Creation or deletion of a drift on a Brownian trajectory

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Summary. We show that a negative drift can be created on a Brownian trajectory by cutting excursions according to a certain Poisson measure. Conversely a negative drift can be annihilated by inserting independent excursions again according to a certain Poisson measure. We first give results in discrete time by considering the random walks as contour processes of Galton-Watson trees and then pass to the limit.

Key words: Galton-Watson trees, pruning, Poisson measure, drift, approximation of diffusion, Markov chains

1 Introduction

1.1 Summary of results and methods

It is possible to create a drift on a linear Brownian motion by cutting some excursions of this trajectory. By excursion we mean a connected part of this trajectory above a certain level. These excursions are chosen according to a certain Poisson measure defined conditionally on the initial trajectory. This is explained in section 2 below but this result is essentially Proposition 4 of [AS]. However the arguments are given there in the setting of random snakes, a subject we want to avoid in the present paper in order to make it readable by a broader public. So we give anew an exposition of theses results, in a slightly more general setting, with more connections to trees and give the ideas of the proofs skipping some details. Our point of view is to approximate (reflected) Brownian motion by random walks, seen as contour processes of Galton-Watson forests. Then the cutting of (discrete) excursions on the contour process amounts to a percolation on the Galton-Watson trees which gives again Galton-Watson trees, with a new offspring law, for which the contour process is a random walk, with a higher downward bias. The version for continous time i.e. Brownian motion as stated in Theorem 3, is obtained by examining the limit of this cutting procedure.

A natural question is to ask whether this operation can be reversed, that is, if a negative drift can be annihilated by adding excursions. The answer is positive and is the subject of section 3. Again the problem is easily solved on random walks interpretating them as the contour processes of trees. The issue is to see how a Galton-Watson tree can be "decorated" by the graft of small trees on certain vertices to give a new Galton-Watson tree with higher progeny. All our Galton-Watson trees have geometric progeny law and we rely simply on a property of these laws. The next step, explained in subsection 3.3, is to consider the continuous-time limit in order to state Theorem 7, our main result. This result specifies how excursions must be added to a Brownian motion with drift to destroy the drift, a procedure that is roughly the converse of the cutting procedure specified in Theorem 3. We conclude this paper with an example of application. Other applications to Brownian snake and super-Brownian motion will be given in [Se].

1.2 Bibliographical notes

The idea of percolation on the edges of a Galton-Watson tree to retain the connected component of the root is exploited for instance by Aldous and Pitman in [AP1] to define what they call a pruning process. They describe the transition rates and give special attention to the Poisson offspring law. This idea of pruning appears also in the setting of the Continuum Random Tree (CRT) in [AP2] where Poissonian pruning leads to a description of a self-similar fragmentation process. This work is related to the results stated here because CRT can be represented by a Brownian excursion: this correspondence is used in [AS2]. Conversely the idea of grafting small trees to a forest in order to obtain a new forest having a law of similar type but with different parameters is central in [PW]. The edges of the trees and forest considered there have variable lengths; a composition rule is proved and the link with Williams decomposition for the Brownian trajectory is explained. Galton-Watson forests with random edge lengths are also studied by Duquesne and Winkel in [DW]. They define a growing family of these trees which is consistent under a Bernoulli percolation on vertices that is described there as "tree coloring". They show the existence of a limit called the Lévy tree in the topological setting of real trees . This setting avoids the coding of trees by real valued processes such as the height process. Our point of view is completely opposite since we seek results on real-valued processes by seeing them as limit of contour processes of forests. In [EW] an operation of tree pruning and regrafting is studied for real trees; for the continuous tree associated to a real valued continuous function, it consists in cutting an excursion in the graph of the function and inserting it to another place.

2 How to create a negative drift ?

In this paper we deal with random rooted trees. We refer to [AP1] for the general terminology of trees, for instance the notions of tree, vertex, edge, root,. . . and also the notion of Galton-Watson tree. We concentrate here on geometric Galton-Watson trees : each "vertex" has, independently of the others, an offspring distributed according to a geometric law $\mathcal{G}(\rho)$ of parameter $\rho \in [1/2, 1)$. We mean that the probability for a vertex to have k children is $\rho\,(1-\rho)^k$ for every $k \in \mathbb{N} = \{0, 1, 2, \ldots\}$. The expectation of this law is $\frac{1}{\rho} - 1$ and then is smaller or equal to 1 for $\rho \in [1/2, 1)$ which implies the a.s. finitness of the tree. For basic facts on Galton-Watson trees such as the previous one the reader can refer to [AP1] Section 2.2. Moreover we let ρ depend on the height of the vertex, that is the number of generations computed from the root, but we still suppose that $\rho(\cdot) \in [1/2, 1)$. In the following we will denote $\mathcal{GW}(\rho(\cdot))$ this inhomogeneous Galton-Watson random tree. Let us perform a percolation with probability $p \in (0, 1)$ on the edges of this tree, that is, each edge is kept with probability p, independently of the others. Again the value of p may depend of the height in the tree i.e. we cut with probability $p(t)$ an edge linking two vertices of respective heights t and $t + 1$. The connected component of the root in the remaining tree is still a inhomogeneous Galton-Watson tree but now the generating function of the number $\tilde{N}$ of children of a vertex can be obtained as below, by conditioning on the number of children N that this vertex had in the original tree :

$$\begin{aligned}
\mathbb{E}\left(s^{\tilde{N}}\right) &= \mathbb{E}\left[\mathbb{E}\left(s^{\tilde{N}}\middle|N\right)\right] \\
&= \mathbb{E}\left[(p\,s + 1 - p)^N\right] \\
&= \frac{\rho}{1 - (1-\rho)(p\,s + 1 - p)} \\
&= \frac{\nu}{1 - (1-\nu)\,s} \text{ where } \nu = \nu(\cdot) = \frac{\rho(\cdot)}{\rho(\cdot) + p(\cdot) - p(\cdot)\rho(\cdot)}\,.
\end{aligned}$$

This means that this (inhomogeneous) Galton-Watson tree has a geometric offspring law of parameter $\nu(\cdot)$ given above.

