_n := \underbrace{I_N \otimes \ldots \otimes I_N}_{n \text{ times}} = \sum_{i_1,\ldots,i_n} e_{i_1} \otimes e_{i_1}^* \otimes \cdots \otimes e_{i_n} \otimes e_{i_n}^*.$$

It is clear that $\Theta_n \in [\mathcal{M}_N^{\otimes n}]^{\mathbb{U}_N}$. Consider now the natural action ρ' of $\mathcal{S}_n \times \mathcal{S}_n$ on $(\mathbb{C}^N \otimes (\mathbb{C}^N)^*)^{\otimes n}$ defined for any permutations σ and τ in $\mathcal{S}_n$ acting respectively on $\{1, \ldots, n\}$ and $\{\bar{1}, \ldots, \bar{n}\}$ by

$$\begin{aligned}\rho'((\sigma,\tau))&(e_{i_1} \otimes e_{i_{\bar{1}}}^* \otimes \cdots \otimes e_{i_n} \otimes e_{i_{\bar{n}}}^*)\\ &= e_{i_{\sigma^{-1}(1)}} \otimes e_{i_{\tau^{-1}(\bar{1})}}^* \otimes \cdots \otimes e_{i_{\sigma^{-1}(n)}} \otimes e_{i_{\tau^{-1}(\bar{n})}}^*.\end{aligned}$$

The actions $\widetilde{\rho}_n$ and ρ' obviously commute. Hence, according to Lemma 1.1, for all (σ, τ) in $\mathcal{S}_n \times \mathcal{S}_n$, $\rho'((\sigma,\tau)) \cdot \Theta_n$ belongs to $[\mathcal{M}_N^{\otimes n}]^{\mathbb{U}_N}$. Note that, since $(\sigma,\tau) = (\sigma\tau^{-1}, id)(\tau,\tau)$ and $\rho'((\tau,\tau)) \cdot \Theta_n = \Theta_n$, then

$$\{\rho'((\sigma,\tau)) \cdot \Theta_n \ ; \ (\sigma,\tau) \in \mathcal{S}_n \times \mathcal{S}_n\} = \{\rho'((\sigma, id)) \cdot \Theta_n \ ; \ \sigma \in \mathcal{S}_n\}.$$

Thus we simply denote $\rho'((\sigma, id))$ by $\rho'(\sigma)$ and we set

$$u_\sigma = \rho'(\sigma) \cdot \Theta_n.$$

Remark that $u_{id} = \Theta_n$. Note also that u_σ corresponds to $\rho_{\mathcal{S}_n}^N(\sigma)$ in [5].

From Theorem 4.3.1 in [8], the set $\{\rho'(\sigma)\cdot\Theta_n \ ; \ \sigma \in \mathcal{S}_n\}$ generates $[\mathcal{M}_N^{\otimes n}]^{GL(N,\mathbb{C})} = [\mathcal{M}_N^{\otimes n}]^{\mathbb{U}_N}$ (see [9]). We now prove that it is a basis when $N \geq n$.

One can easily see that the adjoint of $\rho'((\sigma,\tau))$ satisfies $\rho'((\sigma,\tau))^* = \rho'((\sigma^{-1},\tau^{-1}))$ so that

$$\langle u_\sigma, u_{\sigma'}\rangle = \langle \Theta_n, u_{\sigma^{-1}\sigma'}\rangle = \langle u_{\sigma'^{-1}\sigma}, \Theta_n\rangle \ .$$

Now from (1) we get:

$$\begin{aligned}\langle \Theta_n, u_\sigma\rangle &= \sum_{i_1,\dots,i_n}\prod_{l=1}^{n}\delta_{i_l,i_{\sigma(l)}} = \sum_{i_1,\dots,i_n} r_\sigma(E_{i_1,i_1},\dots,E_{i_n,i_n})\\ &= r_\sigma(I_N,\dots,I_N) = N^{\gamma(\sigma)},\end{aligned}$$

so that

$$\langle u_\sigma, u_{\sigma'}\rangle = N^{\gamma(\sigma^{-1}\sigma')}.$$

Let $G = (\langle u_\sigma, u_{\sigma'}\rangle)_{\sigma,\sigma'\in\mathcal{S}_n\times\mathcal{S}_n}$ be the Gramm matrix of $\{u_\sigma \ ; \ \sigma\in\mathcal{S}_n\}$. Let $a = (a_\sigma)_{\sigma\in\mathcal{S}_n}$ and $b = (b_\sigma)_{\sigma\in\mathcal{S}_n}$ be in $\mathbb{C}^{n!}$. We have:

$$\begin{aligned}Ga = b &\Leftrightarrow \sum_{\sigma'\in\mathcal{S}_n}\langle u_\sigma, u_{\sigma'}\rangle a_{\sigma'} = b_\sigma \quad \forall\sigma\in\mathcal{S}_n\\ &\Leftrightarrow \sum_{\sigma'\in\mathcal{S}_n} N^{\gamma(\sigma'^{-1}\sigma)}a_{\sigma'} = b_\sigma \quad \forall\sigma\in\mathcal{S}_n\\ &\Leftrightarrow b = a * N^\gamma\\ &\Leftrightarrow a = b*(N^\gamma)^{(-1)}\end{aligned}$$

when $N\geq n$ since in that case N^γ is $*$-invertible. Therefore G is invertible when $N \geq n$ and $\{u_\sigma \ ; \ \sigma\in\mathcal{S}_n\}$ is a free system of vectors of $[(\mathcal{M}_N)^{\otimes n}]^{\mathbb{U}_N}$.□

Here are some basic properties satisfied by the $u_\sigma, \sigma\in\mathcal{S}_n$, which can be easily proved. For any σ and τ in $\mathcal{S}_n$ and $A_1,\dots,A_n\in\mathcal{M}_N$,

$$u_\sigma^* = u_{\sigma^{-1}}, \tag{4}$$

$$u_\sigma.u_\tau = u_{\sigma\tau}, \tag{5}$$

$$\langle u_\sigma.(A_1\otimes\cdots\otimes A_n), u_\tau\rangle = \langle A_1\otimes\cdots\otimes A_n, u_{\sigma^{-1}\tau}\rangle, \tag{6}$$

$$\langle (A_1\otimes\cdots\otimes A_n).u_\sigma, u_\tau\rangle = \langle A_1\otimes\cdots\otimes A_n, u_{\tau\sigma^{-1}}\rangle,$$

the two last ones coming from (3), (4) and (5).

Moreover, for any $k<n$, if π in $\mathcal{S}_n$ is such that $\pi = (n)\cdots(k+1)\rho$ for some ρ in $\mathcal{S}_k$, then

$$u_\pi = u_\rho \otimes \underbrace{I_N\otimes\ldots\otimes I_N}_{n-k \text{ times}} \tag{7}$$

and more generally, if $\pi = \rho_1\rho_2$ with $\rho_1 \in \mathcal{S}\{1,\dots,k\}$ and $\rho_2 \in \mathcal{S}\{k+1,\dots,n\}$, then

$$u_\pi = u_{\rho_1} \otimes u_{\rho_2}. \tag{8}$$

Lastly note the following straightforward equality:

$$\rho'((\sigma,\sigma)) \cdot u_\pi = u_{\sigma\pi\sigma^{-1}}. \tag{9}$$

Here is an immediate interpretation of the generalized moments in terms of Hermitian products with the u_π.

Lemma 2.1 *For any $A_1 \otimes \cdots \otimes A_n$ in $\mathcal{M}_N^{\otimes n}$ and any $\pi \in \mathcal{S}_n$*

$$r_\pi(A_1,\dots,A_n) = \langle A_1 \otimes \dots \otimes A_n, u_\pi \rangle. \tag{10}$$

Proof: We have:

$$\begin{aligned}
\langle A_1 \otimes \dots \otimes A_n, u_\pi \rangle &= \sum_{i_1,\dots,i_n} \mathrm{Tr}(A_1 E_{i_1 i_{\pi^{-1}(1)}}) \cdots \mathrm{Tr}(A_n E_{i_n i_{\pi^{-1}(n)}}) \\
&= \sum_{i_1,\dots,i_n} (A_1)_{i_{\pi^{-1}(1)} i_1} \cdots (A_n)_{i_{\pi^{-1}(n)} i_n} \\
&= \sum_{j_1,\dots,j_n} (A_1)_{j_1 j_{\pi(1)}} \cdots (A_n)_{j_n j_{\pi(n)}} \\
&= r_\pi(A_1,\dots,A_n).
\end{aligned}$$

□

2.4 Geometrical interpretation of the $\mathbb{U}$-cumulants

In [5] the authors introduce the linear map Π of $\mathcal{M}_N^{\otimes n}$ on $[\mathcal{M}_N^{\otimes n}]^{\mathbb{U}_N}$ defined for any $A_1 \otimes \cdots \otimes A_n$ by:

$$\Pi(A_1 \otimes \dots \otimes A_n) := \int_{\mathbb{U}_N} UA_1U^* \otimes \dots \otimes UA_nU^* dU = \int_{\mathbb{U}_N} \rho_n(U)(A_1 \otimes \cdots \otimes A_n) dU$$

where integration is performed with respect to the Haar measure on $\mathbb{U}_N$. Note that they call it the conditional expectation onto $[\mathcal{M}_N^{\otimes n}]^{\mathbb{U}_N}$ and denote it by $\mathbb{E}(A_1 \otimes \dots \otimes A_n)$ but we prefer to adopt the previous notation $\Pi(A_1 \otimes \dots \otimes A_n)$ in order to stay faithful to our notations of the expectation in [1] and [3] and also to underline the property of orthogonal projection mentioned in [5] instead of conditional expectation. Indeed it is easy to verify that for any $\mathbf{B} \in [\mathcal{M}_N^{\otimes n}]^{\mathbb{U}_N}$ and any $\mathbf{A} \in \mathcal{M}_N^{\otimes n}$,

$$\langle \Pi(\mathbf{A}), \mathbf{B} \rangle = \int_{\mathbb{U}_N} \langle \rho_n(U)(\mathbf{A}), \mathbf{B} \rangle dU = \int_{\mathbb{U}_N} \langle \mathbf{A}, \rho_n(U^*)(\mathbf{B}) \rangle dU = \langle \mathbf{A}, \mathbf{B} \rangle.$$

We first get the following proposition in the same spirit as formula (10) in [5]. It will be one of the key tools when recovering of the properties of Section 2.2.

