6

Expectations with Respect to the Ground State of the Harmonic Oscillator

We consider a harmonic oscillator with a finite number of degrees of freedom. The classical action for the time interval $[0, t]$ is given by (5.1) with $V = 0$. The corresponding action for the whole trajectory is given by

$$S_0(\gamma) = \frac{1}{2} \int_{-\infty}^{\infty} \dot{\gamma}(\tau)^2 \, d\tau - \frac{1}{2} \int_{-\infty}^{\infty} \gamma A^2 \gamma \, d\tau, \tag{6.1}$$

where $\gamma(\tau)$ and A^2 are as in (5.1) and we have set, for typographical reasons, $m = 1$. Let now $\mathcal{H}$ be the real Hilbert space of real square integrable functions on $\mathbb{R}$ with values in $\mathbb{R}^n$ and norm given by

$$|\gamma|^2 = \int_{-\infty}^{\infty} \dot{\gamma}(\tau)^2 \, d\tau + \int_{-\infty}^{\infty} \gamma(\tau)^2 \, d\tau. \tag{6.2}$$

Let B be the symmetric operator in $\mathcal{H}$ given by

$$(\gamma, B\gamma) = \int_{-\infty}^{\infty} \left(\dot{\gamma}(\tau)^2 - \gamma A^2 \gamma \right) \, d\tau \tag{6.3}$$

with domain $D(B)$ equal to the functions γ in $\mathcal{H}$ with compact support. We then have, for any $\gamma \in D(B)$, that

$$S_0(\gamma) = \frac{1}{2}(\gamma, B\gamma), \tag{6.4}$$

where $(,)$ is the inner product in $\mathcal{H}$. The Fourier transform of an element γ in $\mathcal{H}$ is given by

$$\hat{\gamma}(p) = 1/\sqrt{2\pi} \int_{-\infty}^{\infty} e^{ipt} \gamma(t) \, dt \tag{6.5}$$

and the mapping $\gamma \to \hat{\gamma}$ is then an isometry of $\mathcal{H}$ onto the real subspace of functions in $L_2\left(\left(p^2+1\right)\ \mathrm{d}p\right)$ satisfying

$$\overline{\hat{\gamma}(p)} = \hat{\gamma}(-p) \tag{6.6}$$

and we have, for any $\gamma \in D(B)$,

$$S_0(\gamma) = \frac{1}{2}(\gamma, B\gamma) = \int_R \overline{\hat{\gamma}(p)} \left(\frac{1}{2}p^2 - \frac{1}{2}A^2\right) \hat{\gamma}(p)\ \mathrm{d}p. \tag{6.7}$$

Moreover the range $R(B)$ of B consists of functions whose Fourier transforms are smooth functions and in $L_2\left[\left(p^2+1\right)\ \mathrm{d}p\right]$. Let D be the real Banach space of functions in $\mathcal{H}$ whose Fourier transforms are continuously differentiable functions with norm given by

$$\|\gamma\| = |\gamma| + \sup_p \left|\frac{\mathrm{d}\hat{\gamma}}{\mathrm{d}p}(p)\right|. \tag{6.8}$$

We have obviously that the norm in D is stronger than the norm in $\mathcal{H}$ and that D contains the range of B. We now define on $D \times D$ the symmetric form

$$\triangle\left(\gamma_1, \gamma_2\right) = \lim_{\varepsilon \to 0} \int_R \overline{\hat{\gamma}_1(p)} \left(p^2 - A^2 + \mathrm{i}\varepsilon\right)^{-1} \hat{\gamma}_2(p) \left(p^2+1\right)^2\ \mathrm{d}p. \tag{6.9}$$

That this limit exists follows from the fact that $\overline{\hat{\gamma}_1(p)}\hat{\gamma}_2(p)$ is continuously differentiable and in $L_1\left[\left(p^2+1\right)\ \mathrm{d}p\right]$. That the form is continuous and bounded on $D \times D$ follows by standard results and (6.8). That the form is symmetric,

$$\triangle\left(\gamma_1, \gamma_2\right) = \triangle\left(\gamma_2, \gamma_1\right),$$

follows from (6.9) and (6.6). In fact the limit (6.9) has the following decomposition into its real and imaginary parts

$$\begin{aligned} \triangle\left(\gamma_1, \gamma_2\right) = P \int_R \overline{\hat{\gamma}_1(p)} \left(p^2 - A^2\right)^{-1} \hat{\gamma}_2(p) \left(p^2+1\right)^2\ \mathrm{d}p \\ -\mathrm{i}\pi \int \overline{\hat{\gamma}_1(p)} \delta\left(p^2 - A^2\right) \hat{\gamma}_2(p) \left(p^2+1\right)^2\ \mathrm{d}p, \end{aligned} \tag{6.10}$$

where the first integral is the principal value and hence real by (6.6). We see therefore that

$$\mathrm{Im}\triangle(\gamma, \gamma) \leq 0. \tag{6.11}$$

Let now $\gamma_1 \in D$ and $\gamma_2 \in D(B)$, then

$$\begin{aligned} \triangle\left(\gamma_1, B\gamma_2\right) &= \lim_{\varepsilon \to 0} \int_R \overline{\hat{\gamma}_1(p)} \left(p^2 - A^2 + \mathrm{i}\varepsilon\right)^{-1} \left(p^2 - A^2\right) \hat{\gamma}_2(p) \left(p^2+1\right)\ \mathrm{d}p \\ &= \int \overline{\hat{\gamma}_1(p)} \hat{\gamma}_2(p) \left(p^2+1\right)\ \mathrm{d}p. \end{aligned}$$