A convenient way to describe a tree is to use the contour process of this tree. To our purpose it is more convenient to work with forests –in our case Galton-Watson forests– than trees. A forest is merely a sequence of independent trees. Such a forest can be seen as a sort of tree by connecting with an edge each root of the independent trees to an added vertex that we call the root of the forest. This would create an (infinite) tree with an infinite number of individuals at the first generation. We will still denote $\mathcal{GW}(\rho(\cdot))$ the forest consisting of independent (inhomogeneous) Galton-Watson trees with $\mathcal{G}(\rho(\cdot))$ offspring law where $\rho(\cdot)$ is a function of the height in the tree. The contour process of such a forest is a nearest-neighbour random walk $(X(k),\, k \in \mathbb{N})$ on $\mathbb{N}$, reflecting at 0 and with transition law given, for $j > 0$, by :

$$\mathbb{P}(X(k+1) = j-1 | X(k) = j) = \rho(j) = 1 - \mathbb{P}(X(k+1) = j+1 | X(k) = j)$$

and whose is denoted $\mathcal{RW}(\rho(\cdot))$. To be more specific we consider for this contour process that the roots of the independent trees belonging to the forest are at height 1 and the passage from one tree to the following consists for the contour process in a passage at 0. How can we interpret the percolation procedure of the tree on the contour process ? Cutting an edge and keeping only the part containing the root amounts to cut an excursion of the contour process. More precisely, let us denote by $(X(t),\, t \geq 0)$ the continuous time process which coincides with $(X(k))$ at all integer times and which is piecewise linear between those times. We denote $\mathcal{E}(X)$ the epigraph of $(X(t),\, t \geq 0)$ i.e. the set of points of $[0, +\infty)^2$ which are under the graph of $(X(t),\, t \geq 0)$:

$$\mathcal{E}(X) = \{(s,t) \in [0,+\infty)^2;\ X(s) > t\}. \tag{1}$$

For each $(s,t) \in \mathcal{E}(X)$ we denote by $[\alpha(X,s,t), \beta(X,s,t)]$ the excursion of X above level t and containing time s :

$$\alpha(X,s,t) = \sup\{s' < s;\ X(s') = t\}, \tag{2}$$
$$\beta(X,s,t) = \inf\{s' > s;\ X(s') = t\}. \tag{3}$$

For a non-negative function b on $\mathbb{N}$, we consider the Poisson point measure Λ with intensity

$$\sum_{(s,t)\in\mathbb{N}^2\cap\mathcal{E}(X)} \frac{b(t)}{\beta(X,s,t) - \alpha(X,s,t) - 1}\ \delta_{(s,t)}$$

where $\delta_{(s,t)}$ denotes the Dirac measure at (s,t). Of course the law of Λ given above must be understood as a conditional law given X. The "part to cut" is defined as

$$\mathcal{C} = \bigcup_{(s,t)\,:\,\Lambda((s,t))\neq 0} [\alpha(X,s,t), \beta(X,s,t)]. \tag{4}$$

We can now state the result of the above discussion.

Proposition 1 *The process $Y(s) = X(A(s))$ where*

$$A(s) = \inf\left\{u;\ \int_0^u \mathbf{1}_{\{v\notin\mathcal{C}\}}\, dv > s\right\} \tag{5}$$

is the contour process of the connected component of the root after percolation at rate $p = e^{-b}$ of the forest having contour X. In other words, assuming that X is a (interpolated) random walk on $\mathbb{N}$ following the law $\mathcal{RW}(\rho(\cdot))$ as defined above then Y is a (interpolated) random walk on $\mathbb{N}$ distributed as $\mathcal{RW}(\nu(\cdot))$ where

$$\nu(\cdot) = \frac{\rho(\cdot)}{\rho(\cdot) + p(\cdot) - p(\cdot)\,\rho(\cdot)}.$$

We now want to see the counterpart of this result on diffusion processes, making such processes appear as limit of random walks. We consider a sequence of random walks on $\frac{1}{\sqrt{N}}\mathbb{N}$, at first indexed by $k \in \frac{1}{N}\mathbb{N}$, that we denote $(X_N(k);\ k \in \frac{1}{N}\mathbb{N})$, which is reflecting at 0 and has the transition law given, for every $j \in \frac{1}{\sqrt{N}}\mathbb{N} \setminus \{0\}$ and every $k \in \frac{1}{N}\mathbb{N}$, by

$$\mathbb{P}\left(X_N(k+\frac{1}{N}) = j - \frac{1}{\sqrt{N}} \Big| X_N(k) = j\right) = \frac{1}{2} + \frac{\theta_N(j)}{2\sqrt{N}}, \tag{6}$$

$$\mathbb{P}\left(X_N(k+\frac{1}{N}) = j + \frac{1}{\sqrt{N}} \Big| X_N(k) = j\right) = \frac{1}{2} - \frac{\theta_N(j)}{2\sqrt{N}} \tag{7}$$

where θ_N is a sequence of continuous non-negative functions on $\mathbb{R}_+$. We extend X_N to continuous time by linear interpolation between consecutive times of $k \in \frac{1}{N}\mathbb{N}$. Such a rescaled reflecting random walk will be denoted $\mathcal{RW}(N, \theta_N)$ from now on. In the case $\theta_N = 0$ we call it, as usual, a standard rescaled reflecting random walk. The following "classical" result, gives the limit in law when (θ_N) converges. It can be deduced from general results on interpolated Markov chains, for instance as stated in [Ku] except that it applies to a non-reflecting process. However we give a short proof in the appendix, as a corollary of Donsker Theorem, for the convenience of the reader and because this proof can easily be generalized to path-valued processes which is the setting of the applications we will develop in [Se]. We recall that a reflecting Brownian motion with non-positive drift $-\theta(\cdot)$ has the law of $(|Z_t|)$ where (Z_t) is a solution of the stochastic differential equation $dZ_t = dB_t - \mathrm{sign}(Z_t)\,\theta(Z_t)\,dt$ where (B_t) is a standard Brownian motion.

Proposition 2 *If the sequence (θ_N) of non-negative continuous functions converges to the continuous function θ on $\mathbb{R}_+$, uniformly on compact sets of $\mathbb{R}_+$, then the law of the process $(X_N(s);\ s \geq 0)$ described above converges weakly to the law of a reflecting Brownian motion with drift $-\theta(\cdot)$.*

The procedure of cutting excursion described above on discrete random walks also makes sense on continuous time processes and we are able to state a continuous time analogue of Proposition 1. The following theorem shows that it is possible to create a negative drift on a linear Brownian motion by cutting certain excursions (case $\theta = 0$) or more generally to increase the negative drift of a Brownian motion.