Proposition 2.4 *Let* $\mathbf{A} = (A_1, \ldots, A_n)$ *and* $\mathbf{B} = (B_1, \ldots, B_n)$ *be two independent sets of* $N \times N$ *matrices such that the distribution of* $\mathbf{A}$ *is invariant under unitary conjugation, i.e., for any deterministic unitary matrix* U, $(UA_1U^*, \ldots, UA_nU^*)$ *and* $(A_1, \ldots, A_n)$ *are identically distributed. Then*

$$\begin{aligned}&\mathbb{E}\left(\Pi(A_1B_1 \otimes \ldots \otimes A_nB_n)\right)\\ &\qquad = \mathbb{E}\left(\Pi(A_1 \otimes \ldots \otimes A_n)\right) . \mathbb{E}\left(\Pi(B_1 \otimes \ldots \otimes B_n)\right).\end{aligned} \tag{11}$$

Proof:

$$\begin{aligned}&\mathbb{E}\left(\Pi(A_1B_1 \otimes \ldots \otimes A_nB_n)\right)\\ &\quad = \mathbb{E}\left(\int U_1A_1B_1U_1^* \otimes \ldots \otimes U_1A_nB_nU_1^*\, dU_1\right)\\ &\quad \stackrel{(a)}{=} \int \mathbb{E}\left(\int U_1U_2A_1U_2^*B_1U_1^* \otimes \ldots \otimes U_1U_2A_nU_2^*B_nU_1^*\, dU_1\right) dU_2\\ &\quad \stackrel{(b)}{=} \mathbb{E}\left(\int\int UA_1U^*U_1B_1U_1^* \otimes \ldots \otimes UA_nU^*U_1B_nU_1^*\, dU_1 dU\right)\\ &\quad \stackrel{(c)}{=} \mathbb{E}\left(\Pi(A_1 \otimes \ldots \otimes A_n)\right) . \mathbb{E}\left(\Pi(B_1 \otimes \ldots \otimes B_n)\right),\end{aligned}$$

where we used the invariance under unitary conjugaison of the distribution of $\mathbf{A}$ in (a), a change of variable U for U_1U_2 in (b) and the independence of $\mathbf{A}$ and $\mathbf{B}$ in (c). □

Here is the main result of the section:

Theorem 2.2 *Let* $A_1, \cdots, A_n$ *be in* $\mathcal{M}_N$, $N \geq n$. *Then the matricial* $\mathbb{U}$-*cumulants of* $(A_1, \cdots, A_n)$, $C_\sigma^U(A_1, \ldots, A_n)$ *with* $\sigma \in \mathcal{S}_n$, *are the coordinates of* $\mathbb{E}\left(\Pi(A_1 \otimes \ldots \otimes A_n)\right)$ *in the basis* $\{u_\sigma, \sigma \in \mathcal{S}_n\}$:

$$\mathbb{E}\left(\Pi(A_1 \otimes \ldots \otimes A_n)\right) = \sum_{\sigma \in \mathcal{S}_n} C_\sigma^U(A_1, \ldots, A_n)u_\sigma.$$

Proof: According to Proposition 2.3, there exist $\{\tilde{C}_\sigma(A_1, \ldots, A_n), \sigma \in \mathcal{S}_n\}$ in $\mathbb{C}$ such that

$$\Pi(A_1 \otimes \ldots \otimes A_n) = \sum_{\sigma \in \mathcal{S}_n} \tilde{C}_\sigma(A_1, \ldots, A_n)u_\sigma.$$

Then, using (10),

$$\begin{aligned}r_\pi(A_1, \ldots, A_n) &= \langle \Pi(A_1 \otimes \ldots \otimes A_n), u_\pi\rangle = \sum_{\sigma \in \mathcal{S}_n} \tilde{C}_\sigma(A_1, \ldots, A_n)\langle u_\sigma, u_\pi\rangle\\ &= \sum_{\sigma \in \mathcal{S}_n} \tilde{C}_\sigma(A_1, \ldots, A_n)N^{\gamma(\sigma^{-1}\pi)} = \tilde{C}(A_1, \ldots, A_n) * N^\gamma(\pi).\end{aligned}$$

Thus,

$$\mathbb{E}\left(r_\pi(A_1,\ldots,A_n)\right) = \mathbb{E}\left(\tilde{C}(A_1,\ldots,A_n)\right) * N^\gamma(\pi).$$

On the other hand, by definition of the $C^U(A_1,\ldots,A_n)$, we have

$$\mathbb{E}\left(r_\pi(A_1,\ldots,A_n)\right) = C^U(A_1,\ldots,A_n) * N^\gamma(\pi).$$

Since N^γ is invertible for the $*$-convolution, we can deduce that for any $\sigma \in \mathcal{S}_n$,

$$\mathbb{E}\left(\tilde{C}_\sigma(A_1,\ldots,A_n)\right) = C^U_\sigma(A_1,\ldots,A_n). \qquad \square$$

The key properties of these cumulants taken from [2] and recalled in Section 2.1 can be recovered using this geometric interpretation.

- Proof of Formula (2) (or Lemma 3.1 in [2]):
  Note that $A_{\sigma(1)} \otimes \cdots \otimes A_{\sigma(n)} = \rho'(\sigma^{-1},\sigma^{-1})(A_1 \otimes \cdots \otimes A_n)$. Thus since the actions ρ_n and ρ' commute we have $\Pi(A_{\sigma(1)} \otimes \cdots \otimes A_{\sigma(n)}) = \rho'(\sigma^{-1},\sigma^{-1})\Pi(A_1 \otimes \cdots \otimes A_n)$. Using (9) and Theorem 2.2, Formula (2) follows from the linear independence of the u_π, $\pi \in \mathcal{S}_n$. $\square$

- Proof of Proposition 2.1:
  On the one hand, from Theorem 2.2 we have

  $$\mathbb{E}(\Pi(A_1 \otimes \ldots \otimes A_k \otimes I_N \otimes \cdots \otimes I_N)) = \sum_{\sigma \in \mathcal{S}_n} C^U_\sigma(A_1,\ldots,A_k,I_N,\ldots,I_N)u_\sigma.$$

  On the other hand, we also have

  $$\begin{aligned}&\mathbb{E}\left(\Pi(A_1 \otimes \ldots \otimes A_k \otimes I_N \otimes \cdots \otimes I_N)\right)\\ &\quad = \mathbb{E}\left(\Pi(A_1 \otimes \ldots \otimes A_k)\right) \otimes I_N \otimes \cdots \otimes I_N\\ &\quad = \left(\sum_{\rho \in \mathcal{S}_k} C^U_\rho(A_1,\ldots,A_k)\, u_\rho\right) \otimes I_N \otimes \cdots \otimes I_N\\ &\quad = \sum_{\substack{\sigma \in \mathcal{S}_n\\ \sigma = (n)\cdots(k+1)\rho\\ \text{for some } \rho \in \mathcal{S}_k}} C^U_\rho(A_1,\ldots,A_k)\, u_\sigma,\end{aligned}$$

  the last equality coming from (7). The result follows by the linear independence of all the u_σ. $\square$

- From the two previous points we easily get Corollary 3.1 in [2] that we recall here:
  Let $V = \{i \in \{1,\ldots,n\}, A_i \neq I_N\} = \{i_1 < \cdots < i_k\}$. *Then*

  $$C^U_\pi(A_1,\ldots,A_n) = \begin{cases} C^U_\rho(A_{i_1},\ldots,A_{i_k}) \text{ if } \pi_{|V^c} = id \text{ and } \pi_{|V} = \rho,\\ 0 \text{ else.}\end{cases}$$

- Proof of Theorem 2.1:
  Write:

$$
\begin{aligned}
&\mathbb{E}\left(r_\pi(A_1B_1,\dots,A_nB_n)\right)\\
&\quad= \mathbb{E}\left(\langle \Pi(A_1B_1\otimes\dots\otimes A_nB_n), u_\pi\rangle\right)\\
&\quad\overset{(a)}{=} \langle \mathbb{E}\left(\Pi(A_1\otimes\dots\otimes A_n)\right).\mathbb{E}\left(\Pi(B_1\otimes\dots\otimes B_n)\right), u_\pi\rangle\\
&\quad\overset{(b)}{=} \sum_{\sigma\in\mathcal{S}_n} C^U_\sigma(\mathbf{A})\langle u_\sigma.\mathbb{E}\big(\Pi(B_1\otimes\dots\otimes B_n)\big), u_\pi\rangle\\
&\quad\overset{(c)}{=} \sum_{\sigma\in\mathcal{S}_n} C^U_\sigma(\mathbf{A})\mathbb{E}\big(\langle\Pi(B_1\otimes\dots\otimes B_n), u_{\sigma^{-1}\pi}\rangle\big)\\
&\quad= \sum_{\sigma\in\mathcal{S}_n} C^U_\sigma(\mathbf{A})\mathbb{E}(r_{\sigma^{-1}\pi}(\mathbf{B})),
\end{aligned}
$$

  where (a) comes from (11), (b) from Theorem 2.2 and (c) from (6). Similarly, developing $\mathbb{E}\left(\Pi(B_1\otimes\dots\otimes B_n)\right)$, we also get

$$
\begin{aligned}
&\mathbb{E}\left(r_\pi(A_1B_1,\dots,A_nB_n)\right)\\
&\quad= \sum_{\sigma\in\mathcal{S}_n} C^U_\sigma(\mathbf{B})\mathbb{E}\big(\langle\Pi(A_1\otimes\dots\otimes A_n), u_{\pi\sigma^{-1}}\rangle\big)\\
&\quad= \sum_{\sigma\in\mathcal{S}_n} C^U_\sigma(\mathbf{B})\mathbb{E}(r_{\pi\sigma^{-1}}(\mathbf{A})) = \sum_{\tau\in\mathcal{S}_n}\mathbb{E}(r_\tau(\mathbf{A}))C^U_{\tau^{-1}\pi}(\mathbf{B}).
\end{aligned}
$$

