Chapter 4
From Anaplectic Analysis to Usual Analysis

It is possible to consider anaplectic analysis on the real line as a special case of a one-parameter family of analyses. The parameter ν is a complex number mod 2, subject to the restriction that it should not be an integer: anaplectic analysis, as considered until now, corresponds to the case when $\nu = -\frac{1}{2}$. There is a natural ν-anaplectic representation of some cover of $SL(2,\mathbb{R})$ in some space $\mathfrak{A}_\nu$, compatible in the usual way with the Heisenberg representation; the ν-anaplectic representation is pseudo-unitarizable in the case when ν is real. Depending on ν, the even or odd part of the ν-anaplectic representation coincides in this case with a representation taken from the unitary dual of the universal cover of $SL(2,\mathbb{R})$, as completely described by Pukanszky [24]. In ν-anaplectic analysis, the spectrum of the harmonic oscillator is the arithmetic sequence $\nu + \frac{1}{2} + \mathbb{Z}$. Much of the theory subsists in the case when $\nu \equiv 0 \bmod 2$, which leads to a nontrivial enlargement of usual analysis. However, as we shall make clear, while the ascending pseudodifferential calculus extends to the case of ν-anaplectic analysis when $\nu \in \mathbb{C}\backslash\mathbb{Z}$, it is impossible to extend it to the usual analysis environment.

4.1 The ν-Anaplectic Representation

The one-parameter generalization of anaplectic analysis to be summed up in the present section was introduced in [38, Sects. 11–12]. The most natural way to characterize the space $\mathfrak{A}_\nu$ of functions on the real line which is basic in ν-anaplectic analysis is probably that which follows from a generalization of Proposition 2.2.11: this characterization will be given in Theorem 4.1.3. However, we prefer to follow the plan of Sect. 2.2, starting from the $\mathbb{C}^4$-realization of functions in $\mathfrak{A}_\nu$.

Definition 4.1.1. Let $\nu \in \mathbb{C}\backslash\mathbb{Z}$ and consider the space of $\mathbb{C}^4$-valued functions $\boldsymbol{f} = (f_0, f_1, f_{i,0}, f_{i,1})$ with the following properties: each component of $\boldsymbol{f}$ is a nice function in the sense of Definition 2.2.1, and the components are linked by the following equations:

$$f_{i,0}(x) = \frac{\Gamma(-\nu)}{(2\pi)^{\frac{1}{2}}} \left[e^{-\frac{i\pi}{2}(\nu+1)} f_0(ix) + e^{\frac{i\pi}{2}(\nu+1)} f_0(-ix) \right],$$
$$f_{i,1}(x) = \frac{\Gamma(-\nu)}{(2\pi)^{\frac{1}{2}}} \left[e^{-\frac{i\pi\nu}{2}} f_1(ix) + e^{\frac{i\pi\nu}{2}} f_1(-ix) \right]. \tag{4.1.1}$$

The space $\mathfrak{A}_\nu$ is the image of the space of functions so defined under the map $\boldsymbol{f} \mapsto u$, where the even (resp., odd) part of u is the even part of f_0 (resp., the odd part of f_1). We shall also refer to $\boldsymbol{f}$ as the $\mathbb{C}^4$-realization of u.

Remark 4.1.1. The space $\mathfrak{A}_\nu$ only depends on ν mod 2: for if $(f_0, f_1, f_{i,0}, f_{i,1})$ is a $\mathbb{C}^4$-realization of u when the parameter ν is considered, the vector $\boldsymbol{h} = (f_0, f_1, h_{i,0}, h_{i,1})$, with

$$h_{i,0} = -((\nu+1)(\nu+2))^{-1} f_{i,0}, \qquad h_{i,1} = -((\nu+1)(\nu+2))^{-1} f_{i,1}, \tag{4.1.2}$$

is a $\mathbb{C}^4$-realization of u in the space $\mathfrak{A}_{\nu+2}$. When using $\mathbb{C}^4$-realizations, we shall always assume that a value of $\nu \in \mathbb{C}\backslash\mathbb{Z}$ has been fixed.

Again, the Phragmén–Lindelöf lemma makes it possible to show that the map $\boldsymbol{f} \mapsto u$ is one to one. Unless $\nu \in -\frac{1}{2} + \mathbb{Z}$, the space $\mathfrak{A}_\nu$ is not invariant under the complex rotation by 90°. However, the following holds.

Proposition 4.1.2. *The map $u \mapsto u_i$, with $u_i(x) = u(ix)$, is a linear isomorphism from $\mathfrak{A}_\nu$ to $\mathfrak{A}_{-\nu-1}$. If $(f_0, f_1, f_{i,0}, f_{i,1})$ is the $\mathbb{C}^4$-realization of u in the $\mathfrak{A}_\nu$-analysis, that of u_i in the $\mathfrak{A}_{-\nu-1}$-analysis is*

$$(h_0, h_1, h_{i,0}, h_{i,1}) = C_\nu \,(f_{i,0}, -i\, f_{i,1}, f_0, -i\, f_1,) \tag{4.1.3}$$

with

$$C_\nu = 2^{\nu+\frac{1}{2}} \frac{\Gamma(\frac{2+\nu}{2})}{\Gamma(\frac{1-\nu}{2})}. \tag{4.1.4}$$

The Heisenberg transformations $e^{2i\pi(\eta Q - yP)}$, with $(y, \eta) \in \mathbb{C}^2$, preserve the space $\mathfrak{A}_\nu$.

Theorem 4.1.3. *Let u be an entire function of one variable satisfying for some pair of constants C, R the estimate $|f(z)| \leq C e^{\pi R |z|^2}$. Define the functions $(\mathcal{Q}u)_j$ and $(\mathcal{K}u)_j$ ($j = 0$ or 1) in the same way as in Proposition 2.2.11. Given $\nu \in \mathbb{C}\backslash\mathbb{Z}$, the following three conditions are equivalent:*

(i) u lies in the space $\mathfrak{A}_\nu$;
(ii) each of the two functions $(\mathcal{Q}u)_0$ and $(\mathcal{Q}u)_1$ extends as an analytic function on the real line, admitting for large $|\sigma|$ a convergent expansion $(\mathcal{Q}u)_j(\sigma) = e^{-\frac{i\pi}{2}(\nu+\frac{1}{2})\,\mathrm{sign}\,\sigma} \sum_{n\geq 0} a_n^{(j)} \sigma^{-n} |\sigma|^{-\frac{1}{2}}$;
(iii) each of the two functions $(\mathcal{K}u)_0$ and $(\mathcal{K}u)_1$, initially defined in a neighborhood of the point $z = 1$ of the unit circle S^1, extends as an analytic function to the universal cover of S^1, satisfying the quasiperiodicity conditions

$$(\mathcal{K}u)_j(e^{i\theta}) = e^{-i\pi(\nu+\frac{1}{2})}(\mathcal{K}u)_j(e^{i(\theta-2\pi)}). \tag{4.1.5}$$

We now consider the ν-analogue of Theorem 2.2.3, searching for the eigenfunctions of the standard harmonic oscillator $L = \pi(Q^2+P^2)$ which lie in $\mathfrak{A}_\nu$. As a consequence of the WKB method, near each of the two endpoints $\pm\infty$ of the real line, the equation $Lf = (\nu+\frac{1}{2})f$ has two solutions: one that behaves like $|x|^\nu e^{-\pi x^2}$ and another like $|x|^{-\nu-1} e^{\pi x^2}$. Since we are assuming that $\nu \neq 0, 1, \dots$, no solution can be rapidly decreasing toward $\pm\infty$ simultaneously. In [21, Chap. 8], one denotes as $f(x) = D_\nu(2\pi^{\frac{1}{2}}x)$ the solution of the equation $Lf = (\nu+\frac{1}{2})f$ normalized by the condition

$$f(x) \sim (2\pi^{\frac{1}{2}}x)^\nu e^{-\pi x^2}, \qquad x \to +\infty: \tag{4.1.6}$$

of course, such a function is very far from lying in $L^2(\mathbb{R})$.