Theorem 3 *Let $(X(t),\ t \geq 0)$ be a Brownian motion reflecting at 0 with continuous drift $-\theta(\cdot)$. Let b be a continuous function on $\mathbb{R}_+$ and Λ be a point measure which is, conditionally on X, a Poisson measure with intensity*

$$\frac{2\,b(t)}{\beta(X,s,t) - \alpha(X,s,t)}\ \mathbf{1}_{\mathcal{E}(X)}(s,t)\ \ ds\,dt$$

where $\mathcal{E}(\cdot)$, α and β are defined by (1,2,3). Also $\mathcal{C}$ is still defined by (4) and we set $Y(t) = X(A(t))$ where $A(\cdot)$ is given by (5).

Then $(Y(t),\ t \geq 0)$ *is a Brownian motion reflecting at 0 with drift* $-(\theta(\cdot)+b(\cdot))$.

Proof. We consider X_N a rescaled reflecting random walk $\mathcal{RW}(N,\theta)$ so that in particular, for $j \in \frac{1}{\sqrt{N}}\mathbb{N} \setminus \{0\}$,

$$\mathbb{P}\left(X_N(k+\frac{1}{N}) = j - \frac{1}{\sqrt{N}}\Big| X_N(k) = j\right) = \rho^N(j) = \frac{1}{2} + \frac{\theta(j)}{2\sqrt{N}}.$$

Let Λ_N be the Poisson point measure with intensity

$$\mu_N = \frac{1}{N\sqrt{N}} \sum_{(s,t)\in(\frac{1}{N}\mathbb{N}\times\frac{1}{\sqrt{N}}\mathbb{N})\cap\mathcal{E}(X_N)} \frac{2\,b(t)}{\beta(X_N,s,t)-\alpha(X_N,s,t)-\frac{1}{N}}\ \delta_{(s,t)}\ .$$

We set

$$\mathcal{C}_N = \bigcup_{(s,t)\,:\,\Lambda_N((s,t))\neq 0} [\alpha(X_N,s,t), \beta(X_N,s,t)]$$

and

$$A_N(s) = \inf\left\{u;\ \int_0^u \mathbf{1}_{\{v\notin\mathcal{C}_N\}}\, dv > s\right\}$$

and, finally, $Y_N = X_N \circ A_N$. We apply Proposition 1 with a change of scale. We deduce that $(Y_N(k),\ k \in \frac{1}{N}\mathbb{N})$ is a random walk on $\frac{1}{\sqrt{N}}\mathbb{N}$ reflecting at 0 and with transition probabilities given, for $j \in \frac{1}{\sqrt{N}}\mathbb{N}$ by

$$\mathbb{P}\left(Y_N(k+\frac{1}{N}) = j - \frac{1}{\sqrt{N}}\Big| Y_N(k) = j\right) = \frac{\rho^N(j)}{1-(1-\rho^N(j))\left(1-e^{-\frac{2b(j)}{\sqrt{N}}}\right)}$$
$$= \frac{1}{2}\left(1+\frac{\theta(j)+b(j)+\varepsilon_N(j)}{\sqrt{N}}\right)$$

where ε_N is a function converging to 0 uniformly on compact sets. We now let $N \to +\infty$. By Proposition 2 we know that Y_N converges in law to a Brownian motion reflecting at 0 with drift $-(\theta(\cdot)+b(\cdot))$. Proposition 2 also applies to X_N. By Skorohod representation Theorem we may suppose that X_N converges to X uniformly on compact sets of $\mathbb{R}_+$, almost surely. Then, skipping technicalities explained in the proof of Proposition 4 of [AS], μ_N is shown to converge to the intensity given in the Theorem and we deduce that $Y_N = X_N \circ A_N$ converges to $Y = X \circ A$. We conclude that the law of Y is as stated.

3 How to create a positive drift ?

We have seen in the previous section that we can create a negative drift on a Brownian trajectory by cutting excursions. Conversely is it possible to reduce or even annihilate a negative drift by adding excursions ? The answer is affirmative as stated in Theorem 7 for total annihilation of the drift and Theorem 9 for reduction of the drift.

3.1 Graft on a Galton-Watson tree

Our first issue is the way to transform a subcritical geometric Galton-Watson tree into another one with bigger progeny expectation. We start with an elementary lemma on the geometric law whose proof is left to the reader.

Lemma 4 *Let $0 < \rho < \nu < 1$, Z_ν and Z_ρ be independent random variables distributed according to the respective laws $\mathcal{G}(\nu)$ and $\mathcal{G}(\rho)$. Let U be an independent Bernoulli variable with expectation $p = \frac{\nu-\rho}{\nu-\nu\rho}$. Then $Z_\nu + U\, Z_\rho$ is distributed according to the geometric law $\mathcal{G}(\rho)$.*

We will now apply this elementary result to the "decoration" of a Galton-Watson tree. Considering a forest $\mathcal{GW}(\nu(\cdot))$ we add to each vertex v, with probability $p(h(v))$ depending of the height $h(v)$ of vertex v, an independent tree $\mathcal{GW}(\rho(h(v)+\cdot))$ rooted at v. In case of addition effectively occuring at vertex v, the added tree is placed at the right of the subtrees already born at vertex v.

Proposition 5 *The forest obtained from the forest $\mathcal{GW}(\nu(\cdot))$ by "decoration" at probability $p(\cdot) = \frac{\nu(\cdot)-\rho(\cdot)}{\nu(\cdot)-\nu(\cdot)\rho(\cdot)}$ using $\mathcal{GW}(\rho(\cdot))$–trees as described above, is a $\mathcal{GW}(\rho(\cdot))$–forest.*

Proof. The independence properties being clearly satisfied, it suffices to prove that a vertex v of the decorated tree has a progeny distributed according to the $\mathcal{G}(\rho(h(v)))$–geometric law. This is obvious if v is supposed to belong to one of the added trees. Otherwise the vertex v belonged to the original forest and the number of its children is $Z_1 + U\, Z_2$ where Z_1 is the original number of children, distributed as $\mathcal{G}(\nu(h(v))$, Z_2 is the number of children possibly added, and U equals 1 if decoration occurs at vertex v. But the lemma shows that this variable is $\mathcal{G}(\rho(h(v))$–geometric.