  □

- Proof of Corollary 2.1:
  Using (11), Theorem 2.2 and then (5), we get

$$
\mathbb{E}\big(\Pi(\mathbf{AB})\big) = \sum_{\sigma,\tau} C^U_\sigma(\mathbf{A})C^U_\tau(\mathbf{B})u_\sigma.u_\tau = \sum_\pi\Big(\sum_\sigma C^U_\sigma(\mathbf{A})C^U_{\sigma^{-1}\pi}(\mathbf{B})\Big)u_\pi.
$$

  The result follows from the linear independence of the u_π. □
  Note that Theorem 2.1 or Corollary 2.1 enable to compute the coordinates of $\mathbb{E}\left\{\Pi(\mathbf{AB})\right\}$ in the basis $\{u_\pi, \pi\in\mathcal{S}_n\}$. This also was the aim of formula (10) in [5].

- The linearizing property followed from Proposition 5.1 in [2]. We propose here a slightly modified version of this proposition:

  Proposition 2.5 *Let $\mathcal{A}$ and $\mathcal{B}$ be two independent sets of $N\times N$ matrices such that the distribution of $\mathcal{A}$ is invariant under unitary conjugation. Let $X_1,\dots,X_n$ be in $\mathcal{A}\cup\mathcal{B}$ and define $V=\{i\in\{1,\dots,n\}, X_i\in\mathcal{A}\}$. Denote X_i by A_i if $i\in V$ and by B_i else. Denote also by $\mathbf{A}_{|V}$ the tuple composed by the $X_i, i\in V$ and by $\mathbf{B}_{|V^c}$ the complementary tuple. We assume that $V\neq\emptyset$ and $V\neq\{1,\dots,n\}$. Then*

$$
C^U_\pi(X_1,\dots,X_n) = \begin{cases} C^U_{\pi_{|V}}(\mathbf{A}_{|V})C^U_{\pi_{|V^c}}(\mathbf{B}_{|V^c}) & \text{if } \pi(V)=V,\\ 0 & \text{else.}\end{cases}
$$

Proof: Without lost of generality, thanks to formula (2), we can assume that $V = \{1, \dots, k\}$, $1 < k < n$, so that $(X_1, \dots, X_n) = (A_1, \dots, A_k, B_{k+1}, \dots, B_n)$. Then write

$$\begin{aligned}
&\mathbb{E}\left(\Pi(X_1 \otimes \dots \otimes X_n)\right) \\
&\quad = \mathbb{E}\left(\Pi(A_1 I_N \otimes \dots \otimes A_k I_N \otimes I_N B_{k+1} \otimes \dots \otimes I_N B_n)\right) \\
&\quad \overset{(a)}{=} \mathbb{E}\left(\Pi(A_1 \otimes \dots \otimes A_k \otimes I_N \otimes \dots \otimes I_N)\right) \\
&\qquad . \, \mathbb{E}\left(\Pi(I_N \otimes \dots \otimes I_N \otimes B_{k+1} \otimes \dots \otimes B_n)\right) \\
&\quad = \{\mathbb{E}\left(\Pi(A_1 \otimes \dots \otimes A_k)\right) \otimes I_N \otimes \dots \otimes I_N\} \\
&\qquad . \, \{I_N \otimes \dots \otimes I_N \otimes \mathbb{E}\left(\Pi(B_{k+1} \otimes \dots \otimes B_n)\right)\} \\
&\quad = \mathbb{E}\left(\Pi(A_1 \otimes \dots \otimes A_k)\right) \otimes \mathbb{E}\left(\Pi(B_{k+1} \otimes \dots \otimes B_n)\right) \\
&\quad \overset{(b)}{=} \sum_{\sigma \in \mathcal{S}\{1,\dots,k\}, \tau \in \mathcal{S}\{k+1,\dots,n\}} C^U_\sigma(A_1, \dots, A_k)\, C^U_\tau(B_{k+1}, \dots, B_n)\, u_{\sigma\tau}
\end{aligned}$$

where $(a), (b)$ respectively come from (11), (8). Thus the coordinates of $\mathbb{E}\left(\Pi(X_1 \otimes \dots \otimes X_n)\right)$ in the basis $\{u_\pi, \pi \in \mathcal{S}_n\}$ are null unless $\pi = \sigma\tau$ with $\sigma \in \mathcal{S}\{1, \dots, k\}$, $\tau \in \mathcal{S}\{k+1, \dots, n\}$. In that case they are $C^U_\sigma(A_1, \dots, A_k)\, C^U_\tau(B_{k+1}, \dots, B_n)$. □

In particular if π is a single cycle we have $C^U_\sigma(X_1, \dots, X_n) = 0$ from which the linearisation property follows.

3 Matricial $\mathbb{O}$-cumulants

In order to underline the parallel with the previous section, we first begin with a summary of the definitions and main results of [3]. Note that this work [3] has been greatly inspired by the paper of Graczyk P., Letac G., Massam H. [7].

3.1 Definitions

Let us introduce some objects. Let $\mathcal{S}_{2n}$ be the group of permutations of $\{1, \dots, n, \bar{1}, \dots, \bar{n}\}$. Denote by $(i\,j)$ the transposition sending i onto j and j onto i. Define

$$\theta := \prod_{i=1}^{n} (i\,\bar{i}),$$

$$H_n = \{h \in \mathcal{S}_{2n}, \theta h = h\theta\}.$$

H_n is the hyperoctahedral group. For $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_n)$ in $\{-1, 1\}^n$, set

$$\tau_\varepsilon = \prod_{i;\, \varepsilon_i = -1} (i\,\bar{i}).$$

For any $\pi \in \mathcal{S}_n$, define the permutation $s_\pi \in \mathcal{S}_{2n}$ as follows: for all $j = 1, \dots, n$,

$$s_\pi(j) = \pi(j) \qquad s_\pi(\bar{j}) = \overline{\pi(j)}.$$

Note that $H_n = \{s_\pi \tau_\varepsilon, (\pi, \varepsilon) \in \mathcal{S}_n \times \{-1,1\}^n\}$. If $\pi \in \mathcal{S}_n$, we still denote by π its extension on $\mathcal{S}_{2n}$ which is equal to the identity on $\{\bar{1}, \cdots, \bar{n}\}$. For ε in $\{-1,1\}^n$ and $\pi \in \mathcal{S}_n$, we define

$$g_{(\varepsilon,\pi)} := \tau_\varepsilon \pi \tau_\varepsilon.$$

Note that it is easy to deduce $g_{(\varepsilon,\pi)}$ from π, since one just has to put a bar on i if $\varepsilon_i = -1$ in the writing of π.

Example: $\pi = (134)(25)$, $\tau_{(1,-1,-1,1,1)} = (2\bar{2})(3\bar{3})$ then $g_{((1,-1,-1,1,1),(134)(25))} = (1\bar{3}4)(\bar{2}5)$.

Definition 3.1 *A pair $(\varepsilon, \pi) \in \{-1;1\}^n \times \mathcal{S}_n$ is particular if for any cycle c of π we have $\varepsilon_i = 1$ when i is the smallest element of c. The permutation $g_{(\varepsilon,\pi)}$ is called particular too.*

There are $K = \frac{(2n)!}{n!2^n}$ particular pairs $(\varepsilon(l), \pi_l)$ which define K particular permutations $g_l = g_{(\varepsilon(l),\pi_l)}$ and it is easy to deduce from Theorem 8 in [7] (see also [3]) that we have the partition

$$\mathcal{S}_{2n} / H_n = \bigcup_{l=1}^{K} g_l H_n.$$

We are going to extend the generalized moments (1) defined on $\mathcal{S}_n$ into two functions defined on $\mathcal{S}_{2n}$, respectively H_n-right and H_n-left invariant:

Definition 3.2 *Let $g_l, l = 1\dots, K$ be the particular permutations of $\mathcal{S}_{2n}$. For any n-tuple $\mathbf{X} = (X_1, \dots, X_n)$ of complex random matrices, set for any $g \in \mathcal{S}_{2n}$*

$$\begin{aligned}
&M^+_{\mathbf{X}}(g) := r_{\pi_l}(X_1^{\varepsilon_1(l)}, \dots, X_n^{\varepsilon_n(l)}) \text{ when } g \in g_l H_n,\\
&\mathbb{M}^+_{\mathbf{X}}(g) := \mathbb{E}\{M^+_{\mathbf{X}}(g)\},\\
&M^-_{\mathbf{X}}(g) := r_{\pi_l}(X_1^{\varepsilon_1(l)}, \dots, X_n^{\varepsilon_n(l)}) \text{ when } g \in H_n g_l,\\
&\mathbb{M}^-_{\mathbf{X}}(g) := \mathbb{E}\{M^-_{\mathbf{X}}(g)\}.
\end{aligned}$$