The following definition generalizes Proposition 2.2.2: in the case when $\nu = -\frac{1}{2}$, the function $\psi^{-\frac{1}{2}}$ to be introduced now and the function ϕ introduced there only differ by some normalizing factor. That the $\mathbb{C}^4$-valued function below is indeed the $\mathbb{C}^4$-realization of some function in $\mathfrak{A}_\nu$, i.e., that the equations (4.1.1) are valid, is a consequence of [21, p. 330].

Proposition 4.1.4. *Let $\nu \notin \mathbb{Z}$ and let ψ^ν be the function in $\mathfrak{A}_\nu$ the $\mathbb{C}^4$-realization of which is the function*

$$\boldsymbol{f}(x) = (D_\nu(2\pi^{\frac{1}{2}}x), 0, D_{-\nu-1}(2\pi^{\frac{1}{2}}x), 0). \tag{4.1.7}$$

One has, for $0 < \theta < 2\pi$,

$$(\mathcal{K}\psi^\nu)_0(e^{-i\theta}) = \frac{2^{\frac{\nu-1}{2}}\pi^{\frac{1}{2}}}{\Gamma(\frac{1-\nu}{2})} e^{\frac{i}{2}(\nu+\frac{1}{2})\theta}. \tag{4.1.8}$$

Theorem 4.1.5. *The set of eigenvalues of the harmonic oscillator in the space $\mathfrak{A}_\nu$ is the arithmetic sequence $\nu+\frac{1}{2}+\mathbb{Z}$. The eigenspace corresponding to an eigenvalue $\nu+2j+\frac{1}{2}$, $j\in\mathbb{Z}$, is generated by the function $\psi^{\nu+2j}$ with the $\mathbb{C}^4$-realization*

$$x \mapsto \left(D_{\nu+2j}(2\pi^{\frac{1}{2}}x), 0, (-1)^j\frac{\Gamma(\nu+2j+1)}{\Gamma(\nu+1)}D_{-\nu-2j-1}(2\pi^{\frac{1}{2}}x), 0\right) \tag{4.1.9}$$

and the eigenspace corresponding to an eigenvalue $\nu+2j+\frac{3}{2}$, $j\in\mathbb{Z}$, is generated by the function $\chi^{\nu+2j+1}$ with the $\mathbb{C}^4$-realization

$$x \mapsto \left(0, D_{\nu+2j+1}(2\pi^{\frac{1}{2}}x), 0, (-1)^j\frac{\Gamma(\nu+2j+2)}{\Gamma(\nu+1)}D_{-\nu-2j-2}(2\pi^{\frac{1}{2}}x)\right). \tag{4.1.10}$$

One has the relations

$$\begin{aligned} A\,\psi^{\nu+2j} &= (\nu+2j)\,\chi^{\nu+2j-1}, & A^*\,\psi^{\nu+2j} &= \chi^{\nu+2j+1},\\ A\,\chi^{\nu+2j+1} &= (\nu+2j+1)\,\psi^{\nu+2j}, & A^*\,\chi^{\nu+2j+1} &= \psi^{\nu+2j+2}. \end{aligned} \tag{4.1.11}$$

Remark 4.1.2. By its definition in Proposition 4.1.4, the function $\psi^{\nu+2j}$ lies in $\mathfrak{A}_{\nu+2j} = \mathfrak{A}_\nu$: however, as seen from Remark 4.1.1, its $\mathbb{C}^4$-realization is not the same in the ν-anaplectic or in the $(\nu+2j)$-anaplectic theory. One should not make a confusion between the odd function $\chi^{\nu+2j+1} \in \mathfrak{A}_\nu$ and the even function $\psi^{\nu+2j+1}$, which does not lie in $\mathfrak{A}_\nu$ but in $\mathfrak{A}_{\nu+1}$, and which will not concern us; in analogy with (4.1.8), one has the equation

$$(\mathcal{K}\chi^{\nu+1})_1(e^{-i\theta}) = e^{\frac{i\pi}{4}} \, \frac{2^{\frac{\nu-1}{2}}\,(\nu+1)}{\Gamma(\frac{1-\nu}{2})} \, e^{\frac{i}{2}(\nu+\frac{1}{2})\theta}. \tag{4.1.12}$$

Equation (11.56) given in [38, Theorem 11.10] is erroneous and should be replaced by the present equation (4.1.12). It is the equation

$$\pi^{\frac{1}{2}} x^2 - \frac{1}{2\pi^{\frac{1}{2}}}\, x \frac{d}{dx} = \frac{\pi^{-\frac{1}{2}}}{2} \, [L + A^{*2} + \frac{1}{2}] \tag{4.1.13}$$

which must take the place of equation (11.58) given there.

Remark 4.1.3. Since [21, p. 326] $D_{-\frac{1}{2}}(2\pi^{\frac{1}{2}} x) = \pi^{-\frac{1}{4}}\, x^{\frac{1}{2}}\, K_{\frac{1}{4}}(\pi x^2)$ for $x > 0$, one sees, comparing $\psi^{-\frac{1}{2}}$ with the function ϕ introduced in Proposition 2.2.2 and using the analytic continuation of the function $K_{\frac{1}{4}}$ as provided, for instance, by [21, p. 69], that $\phi = 2^{\frac{1}{2}}\, \pi^{-\frac{1}{4}}\, \psi^{-\frac{1}{2}}$. More generally, the function $\psi^{-\frac{1}{2}+2j}$ is a multiple of the function denoted as ϕ^{2j} in (2.2.10) and the function $\chi^{-\frac{1}{2}+2j+1}$ is a multiple of the function ϕ^{2j+1}. The coefficients of proportionality can be obtained from (4.1.11) and Lemma 2.2.9.

Contrary to that of the $\mathbb{C}^4$-realization of a function in $\mathfrak{A}_\nu$, Definition 4.1.6 depends only on ν mod 2.

Definition 4.1.6. Given $u \in \mathfrak{A}_\nu$, we set

$$\mathrm{Int}\,[u] = e^{\frac{i\pi}{2}(\nu+\frac{1}{2})} \left[2\cos\frac{\pi\nu}{2} \int_0^\infty f_0(x)\,dx + \frac{(2\pi)^{\frac{1}{2}}}{\Gamma(-\nu)} \int_0^\infty f_{i,0}(x)\,dx \right]. \tag{4.1.14}$$

The ν-anaplectic Fourier transformation $\mathcal{F}^\nu_{\mathrm{ana}}$ is defined by the equation

$$(\mathcal{F}^\nu_{\mathrm{ana}}\, u)(x) = \mathrm{Int}\,[y \mapsto e^{-2i\pi xy}\, u(y)], \qquad u \in \mathfrak{A}_\nu. \tag{4.1.15}$$

Theorem 4.1.7. *The linear form* Int *is invariant under the (real or complex) Heisenberg translations $\pi(y,0)$, and the ν-anaplectic Fourier transformation is a linear automorphism of the space $\mathfrak{A}_\nu$. In terms of the $\mathcal{K}$-transform of u, one has*

$$\mathrm{Int}\,[u] = 2^{\frac{1}{2}}\,(\mathcal{K}u)_0(e^{-i\pi}) \tag{4.1.16}$$

and the equations

$$\begin{aligned}(\mathcal{K}(\mathcal{F}^\nu_{\mathrm{ana}}\, u))_0(z) &= e^{i\pi(\nu+\frac{1}{2})}\,(\mathcal{K}u)_0(e^{i\pi}z),\\ (\mathcal{K}(\mathcal{F}^\nu_{\mathrm{ana}}\, u))_1(z) &= e^{i\pi\nu}\,(\mathcal{K}u)_1(e^{i\pi}z).\end{aligned} \tag{4.1.17}$$

For every $\nu \in \mathbb{C}\backslash\mathbb{Z}$, there is a ν-version of the anaplectic representation. The following generalizes Theorem 2.2.3: however, unless $\nu \in -\frac{1}{2}+\mathbb{Z}$, it is necessary to substitute for $SL(2,\mathbb{R})$ a cover of this group to get a genuine representation. Note that elements such as $\left(\begin{smallmatrix}1&0\\c&1\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}a&0\\0&a^{-1}\end{smallmatrix}\right)$ of $SL(2,\mathbb{R})$ are naturally associated to elements, denoted in the same way, of the universal cover of $SL(2,\mathbb{R})$; in what follows, $\exp\frac{\pi}{2}\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ denotes the element of the cover $G^{(N)}$ under consideration, which is the end of the path originating at the identity and covering the path $t \mapsto \left(\begin{smallmatrix}\cos t&\sin t\\-\sin t&\cos t\end{smallmatrix}\right)$ of $SL(2,\mathbb{R})$.