3.2 Translation to random walks

We can translate this result in the language of random walks. The contour process of a $\mathcal{GW}(\nu(\cdot))$–forest is a random walk $(X(k),\ k \in \mathbb{N})$ on $\mathbb{N}$, reflecting at 0 whose law is, as denoted before, $\mathcal{RW}(\nu(\cdot))$. Moreover we prolong this random walk into a continuous time process $(X(s),\ s \geq 0)$ by linear interpolation between consecutive integer times. For each $(s,t) \in \mathbb{N} \times \mathbb{N}$, we define $E_{(s,t)}$ as the contour process of a $\mathcal{GW}(\rho(t+\cdot))$–tree i.e. a reflecting random walk that is going down with probability $\rho(t+\cdot)$ and stopped after a number of returns to 0 which is equal to the progeny of the first generation thus distributed as $\mathcal{G}(\rho(t))$. These processes are supposed to be independent. We set

$$p(t) = \frac{\nu(t) - \rho(t)}{\nu(t) - \nu(t)\,\rho(t)}.$$

Conditionally on $(X(s))$, for every $s \in \mathbb{N}$ such that $X(s+1) = X(s) - 1$, with probability $p(X(s))$, the walk $E_{(s,X(s))}$ is inserted in the graph of X at $(s, X(s))$. Let Y be the walk obtained after these insertions have been done altogether(they are of finite number on a bounded time interval). The reader wishing to see formulas describing this procedure can refer to the proof of Proposition 7. We let the reader check that this procedure of insertion in the graph of X to obtain Y is the translation into the language of contour process of the "decoration" procedure of a forest described in Proposition 5. Therefore we can conclude on the following result.

Proposition 6 *Y is the contour of a $\mathcal{GW}(\rho(\cdot))$–forest and as a consequence is a reflecting random walk with law $\mathcal{RW}(\rho(\cdot))$.*

3.3 From discrete time to continous time

The problem now consists in stating a continuous time analogue of Proposition 6.

Theorem 7 *Let $(X(t),\ t \geq 0)$ be a Brownian motion reflecting at 0 with continuous non-positive drift $-\theta(\cdot)$. We define, conditionally on X, a Poisson point measure Λ on $\mathbb{R}_+ \times C(\mathbb{R}_+, \mathbb{R}_+)$ with intensity*

$$2\,\theta(X(s))\;ds\;n(de) \tag{8}$$

where $n(\cdot)$ denotes the Itô measure of positive excursions of Brownian motion. Let $\sigma(e)$ denote the duration (length) of an excursion e. The function

$$A_u = u + \int_{\{s \leq u\}} \sigma(e)\;\Lambda(ds\,de)$$

is increasing right-continuous and has a jump $A_u - A_{u-} = \sigma(e_u)$ for every u such that $\Lambda(\{(u, e_u)\}) \neq 0$. We define $(Y(v))_{v \geq 0}$ by $Y(v) = X(u)$ if $v = A_u$ and $Y(v) = X(u) + e_u(v - A_{u-})$ for $A_{u-} \leq v < A_u$.

Then $(Y(t),\ t \geq 0)$ is a Brownian motion reflecting at 0.

Proof. We let $(X_N(s),\ s \in \frac{1}{N}\mathbb{N})$ be a $\mathcal{RW}(N, \theta)$ random walk. Our first goal is to apply Proposition 6 to X_N so that we now set

$$\nu^N(t) = \frac{1}{2} + \frac{\theta(t)}{2\sqrt{N}}, \quad \rho^N(t) = \frac{1}{2}$$

and

$$p^N(t) = \frac{\nu^N(t) - \rho^N(t)}{\nu^N(t) - \nu^N(t)\,\rho^N(t)} = \frac{2\,\theta(t)/\sqrt{N}}{1 + \theta(t)/\sqrt{N}}\;. \tag{9}$$

We let $(U_s,\ s \in \frac{1}{N}\mathbb{N})$ be a family of independent uniform variables on $(0,1)$ and $(B(N,s),\ s \in \frac{1}{N}\mathbb{N})$ be independent copies of a rescaled reflecting standard $\mathcal{RW}(N, 0)$ random walk B^N, stopped at the time of the g-th return at 0 where

g is an independent random variable with law $\mathcal{G}(1/2)$. We consider the point measure Λ^N on $\mathbb{R}_+ \times C(\mathbb{R}_+, \mathbb{R}_+)$ given by

$$\Lambda^N = \sum_{s \in \frac{1}{N}\mathbb{N}} \mathbf{1}_{\{X_N(s+\frac{1}{N}) = X_N(s) - \frac{1}{\sqrt{N}}\}} \mathbf{1}_{\{U_s \le p^N(X_N(s))\}} \, \delta_{(s, B(N,s))}. \tag{10}$$

We set

$$A_u^N = u + \int_{\{s \le u\}} \Lambda^N(ds\, de) \;\; \sigma(e)$$

where $\sigma(e)$ is the time of the last return to 0 of e. We define $(Y^N(v))_{v \ge 0}$ by $Y^N(v) = X^N(u)$ if $v = A_u^N$ and $Y^N(v) = X^N(u) + e_u(v - A_u^N)$ if

$$A_{u-}^N \le v < A_u^N = A^N(u-) + \sigma(e_u) \text{ where } \Lambda^N(\{(u, e_u)\}) \ne 0.$$

The effect of this time change is to insert, at point $(s, X_N(s))$ preceding a descent of X_N, with probability $p^N(X_N(s))$, a rescaled reflecting standard random walk stopped after a number of return to 0 distributed according to $\mathcal{G}(1/2)$. It follows from Proposition 6 that $(Y^N(s))$ is a rescaled reflecting standard random walk. We now let $N \to +\infty$. As before, we may suppose that $(X_N(s))$ converges uniformly on every compact to $(X(s))$, almost surely, where $(X(s))$ is a Brownian motion with drift $-\theta(\cdot)$. Also Y^N converges in law to a Brownian motion, reflecting at 0.