Note that $M^+_{(I_N,\dots,I_N)} = M^-_{(I_N,\dots,I_N)}$ and we will denote this H_n-bi-invariant function by M_{I_N}. Note also that

$$M_{I_N}(g_{(\varepsilon,\pi)}) = N^{\gamma_n(\pi)}. \tag{12}$$

We denote by $\mathcal{A}^-$ the space of H_n-left invariant functions on $\mathcal{S}_{2n}$, by $\mathcal{A}^+$ the space of H_n-right invariant functions and by $\mathcal{A}_0$ the space of H_n-bi-invariant functions. For any ϕ in $\mathcal{A}^+$ and any ψ in $\mathcal{A}^-$, define the convolution $\circledast$ on $\mathcal{A}^+ \times \mathcal{A}^-$ by

$$\phi \circledast \psi(g) := \frac{1}{|H_n|}\, \phi * \psi(g) = \sum_{l=1}^{K} \phi(g_l)\psi(g_l^{-1}g),$$

where $*$ stands for the classical convolution on $\mathcal{S}_{2n}$.
We showed in [3] that M_{I_N} is $\circledast$-invertible when $n \leq N$ and its $\circledast$-inverse relies on the Weingarten function Wg introduced in [5]. Denoting by $(M_{I_N})^{\circledast(-1)}$ this inverse function, we introduced two cumulant functions $C^{O+}_{\mathbf{X}}$, $C^{O-}_{\mathbf{X}}$: $\mathcal{S}_{2n} \to \mathbb{C}$ by setting

$$C^{O+}_{\mathbf{X}} = \mathbb{M}^{+}_{\mathbf{X}} \circledast (M_{I_N})^{\circledast(-1)},$$
$$C^{O-}_{\mathbf{X}} = (M_{I_N})^{\circledast(-1)} \circledast \mathbb{M}^{-}_{\mathbf{X}}.$$

(We slightly modified the notation $C^{O\pm}(\mathbf{X})$ we adopted in the introduction and for the $\mathbb{U}$-cumulant functions $C^{U}(\mathbf{X})$ in order to lighten the indices when we consider for instance $C^{O\pm}_{\mathbf{X}}(g_{(\varepsilon,\pi)})$, that seems more readable than $C^{O\pm}_{g_{(\varepsilon,\pi)}}(\mathbf{X})$).)
Note that

$$C^{O+}_{\mathbf{X}}(g) = C^{O-}_{\mathbf{X}}(\theta g^{-1}\theta).$$

These functions are respectively H_n-right and H_n-left invariant and coincide on the $g_{(\varepsilon,\pi)}$, $(\varepsilon,\pi) \in \{-1,1\}^n \times \mathcal{S}_n$.

Definition 3.3 *The functions $C^{O+}_{\mathbf{X}}$ and $C^{O-}_{\mathbf{X}}$ are respectively called the right and left $\mathbb{O}$-cumulant functions of order n.*

Thus, for example,

$$C^{O+}_{X}((1)) = \frac{1}{N}\mathbb{E}(\mathrm{Tr}(X)),$$
$$C^{O+}_{(X_1,X_2)}((1)(2)) = \frac{(N+1)\mathbb{E}\{\mathrm{Tr}(X_1)\mathrm{Tr}(X_2)\} - \mathbb{E}\{\mathrm{Tr}(X_1X_2)\} - \mathbb{E}\{\mathrm{Tr}(X_1\,{}^tX_2)\}}{N(N-1)(N+2)},$$
$$C^{O+}_{(X_1,X_2)}((1\,2)) = \frac{-\mathbb{E}\{\mathrm{Tr}(X_1)\mathrm{Tr}(X_2)\} + (N+1)\mathbb{E}\{\mathrm{Tr}(X_1X_2)\} - \mathbb{E}\{\mathrm{Tr}(X_1\,{}^tX_2)\}}{N(N-1)(N+2)}.$$

The analogues of formula (2) and Proposition 2.1 are the following:

Lemma 3.1 *If $\mathbf{X}^{\varepsilon} = (X_1^{\varepsilon_1}, \cdots, X_n^{\varepsilon_n})$ and if $\mathbf{X}_{\pi} = (X_{\pi(1)}, \cdots, X_{\pi(n)})$, then*

$$M^{+}_{\mathbf{X}^{\varepsilon}}(g) = M^{+}_{\mathbf{X}}(\tau_{\varepsilon} g) \quad \text{and} \quad M^{+}_{\mathbf{X}_{\pi}}(g) = M^{+}_{\mathbf{X}}(s_{\pi} g).$$
$$C^{O+}_{\mathbf{X}^{\varepsilon}}(g) = C^{O+}_{\mathbf{X}}(\tau_{\varepsilon} g) \quad \text{and} \quad C^{O+}_{\mathbf{X}_{\pi}}(g) = C^{O+}_{\mathbf{X}}(s_{\pi} g). \tag{13}$$

Proposition 3.1 *Let $X_1, \cdots, X_k$ be k $N \times N$ matrices. Then*

$$C^{O+}_{(X_1,\cdots,X_k,I_N,\cdots,I_N)}(g) = \begin{cases} C^{O+}_{(X_1,\cdots,X_k)}(g') & \text{if there exists } g' \text{ in } \mathcal{S}_{2k} \text{ such that } g \in g'H_n \\ 0 & \text{else.} \end{cases}$$

3.2 Fundamental properties

3.2.1 Mixed moments of independent tuples

In [3] we established the general convolution formula for mixed moments involving the cumulant functions C^{O+} or C^{O-}.

Theorem 3.1 *Let $\mathcal{X}$ and $\mathcal{B}$ be two independent sets of $N \times N$ random matrices such that $\mathcal{B}$ is deterministic and $\mathcal{X}$ is random whose distribution is invariant under orthogonal conjugation. Then for any $1 \leq n \leq N$, $\mathbf{X} = (X_1, \ldots, X_n)$ a n-tuple in $\mathcal{X}$, $\mathbf{B} = (B_1, \ldots, B_n)$ in $\mathcal{B}$, and for any $(\varepsilon, \varepsilon', \pi) \in \{-1; 1\}^n \times \{-1; 1\}^n \times \mathcal{S}_n$,*

$$\mathbb{E}\{r_\pi(B_1^{\varepsilon_1} X_1^{\varepsilon'_1}, \ldots, B_n^{\varepsilon_n} X_n^{\varepsilon'_n})\} = \left(M_{\mathbf{B}}^{+} \circledast C_{\mathbf{X}}^{O-}\right)(\tau_\varepsilon \pi \tau_{\varepsilon'}) = \left(C_{\mathbf{B}}^{O+} \circledast \mathbb{M}_{\mathbf{X}}^{-}\right)(\tau_\varepsilon \pi \tau_{\varepsilon'}).$$

In particular, we have

$$\begin{aligned}\mathbb{E}\{r_\pi(B_1 X_1, \ldots, B_n X_n)\} &= \left(M_{\mathbf{B}}^{+} \circledast C_{\mathbf{X}}^{O-}\right)(\pi) \\ &= \left(C_{\mathbf{B}}^{O+} \circledast \mathbb{M}_{\mathbf{X}}^{-}\right)(\pi).\end{aligned}$$

3.2.2 Linearizing property

Note that unlike the $\mathbb{U}$-cumulants the $\mathbb{O}$-cumulants $C_{\mathbf{X}}^{O\pm}(\pi)$ of a matrix X do not depend only on the class of conjugation of π (Nevertheless, when X is symmetric, $M_{\mathbf{X}}^{\pm}$ and $C_{\mathbf{X}}^{\pm}$ are bi-invariant). Thus the linearizing property has the following meaning.

Proposition 3.2 *Let A and B be two independent $N \times N$ matrices such that the distribution of A is invariant under orthogonal conjugation. Then for any single cycle π in $\mathcal{S}_n$ and any $\varepsilon \in \{-1, 1\}^n$,*

$$C_{A+B}^{O+}(g_{(\varepsilon,\pi)}) = C_A^{O+}(g_{(\varepsilon,\pi)}) + C_B^{O+}(g_{(\varepsilon,\pi)}).$$

3.2.3 Asymptotic behavior

We now come to the asymptotic behavior of the moment and cumulant functions. We need the following normalization:

Definition 3.4 *Let $\mathbf{X}$ be a n-tuple of $N \times N$ complex random matrices. The functions defined for all $g \in \mathcal{S}_{2n}$ by:*

$$\begin{aligned}\mathbb{M}_{\mathbf{X}}^{\pm(N)}(g) &:= \frac{1}{N^{\tilde{\gamma}_n(g)}} \mathbb{M}_{\mathbf{X}}^{\pm}(g) \\ (C_{\mathbf{X}}^{O\pm})^{(N)}(g) &:= N^{n-\tilde{\gamma}_n(g)} C_{\mathbf{X}}^{O\pm}(g)\end{aligned}$$

where

$$\tilde{\gamma}_n(g) = \gamma_n(\pi) \quad \text{if} \quad g \in g_{(\varepsilon,\pi)} H_n$$

are respectively called the normalized right/left moment and $\mathbb{O}$-cumulant functions of $\mathbf{X}$ on $\mathcal{S}_{2n}$.