Theorem 4.1.8. *Assume that $N=\infty$ or that N is a positive integer such that $N\left(\frac{\nu}{2}+\frac{1}{4}\right) \in \mathbb{Z}$. There exists a unique representation* Ana_ν *of the N-fold cover $G^{(N)}$ of $G = SL(2,\mathbb{R})$ in the space $\mathfrak{A}_\nu$ with the following properties:*

(i) if $g = \left(\begin{smallmatrix}1&0\\c&1\end{smallmatrix}\right)$, one has $(\mathrm{Ana}_\nu(g)\,u)(x) = u(x)\,e^{i\pi cx^2}$;
(ii) if $g = \left(\begin{smallmatrix}a&0\\0&a^{-1}\end{smallmatrix}\right)$ with $a>0$, one has $(\mathrm{Ana}_\nu(g)\,u)(x) = a^{-\frac{1}{2}}\,u(a^{-1}x)$;
(iii) one has $\mathrm{Ana}_\nu\left(\exp\frac{\pi}{2}\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)\right) = e^{-i\pi(\nu+\frac{1}{2})}\,\mathcal{F}^\nu_{\mathrm{ana}}$.

This representation combines with the Heisenberg representation in the way expressed by (2.2.18)*, only replacing* Ana *by* Ana_ν.

The ν-anaplectic representation can be defined globally with the help of $\mathcal{K}$-transforms of functions in Ana_ν. Assume that $g \in G^{(N)}$ lies above the matrix $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) \in G$, and set $\left(\begin{smallmatrix}\alpha&\beta\\\bar\beta&\bar\alpha\end{smallmatrix}\right) = S\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)S^{-1}$ with $S = 2^{-\frac{1}{2}}\left(\begin{smallmatrix}1&i\\i&1\end{smallmatrix}\right)$. On the other hand, associate with $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ the matrix

$$\begin{pmatrix}\lambda&\mu\\\bar\mu&\bar\lambda\end{pmatrix} = \frac{1}{2}\begin{pmatrix}a-ib+ic+d & -a-ib-ic+d\\ -a+ib+ic+d & a+ib-ic+d\end{pmatrix} \tag{4.1.18}$$

and the transformation $z \mapsto \frac{\lambda z+\mu}{\bar\mu z+\bar\lambda}$ of S^1: this extends as a homomorphism $g \mapsto [g]$ of $G^{(N)}$ into the group of analytic automorphisms of the N-fold cover $\Sigma^{(N)}$ of $\Sigma = S^1$. Denoting as $\frac{[g^{-1}]_*\,d\theta}{d\theta}$ the Radon–Nikodym derivative of the transformation $[g^{-1}]$ with respect to the "rotation"-invariant measure of $\Sigma^{(N)}$, one can characterize the ν-anaplectic representation by the pair of equations

$$\begin{aligned}(\mathcal{K}\,\mathrm{Ana}_\nu(g)\,u)_0(z) &= \left(\frac{[g^{-1}]_*\,d\theta}{d\theta}\,(z)\right)^{\frac{1}{4}}\,(\mathcal{K}\,u)_0([g^{-1}]\,(z)),\\ (\mathcal{K}\,\mathrm{Ana}_\nu(g)\,u)_1(z) &= \left(\frac{[g^{-1}]_*\,d\theta}{d\theta}\,(z)\right)^{\frac{1}{4}}\left[\alpha - i\beta\,([g^{-1}]\,(z))^{-1}\right].(\mathcal{K}\,u)_1([g^{-1}]\,(z)).\end{aligned} \tag{4.1.19}$$

Finally, when ν is real (and not an integer), there is on $\mathfrak{A}_\nu$ a nondegenerate pseudoscalar product invariant under the ν-anaplectic representation as well as under the Heisenberg representation (considering this time only the operators $e^{2i\pi(\eta Q - yP)}$ with $(y,\eta) \in \mathbb{R}^2$).

Theorem 4.1.9. *Let $\nu \in \mathbb{R}\backslash\mathbb{Z}$. The ν-anaplectic representation and the Heisenberg representation are both pseudo-unitary with respect to the pseudoscalar product on* Ana_ν *defined, in terms of the $\mathbb{C}^4$-realization $(f_0, f_1, f_{i,0}, f_{i,1})$ of u, by the equation*

$$(u\,|\,u) = 2^{\frac{1}{2}} \int_0^\infty \left[|f_0|^2 + |f_1|^2 + \frac{\Gamma(\nu+1)}{\Gamma(-\nu)} \left(|f_{i,0}|^2 - |f_{i,1}|^2\right) \right] dx. \tag{4.1.20}$$

This pseudoscalar product is nondegenerate and only depends on ν mod 2. In terms of the $\mathcal{K}$-realization, setting

$$(\mathcal{K}u)_0(z) = z^{-\frac{1}{2}(\nu+\frac{1}{2})} \sum_{j\in\mathbb{Z}} c_j z^{-j},$$
$$(\mathcal{K}u)_1(z) = z^{-\frac{1}{2}(\nu+\frac{1}{2})} \sum_{j\in\mathbb{Z}} c'_j z^{-j}, \tag{4.1.21}$$

one has

$$(u\,|\,u) = \frac{\pi^{\frac{1}{2}}}{\cos^2 \frac{\pi\nu}{2}} \sum_{j\in\mathbb{Z}} \Gamma(\frac{\nu}{2}+j+1) \left[\frac{2}{\Gamma(\frac{\nu+1}{2}+j)} |c_j|^2 + \frac{\pi}{2\Gamma(\frac{\nu+3}{2}+j)} |c'_j|^2 \right]. \tag{4.1.22}$$

In particular,

$$(\psi^{\nu+2j}\,|\,\psi^{\nu+2j}) = \frac{1}{2}\Gamma(\nu+2j+1),$$
$$(\chi^{\nu+2j+1}\,|\,\chi^{\nu+2j+1}) = \frac{1}{2}\Gamma(\nu+2j+2). \tag{4.1.23}$$

Proof. There would be no need to redo the proof, in principle given in [38] as the proof of Theorem 12.4 there. However, the second term of the sum on the right-hand side of (4.1.22) does not agree with the one given there. This is the only place where we had to use the former version of (4.1.12) which, as pointed out in Remark 4.1.2, was erroneous. Now, the coefficients of the identity under study have been defined, in loc.cit., so as to make (4.1.23) valid: this leads to the corrected version (4.1.22). □

Remark 4.1.4. (i) That the pseudoscalar product only depends on ν mod 2 may seem surprising: only, do not forget (cf. (4.1.2)) that the $\mathbb{C}^4$-realization of a function $u \in \mathfrak{A}_\nu$ depends on ν, not only on ν mod 2.

(ii) In the case when $\nu \in]-1, 0[+2\mathbb{Z}$, the pseudoscalar product is positive definite when restricted to the even part of $\mathfrak{A}_\nu$; when $\nu \in]0, 1[+2\mathbb{Z}$, it is positive definite on the odd part of this space.

(iii) In Proposition 2.2.12, it has been shown that the even part of the anaplectic representation is unitarily equivalent to some representation of $SL(2,\mathbb{R})$ taken from the complementary series of that group. It can be shown that, in the case when $\nu \in]-1, 0[+2\mathbb{Z}$ (resp. $\nu \in]0, 1[+2\mathbb{Z}$), the restriction of the ν-anaplectic representation to the even (resp., odd) part of Ana_ν coincides with a representation taken from the unitary dual of the universal cover of $SL(2,\mathbb{R})$ [24]: details can be found in the last section of [38].