From now on we denote $C^*(\mathbb{R}_+, \mathbb{R}_+)$ the set of the $e \in C(\mathbb{R}_+, \mathbb{R}_+)$ such that there exists $\sigma(e) = \inf\{s;\ \forall s' \ge s,\ e(s') = 0\}$. Let us consider an interval $p \in \mathbb{Z}_+$ and $K_1, \dots, K_p$ be disjoint Borel subsets of $C^*(\mathbb{R}_+, \mathbb{R}_+) \cap \{\sigma \ge \eta\}$. We thus have for every $i \le p$, $n(K_i) < +\infty$ and we suppose moreover that $n(\partial K_i) = 0$. Let $\lambda_1, \dots, \lambda_p$ be positive real numbers. Let us define, for any interval $[a,b]$ of $\mathbb{R}_+$, the set $D_N(a,b)$ consisting in the times of descents of X_N over $[a,b]$:

$$D_N(a,b) = \left\{ s \in [a,b] \cap \frac{1}{N}\mathbb{N};\;\; X_N(s + \frac{1}{N}) = X_N(s) - \frac{1}{\sqrt{N}} \right\}.$$

Then, for $0 < t_1 < t_2$, we have

$$\log \mathbb{E}\left[\exp - \left(\sum_{i=1}^p \lambda_i \, \Lambda^N([t_1, t_2] \times K_i) \right) \Big| X_N \right] \tag{11}$$

$$= \sum_{s \in D_N(t_1,t_2)} \log\left(1 + \sum_{i=1}^p (e^{-\lambda_i} - 1)\, \mathbb{P}[B(N,s) \in K_i]\, \mathbb{P}[U_s \le p^N(X_N(s)) \Big| X_N] \right)$$

But, as $N \to +\infty$,

$$\mathbb{P}(B(N,s) \in K_i)\, \mathbb{P}(U_s \le p^N(X_N(s))\, | X_N) \sim \frac{4\,\theta(X(s))\;\; n(K_i)}{N}. \tag{12}$$

We have used Equation (9) which shows

$$p^N(X_N(s)) \sim \frac{2}{\sqrt{N}}\,\theta(X(s))$$

and Lemma 13 below which asserts that

$$\sqrt{N}\;P(B^N \in K_i) \to 2\,n(K_i).$$

At this point we need the following lemma.

Lemma 8 *Let $0 < t_1 < t_2$ and φ be a continuous function on $[t_1, t_2]$. Then, as $N \to +\infty$, almost surely,*

$$\frac{1}{N}\sum_{s\in D_N(t_1,t_2)} \varphi(s) \to \frac{1}{2}\int_{t_1}^{t_2} \varphi(s)\,ds$$

Proof of the lemma. For any interval $[a, b]$ of $\mathbb{R}_+$, we denote $\#D_N(a, b)$ the number of descents of X_N over $[a, b]$ i.e. the number of s in $D_N(a, b)$. We introduce a partition $[t_1 = s_0 < s_1 < \ldots < s_{k+1} = t_2]$ of the interval $[t_1, t_2]$. By immediate bounds,

$$\frac{1}{N}\sum_{s\in D_N(t_1,t_2)} \varphi(s) \le \frac{1}{N}\sum_{i=0}^{k} \#D_N(s_i, s_{i+1}) \sup_{[s_i,s_{i+1}]} \varphi. \tag{13}$$

But an elementary count of climbs and descents gives

$$2\,\#D_N(s_i, s_{i+1}) = N\,(s_{i+1} - s_i) - \sqrt{N}(X_N(s_{i+1}) - X_N(s_i))$$

which implies that, almost surely,

$$\#D_N(s_i, s_{i+1}) \sim N\,\frac{s_{i+1} - s_i}{2}.$$

From Inequality (13), we deduce

$$\limsup_{N\to+\infty} \frac{1}{N}\sum_{s\in D_N(t_1,t_2)} \varphi(s) \le \frac{1}{2}\sum_{i=0}^{k}(s_{i+1} - s_i) \sup_{[s_i,s_{i+1}]} \varphi.$$

When the stepsize of the considered subdivision goes to 0, the right-hand side above converges to $\int_{t_1}^{t_2} \varphi/2$. By symmetrical bounds we obtain obviously

$$\liminf_{N\to+\infty} \frac{1}{N}\sum_{s\in D_N(t_1,t_2)} \varphi(s) \ge \frac{1}{2}\int_{t_1}^{t_2} \varphi$$

and the proof of the lemma is complete.

Coming back to the proof of the theorem and more precisely Equation (11), we see, using (12) and a Taylor expansion of the logarithm, that the right-hand side in (11) has the same asymptotic behaviour as

$$\sum_{s\in D_N(t_1,t_2)} \sum_{i=1}^{p} (e^{-\lambda_i}-1)\frac{4\,\theta(X(s))\ \ n(U_i)}{N}.$$

To see this, the reader can note that (12) holds uniformly for $s \in [t_1, t_2]$. But the previous lemma implies that this quantity converges, as $N \to +\infty$, to

$$2\int_{t_1}^{t_2} ds\ \theta(X(s))\ \sum_{i=1}^{p}(e^{-\lambda_i}-1)\ n(U_i).$$

So, it follows from the preceding derivation and from obvious independence properties that, for a function $\varphi :\ \mathbb{R}_+ \times C^*(\mathbb{R}_+, \mathbb{R}_+) \to \mathbb{R}_+$ of the type

$$\varphi(s,e) = \sum_{i,j} \lambda_{i,j}\ \mathbf{1}_{[t_i,t_{i+1})}(s)\ \mathbf{1}_{U_j}(e)$$

with $U_j \subset \{e; \sigma(e) \le \eta\}$ for $\eta > 0$, we have

$$\begin{aligned}\lim_{N\to+\infty} \mathbb{E}&\left[\exp\int_0^t \varphi(s,e)\ \Lambda^N(ds\,de)\Big|X_N\right]\\ &= \exp\left[2\int n(de)\int_0^t ds\ \theta(X(s))\ \left(e^{\varphi(s,e)}-1\right)\right].\end{aligned}$$