Proposition 3.3 *Let $\mathcal{X} = \{X_i, i \in \mathbb{N}^*\}$ be a set of $N \times N$ complex random matrices and let $\mathrm{x} = \{x_i, i \in \mathbb{N}^*\}$ be a set of noncommutative random variables in some noncommutative probability space $(\mathcal{A}, \phi)$. Denote by k the corresponding free cumulant functions. Then for all $n, i_1, \ldots, i_n \in \mathbb{N}^*$, the two following assertions are equivalent:*

$$i)\ \forall\, \varepsilon, \pi\ \mathbb{M}^{\pm(N)}_{X_{i_1},\ldots,X_{i_n}}(g_{(\varepsilon,\pi)}) \underset{N\to\infty}{\longrightarrow} \phi_\pi(x_{i_1}^{\varepsilon_1}, \ldots, x_{i_n}^{\varepsilon_n}),$$

$$ii)\ \forall\, \varepsilon, \pi\ (C^{O\pm}_{X_{i_1},\ldots,X_{i_n}})^{(N)}(g_{(\varepsilon,\pi)}) \underset{N\to\infty}{\longrightarrow} k_\pi(x_{i_1}^{\varepsilon_1}, \ldots, x_{i_n}^{\varepsilon_n}).$$

3.3 Action of the orthogonal group on the space of complex matrices

We start again with some basic generalities and notations. Endow now $\mathbb{C}^N$ with the symmetric non degenerate bilinear form $\widetilde{B}(\sum_i u_i e_i, \sum_i v_i e_i) = \sum_i u_i v_i$ so that $(e_1, \ldots, e_N)$ is $\widetilde{B}$-orthonormal. Then the tensor product $\mathbb{C}^N \otimes \mathbb{C}^N$ is endowed with the bilinear form

$$\widetilde{B}_2(u_1 \otimes v_1, u_2 \otimes v_2) = \widetilde{B}(u_1, u_2)\widetilde{B}(v_1, v_2)$$

and $e_i \otimes e_j$, $i, j = 1, \ldots, N$ is a $\widetilde{B}_2$-orthonormal basis of $(\mathbb{C}^N)^{\otimes 2}$.
The orthogonal group $\mathbb{O}_N$ acts on $(\mathbb{C}^N)^{\otimes 2}$ as follows:

$$\widetilde{\rho}(O)(e_i \otimes e_j) = Oe_i \otimes Oe_j.$$

On the other hand, endow $\mathcal{M}_N$ now with the symmetric non degenerate bilinear form $B(X, Y) = \mathrm{Tr}(XY^t)$. Here again, $\mathcal{M}_N$ and $\mathbb{C}^N \otimes \mathbb{C}^N$ are isomorphic vector spaces when we identify any $X = (X_{ij})_{1\le i,j\le N} \in \mathcal{M}_N$ with $\tilde{X} = \sum_{1\le i,j\le N} X_{ij} e_i \otimes e_j$ (and hence $\tilde{E}_{a,b} = e_a \otimes e_b$). The action $\widetilde{\rho}$ gives on $\mathcal{M}_N$

$$\rho(O)(X) = OXO^t.$$

Note also that the inner product XY in $\mathcal{M}_N$ corresponds to the product defined by

$$(u_1 \otimes v_1).(u_2 \otimes v_2) = \widetilde{B}(v_1, u_2)\, u_1 \otimes v_2, \tag{14}$$

and the transposition X^t to the following rule: $(u \otimes v)^t = v \otimes u$.

Now for any n, the tensor products $\mathcal{M}_N^{\otimes n}$ and $(\mathbb{C}^N \otimes \mathbb{C}^N)^{\otimes n} = (\mathbb{C}^N)^{\otimes 2n}$ are isomorphic through the map: $X = X_1 \otimes \cdots \otimes X_n \mapsto \tilde{X} = \tilde{X}_1 \otimes \cdots \otimes \tilde{X}_n$ and with bilinear forms

$$\begin{aligned} B_n(X_1 \otimes \cdots \otimes X_n, Y_1 \otimes \cdots \otimes Y_n) &= \prod_{i=1}^n \mathrm{Tr}(X_i Y_i^t) = \prod_{i=1}^n \widetilde{B}_2(\widetilde{X}_i, \widetilde{Y}_i) \\ &= \widetilde{B}_{2n}(\widetilde{X}_1 \otimes \cdots \otimes \widetilde{X}_n, \widetilde{Y}_1 \otimes \cdots \otimes \widetilde{Y}_n). \end{aligned}$$

Here again the following actions of $\mathbb{O}_N$ are equivalent:

$$\begin{aligned} &\text{on } (\mathbb{C}^N)^{\otimes 2n} && \widetilde{\rho}_n(O)(e_{i(1)} \otimes e_{i(\bar{1})} \cdots \otimes e_{i(n)} \otimes e_{i(\bar{n})}) \\ & && = Oe_{i(1)} \otimes Oe_{i(\bar{1})} \otimes \cdots \otimes Oe_{i(n)} \otimes Oe_{i(\bar{n})}, \\ &\text{on } \mathcal{M}_N^{\otimes n} && \rho_n(O)(X_1 \otimes \cdots \otimes X_n) = OX_1O^t \otimes \cdots \otimes OX_nO^t. \end{aligned}$$

Denote by $[V]^{\mathbb{O}_N}$ the subspace of $\mathbb{O}_N$-invariant vectors of V with $V = \mathcal{M}_N^{\otimes n}$ or $(\mathbb{C}^N)^{\otimes 2n}$. Then $[\mathcal{M}_N^{\otimes n}]^{\mathbb{O}_N}$ and $[(\mathbb{C}^N)^{\otimes 2n}]^{\mathbb{O}_N}$ are still isomorphic and we identify $\mathcal{M}_N^{\otimes n}$ and $(\mathbb{C}^N)^{\otimes 2n}$. We also simply denote the bilinear form B_n or $\widetilde{B}_{2n}$ by $\langle .,. \rangle$ (even if it is not a scalar nor a Hermitian product). Note lastly that the inner product in $\mathcal{M}_N^{\otimes n}$ is defined by

$$(X_1 \otimes \cdots \otimes X_n).(Y_1 \otimes \cdots \otimes Y_n) = X_1Y_1 \otimes \cdots \otimes X_nY_n,$$

and the transposition by $(X_1 \otimes \cdots \otimes X_n)^t = X_1^t \otimes \cdots \otimes X_n^t$. They satisfy for any $u, v, w \in \mathcal{M}_N^{\otimes n}$:

$$\langle u.v, w \rangle = \langle v, u^t.w \rangle = \langle u, w.v^t \rangle. \tag{15}$$

In order to present a basis of $[\mathcal{M}_N^{\otimes n}]^{\mathbb{O}_N}$ in Proposition 3.4 below, we need to introduce the second action of group. We always use the notation

$$\Theta_n := \underbrace{I_N \otimes \ldots \otimes I_N}_{n \text{ times}} = \sum_{i_1,\ldots,i_n} e_{i_1} \otimes e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes e_{i_n}$$

and we now consider the natural action ρ' of $\mathcal{S}_{2n}$ on $(\mathbb{C}^N)^{\otimes 2n}$ defined for any permutation g in $\mathcal{S}_{2n}$ acting on $\{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$ by

$$\begin{aligned} &\rho'(g)(e_{i(1)} \otimes e_{i(\bar{1})} \otimes \cdots \otimes e_{i(n)} \otimes e_{i(\bar{n})}) \\ &\quad = e_{i(g^{-1}(1))} \otimes e_{i(g^{-1}(\bar{1}))} \otimes \cdots \otimes e_{i(g^{-1}(n))} \otimes e_{i_(g^{-1}(\bar{n}))}. \end{aligned} \tag{16}$$

Note first that

$$\langle \rho'(g)u, v \rangle = \langle u, \rho'(g^{-1})v \rangle. \tag{17}$$

Now the actions ρ and ρ' commute. Hence, according to Lemma 1.1,

$$\{\rho'(g) \cdot \Theta_n; g \in \mathcal{S}_{2n}\} \subset [\mathcal{M}_N^{\otimes n}]^{\mathbb{O}_N}.$$

But writing

$$\begin{aligned} \rho'(g) \cdot \Theta_n = &\sum_{i(1),\ldots,i(n),i(\bar{1}),\ldots,i(\bar{n})} \left(\prod_{l=1}^{n} \delta_{i(l)i(\bar{l})} \right) \\ &\cdot e_{i(g^{-1}(1))} \otimes e_{i(g^{-1}(\bar{1}))} \otimes \ldots e_{i(g^{-1}(n))} \otimes e_{i(g^{-1}(\bar{n}))}, \end{aligned}$$

it is easy to see that

$$\begin{aligned}\rho'(g)\cdot\Theta_n=\Theta_n &\iff \forall l,\ g^{-1}(l)=\overline{g^{-1}(\bar{l})}=\theta g^{-1}\theta(l) \quad \text{(where } \theta=\textstyle\prod_{i=1}^n(i\,\bar{i}).)\\ &\iff \theta=g\theta g^{-1}\\ &\iff g\in H_n,\end{aligned}$$

so that $g\mapsto\rho'(g)\cdot\Theta_n$ is H_n-right invariant. Actually Theorem 4.3.4 in [8] makes this first result more precise:

Lemma 3.2 *Let $\Xi_n\subset\mathcal{S}_{2n}$ be a collection of representatives for the cosets $\mathcal{S}_{2n}/H_n$. Then*

$$[\mathcal{M}_N^{\otimes n}]^{\mathbb{O}_N}=Span\{\rho'(g)\cdot\Theta_n; g\in\Xi_n\}.$$

We will use the parametrization of $\mathcal{S}_{2n}/H_n$ by the subset $\mathcal{P}_{2n}$ of $\mathcal{S}_{2n}$ composed with the pairings of $\{1,\dots,2n\}$. Let

$$\begin{aligned}\eta:\ &\mathcal{S}_{2n}\to\mathcal{P}_{2n}\\ &g\quad\mapsto\eta(g)=g\theta g^{-1}=\prod_{i=1}^n(g(i)\,g(\bar{i})).\end{aligned}$$