(iv) As seen from (4.1.8) and (4.1.17), one has

$$\mathcal{F}^{\nu}_{\mathrm{ana}}\,\psi^\nu = e^{\frac{i\pi}{2}(\nu+\frac{1}{2})}\,\psi^\nu : \tag{4.1.24}$$

in view of the condition (iii) from Theorem 4.1.8 and of (2.2.30), still valid when Ana_ν is substituted for Ana provided that one interprets the matrix $\left(\begin{smallmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{smallmatrix}\right)$ as $\exp t\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$, one obtains

$$\exp\left(-\frac{i\pi}{2}L\right)\psi^\nu = e^{-\frac{i\pi}{2}(\nu+\frac{1}{2})}\,\psi^\nu, \tag{4.1.25}$$

a form of the eigenvalue equation $L\psi^\nu = (\nu+\frac{1}{2})\,\psi^\nu$.

More generally, for any point z in the upper half-plane Π, we consider (cf. Proposition 2.2.8) the operators $A_z = \pi^{\frac{1}{2}}(Q-\bar{z}P)$ and $L_z = A_zA_z^* - \frac{\mathrm{Im}\,z}{2}$. Let $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in SL(2,\mathbb{R})$ be given and let g be any point of the group $G^{(N)}$ introduced in Theorem 4.1.8 lying above this matrix: one then has

$$\begin{aligned}\mathrm{Ana}_\nu(g)\,A_z\,\mathrm{Ana}_\nu(g^{-1}) &= (c\bar{z}+d)\,A_{\frac{az+b}{cz+d}}\,,\\ \mathrm{Ana}_\nu(g)\,L_z\,\mathrm{Ana}_\nu(g^{-1}) &= |c\bar{z}+d|^2\,L_{\frac{az+b}{cz+d}}\,.\end{aligned} \tag{4.1.26}$$

These identities, the ν-anaplectic analogue of (2.2.22), can be proved in the same way, as a consequence of the analogue of (2.2.19).

If $z=x+iy\in\Pi$ and g_z is a point of $G^{(N)}$ above the matrix $\begin{pmatrix}y^{\frac{1}{2}} & y^{-\frac{1}{2}}x\\ 0 & y^{-\frac{1}{2}}\end{pmatrix}$, finally if one sets

$$\phi_z^{\nu,k} = \begin{cases}\mathrm{Ana}_\nu(g_z)\,\psi^{\nu+k} & \text{if } k \text{ is even}\\ \mathrm{Ana}_\nu(g_z)\,\chi^{\nu+k} & \text{if } k \text{ is odd,}\end{cases} \tag{4.1.27}$$

one obtains a full set of ν-anaplectic eigenfunctions of L, $\phi_z^{\nu,k}$ corresponding to the eigenvalue $\nu+k$. However, defining g_z without any ambiguity requires more care. Set

$$z = \begin{pmatrix}\alpha & 0\\ \gamma & \alpha^{-1}\end{pmatrix}.i = \begin{pmatrix}\alpha & 0\\ \gamma & \alpha^{-1}\end{pmatrix}\begin{pmatrix}\frac{y}{|z|} & \frac{x}{|z|}\\ -\frac{x}{|z|} & \frac{y}{|z|}\end{pmatrix}.i : \tag{4.1.28}$$

since the last matrix on the right-hand side is orthogonal, its associated fractional-linear transformation fixes the point i, so that this equation is valid provided that $z=\frac{\alpha^2}{\alpha\gamma-i}$, in other words $\alpha = y^{-\frac{1}{2}}|z|$, $\gamma = y^{-\frac{1}{2}}|z|^{-1}x$.

To define g_z, it suffices to cover each of the two matrices on the right-hand side of (4.1.28) by a well-defined element of the group $G^{(N)}$. For the first one, it is obvious how to do this, only splitting it as the product $\left(\begin{smallmatrix}1 & 0\\ \gamma\alpha^{-1} & 1\end{smallmatrix}\right)\left(\begin{smallmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{smallmatrix}\right)$: the associated element of the ν-anaplectic representation is the map $u\mapsto v$ with

$$v(t) = \alpha^{-\frac{1}{2}}\,e^{i\pi\frac{\gamma}{\alpha}t^2}\,u(\alpha^{-1}t). \tag{4.1.29}$$

Since $y > 0$, one can uniquely set $y + ix = e^{i\theta}$ with $|\theta| < \frac{\pi}{2}$, which gives a meaning to fractional powers of $\frac{-iz}{|z|} = e^{-i\theta}$. Writing the last matrix on the right-hand side of (4.1.28) as $\exp\theta\,\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$, one can regard it as an element of $G^{(N)}$. To see the effect of the associated element of the ν-anaplectic representation on $\psi^{\nu+k}$ (or $\chi^{\nu+k}$, depending on the parity of k), we use the equation

$$\exp(-i\theta L) = \mathrm{Ana}_\nu\left(\exp\theta\,\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)\right), \tag{4.1.30}$$

the proof of which is absolutely identical to that of (2.2.30). Since (for even k) $L\,\psi^{\nu+k} = (\nu+\frac{1}{2}+k)\,\psi^{\nu+k}$, we finally obtain

$$\phi_z^{\nu,k}(t) = \left(\frac{-iz}{|z|}\right)^{\nu+\frac{1}{2}+k}\,\frac{(\mathrm{Im}\,z)^{\frac{1}{4}}}{|z|^{\frac{1}{2}}}\;\psi^{\nu+k}\left(\frac{(\mathrm{Im}\,z)^{\frac{1}{2}}t}{|z|}\right)\,e^{i\pi\frac{\mathrm{Re}\,z}{|z|^2}t^2} \tag{4.1.31}$$

if k is even, and a fully similar equation, only replacing $\psi^{\nu+k}$ by $\chi^{\nu+k}$, if k is odd.

We finally generalize Lemma 2.2.9.

Lemma 4.1.10. *For every $z \in \Pi$ and $k \in \mathbb{Z}$, one has*

$$A_z\,\phi_z^{\nu,k} = (\nu+k)\,(\mathrm{Im}\,z)^{\frac{1}{2}}\,\phi_z^{\nu,k-1}\,, \qquad A_z^*\,\phi_z^{\nu,k} = (\mathrm{Im}\,z)^{\frac{1}{2}}\,\phi_z^{\nu,k+1}. \tag{4.1.32}$$

Also,

$$\begin{aligned}\left[4i\,(\mathrm{Im}\,z)\frac{\partial}{\partial z} - \nu - \frac{1}{2} - k\right]\phi_z^{\nu,k} &= -(\nu+k)(\nu+k-1)\,\phi_z^{\nu,k-2}\,,\\ \left[4i\,(\mathrm{Im}\,z)\frac{\partial}{\partial \bar z} - \nu - \frac{1}{2} - k\right]\phi_z^{\nu,k} &= -\phi_z^{\nu,k+2}.\end{aligned} \tag{4.1.33}$$

Proof. The proof of the first part is based on (4.1.11) and (4.1.26). Note that, in the case when $\nu = -\frac{1}{2}$, comparing this pair of equations with the corresponding pair (2.2.34) from Lemma 2.2.9 makes it possible to obtain the coefficients of proportionality between the functions $\phi_z^{-\frac{1}{2},k}$ and ϕ_z^k, starting from Remark 4.1.3.