We deduce the convergence in law of Λ^N toward a Poisson measure Λ on $\mathbb{R}_+ \times C(\mathbb{R}_+, \mathbb{R}_+)$ with intensity

$$2\,\theta(X(s))\ ds\ n(de).$$

By Skorokhod representation theorem, we may even suppose that the Λ^N are such that, almost surely, the measure Λ^N converges weakly to the measure Λ, when restricted to any set $[0,t] \times \{\sigma > \eta\}$ where $\eta, t > 0$. It follows that we can suppose that on each such set, the atoms of Λ^N converge to the atoms of Λ. This implies that

$$u \to \int_{\{s\le u\}} \Lambda^N(ds\,de)\ \mathbf{1}_{\{\sigma(e)\ge\eta\}}\ \sigma(e)$$

converges to

$$u \to \int_{\{s\le u\}} \Lambda(ds\,de)\ \mathbf{1}_{\{\sigma(e)\ge\eta\}}\ \sigma(e)$$

in the Skorokhod topology on càdlàg functions. Moreover

$$\begin{aligned}&\mathbb{E}\left(\int_{\{s\le u\}} \Lambda^N(ds,de)\ \mathbf{1}_{\{\sigma(e)\le\eta\}}\ \sigma(e)\right)\\ &\le \mathbb{E}\left(\sum_{s\le u, s\in\frac{1}{N}\mathbb{N}} \sigma(B(N,s))\ \mathbf{1}_{\{\sigma(B(N,s)\le\eta\}}\ p(X_N(s))\right)\\ &\le c\,\sqrt{N}\,\mathbb{E}\left(\sigma(B^N)\ \mathbf{1}_{\{\sigma(B^N)\le\eta\}}\right)\end{aligned}$$

and this last quantity is small for all N, provided η is chosen small enough, by Lemma 14. We can thus neglect small durations up to a set of small probability. We deduce the convergence in probability of Y^N as defined above to Y as defined in the statement of the theorem. But we know that the limit in law of Y^N is Brownian motion so we can conclude on the law of Y and the proof is complete.

At the price of a complexification of the notations, the ideas of the previous proof show that we can also reduce a drift $-\theta$ to $b-\theta \leq 0$ by adding excursions of a Brownian motion (subjected itself to the drift $b-\theta$).

Theorem 9 *Let $(X(t),\ t \geq 0)$ be a Brownian motion reflecting at 0 with continuous non-positive drift $-\theta(\cdot)$. Let b be a non-negative continuous function on $\mathbb{R}_+$ such that $b \leq \theta$. We define, conditionally on X, a Poisson point measure Λ on $\mathbb{R}_+ \times C(\mathbb{R}_+, \mathbb{R}_+)$ with intensity*

$$2\, b(X(s))\ \ ds\, n_{(X(s))}(de) \tag{14}$$

where $n_{(t)}$ denotes the Itô measure of positive excursions of Brownian motion with drift $(b-\theta)(t+\cdot)$. The function

$$A_u = u + \int_{\{s \leq u\}} \sigma(e)\ \Lambda(ds\, de)$$

is increasing right-continuous and has a jump $A_u - A_{u-} = \sigma(e_u)$ for every u such that $\Lambda(\{(u, e_u)\}) \neq 0$. We define $(Y(v))_{v \geq 0}$ by $Y(v) = X(u)$ if $v = A_u$ and $Y(v) = X(u) + e_u(v - A_{u-})$ for $A_{u-} \leq v < A_u$.

Then $(Y(t),\ t \geq 0)$ is a Brownian motion reflecting at 0 with drift $-\theta(\cdot) + b(\cdot)$.

3.4 Extension and applications

Theorem 7 remains true if $(X(t))$ is a Brownian motion starting from $x \in \mathbb{R}$ with continuous non-positive drift $-\theta(\cdot)$ and in this case $(Y(t))$ is a Brownian motion (starting from $x \in \mathbb{R}$).

Indeed we can look at Theorem 7 when applied on $[T_x, T_0']$ where T_x is the hitting time of $x > 0$ and T_0' the following return to 0. As x is arbitrary and by translation invariance this prove the result mentioned above for a Brownian motion starting from $x \in \mathbb{R}$ up to the hitting time of any lower value. We give an application establishing a connection between Brownian motion and Brownian motion with drift.

Proposition 10 *Let $\mathbb{P}_x$ and $\mathbb{P}_x^\theta$ denote respectively the law of Brownian motion and Brownian motion with continuous non-positive drift $-\theta(\cdot)$, both starting at $x > 0$. Let T_0 denote the hitting time of 0. Then we have*

$$\mathbb{E}_x^\theta \left[\exp - \int_0^{T_0} g(B_s)\, ds \right] = \mathbb{E}_x \left[\exp - \int_0^{T_0} f(B_s)\, ds \right] \tag{15}$$

when f, g are continuous non-negative functions such that $f + \theta\, \tilde{f} = g$ where $\tilde{f}$ denotes the function

$$\tilde{f}(x) = 2 \int n(de) \left(1 - \exp - \int_0^\sigma f(x + e_r)\, dr \right)$$

which is a solution of the Ricatti differential equation $y' = -2\, f + y^2$.

Proof. The Equality (15) is a straightforward consequence of Theorem 7 and the classical exponential formula for Poisson measures. We then sketch the proof that $\tilde{f}$ satisfies the given Ricatti equation:

$$\begin{aligned}
\frac{1}{2}\, \tilde{f}(x) &= \int n(de) \int_0^\sigma \exp - \left(\int_0^s f(x + e_r)\, dr \right) f(x + e_s)\, ds \\
&= \int_0^{+\infty} dy\, f(x + y) \\
&\qquad \cdot \exp -2 \left[\int_0^y dh \int n(de) \left(1 - \exp - \int_0^\sigma f(x + h + e_u)\, du \right) \right] \\
&= \int_0^{+\infty} dy\, f(x + y)\, \exp -2 \int_0^y dh\, \tilde{f}(x + h) \\
&= \int_x^{+\infty} dy\, f(y)\, \exp -2 \int_x^y dh\, \tilde{f}(h)
\end{aligned}$$

The first equality is elementary calculus, the third one uses only the definition of $\tilde{f}$ and the fourth one is a change of variables. The second one involves more sophisticated arguments; first Bismut's description of the Brownian excursion under Itô measure; then we use the excursions above the future infimum of $(e(r), r \le s)$ which is a three dimensional Bessel process run up to a hitting time; these excursions have the same intensity as the excursions of a reflected Brownian motion and we finish with the exponential formula. Finally the last equality leads easily to the Ricatti equation.

As a (trivial) example consider the case of constant θ and f so that g is also constant. We obtain, using the well-known Laplace transform of T_0 under $\mathbb{P}_x$,

$$\mathbb{E}_x^\theta \left[e^{-g\, T_0} \right] = \mathbb{E}_x \left[e^{-f\, T_0} \right] = e^{-x \sqrt{2\, f}} = e^{-x \left(\sqrt{\theta^2 + 2g} - \theta \right)}$$

as could also be obtained by an application of Girsanov Theorem.