Clearly $\eta(g)=\eta(g')\iff g'\in gH_n$. We thus get a bijection from $\mathcal{S}_{2n}/H_n$ onto $\mathcal{P}_{2n}$ (see Proposition 17, [7] or Lemma 4.1, [2] for more details). Therefore we set for any $p\in\mathcal{P}_{2n}$:

$$u_p=\rho'(g)\cdot\Theta_n\quad\text{if}\quad\eta(g)=p.\tag{18}$$

The vector u_p corresponds to $\rho_B(p)$ in [5]. Note that $\eta(id)=\theta$ and $u_\theta=\Theta_N$. The $u_p, p\in\mathcal{P}_{2n}$, satisfy the following properties:

Lemma 3.3 *1. For all $A_1,\cdots,A_n$ in $\mathcal{M}_N$, for any $\pi\in\mathcal{S}_n$ and $\varepsilon\in\{-1,1\}^n$, we have:*

$$r_\pi(A_1^{\varepsilon_1},\cdots,A_n^{\varepsilon_n})=\langle A_1\otimes\cdots\otimes A_n,u_{\eta(g_{(\varepsilon,\pi)})}\rangle,$$

and more generally:

$$M^+_{\mathbf{A}}(g)=\langle A_1\otimes\cdots\otimes A_n,u_{\eta(g)}\rangle\tag{19}$$

2.

$$\langle\Theta_n,u_{\eta(g_{(\varepsilon,\pi)})}\rangle=N^{\gamma(\pi)}=M_{I_N}(g_{(\varepsilon,\pi)})$$

and hence

$$\langle u_{\eta(g)},u_{\eta(g')}\rangle=\langle\Theta_n,u_{\eta(g^{-1}g')}\rangle=M_{I_N}(g^{-1}g').\tag{20}$$

Proof: 1.) Write $\mathbf{j}=(j(1),\cdots,j(n),j(\bar{1}),\cdots,j(\bar{n}))$ a $2n$-tuple of integers in $\{1,\dots,N\}$ and

$$u_{\eta(g)}=\sum_{\mathbf{j}}\left(\prod_{l=1}^n\delta_{j(l)j(\bar{l})}\right)\bigotimes_{l=1}^n\left(e_{j(g^{-1}(l))}\otimes e_{j(g^{-1}(\bar{l}))}\right).$$

Thus

$$
\begin{aligned}
&\langle A_1 \otimes \cdots \otimes A_n, u_{\eta(g)} \rangle \\
&= \sum_{\mathbf{i},\mathbf{j}} \left(\prod_{k=1}^{n} (A_k)_{i(k)i(\bar{k})} \right) \left(\prod_{l=1}^{n} \delta_{j(l)j(\bar{l})} \right) \\
&\quad \cdot \langle \bigotimes_{l=1}^{n} \left(e_{i(l)} \otimes e_{i(\bar{l})} \right), \bigotimes_{l=1}^{n} \left(e_{j(g^{-1}(l))} \otimes e_{j(g^{-1}(\bar{l}))} \right) \rangle \\
&= \sum_{\mathbf{i},\mathbf{j}} \left(\prod_{k=1}^{n} (A_k)_{i(k)i(\bar{k})} \right) \left(\prod_{l=1}^{n} \delta_{j(l)j(\bar{l})} \right) \left(\prod_{l=1}^{n} \delta_{i(l)j(g^{-1}(l))} \right) \left(\prod_{l=1}^{n} \delta_{i(\bar{l})j(g^{-1}(\bar{l}))} \right).
\end{aligned}
$$

Thus for any s in $\{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$, $i(s) = j(g^{-1}(s)) = j(\theta g^{-1}(s))$, and setting $s = g(t)$ we get $i(g(t))) = i(g\theta(t)) = j(t)$ for all t in $\{1, \ldots, n, \bar{1}, \ldots, \bar{n}\}$. Hence

$$
\langle A_1 \otimes \cdots \otimes A_n, u_{\eta(g)} \rangle = \sum_{\mathbf{i}} \left(\prod_{k=1}^{n} (A_k)_{i(k)i(\bar{k})} \right) \left(\prod_{l=1}^{n} \delta_{i(g(l))i(g(\bar{l}))} \right)
$$

In particular for $g = g_{(\varepsilon,\pi)}$, this is formula (18) in [3] (or formula (2.10) in [7]) which gives $r_\pi(A_1^{\varepsilon_1}, \cdots, A_n^{\varepsilon_n})$. Now (19) comes from definition 3.2.
2.) The first line follows by taking the A_i equal to I_N and from the definition of M_{I_N} (see (12)). The second one comes from (17). □

The following proposition is essential for our purpose. It relies on a result in [5] that we found in a different way in [3] from mixed moments.

Proposition 3.4 *The set $\{u_p; p \in \mathcal{P}_{2n}\}$ is a basis of $[\mathcal{M}_N^{\otimes n}]^{\mathbb{O}_N}$ (when $N \geq n$).*

Proof: let $p_l = \eta(g_l), l = 1, \ldots, K$ and $G = (\langle u_{p_k}, u_{p_l} \rangle)_{k,l=1}^{K}$ be the Gramm-matrix of $\{u_p,\ p \in \mathcal{P}_{2n}\}$. It exactly corresponds to the matrix of the operator $\tilde{\Phi}$ in [5] which is shown to be invertible with inverse operator the Weingarten function Wg (see Proposition 3.10 in [5]). □

Here are some differences with the unitary case which can explain the intricate development we did for the $\mathbb{O}$-cumulants. We give the proof below.

1. We have $u^t_{\eta(g_{(\varepsilon,\pi)})} = u_{\eta(g^{-1}_{(\varepsilon,\pi)})}$. In particular

$$
u^t_{\eta(\pi)} = u_{\eta(\pi^{-1})}. \tag{21}
$$

But in general $u^t_{\eta(g)} \neq u_{\eta(g^{-1})}$. Instead we have

$$
u^t_{\eta(g)} = u_{\eta(\theta g)}. \tag{22}
$$

In fact define the transposition in $\mathcal{P}_{2n}$ by setting $p^t = \eta(\theta g)$ for $p = \eta(g)$. Then $u_p^t = u_{p^t}$. Note that this corresponds to the parametrization of $\mathcal{P}_{2n}$ by $H_n \diagdown \mathcal{S}_{2n} = \bigcup_{l=1}^K H_n g_l$. Indeed consider $\eta^- : g \mapsto \eta^-(g) := \eta(\theta g^{-1})$ from $\mathcal{S}_{2n}$ on $\mathcal{P}_{2n}$. It induces a one-to-one mapping from $H_n \diagdown \mathcal{S}_{2n}$ onto $\mathcal{P}_{2n}$ such that $\eta^-(g_{(\varepsilon,\pi)}) = \eta(g_{(\varepsilon,\pi)})$. Then $u^t_{\eta(g)} = u_{\eta^-(g^{-1})}$. Consequently

$$M^-_{\mathbf{A}}(g) = \langle A_1 \otimes \cdots \otimes A_n, u_{\eta^-(g)} \rangle = \langle A_1 \otimes \cdots \otimes A_n, u_{\eta(\theta g^{-1})} \rangle. \quad (23)$$

2. In general $u_{\eta(g_1)}.u_{\eta(g_2)} \neq u_{\eta(g_1 g_2)}$, but

$$u_{\eta(\pi)}.u_{\eta(g)} = u_{\eta(\pi g)} \quad \text{and} \quad u_{\eta(g)}.u_{\eta(\pi)} = u_{\eta(\theta \pi^{-1} \theta g)}. \quad (24)$$

Here again this relation could be understood by introducing the inner product in $\mathbb{C}(\mathcal{P}_{2n}) = \{ \sum_{p \in \mathcal{P}_{2n}} a_p p ; a_p \in \mathbb{C} \}$ described in [8] Section 10.1.2, for which $\mathbb{C}(\mathcal{P}_{2n})$ is called the Brauer algebra. This product is of the form $p.q = N^{\alpha(p,q)} r(p,q)$ with $\alpha(p,q) \in \mathbb{N}$ and $r(p,q) \in \mathcal{P}_{2n}$ and we get

$$u_p.u_q = N^{\alpha(p,q)} u_{r(p,q)}. \quad (25)$$

As we do not use it in the following, we choose not to detail it here.

3.

$$\begin{aligned} \langle u_{\eta(\pi)}.(A_1 \otimes \cdots \otimes A_n), u_{\eta(g)} \rangle &= \langle A_1 \otimes \cdots \otimes A_n, u_{\eta(\pi^{-1} g)} \rangle = M^+_{\mathbf{A}}(\pi^{-1} g), \\ \langle u_{\eta(g)}.(A_1 \otimes \cdots \otimes A_n), u_{\eta(\pi)} \rangle &= \langle A_1 \otimes \cdots \otimes A_n, u_{\eta(\theta \pi^{-1} g)} \rangle \quad (26) \\ &= \langle A_1 \otimes \cdots \otimes A_n, u_{\eta^-(g^{-1}\pi)} \rangle = M^-_{\mathbf{A}}(g^{-1}\pi). \end{aligned}$$

4. If $p \in \mathcal{P}(\{1, \ldots, k, \bar{1}, \ldots, \bar{k}\})$, if $\theta_k = \prod_{l=n-k}^n (l\ \bar{l})$, then

$$u_{p\theta_k} = u_p \otimes I_N \cdots \otimes I_N \quad (27)$$

and more generally if $p \in \mathcal{P}(\{1, \ldots, k, \bar{1}, \ldots, \bar{k}\})$ and $q \in \mathcal{P}(\{k+1, \ldots, n, \overline{k+1}, \ldots, \bar{n}\})$, then the juxtaposition pq is in $\mathcal{P}_{2n}$ and

$$u_{pq} = u_p \otimes u_q. \quad (28)$$

Proof: 1.) We simply write:

$$\begin{aligned} u^t_{\eta(g)} &= \sum_{\mathbf{j}} \left(\prod_{l=1}^n \delta_{j(l) j(\bar{l})} \right) \bigotimes_{l=1}^n \left(e_{j(g^{-1}(\bar{l}))} \otimes e_{j(g^{-1}(l))} \right) \\ &= \sum_{\mathbf{j}} \left(\prod_{l=1}^n \delta_{j(l) j(\bar{l})} \right) \bigotimes_{l=1}^n \left(e_{j(g^{-1}\theta(l))} \otimes e_{j(g^{-1}\theta(\bar{l}))} \right) \\ &= u_{\eta(\theta g)} \end{aligned}$$

Now $\eta(\theta g) = \eta(\theta g \theta) \neq \eta(g^{-1})$ in general. For instance if $g = (12\bar{2})$, then $\theta g \theta = (\bar{1}\bar{2}2)$ and $\eta(\theta g \theta) = (1\bar{2})(\bar{1}2)$. On the other hand $g^{-1} = (1\bar{2}2)$ and $\eta(g^{-1}) = (12)(\bar{1}\bar{2}) \neq \eta(\theta g \theta)$.