So that

$$\triangle\left(\gamma_1, B\gamma_2\right) = \left(\gamma_1, \gamma_2\right). \tag{6.12}$$

We have now verified that $\mathcal{H}$, D, B and $\triangle$ satisfy the conditions in the Definition 4.1 for the Fresnel integral with respect to $\triangle$.

Hence for any function $f \in \mathcal{F}(D^*)$ we have that

$$\int_{\mathcal{H}}^{\triangle} \mathrm{e}^{\frac{\mathrm{i}}{2}(\gamma, B\gamma)} f(\gamma)\ \mathrm{d}\gamma \tag{6.13}$$

is well defined and given by (4.12). It follows from (6.8) that γ_t, given by

$$(\gamma_t, \gamma) = \gamma(t),$$

is in $D \times \mathbb{R}^n$, since

$$\hat{\gamma}_t(p) = \sqrt{\frac{1}{2\pi}} \cdot \frac{\mathrm{e}^{\mathrm{i}pt}}{p^2 + 1}. \tag{6.14}$$

So that

$$f(\gamma) = \mathrm{e}^{\mathrm{i}\sum\limits_{j=1}^{n} \alpha_j \cdot \gamma(t_j)} \tag{6.15}$$

is in $\mathcal{F}(D^*)$.

Hence we may compute (6.13) with $f(\gamma)$ given by (6.15) and we get

$$\int_{\mathcal{H}}^{\triangle} \mathrm{e}^{\frac{\mathrm{i}}{2}(\gamma, B\gamma)} f(\gamma)\ \mathrm{d}\gamma = \mathrm{e}^{\frac{\mathrm{i}}{2}\sum_{jk=1}^{n} \alpha_j \triangle\left(\gamma_{t_j}, \gamma_{t_k}\right)\alpha_k}. \tag{6.16}$$

From the definition of $\triangle$ we easily compute

$$\triangle\left(\gamma_s, \gamma_t\right) = \frac{1}{2\mathrm{i}A} \mathrm{e}^{-\mathrm{i}|t-s|A}. \tag{6.17}$$

Hence we get that

$$\int_{\mathcal{H}}^{\triangle} \mathrm{e}^{\frac{\mathrm{i}}{2}(\gamma, B\gamma)} \mathrm{e}^{\mathrm{i}\sum\limits_{j=1}^{n} \alpha_j \gamma(t_j)}\ \mathrm{d}\gamma = \mathrm{e}^{-\frac{1}{2}\sum\limits_{j,k=1}^{n} \alpha_j (2A)^{-1} \mathrm{e}^{-\mathrm{i}|t_j - t_k|A} \alpha_k}. \tag{6.18}$$

Let now Ω_0 be the vacuum i.e. the function given by (5.21), and let us set in this section

$$H_0 = -\frac{1}{2}\triangle + \frac{1}{2} x A^2 x - \frac{1}{2}\mathrm{tr}A, \tag{6.19}$$

where we have changed the notation so that

$$H_0 \Omega_0 = 0\,. \tag{6.20}$$

Let $t_1 \leq \ldots \leq t_n$, then we get from (6.18) and (5.32) that

$$\begin{aligned}\left(\Omega_0, \mathrm{e}^{\mathrm{i}\alpha_1 x(t_1)} \ldots \mathrm{e}^{\mathrm{i}\alpha_n x(t_n)} \Omega_0\right) &= \int_{\mathcal{H}}^{\triangle} \mathrm{e}^{\frac{\mathrm{i}}{2}(\gamma, B\gamma)} \mathrm{e}^{\mathrm{i}\sum_{j=1}^{n} \alpha_j \gamma(t_j)} \,\mathrm{d}\gamma \\ &= \int_{\mathcal{H}}^{\triangle} \mathrm{e}^{\mathrm{i}\int_{-\infty}^{\infty}\left(\frac{1}{2}\dot{\gamma}^2 - \frac{1}{2}\gamma A^2 \gamma\right)\,\mathrm{d}\tau} \mathrm{e}^{\mathrm{i}\sum_{j=1}^{n} \alpha_j \gamma(t_j)} \,\mathrm{d}\gamma, \end{aligned} \tag{6.21}$$

where[1]

$$\mathrm{e}^{\mathrm{i}\alpha x(t)} = \mathrm{e}^{-\mathrm{i}tH_0} \mathrm{e}^{\mathrm{i}\alpha x} \mathrm{e}^{\mathrm{i}tH_0}.$$