More sophisticated applications, in the setting of super-processes will be given in [Se].

4 Appendix

4.1 Proof for Proposition 2

To simplify notation we restrict ourselves to the convergence of $(X_N(s), s \in [0,1])$. We denote (U_N) a reflecting and rescaled standard walk $\mathcal{RW}(N,0)$. Let F be a continuous function on $C([0,1],\mathbb{R}_+)$. By the definition of the law of X_N given by Formulas (6, 7), we have

$$\mathbb{E}\left[F(X_N)\right] = \mathbb{E}\Bigg[F(U_N) \times \prod_{k=0}^{N-1}\Bigg(1 - \mathbf{1}_{\{U_N(\frac{k}{N})\neq 0\}}\left(U_N(\frac{k+1}{N}) - U_N(\frac{k}{N})\right)\,\theta_N(U_N(\frac{k}{N}))\Bigg)\Bigg].$$

We introduce a reflecting Brownian motion $(B_s)_{s\in\mathbb{R}_+}$, starting from $B_0 = 0$ and the stopping times : $T_0^N = 0$,

$$T_{k+1}^N = \inf\left\{s > T_k^N,\ |B_s - B_{T_k^N}| = \frac{1}{\sqrt{N}}\right\}.$$

It is clear that $(B_{T_k^N},\ 0 \le k \le N)$ is identically distributed as $(U_N(k/N), 0 \le k \le N)$. We set

$$B_s^N = B_{T_k^N} + (N\,s - k)(B_{T_{k+1}^N} - B_{T_k^N}) \text{ for } s \in [k/N, (k+1)/N).$$

We get

$$\mathbb{E}\left[F(X_N)\right] = \mathbb{E}\left[F(B^N)\ \exp(L_N)\right]$$

where

$$\begin{aligned}
L_N &= \sum_{k=0}^{N-1} \log\left(1 - \mathbf{1}_{\{B_{T_k^N}\neq 0\}}\ (B_{T_{k+1}^N} - B_{T_k^N})\,\theta_N(B_{T_k^N})\right) \\
&= -\sum_{k=0}^{N-1} \mathbf{1}_{\{B_{T_k^N}\neq 0\}}\ (B_{T_{k+1}^N} - B_{T_k^N})\,\theta_N(B_{T_k^N}) \\
&\qquad -\frac{1}{2}\,\frac{1}{N}\sum_{k=0}^{N-1} \mathbf{1}_{\{B_{T_k^N}\neq 0\}}\,\theta_N(B_{T_k^N})^2 + R_N
\end{aligned}$$

with R_N being a remainder which converges to 0 in probability. By the Markov property for B and the scaling property of Brownian motion we can write, for $k \le N$,

$$T_k^N = \frac{1}{N}\sum_{j=1}^{k} V_j$$

where $V_1, V_2, \ldots$ are independent and distributed as the hitting time of $\{1, -1\}$ for a Brownian motion starting from 0. In particular $\mathbb{E}(V_1) = 1$. By Kolmogorov's Lemma (or Doob's inequality) we deduce for $\varepsilon > 0$, that

$$\mathbb{P}\left[\sup_{k \le N} |T_k^N - \frac{k}{N}| \ge \varepsilon\right] = \mathbb{P}\left[\frac{1}{N}\sup_{k \le N}\left|\sum_{j=1}^{k}(V_j - 1)\right| \ge \varepsilon\right] \le \frac{1}{\varepsilon^2 N}\mathrm{Var}(V_1)\,.$$

This shows that $\sup_{k \le N} |T_k^N - \frac{k}{N}|$ converges to 0 in probability as $N \to +\infty$ and thus almost surely along a subsequence. Then it follows that $B_s^N \to B_s$, uniformly in s, almost surely when $N \to +\infty$ along the previous subsequence.

Noting $T_N^N \to 1$, a. s., it follows from standard arguments (see for instance [RY] Proposition IV.2.13) that, a. s., for $N \to +\infty$ along a subsequence,

$$L_N \to L = -\int_0^1 \theta(B_s)\, \mathbf{1}_{\{B_s \neq 0\}}\, dB_s - \frac{1}{2}\int_0^1 \theta(B_s)^2\, ds.$$

Since the extraction of a converging subsequence can be made from any sequence along which N goes to infinity, we claim that $F(B^N)\exp L_N \to F(B)\exp L$ in probability. It is easy to prove, by using induction and the Markov property for B, that

$$\sup_N \mathbb{E}\left[\left(F(B^N)\ \exp L_N\right)^2\right] < +\infty.$$

We conclude that $\mathbb{E}\left[F(X_N)\right] \to \mathbb{E}\left[F(B)\ \exp L\right]$ and this is, via Girsanov Theorem, the desired result.

4.2 Convergence of discrete excursions and walks

Lemma 11 *Let $(e^N(s),\ s \ge 0)$ be an excursion of the rescaled reflecting standard random walk $\mathcal{RW}(N, 0)$. Let $\eta > 0$ and F be a bounded continuous function on $C^*(\mathbb{R}_+, \mathbb{R}_+)$ null on $\{\sigma < \eta\}$.*

Then, we have

$$\sqrt{N}\,\mathbb{E}\left[F(e^N)\right] \stackrel{N \to +\infty}{\longrightarrow} 2\int F(e)\, n(de)$$

Proof. Let B be a standard (non-rescaled) random walk on $\mathbb{N}$ starting from 0 and stopped at its first return to 0, denoted $\sigma(B)$ so that e^N is the renormalization of B by $1/N$ in time and $1/\sqrt{N}$ in space. A classical exercise on reflection principle gives that

$$\mathbb{P}(\sigma(B) = 2n) = \binom{2n}{n}\frac{2^{-2n}}{2n-1} \sim \frac{1}{\sqrt{\pi\, n}\, 2n}. \qquad (16)$$

We deduce that

$$\mathbb{P}\left(\sigma(e^N) \geq \eta\right) \sim \frac{\sqrt{2}}{\sqrt{\eta\, N\, \pi}}.$$

It is well known (see for instance [RY] Proposition XII.2.8) that

$$n\left(\sigma(e) \geq \eta\right) = \frac{1}{\sqrt{2\pi\, \eta}}.$$

So, it suffices to prove that

$$\mathbb{E}\left[F(e^N)\middle|\sigma(e^N) \geq \eta\right] \overset{N\to+\infty}{\longrightarrow} \int F(e)\, n(de|\sigma(e) \geq \eta).$$

This is a conditioned version of Donsker invariance Theorem for which we refer to [Ka].