Nevertheless $\eta(\theta g_{(\varepsilon,\pi)}) = \eta(g^{-1}_{(\varepsilon,\pi)})$ since $\theta g_{(\varepsilon,\pi)}\theta = \tau_\varepsilon\theta\pi\theta\tau_\varepsilon = \tau_\varepsilon\pi^{-1}\tau_\varepsilon$ $(\tau_\varepsilon s_\pi \tau_\varepsilon) \in g^{-1}_{(\varepsilon,\pi)}H_n$.

2.) Take $g = (1\bar{2})$ so that $g^2 = id$, $\eta(g^2) = \theta$ and $u_\theta = I_N \otimes I_N$. Now $u_{\eta(g)} = \sum_{i_1,i_2} e_{i_2} \otimes e_{i_1} \otimes e_{i_2} \otimes e_{i_1}$ and therefore, with (14), $u_{\eta(g)}.u_{\eta(g)} = \sum_{i_1,i_2,j_1,j_2} (\delta_{i_1 j_2}) e_{i_2} \otimes e_{j_1} \otimes e_{i_2} \otimes e_{j_1} = N u_{\eta(g)} \neq I_N \otimes I_N$.
Now write

$$
\begin{aligned}
u_\pi . u_{\eta(g)} = & \left(\sum_{i(1),\ldots,i(n)} \bigotimes_{l=1}^{n} e_{i(\pi^{-1}(l))} \otimes e_{i(l)} \right) \\
& \cdot \left(\sum_{\mathbf{j}} \left(\prod_{l=1}^{n} \delta_{j(l)j(\bar{l})} \right) \bigotimes_{l=1}^{n} \left(e_{j(g^{-1}(l))} \otimes e_{j(g^{-1}(\bar{l}))} \right) \right) \\
& = \sum_{i(1),\ldots,i(n),\mathbf{j}} \left(\prod_{l=1}^{n} \delta_{j(l)j(\bar{l})} \right) \left(\prod_{l=1}^{n} \delta_{i(l)j(g^{-1}(l))} \right) \bigotimes_{l=1}^{n} \left(e_{i(\pi^{-1}(l))} \otimes e_{j(g^{-1}(\bar{l}))} \right) \\
& = \sum_{\mathbf{j}} \left(\prod_{l=1}^{n} \delta_{j(l)j(\bar{l})} \right) \bigotimes_{l=1}^{n} \left(e_{j(g^{-1}\pi^{-1}(l))} \otimes e_{j(g^{-1}(\bar{l}))} \right) \\
& = u_{\eta(\pi g)}.
\end{aligned}
$$

For the second one we have

$$
\begin{aligned}
u_{\eta(g)} . u_\pi = & \left(\sum_{\mathbf{j}} \left(\prod_{l=1}^{n} \delta_{j(l)j(\bar{l})} \right) \bigotimes_{l=1}^{n} \left(e_{j(g^{-1}(l))} \otimes e_{j(g^{-1}(\bar{l}))} \right) \right) \\
& \cdot \left(\sum_{i(1),\ldots,i(n)} \bigotimes_{l=1}^{n} e_{i(\pi^{-1}(l))} \otimes e_{i(l)} \right) \\
& = \sum_{i(1),\ldots,i(n),\mathbf{j}} \left(\prod_{l=1}^{n} \delta_{j(l)j(\bar{l})} \right) \left(\prod_{l=1}^{n} \delta_{j(g^{-1}(\bar{l}))i(\pi^{-1}(l))} \right) \bigotimes_{l=1}^{n} \left(e_{j(g^{-1}(l))} \otimes e_{i(l)} \right) \\
& = \sum_{\mathbf{j}} \left(\prod_{l=1}^{n} \delta_{j(l)j(\bar{l})} \right) \bigotimes_{l=1}^{n} \left(e_{j(g^{-1}(l))} \otimes e_{j(g^{-1}(\overline{\pi(\bar{l})}))} \right) \\
& \sum_{\mathbf{j}} \left(\prod_{l=1}^{n} \delta_{j(l)j(\bar{l})} \right) \bigotimes_{l=1}^{n} \left(e_{j(g^{-1}(l))} \otimes e_{j(g^{-1}(\theta\pi\theta(\bar{l})))} \right) \\
& = u_{\eta(\theta\pi^{-1}\theta g)}.
\end{aligned}
$$

3.) comes from (15), (21) or (22), and (24). Finally Property 4.) is clear from the definition of the u_p. □

3.4 Geometrical interpretation of the $\mathbb{O}$-cumulants

Consider now, as in [5], the orthogonal projection Π of $\mathcal{M}_N^{\otimes n}$ onto $[\mathcal{M}_N^{\otimes n}]^{\mathbb{O}_N}$ defined by

$$\Pi(A_1\otimes\ldots\otimes A_n) := \int_{\mathbb{O}_N} OA_1O^t\otimes\ldots\otimes OA_nO^t\, dO = \int_{\mathbb{O}_N} \rho_n(O)(A_1\otimes\cdots\otimes A_n)dO$$

where integration is performed with respect to the Haar measure on $\mathbb{O}_N$. As was the case for the unitary case, it corresponds to the conditional expectation on $[\mathcal{M}_N^{\otimes n}]^{\mathbb{O}_N}$ (which is still denoted by $\mathbb{E}(A)$ in [5]).
Note first that Π commutes with the action of ρ': for any A in $\mathcal{M}_N^{\otimes n}$ and g in $\mathcal{S}_{2n}$,

$$\rho'(g)\Pi(A) = \Pi\rho'(g)(A). \tag{29}$$

Here is Proposition 2.4 which we have completely translated for models invariant under orthogonal conjugation. Its proof can be carried on in a very similar way.

Proposition 3.5 *Let $\mathbf{A} = (A_1, \ldots, A_n)$ and $\mathbf{B} = (B_1, \ldots, B_n)$ be two independent sets of $N \times N$ matrices such that the distribution of $\mathbf{A}$ is invariant under orthogonal conjugation. Then*

$$\begin{aligned}\mathbb{E}\left(\Pi(A_1B_1 \otimes \ldots \otimes A_nB_n)\right) = \mathbb{E}\left(\Pi(A_1 \otimes \ldots \otimes A_n)\right)\\ .\ \mathbb{E}\left(\Pi(B_1 \otimes \ldots \otimes B_n)\right).\end{aligned} \tag{30}$$

Now we get:

Theorem 3.2 *Let $g_l, l = 1, \ldots, K$ be all the particular permutations of $\mathcal{S}_{2n}$; denote by p_l the pairing $\eta(g_l)$. For any $A_1, \cdots, A_n$ in $\mathcal{M}_N$, denote by $C^O_{\mathbf{A}}(g_l)$ the matricial $\mathbb{O}$-cumulants $C^{O\pm}_{\mathbf{A}}(g_l)$ of $\mathbf{A} = (A_1, \cdots, A_n)$. Then $\{C^O_{\mathbf{A}}(g_l), l = 1, \ldots, K\}$ is the set of coordinates of $\mathbb{E}\left(\Pi(A_1 \otimes \ldots \otimes A_n)\right)$ in the basis $\{u_{p_l}, l = 1, \ldots, K\}$:*

$$\mathbb{E}\left(\Pi(A_1 \otimes \ldots \otimes A_n)\right) = \sum_{l=1}^{K} C^O_{\mathbf{A}}(g_l)u_{p_l}. \tag{31}$$

Proof: As $\{u_l, l = 1, \ldots, K\}$ is a basis of $[\mathcal{M}_N^{\otimes n}]^{\mathbb{O}_N}$, we can write $\mathbb{E}\big(\Pi(A_1 \otimes \ldots \otimes A_n)\big) = \sum_{l=1}^{K} \alpha_l(\mathbf{A})u_{p_l}$, and hence, using (19),

$$\begin{aligned}\mathbb{M}^+_{\mathbf{A}}(g_k) &= \mathbb{E}(\langle\Pi(A_1 \otimes \ldots \otimes A_n), u_{p_k}\rangle)\\ &= \sum_{l=1}^{K} \alpha_l(\mathbf{A})\langle u_{p_l}, u_{p_k}\rangle = \sum_{l=1}^{K} \alpha_l(\mathbf{A})M_{I_N}(g_l^{-1}g_k)\end{aligned}$$

from (20). Define $\tilde{C}_{\mathbf{A}}$ on $\mathcal{S}_{2n}$ by $\tilde{C}_{\mathbf{A}}(g) = \alpha_l(\mathbf{A})$ if $g \in g_lH_n$ so that the previous equality gives $\mathbb{M}^+_{\mathbf{A}}(g_k) = \tilde{C}_{\mathbf{A}} \circledast M_{I_N}(g_k)$. Since M_{I_N} is $\circledast$-invertible, it follows that $\tilde{C}_{\mathbf{A}} = C^{O+}_{\mathbf{A}}$ and hence $\alpha_l(\mathbf{A}) = C^O_{\mathbf{A}}(g_l)$. □

We now review the properties of the $\mathbb{O}$-cumulants expressed in Sections 3.1 and 3.2.