For the second part, we still follow the proof of Lemma 2.2.9, interpreting this time J as the element $\exp\frac{\pi}{2}\left(\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right)$ of $G^{(N)}$ rather than as a matrix. Assuming for instance that k is even (if not, it suffices to change ψ to χ), (2.2.41) becomes

$$\begin{aligned}A^2\,\psi^{\nu+k} &= 2\pi\,Q\,(Q+iP)\,\psi^{\nu+k} - (\nu+k)\,\psi^{\nu+k},\\ A^{*2}\,\psi^{\nu+k} &= 2\pi\,Q\,(Q-iP)\,\psi^{\nu+k} - (\nu+1+k)\,\psi^{\nu+k},\end{aligned} \tag{4.1.34}$$

and (2.2.43) becomes

$$\begin{aligned}\mathrm{Ana}_\nu(\tilde g_z)\,A^2\,\psi^{\nu+k} &= \left(4iy\frac{\partial}{\partial z} - \nu - \frac{1}{2} - k\right)\mathrm{Ana}_\nu(\tilde g_z)\,\psi^{\nu+k},\\ \mathrm{Ana}_\nu(\tilde g_z)\,A^{*2}\,\psi^{\nu+k} &= \left(4iy\frac{\partial}{\partial \bar z} - \nu - \frac{1}{2} - k\right)\mathrm{Ana}_\nu(\tilde g_z)\,\psi^{\nu+k}.\end{aligned} \tag{4.1.35}$$

Also, one has

$$\mathrm{Ana}_\nu(J)\,\psi^{\nu+k} = e^{-\frac{i\pi}{2}(\nu+\frac{1}{2}+k)}\,\psi^{\nu+k}. \tag{4.1.36}$$

Equations (4.1.33) follow. □

4.2 Ascending Pseudodifferential Calculus in ν-Anaplectic Analysis

We here briefly show how the ascending pseudodifferential analysis extends to the ν-anaplectic case, under the assumption that $\nu \in \mathbb{R}\backslash\mathbb{Z}$. Then, we shall examine which part of ν-anaplectic analysis can survive when $\nu \equiv 0 \bmod 2$, and why this case contains usual analysis.

As a first step, we start with a generalization of Proposition 3.3.1, to the effect that the operator $Q+iP$ is an automorphism of $\mathfrak{A}_\nu$ for every $\nu \in \mathbb{C}\backslash\mathbb{Z}$. This is proved with the help of the following formulas, in which $\boldsymbol{f} = (f_0, f_1, f_{i,0}, f_{i,1})$ is the $\mathbb{C}^4$-realization of some function $u \in \mathfrak{A}_\nu$ and $\boldsymbol{g} = (g_0, g_1, g_{i,0}, g_{i,1})$ is the $\mathbb{C}^4$-realization of the function $u_1 = (Q+iP)^{-1}\,u$:

$$\begin{aligned}
g_0(x) &= 2\pi\left[\int_0^x e^{-\pi(x^2-y^2)}\,f_1(y)\,dy - 2^{\nu+\frac{1}{2}}\,\frac{\Gamma(\frac{2+\nu}{2})}{\Gamma(\frac{1-\nu}{2})}\,e^{-\pi x^2}\int_0^\infty e^{-\pi y^2}\,f_{i,1}(y)\,dy\right],\\
g_{i,0}(x) &= -2\pi\,e^{\pi x^2}\int_x^\infty e^{-\pi y^2}\,f_{i,1}(y)\,dy,\\
g_1(x) &= 2\pi\left[\int_0^x e^{-\pi(x^2-y^2)}\,f_0(y)\,dy + 2^{\nu+\frac{1}{2}}\,\frac{\Gamma(\frac{1+\nu}{2})}{\Gamma(-\frac{\nu}{2})}\,e^{-\pi x^2}\int_0^\infty e^{-\pi y^2}\,f_{i,0}(y)\,dy\right],\\
g_{i,1}(x) &= 2\pi\,e^{\pi x^2}\int_x^\infty e^{-\pi y^2}\,f_{i,0}(y)\,dy.
\end{aligned} \tag{4.2.1}$$

No change has to be done in the proof of Proposition 3.3.1, except for the use of (4.1.1) in place of (2.2.2).

We then extend Definition 3.1.2 to the environment of ν-anaplectic analysis and start with observing that all of Sect. 3.1 extends without modification: indeed, the lemmas in this section are only concerned with analysis on Π, once the invertibility of A_z has been established.

Extending Sects. 3.2–3.4, however, requires a few inessential modifications. First, since pseudoscalar products play a basic role, we shall assume, so as to take advantage of (4.1.20), that ν is real (and not an integer). That the operators Q and P are still self-adjoint comes from the fact that (2.2.6) and (2.2.7) extend without modification to ν-anaplectic analysis. Lemma 3.2.1 extends as the equation

$$(A_z^{-m-1}\,\phi_\zeta^{\nu,k}\,|\,\phi_\zeta^{\nu,j}) = \overline{C_{\nu,m}^{j,k}}\,(\mathrm{Im}\,\zeta)^{\frac{m+1}{2}}\,(z-\zeta)^{\frac{-m-1+j-k}{2}}\,(z-\bar{\zeta})^{\frac{-m-1-j+k}{2}}, \tag{4.2.2}$$

and one has $C_{\nu,m}^{k+m+1,k} = \frac{1}{2}\Gamma(\nu+k+1)$. The proof of this latter equation is based on (4.1.32) and (4.1.20): note that $C^{j,k}_{-\frac{1}{2},m}$ is not the same as the coefficient $C_m^{j,k}$ from Sect. 3.2, in view of the different normalizations of the eigenfunctions of L.

It is not absolutely necessary to extend Lemmas 3.2.2 and 3.2.3, in which it would suffice, anyway, to change some coefficients. The essential Lemma 3.2.4, Proposition 3.2.5, Theorem 3.2.6, and Lemma 3.2.7 extend without any modification. Some is required, however, in the proof of Theorem 3.2.8, which depends on some explicit equalities involving the coefficients $F_m^{k+m+1,k}$ introduced in Lemma 3.2.2 and the coefficients γ_k, γ_k^* from Lemma 2.2.9. Setting $C_{\nu,m}^{j,k} = \frac{(-2i)^{m+1}}{m!}\, F_{\nu,m}^{j,k}$ so as to extend (3.2.12) and setting

$$\gamma_{\nu,k} = \nu + k\,, \qquad \gamma^*_{\nu,k} = 1 \tag{4.2.3}$$

so that the first part of Lemma 4.1.10 should extend the corresponding part of Lemma 2.2.9 with the same notation, we first have to verify, so as to generalize (3.2.75), that

$$\frac{F_{\nu,m_0}^{k+m_0+2,k+1}}{F_{\nu,m_0}^{k+m_0+1,k}} = \frac{\gamma_{\nu,k+1}}{\gamma^*_{\nu,k+m_0+1}}, \tag{4.2.4}$$

an immediate task. Following the proof of Theorem 3.2.8, we next come to (3.2.83), which must be generalized in the obvious way: since $\gamma_{\nu,j+1}\,\gamma^*_{\nu,j} = \nu + j + 1$, this is again immediate. Equation (3.2.88) has to be changed in the obvious way, only inserting the subscript ν where needed: only, since (2.2.36) has to be replaced by (4.1.33), the right-hand side of the equation that replaces (3.2.88) is now (compare (3.2.90))

$$\begin{aligned}(\mathrm{Im}\;\zeta)^{\frac{1}{2}} &\left([k+\nu+\frac{1}{2}+m_0+1+\alpha+\beta-4i\,(\mathrm{Im}\;\zeta)\,\frac{\partial}{\partial\bar\zeta}]\;\phi_\zeta^{\nu,k+m_0+1+\alpha+\beta}\,|\,C\,\phi_\zeta^{\nu,k}\right)\\ &-(\mathrm{Im}\;\zeta)^{\frac{1}{2}}\left(\phi_\zeta^{k+\nu+1+m_0+1+\alpha+\beta}\,|\,C\,[k+\nu+\frac{1}{2}-4i\,(\mathrm{Im}\;\zeta)\,\frac{\partial}{\partial\zeta}]\,\phi_\zeta^{\nu,k}\right).\end{aligned} \tag{4.2.5}$$

Since the shift $k \mapsto k+\nu+\frac{1}{2}$ occurs on both sides, (3.2.91) does not have to be modified, so that Theorem 3.2.8 extends.