Lemma 12 *Let $(B^N(s),\ s \geq 0)$ be a standard rescaled reflecting random walk $\mathcal{RW}(N,0)$, stopped at the time of the g-th return at 0. Let $\eta > 0$ and F be a bounded continuous function on $C^*(\mathbb{R}_+, \mathbb{R}_+)$ null on $\{\sigma < \eta\}$.*

Then, we have

$$\sqrt{N}\, \mathbb{E}\left[F(B^N)\right] \overset{N\to+\infty}{\longrightarrow} g\, 2 \int F(e)\, n(de)\,. \tag{17}$$

Proof. For simplicity of notations let us suppose in fact that F vanishes on $\{\sigma < g\, \eta\}$. We denote $e_1^N, \ldots, e_g^N$ the excursions of B^N. We have to work on the event that at least one of these excursions has a duration greater than η. From the proof of Lemma 11, we recall that

$$\mathbb{P}\left(\sigma(e_i^N) \geq \eta\right) \leq \frac{c}{\sqrt{N}}$$

so the event that two excursions are of duration larger than η is of order $1/N$ and can be asymptotically neglected. We set $H(x.) = \sup_s |x_s|$ for $x. \in C^*(\mathbb{R}_+, \mathbb{R}_+)$. The renormalization done on e^N shows that

$$\mathbb{P}\left(H(e^N) \geq \varepsilon(N)\right) \to 0 \text{ if } \sqrt{N}\, \varepsilon(N) \to +\infty$$

and similarily

$$\mathbb{P}\left(\sigma(e^N) \geq \varepsilon(N)\right) \to 0 \text{ if } N\, \varepsilon(N) \to +\infty\,.$$

From now on, we fix $\varepsilon(N) \to 0$ such that the first (hence both) of the above conditions hold. We work on one of the g events

$$\left\{\sigma(e_i^N) \geq \eta,\ \forall j \neq i,\ H(e_j^N) \leq \varepsilon(N),\ \sigma(e_j^N) \leq \varepsilon(N)\right\}$$

where $i \in \{1, \ldots, g\}$. On such an event, we have

$$|B^N(s) - e_i^N(s)| \leq \varepsilon(N) + \sup_{s \leq g\varepsilon(N)} |e_i^N(s)| + \sup_{s,\, u \leq g\varepsilon(N)} \left|e_i^N(s+u) - e_i^N(s)\right|, \tag{18}$$

because $|B^N(s) - e_i^N(s)|$ is smaller than

$$\sup_{r,\, j>i} |e_j^N(r)| \text{ if } s \geq \sum_{j\leq i} \sigma(e_j^N),$$

or is lower than

$$\sup_r \left| e_i^N \left(r + \sum_{j<i} \sigma(e_j^N) \right) - e_i^N(r) \right| \text{ if } \sum_{j<i} \sigma(e_j^N) \leq s \leq \sum_{j\leq i} \sigma(e_j^N),$$

or is lower than

$$\sup_{r,\, j<i} |e_j^N(r)| + \sup_{r \leq \sum_{j<i} \sigma(e_j^N)} |e_i^N(r)| \text{ if } s \leq \sum_{j<i} \sigma(e_j^N).$$

But under $\sqrt{N}\,\mathbb{P}$ restricted to $\{\sigma \geq \eta\}$, e_i^N converges in distribution to n restricted to $\{\sigma \geq \eta\}$. It follows that the right-hand-side of (18) converges to 0 in probability.

We deduce from these facts that the left-hand side in (17) has the same limit as $g\sqrt{N}\,\mathbb{E}\left[F(e^N)\right]$ and this one is given by Lemma 11.

Lemma 13 *Let* $(B^N(s),\ s \geq 0)$ *be a standard rescaled reflecting random walk* $\mathcal{RW}(N,0)$*, stopped at the time of the g-th return at 0 where g is an independent random variable with law* $\mathcal{G}(\frac{1}{2})$*. Then, for any mesurable* $U \subset C^*(\mathbb{R}_+,\mathbb{R}_+) \cap \{\sigma > \eta\}$ *with* $\eta > 0$ *such that* $n(\partial U) = 0$*,*

$$\sqrt{N}\,\mathbb{P}[B^N \in U] \overset{N\to+\infty}{\longrightarrow} 2\, n(U)\,.$$

Proof. We first randomize g in the limit (17) according to the law $\mathcal{G}(\frac{1}{2})$ as specified here. But this law has mean 1 so (17) is now re-expressed in our new setting by replacing "g" by 1. A reformulation of this result of limit in the language of sets is the above statement.

Lemma 14 *Let* $(B^N(s),\ s \geq 0)$ *be a standard rescaled reflecting random walk* $\mathcal{RW}(N,0)$*, stopped at the time of the g-th return at 0 where g is an independent random variable with law* $\mathcal{G}(\frac{1}{2})$*. Let* $\varepsilon > 0$*.*

Then there exists $\eta > 0$ *such that, for every* N*,*

$$\sqrt{N}\,\mathbb{E}\left(\sigma(B^N)\,\mathbf{1}_{\{\sigma(B^N)\leq\eta\}}\right) \leq \varepsilon.$$

Proof. By conditioning by the value of g, it suffices to prove the same result with B^N replaced by e^N. As before, we denote by B a standard non-rescaled random walk on $\mathbb{N}$ starting from 0 and stopped at its first return to 0 i.e. e^N is the renormalization of B by $1/N$ in time and $1/\sqrt{N}$ in space. We now have to find $\eta > 0$ such that, for every N,

$$\frac{1}{\sqrt{N}}\,\mathbb{E}\left[\sigma(B)\,\mathbf{1}_{\{\sigma(B)\leq\eta N\}}\right] \leq \varepsilon.$$

From Formula (16) we deduce

$$\mathbb{E}\left(\sigma(B)\,\mathbf{1}_{\{\sigma(B)\leq \eta N\}}\right) \leq c \sum_{n=1}^{\eta N/2} \frac{1}{\sqrt{n}}\,.$$

But this quantity behaves like $\sqrt{\eta N}$ and the proof of the lemma is complete.

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