- Proof of lemma 3.1:
Note that $\mathbf{X}^{\varepsilon} = \rho'(\tau_{\varepsilon})\mathbf{X}$ and $\mathbf{X}_{\pi} = \rho'((s_{\pi})^{-1})\mathbf{X}$. Then use (19), (17) and the definition (18). We get the expression of $M^{+}_{\mathbf{X}^{\varepsilon}}(g)$ and $M^{+}_{\mathbf{X}_{\pi}}(g)$. Then use (29) in writing

$$\begin{aligned}
\mathbb{E}\left\{\Pi(\mathbf{X}^{\varepsilon})\right\} &= \rho'(\tau_{\varepsilon})\mathbb{E}\left\{\Pi(\mathbf{X})\right\} \\
&= \sum_{l=1}^{K} C^{O+}_{\mathbf{X}}(g_l)\rho'(\tau_{\varepsilon})u_{p_l} \\
&= \sum_{l=1}^{K} C^{O+}_{\mathbf{X}}(g_l)u_{\eta(\tau_{\varepsilon} g_l)} \\
&= \sum_{k=1}^{K} C^{O+}_{\mathbf{X}}(\tau_{\varepsilon} g_k)u_{\eta(g_k)}, \qquad (32)
\end{aligned}$$

what gives $C^{O+}_{\mathbf{X}^{\varepsilon}}$. And a similar development can be led with $\mathbf{X}_{\pi}$.
- Proof of Proposition 3.1: It is the same to the proof of Proposition 2.1 in using (27).
- Proof of Theorem 3.1:

$$\begin{aligned}
\mathbb{E}\{r_{\pi}(B_1^{\varepsilon_1}X_1^{\varepsilon'_1},\ldots,B_n^{\varepsilon_n}X_n^{\varepsilon'_n})\} &= \mathbb{E}\big(\langle\Pi(B_1^{\varepsilon_1}X_1^{\varepsilon'_1}\otimes\ldots\otimes B_n^{\varepsilon_n}X_n^{\varepsilon'_n}),u_{\eta(\pi)}\rangle\big) \\
&\overset{(a)}{=} \langle\mathbb{E}\big(\Pi(\mathbf{B}^{\varepsilon})\big).\mathbb{E}\big(\Pi(\mathbf{X}^{\varepsilon'})\big),u_{\eta(\pi)}\rangle \\
&\overset{(b)}{=} \sum_{l=1}^{K} C^{O+}_{\mathbf{B}}(g_l)\langle u_{\eta(\tau_{\varepsilon} g_l)}.\mathbb{E}\big(\Pi(\mathbf{X}^{\varepsilon'})\big),u_{\eta(\pi)}\rangle \\
&\overset{(c)}{=} \sum_{l=1}^{K} C^{O+}_{\mathbf{B}}(g_l)\langle\mathbb{E}\big(\Pi(\mathbf{X}^{\varepsilon'})\big),u_{\eta(\theta\pi^{-1}\tau_{\varepsilon} g_l)}\rangle \\
&\overset{(d)}{=} \sum_{l=1}^{K} C^{O+}_{\mathbf{B}}(g_l)\langle\mathbb{E}\big(\Pi(\mathbf{X})\big),u_{\eta(\theta\tau_{\varepsilon'}\pi^{-1}\tau_{\varepsilon} g_l)}\rangle \\
&\overset{(e)}{=} \sum_{l=1}^{K} C^{O+}_{\mathbf{B}}(g_l)\langle\mathbb{E}\big(\Pi(\mathbf{X})\big),u_{\eta^{-}(g_l^{-1}\tau_{\varepsilon}\pi\tau_{\varepsilon'})}\rangle \\
&= \sum_{l=1}^{K} C^{O+}_{\mathbf{B}}(g_l)\mathbb{M}^{-}_{\mathbf{X}}(g_l^{-1}\tau_{\varepsilon}\pi\tau_{\varepsilon'}) \\
&= C^{O+}_{\mathbf{B}} \circledast \mathbb{M}^{-}_{\mathbf{X}}(\tau_{\varepsilon}\pi\tau_{\varepsilon'}),
\end{aligned}$$

(a) comes from (30), (b) from (32), (c) from (26), (d) uses $\theta\tau_{\varepsilon} = \tau_{\varepsilon}\theta$ and finally (e) comes from (23).
We conduct the second equality in an identical way.

- Lastly the linearizing property can be led in a very similar manner as for the $\mathbb{U}$-cumulants. Just translate Proposition 2.5 in using (28) and Proposition (3.5).

Note that Theorem 3.1 here again gives $\langle \mathbb{E}(\mathbf{AB}), u_p \rangle$ but only for the particular $p = \eta(\pi)$, $\pi \in \mathcal{S}_n$. Similarly formula (19) in [5] only gave $\langle \mathbb{E}(\mathbf{AB}), \Theta_n \rangle$. Actually it is impossible to get $\langle \mathbb{E}(\mathbf{AB}), u_p \rangle$ for all p as a convolution formula, although we did it for $\mathbb{U}$-invariant models. This is due to the structure of $\mathcal{P}_{2n}$ as Brauer algebra that we briefly mentioned in (25). In fact we have:

$$\begin{aligned}\mathbb{E}(\Pi(\mathbf{AB})) &= \sum_{k,l} C^O_{\mathbf{A}}(g_k) C^O_{\mathbf{B}}(g_l) u_{p_k}.u_{p_l} \\ &= \sum_{k,l} C^O_{\mathbf{A}}(g_k) C^O_{\mathbf{B}}(g_l) N^{\alpha(p_k,p_l)} u_{r(p_k,p_l)}.\end{aligned}$$

3.5 About matricial *Sp*-cumulants

Let us end this section with some words about the symplectic case. Here N is even. Recall that if $J = \begin{pmatrix} 0 & I_{\frac{N}{2}} \\ -I_{\frac{N}{2}} & 0 \end{pmatrix}$, then $Sp(N) = \{T \in GL(N, \mathbb{C});\ T^t J T = J\}$. Now identify $\mathcal{M}_N$ and $\mathbb{C}^N \otimes \mathbb{C}^N$ through

$$X = (X_{ij})_{1 \le i,j \le N} \in \mathcal{M}_N \mapsto \tilde{X} = \sum_{1 \le i,j \le N} X_{ij}\ e_i \otimes J^{-1} e_j. \tag{33}$$

Endow $\mathcal{M}_N^{\otimes n}$ with the non degenerate skew-symmetric bilinear form

$$\Omega_n(X_1 \otimes \cdots \otimes X_n, Y_1 \otimes \cdots \otimes Y_n) = \prod_{i=1}^n \mathrm{Tr}(X_i Y_i^*)$$

where $Y_i^* = J Y_i^t J^{-1}$ and consider both following group actions: first the action of $Sp(N)$ defined by $\rho(T)(X_1 \otimes \cdots \otimes X_n) = T X_1 T^* \otimes \cdots \otimes T X_n T^*$, second the action of $\mathcal{S}_{2n}$ corresponding to (16) on $(\mathbb{C}^N \otimes \mathbb{C}^N)^{\otimes n}$ via the previous identification (33) and which we still denote by ρ'.
Then the fit basis of $[\mathcal{M}_N^{\otimes n}]^{Sp(N)}$ is composed by the vectors $u_p, p \in \mathcal{P}_{2n}$ now defined by

$$u_p = \mathsf{sgn}(g) \rho'(g) \cdot \Theta_n \quad \text{if} \quad \eta(g) = p$$

where $\mathsf{sgn}(g)$ denotes the signature of the permutation g in $\mathcal{S}_{2n}$ and where $\Theta_n = I_N \otimes \cdots \otimes I_N$. It can be proved that, denoting A_i^* by A_i^{-1},

$$\Omega_n(A_1 \otimes \cdots \otimes A_n, u_{\eta(g_{(\varepsilon,\pi)})}) = \mathsf{sgn}(\pi) r_\pi(A_1^{\varepsilon_1}, \cdots, A_n^{\varepsilon_n}).$$

We thus are led to introduce:

$$
\begin{aligned}
M_{\mathbf{X}}^{Sp+}(g) &:= \Omega_n(\mathbf{X}, u_{\eta(g)}) \\
M_{\mathbf{X}}^{Sp-}(g) &:= \Omega_n(\mathbf{X}, u_{\eta^-(g)}) = \Omega_n(\mathbf{X}, u^*_{\eta(g^{-1})}) \\
\mathbb{M}_{\mathbf{X}}^{Sp+}(g) &:= \mathbb{E}\{M_{\mathbf{X}}^{Sp+}(g)\} \\
\mathbb{M}_{\mathbf{X}}^{Sp-}(g) &:= \mathbb{E}\{M_{\mathbf{X}}^{Sp-}(g)\}, \\
C_{\mathbf{X}}^{Sp+}(g) &:= \{\mathbb{M}_{\mathbf{X}}^{Sp+} \circledast (M_{I_N}^{Sp-})^{\circledast(-1)}\}(g) \\
C_{\mathbf{X}}^{Sp-}(g) &:= \{(M_{I_N}^{Sp+})^{\circledast(-1)} \circledast \mathbb{M}_{\mathbf{X}}^{Sp-}\}(g).
\end{aligned}
$$

With these definitions the geometrical interpretation of the Sp-cumulants as in (31) holds true and similar properties as those exposed in Section 3.2 can be proved like in Section 3.4.

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