In view of the set of equations (4.2.1), some coefficients have to be modified in the formulas from Sect. 3.3. If $u_2 = A_z^{-2}u$, keeping the same notation as in Proposition 3.3.2, one sees that no modification whatsoever occurs in the equations for $g_{i,0}$ and $g_{i,1}$. However, the other two equations are to be replaced by the following:

$$\begin{aligned}g_0(x) = -\frac{4\pi}{\bar z^2}\Bigg[&\int_0^x (x-y)\,e^{\frac{i\pi}{z}(x^2-y^2)}\,f_0(y)\,dy\\ &+\int_0^\infty 2^{\nu+\frac{1}{2}}\left[\frac{\Gamma(\frac{1+\nu}{2})}{\Gamma(-\frac{\nu}{2})}\,x-\frac{\Gamma(\frac{2+\nu}{2})}{\Gamma(\frac{1-\nu}{2})}\,y\right]e^{\frac{i\pi}{z}(x^2+y^2)}\,f_{i,0}(y)\,dy\Bigg]\end{aligned} \tag{4.2.6}$$

and

$$g_1(x) = -\frac{4\pi}{\bar{z}^2}\left[\int_0^x (x-y)\, e^{\frac{i\pi}{\bar{z}}(x^2-y^2)} f_1(y)\, dy \right.$$
$$\left. - \int_0^\infty 2^{\nu+\frac{1}{2}} \left[\frac{\Gamma(\frac{2+\nu}{2})}{\Gamma(\frac{1-\nu}{2})}\, x - \frac{\Gamma(\frac{1+\nu}{2})}{\Gamma(-\frac{\nu}{2})}\, y\right] e^{\frac{i\pi}{\bar{z}}(x^2+y^2)} f_{i,1}(y)\, dy\right]. \tag{4.2.7}$$

Then,

$$u_2(x) = -\frac{4\pi}{\bar{z}^2}\left[\int_0^x (x-y)\, e^{\frac{i\pi}{\bar{z}}(x^2-y^2)}\, u(y)\, dy \right.$$
$$\left. - 2^{\nu+\frac{1}{2}}\, \frac{\Gamma(\frac{2+\nu}{2})}{\Gamma(\frac{1-\nu}{2})} \int_0^\infty [y\, f_{i,0}(y) + x\, f_{i,1}(y)]\, e^{\frac{i\pi}{\bar{z}}(x^2+y^2)}\, dy\right]. \tag{4.2.8}$$

Consequently, (3.3.21) becomes

$$(\mathrm{Op}_1^{\mathrm{asc}}(h)\, u)(x) = -\pi \left[\int_0^x (x-y)\, k(x^2-y^2)\, u(y)\, dy \right.$$
$$\left. - 2^{\nu+\frac{1}{2}}\, \frac{\Gamma(\frac{2+\nu}{2})}{\Gamma(\frac{1-\nu}{2})} \int_0^\infty k(x^2+y^2)\, [y\, f_{i,0}(y) + x\, f_{i,1}(y)]\, dy\right], \tag{4.2.9}$$

but there is nothing to change in the proof of Theorem 3.3.4.

Finally, the results of Sect. 3.4 concerning the composition formula remain valid in ν-anaplectic analysis. The coefficient $(\frac{i}{\pi})^p$ on the right-hand side of the main formula (3.4.38) does not depend on ν, since its value is based on the result of Lemma 3.4.4: now, in the proof of that lemma, the constant $C_m^{j,k}$, which is to be replaced by $C_{\nu,m}^{j,k}$, disappears since it occurs on both sides of the equation one is taking advantage of.

Remark 4.2.1. As recalled in Remark 4.1.4, the even (resp., odd) part of the ν-anaplectic representation (if $\nu \in]-1, 0[+2\mathbb{Z}$, resp. $]0, 1[+2\mathbb{Z}$) is unitarily equivalent to a representation taken from the complementary series of the universal cover of $SL(2,\mathbb{R})$. This is a two-parameter series [24], and we are here considering only a one-parameter subfamily. If one considers instead another one-parameter family, to wit that which corresponds to representations of $SL(2,\mathbb{R})$ (as opposed to the universal cover of that group), the situation is totally different. In that case, it is in general not possible to combine two representations (as we have done with $\pi_{-\frac{1}{2},0}$ and $\pi_{\frac{1}{2},1}$ in Proposition 2.2.12) so as to let the Heisenberg group show in the picture. However, one can still define a pseudodifferential analysis acting on the space of just one representation, using the one-sheeted hyperboloid as a phase space. Going from the principal series of $SL(2,\mathbb{R})$ to the complementary series by complex continuation (once both representations have been embedded into the full nonunitary principal series), one sees from [34, Theorem 4.2] that the coefficients of the Rankin–Cohen products which occur in the expansion of the composition of symbols certainly depend on the parameter that specifies the representation within its series.

The preceding considerations, to the effect that the anaplectic analyses corresponding to two distinct values of ν lead to the same sharp composition of symbols, show that the structure of the composition formula is far from revealing the details of a quantization theory in general. We now show that a certain limit of the ν-anaplectic series of analyses contains a part which fits with analysis of the usual kind.

When ν is an even integer, it is possible to save most of the ν-anaplectic analysis. It would be pleasant to set $\nu = 0$, but (4.1.1) would then cease to be meaningful: instead, we set $\nu = -2$, in which case (4.1.1) becomes

$$\begin{aligned} f_{i,0}(x) &= \frac{i}{(2\pi)^{\frac{1}{2}}}\,[f_0(ix) - f_0(-ix)]\,, \\ f_{i,1}(x) &= -\frac{1}{(2\pi)^{\frac{1}{2}}}\,[f_1(ix) + f_1(-ix)]\,. \end{aligned} \tag{4.2.10}$$

A function u on the real line lies in $\mathfrak{A}_0 : = \mathfrak{A}_{-2}$ if $u = (f_0)_{\text{even}} + (f_1)_{\text{odd}}$ for a pair (f_0, f_1) of *nice* functions satisfying (4.2.10). The map $(f_0, f_1, f_{i,0}, f_{i,1}) \mapsto u$ is still one to one since if f_0 is odd and both f_0 and $f_{i,0}$, linked by (4.2.10), are nice, it follows from the Phragmén–Lindelöf lemma that $f_0 = 0$ [38, p. 6]; the same goes if f_1 is even and both f_0 and $f_{i,0}$ are nice. That not much remains of the anaplectic theory in the case when ν is an odd integer is due to the fact that, though one might certainly set $\nu = -1$ in (4.1.1), the map $\boldsymbol{f} \mapsto u$ would then cease to be one to one, as is easily ascertained: anyway, turning – as is possible – around this difficulty, one does not get anything essentially new in view of Proposition 4.1.2.

We thus assume, from now on, that $\nu \equiv 0 \bmod 2$ and fix $\nu = -2$ when dealing with the $\mathbb{C}^4$-realization, though the space $\mathfrak{A}_0$ could of course also be identified with the space $\mathfrak{A}_{-2j}$ defined in the usual way after an arbitrary number $j = 1, 2, \dots$ has been chosen.