Theorem 6.1. *Let $\mathcal{H}$ be the real Hilbert space of real continuous and square integrable functions such that the norm given by*

$$|\gamma|^2 = \int_{-\infty}^{\infty} \dot{\gamma}(\tau)^2 \,\mathrm{d}\tau + \int_{-\infty}^{\infty} \gamma(\tau)^2 \,\mathrm{d}\tau$$

is finite. Let B be the symmetric operator with domain equal to the functions in $\mathcal{H}$ with compact support and given by

$$(\gamma, B\gamma) = 2S_0(\gamma) = \int_{-\infty}^{\infty} \left(\dot{\gamma}(\tau)^2 - \gamma A^2 \gamma\right) \,\mathrm{d}\tau,$$

and let D be the real Banach space of functions in $\mathcal{H}$ with differentiable Fourier transforms and norm given by (6.8), and let $\triangle$ be given by (6.9). Then ($\mathcal{H}$, D, B, $\triangle$) satisfies the condition of Definition 4.1 for the integral normalized with respect to $\triangle$. Let f, g and V be in $\mathcal{F}(\mathbb{R}^n)$, then $f(\gamma(0))g(\gamma(t))$ and $\exp\left[-\mathrm{i}\int_0^t V(\gamma(\tau))\,\mathrm{d}\tau\right]$ are all in $\mathcal{F}(D^*)$ and

$$\left(f\Omega_0, \mathrm{e}^{-\mathrm{i}tH} g\Omega_0\right) = \int_{\mathcal{H}}^{\triangle} \mathrm{e}^{\mathrm{i}S_0(\gamma)} \mathrm{e}^{\mathrm{i}\int_0^t V(\gamma(\tau))\,\mathrm{d}\tau} \overline{f}(\gamma(0)) g(\gamma(t)) \,\mathrm{d}\gamma,$$

where

$$H = H_0 + V$$

[1] This is (5.33) written in a different way.

Proof. The first part of the theorem is already proved. Let therefore f be in $\mathcal{F}(\mathbb{R}^n)$ i.e.

$$f(x) = \int e^{i\alpha x} \, d\nu(\alpha), \tag{6.22}$$

then

$$f(\gamma(0)) = \int e^{i\alpha\gamma(0)} \, d\nu(\alpha),$$

which is in $\mathcal{F}(D^*)$ by the definition of $\mathcal{F}(D^*)$, since $\gamma(0) = (\gamma_0, \gamma)$ and we already proved that $\gamma_0 \in D$. Hence also $g(\gamma(t))$ is in $\mathcal{F}(D^*)$. Now

$$\int_0^t V(\gamma(\tau)) \, d\tau = \int_0^t \int e^{i\alpha\gamma(\tau)} \, d\mu(\alpha) \, d\tau \tag{6.23}$$

is again in $\mathcal{F}(D^*)$ and therefore also $\exp\left[-i\int_0^t V(\gamma(\tau)) \, d\tau\right]$ belongs to $\mathcal{F}(D^*)$ by Proposition 4.1 (which states that $\mathcal{F}(D^*)$ is a Banach algebra). Since, also by Proposition 4.1, the Fresnel integral with respect to $\triangle$ is a continuous linear functional on this Banach algebra we have

$$\begin{aligned} &\int_{\mathcal{H}}^{\triangle} e^{iS_0(\gamma)} e^{-i\int_0^t V(\gamma(\tau)) \, d\tau} \overline{f}(\gamma(0)) g(\gamma(t)) \, d\gamma \\ &= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_0^t \dots \int_0^t \int_{\mathcal{H}}^{\triangle} e^{iS_0(\gamma)} V(\gamma(t_1)) \dots V(\gamma(t_n)) \, d\gamma \, dt_1 \dots dt_n. \end{aligned} \tag{6.24}$$

Utilizing now (6.23), (6.21) and the perturbation expansion (5.35) the theorem is proved. □

Theorem 6.2. *Let the notations be the same as in Theorem 6.1, and let* $t_1 \le \dots \le t_m$*, then for* $f_i \in \mathcal{S}(\mathbb{R}^n)$*,* $i = 1, \dots m$

$$\begin{aligned} &\left(\Omega_0, f_1 e^{-i(t_2-t_1)H} f_2 e^{-i(t_3-t_2)H} f_3 \dots e^{-i(t_m-t_{m-1})H} f_m \Omega_0\right) \\ &= \int_{\mathcal{H}}^{\triangle} e^{iS_0(\gamma)} e^{-i\int_{t_1}^{t_m} V(\gamma(\tau)) \, d\tau} \prod_{j=1}^{m} f_j(\gamma(t_j)) \, d\gamma. \end{aligned}$$

This theorem is proved by the series expansion of the function

$$\exp\left(-i\int_{t_1}^{t_m} V(\gamma(\tau)) \, d\tau\right)$$

and the fact that this series converges in $\mathcal{F}(D^*)$*, in the same way as in the proof of Theorem 6.1.*

Notes

This section, first presented in the first edition of this book, is geared towards quantum field theory (looking at nonrelativistic quantum mechanics as a "zero dimensional" quantum field theory). Formulae like (6.21) are typical of this view, see e.g. [425] for similar formulae in the "Euclidean approach" to quantum fields.