Some special function formulas will help: with the notation of [21], the error functions Erf and Erfc are defined as

$$\begin{aligned} \mathrm{Erf}(x) &= \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt, \\ \mathrm{Erfc}(x) &= \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\,dt = 1 - \mathrm{Erf}(x). \end{aligned} \tag{4.2.11}$$

On the other hand, $(H_k)_{k\geq 0}$ is the sequence of Hermite polynomials, characterized by the eigenvalue equation

$$\left[\frac{d^2}{dt^2} + (2k+1-t^2)\right]\left(e^{-\frac{t^2}{2}}\,H_k(t)\right) = 0 \tag{4.2.12}$$

together with the normalizing conditions

$$\int_{-\infty}^{\infty} e^{-t^2}\,[H_k(t)]^2\,dt = 2^k\,k\,!\,\pi^{\frac{1}{2}}\,, \qquad H_k(+\infty) = +\infty. \tag{4.2.13}$$

From [21, p. 331], the entire function $D_k(2\pi^{\frac{1}{2}}x)$ is, for every $k \in \mathbb{Z}$, a nice function in the sense of Definition 2.2.1 since, as $x \to +\infty$, it is equivalent to a constant times $x^k\, e^{-\pi x^2}$. For $k \geq 0$, one has (loc.cit., p. 326)

$$D_k(2\pi^{\frac{1}{2}}x) = 2^{-\frac{k}{2}}\, e^{-\pi x^2}\, H_k(\sqrt{2\pi}\, x),$$
$$D_{-k-1}(2\pi^{\frac{1}{2}}x) = \frac{(-1)^k}{k!}\, (\frac{\pi}{2})^{\frac{1}{2}}\, e^{-\pi x^2} \left(\frac{1}{2\pi^{\frac{1}{2}}}\frac{d}{dx}\right)^k \left[e^{2\pi x^2}\, \mathrm{Erfc}(\sqrt{2\pi}\, x)\right]. \quad (4.2.14)$$

In particular,

$$D_0(2\pi^{\frac{1}{2}}x) = e^{-\pi x^2},$$
$$D_1(2\pi^{\frac{1}{2}}x) = 2\pi^{\frac{1}{2}}\, x e^{-\pi x^2},$$
$$D_{-1}(2\pi^{\frac{1}{2}}x) = (\frac{\pi}{2})^{\frac{1}{2}}\, e^{\pi x^2}\, \mathrm{Erfc}(\sqrt{2\pi}\, x),$$
$$D_{-2}(2\pi^{\frac{1}{2}}x) = e^{-\pi x^2} - 2^{\frac{1}{2}}\, \pi x e^{\pi x^2}\, \mathrm{Erfc}(\sqrt{2\pi}\, x). \quad (4.2.15)$$

With the help of Definition 4.1.1 and (4.1.9) and (4.1.10), one finds

$$\psi^0(x) = e^{-\pi x^2},$$
$$\chi^1(x) = 2\pi^{\frac{1}{2}}\, x e^{-\pi x^2},$$
$$\psi^{-2}(x) = e^{-\pi x^2} + 2^{\frac{1}{2}}\, \pi x e^{\pi x^2}\, \mathrm{Erf}(\sqrt{2\pi}\, x),$$
$$\chi^{-1}(x) = -(\frac{\pi}{2})^{\frac{1}{2}}\, e^{\pi x^2}\, \mathrm{Erf}(\sqrt{2\pi}\, x). \quad (4.2.16)$$

It is of course not necessary to restart the ν-anaplectic theory from scratch when $\nu = -2$, since we may regard it as a limiting case: looking at the coefficients which occur on the right-hand sides of (4.1.9) and (4.1.10), one should observe that $\frac{\Gamma(\nu+2j+1)}{\Gamma(\nu+1)}$ and $\frac{\Gamma(\nu+2j+2)}{\Gamma(\nu+1)}$, as functions of ν, are regular at $\nu = -2$ for every $j \in \mathbb{Z}$: each of them vanishes at this point if and only if $j \geq 1$. One may then apply the definitions of Theorem 4.1.5, obtaining the following.

Theorem 4.2.1. *In the space $\mathfrak{A}_0$, the eigenvalues of the harmonic oscillator $L = \pi(Q^2 + P^2)$ are the numbers $\frac{1}{2} + j$, $j \in \mathbb{Z}$. Every eigenvalue is simple. When $j \geq 0$, the corresponding eigenspace is the same as that obtained in usual analysis, i.e., it is generated by the function $A^{*j}\, \psi^0$; when $j \leq -1$, the corresponding eigenspace is generated by the function $A^{-j-1}\, \chi^{-1}$.*

We shall now analyze which part of the ν-anaplectic theory subsists in the case under study, at the same time showing that the analysis obtained extends the usual one. Let us emphasize at once that the theory obtained has nothing to do with the direct sum of two parts: one which would correspond to some space containing the usual Hermite functions and another one which would correspond to the negative part (based on the use of some version of the error function) of the spectrum of the harmonic oscillator. For, as will be seen, one can define the first space so as to make it invariant both under the Heisenberg representation and the 0-anaplectic representation. But nothing comparable can occur on the other side since, starting from any (anaplectic) eigenfunction of the harmonic oscillator corresponding to a negative eigenvalue and applying a suitable power of the raising operator $A^* = \pi^{\frac{1}{2}}(Q - iP)$, one can reach usual Hermite functions. Indeed, note that (4.1.11) are still valid, whether one substitutes 0 or -2 (or any even integer) for ν.

There is nothing wrong with using the value $\nu = -2$ so far as the linear form Int is concerned, and we may use the results of Sect. 4.1.

Proposition 4.2.2. *In* 0*-anaplectic (this is by definition the same as* (-2)*-anaplectic) analysis, the linear form* Int *vanishes on odd functions. It coincides with the linear form* $u \mapsto e^{\frac{i\pi}{4}} \int_{-\infty}^{\infty} u(x)\,dx$ *on the subspace of* $\mathfrak{A}_0$ *generated by the usual Hermite functions* ψ^{2j}, $j \geq 0$*. The functions* ψ^{-2j-2}, $j \geq 0$*, are not integrable on the real line, but one has the equation*

$$\mathrm{Int}\,[\psi^{-2j-2}] = \frac{(-1)^{j+1}\, e^{\frac{i\pi}{4}}}{1.3\ldots(2j+1)}. \tag{4.2.17}$$

Proof. That Int vanishes on odd functions is a consequence of Definition 8.6 (applied with $\nu = -2$) together with the fact that the components $f_0, f_{i,0}$ of the $\mathbb{C}^4$-realization of an odd function are zero. Since the ν-anaplectic representation combines with the Heisenberg representation in the usual way, one has, denoting $\mathcal{F}_{\mathrm{ana}}^{\nu}$ as $\mathcal{F}_{\mathrm{ana}}^{(0)}$ when $\nu \equiv 0 \bmod 2$, $\mathcal{F}_{\mathrm{ana}}^{(0)} A^* = -iA^* \mathcal{F}_{\mathrm{ana}}^{(0)}$: since, for $j \geq 0$, $\psi^{2j} = (A^*)^{2j+2}\,\psi^{-2}$, and, as displayed in (4.1.24), $\mathcal{F}_{\mathrm{ana}}^{(0)}\,\psi^{-2} = -e^{\frac{i\pi}{4}}\,\psi^{-2}$, one obtains

$$\mathcal{F}_{\mathrm{ana}}^{(0)}\,\psi^{2j} = (-iA^*)^{2j+2}\,\mathcal{F}_{\mathrm{ana}}^{(0)}\,\psi^{-2} = (-1)^j\,e^{\frac{i\pi}{4}}\,(A^*)^{2j+2}\,\psi^{-2} = (-1)^j\,e^{\frac{i\pi}{4}}\,\psi^{2j}: \tag{4.2.18}$$

consequently,

$$\mathrm{Int}\,[\psi^{2j}] = \left(\mathcal{F}_{\mathrm{ana}}^{(0)}\,\psi^{2j}\right)(0) = (-1)^j\,e^{\frac{i\pi}{4}}\,\psi^{2j}(0). \tag{4.2.19}$$

The same calculation, using the usual integral on the line instead of the linear form Int and the usual Fourier transformation in place of $\mathcal{F}_{\mathrm{ana}}^{(0)}$ leads to the same result, save for the factor $e^{\frac{i\pi}{4}}$.

The functions ψ^{-2j-2} can never be integrable on the line, since they are formal eigenfunctions of the harmonic oscillator corresponding to negative eigenvalues. Still, using (4.1.11), one can write

$$\psi^{-2j-2} = \frac{1}{(2j+1)\,!}\,A^{2j}\,\psi^{-2}, \qquad j \geq 1, \tag{4.2.20}$$

from which one obtains

$$\begin{aligned}\mathcal{F}^{(0)}_{\mathrm{ana}}\,\psi^{-2j-2} &= \frac{1}{(2j+1)\,!}\,(iA)^{2j}\,\mathcal{F}^{(0)}_{\mathrm{ana}}\,\psi^{-2}\\ &= \frac{(-1)^j}{(2j+1)\,!}\left(-e^{\frac{i\pi}{4}}\,A^{2j}\,\psi^{-2}\right) = (-1)^{j+1}\,e^{\frac{i\pi}{4}}\,\psi^{-2j-2},\end{aligned} \tag{4.2.21}$$

so that

$$\mathrm{Int}\,[\psi^{-2j-2}] = (-1)^{j+1}\,e^{\frac{i\pi}{4}}\,\psi^{-2j-2}(0); \tag{4.2.22}$$

finally, one has [21, p. 324] $D_{-2j-2}(0) = (1.3.\dots(2j+1))^{-1}$, and ψ^{-2j-2} is by definition the even part of the function $x \mapsto D_{-2j-2}(2\pi^{\frac{1}{2}}\,x)$. $\square$

We now turn to the consideration of the 0-anaplectic representation Ana_0. To take benefit from the fact that the case when $\nu = -2$ is a limit of cases where Theorem 4.1.8 is already known to apply, we must first interpret Ana_0 as a representation of the universal cover $G^{(\infty)}$ of G. Consider the linear space E generated by functions $u(x) = p(x)\,\exp\left(-\pi\,q(x)\right)$, where p and q are complex polynomials, q is of degree 2 and has a top-order coefficient with a positive real part: it contains the Hermite functions. On the other hand, for any $u \in E$, the vector $(u_{\mathrm{even}}, u_{\mathrm{odd}}, 0, 0)$ is a $\mathbb{C}^4$-realization of u, so that $E \subset \mathfrak{A}_0$: note that the simple recipe just indicated toward the construction of a $\mathbb{C}^4$-realization only works when $\nu \equiv 0 \bmod 2$. We now consider the effect on functions in E of operators from the representation Ana_0 or from the metaplectic representation $\mathrm{Met}^{(1)}$.

Proposition 4.2.3. *On the space E defined as the smallest linear space of functions on the real line containing the standard Gaussian function and stable under transformations $u \mapsto u_1$ with $u_1(x) = u(ax)$, $a > 0$ or $u_1(x) = u(x)\,e^{i\pi cx^2}$, $c \in \mathbb{R}$, as well as under the operators Q and P, the representation* Ana_0 *agrees with the metaplectic representation.*

Proof. Again, we shall use generators. On matrices g such as $\left(\begin{smallmatrix}1&0\\c&1\end{smallmatrix}\right)$ or $\left(\begin{smallmatrix}a&0\\0&a^{-1}\end{smallmatrix}\right)$ (identified with elements of $G^{(\infty)}$), $\mathrm{Met}^{(1)}(g)$ and $\mathrm{Ana}_0(g)$ coincide. Consider now the element $g = \exp\frac{\pi}{2}\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right) \in G^{(\infty)}$ introduced in Theorem 4.1.8, and denote by the same letter its canonical image in $G^{(2)} = \widetilde{\mathrm{Sp}}(1,\mathbb{R})$: according to the definitions recalled in Sect. 2.1, one has $\mathrm{Met}^{(1)}(g) = e^{-\frac{i\pi}{4}}\,\mathcal{F}$ whereas, from Theorem 4.1.8, $\mathrm{Ana}_0(g) = e^{-\frac{i\pi}{2}}\,\mathcal{F}^0_{\mathrm{ana}}$. Since ψ^0, the standard Gaussian function, is invariant under $\mathcal{F}$ but, according to (4.1.24), is multiplied by $e^{\frac{i\pi}{4}}$ under $\mathcal{F}^0_{\mathrm{ana}}$, one sees that ψ^0 has the same images under $\mathrm{Met}^{(1)}(g)$ and under $\mathrm{Ana}_0(g)$: as the operator A^* undergoes the same transformation under adjunction by $\mathcal{F}$ or by $\mathcal{F}^0_{\mathrm{ana}}$, one sees that $\mathrm{Met}^{(1)}(g)$ and $\mathrm{Ana}_0(g)$ agree on the space generated by Hermite functions and, as an easy extension, on E. $\square$

Of course, the novelty of the 0-anaplectic representation is that it makes sense on a space much larger than E. When $\nu \equiv 0 \bmod 2$, it is no longer possible to define an invariant nondegenerate pseudoscalar product. Indeed, the coefficient $\frac{\Gamma(\nu+1)}{\Gamma(-\nu)}$ which occurs on the right-hand side of (4.1.20) is infinite when $\nu = -2$. Note that, in the case when $u \in E$, one has $f_{i,0} = f_{i,1} = 0$ as already mentioned, and $(u\,|\,u)$, reduced to the sum of its first two terms, is then well defined and coincides with $2^{-\frac{1}{2}}\,\|\,u\,\|^2_{L^2(\mathbb{R})}$. In another direction, for every $u \in \mathfrak{A}_0$, one may define instead $((u\,|\,u))$ as the "infinite part" of (4.1.20), i.e., as

$$((u\,|\,u)) = 2^{\frac{1}{2}} \int_0^\infty (|f_{i,0}|^2 - |f_{i,1}|^2)\, dx. \tag{4.2.23}$$

The pseudoscalar product so defined is still invariant both under the 0-anaplectic representation and under the Heisenberg representation. The eigenstates of the harmonic oscillator are still pairwise orthogonal. After having multiplied by $\frac{\Gamma(-\nu)}{\Gamma(\nu+1)}$ the right-hand sides of (4.1.23) which led to the normalization used in Theorem 4.1.9, one obtains $((\psi^0\,|\,\psi^0)) = ((\psi^2\,|\,\psi^2)) = \cdots = 0$ and $((\chi^1\,|\,\chi^1)) = ((\chi^3\,|\,\chi^3)) = \cdots = 0$, i.e., the pseudoscalar product under consideration vanishes when considered on Hermite functions: it also vanishes when considered on a pair of functions, at least one of which lies in E. On the other hand, one verifies from the same calculation that

$$((\psi^{-2k-2}\,|\,\psi^{-2k-2})) = \frac{1}{2\,(2k+1)\,!}, \qquad ((\chi^{-2k-1}\,|\,\chi^{-2k-1})) = -\frac{1}{2\,(2k)\,!} \tag{4.2.24}$$

if $k = 0, 1, \ldots$.

In ν-analysis with $\nu \in \mathbb{C}\backslash\mathbb{Z}$, one obtains from (4.1.11) that

$$(Q+iP)^{-1}\,\chi^{\nu+2j-1} = \frac{\pi^{\frac{1}{2}}}{\nu+2j}\,\psi^{\nu+2j},$$
$$(Q+iP)^{-1}\,\psi^{\nu+2j} = \frac{\pi^{\frac{1}{2}}}{\nu+2j+1}\,\chi^{\nu+2j+1}. \tag{4.2.25}$$

In the case when $\nu \equiv 0 \bmod 2$, $(Q+iP)^{-1}$ ceases to be well defined since the operator $Q+iP$ is neither onto nor one to one as an endomorphism of $\mathfrak{A}_0$. When attention is restricted to usual analysis, one can of course define a *right inverse* $(Q+iP)^{-1}$ of this operator by the equations

$$(Q+iP)^{-1}\,\chi^{2j-1} = \frac{\pi^{\frac{1}{2}}}{2j}\,\psi^{2j}, \qquad j \geq 1,$$
$$(Q+iP)^{-1}\,\psi^{2j} = \frac{\pi^{\frac{1}{2}}}{2j+1}\,\chi^{2j+1}, \qquad j \geq 0: \tag{4.2.26}$$

these equations define $(Q+iP)^{-1}$ on the linear space generated by Hermite functions, and the operator can be extended to $\mathcal{S}(\mathbb{R})$ or to $L^2(\mathbb{R})$ by means of the equation

$$A^{-1}:\ = A^*\,(L+\frac{1}{2})^{-1} = (L-\frac{1}{2})^{-1}\,A^*. \tag{4.2.27}$$

However, it is not possible to base on the use of such an operator (and of its conjugates under operators of the metaplectic representation) an ascending pseudodifferential analysis, by a generalization of Definition 3.1.2. What goes wrong is that the analogue of Lemma 3.1.5 does not hold any more: equations (3.1.20) still hold, but only for nonnegative exponents.