# Applications of Representation Theory to Harmonic Analysis of Lie Groups (and Vice Versa)

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These notes began as lectures that I intended to deliver in Edinburgh in April, 1999. Unfortunately I was not able to leave Australia at the time. Since then there has been progress on many of the topics, some of which is reported here, and I have added another lecture, on uniformly bounded representations, so that these notes are expanded on the original version in several ways.

I have tried to make these notes an understandable introduction to the subject for mathematicians with little experience of analysis on Lie groups or Lie theory. I aimed to present a wide panorama of different aspects of harmonic analysis on semisimple groups and symmetric spaces, and to try to illuminate some of the links between these aspects; I may well not have succeeded in this aim. Many readers will find much of what is written here to be elementary, and others may well disagree with my perspective. I apologise in advance to both the neophytes for whom my outline is too sketchy and to the experts for whom these notes are worthless.

I had hoped to produce an extensive bibliography, but I have not found the time to do so. Consequently I must bear the responsibility for the many omissions of important references in the subject.

Whoever wishes to delve into this subject more deeply will need a more complete introduction. There are many possibilities; the books of S. Helgason [59, 60, 62] and of A.W. Knapp [71] come to mind immediately as essential reading.

# 1 Basic Facts of Harmonic Analysis on Semisimple Groups and Symmetric Spaces

I will deal with noncompact classical algebraic semisimple Lie groups, such as $\mathrm{SO}(p,q)$, $\mathrm{SU}(p,q)$, $\mathrm{Sp}(p,q)$, $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{SL}(n,\mathbb{C})$, and $\mathrm{SL}(n,\mathbb{H})$. The definitions of these may be found in [59, pp. 444–447] or [71, pp. 3–6].

All noncompact algebraic semisimple Lie groups have various standard subgroups and decompositions. I begin by describing these, then describe families of unitary representations parametrised by representations of some of these subgroups. Finally, I discuss the Plancherel formula. The fact that most of the important representations are parametrised by representations of subgroups allows arguments involving induction on the rank of the group.

## 1.1 Structure of Semisimple Lie Algebras

First, fix a *Cartan involution* $\theta$ of the Lie algebra $\mathfrak{g}$ of the group $G$, and write $\mathfrak{k}$ and $\mathfrak{p}$ for the $+1$ and $-1$ eigenspaces of $\theta$. Then $\mathfrak{k}$ is a maximal compact subalgebra of $\mathfrak{g}$, and $\mathfrak{p}$ is a subspace; $[X,Y] \in \mathfrak{k}$ for all $X, Y \in \mathfrak{p}$. Since $\theta$ is an involution, we have the Cartan decomposition of the Lie algebra:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

In this and future formulae about the Lie algebra, $\oplus$ means "vector space direct sum". All Cartan involutions are conjugate in the group of Lie algebra automorphisms of $\mathfrak{g}$, which is a finite extension of the group generated by $\{\exp \operatorname{ad} X : X \in \mathfrak{g}\}$. The Cartan involution $\theta$ extends to an automorphism $\Theta$ of the group $G$, whose fixed point set is a maximal compact subgroup $K$ of $G$.

Next choose a maximal subalgebra of $\mathfrak{p}$; this is abelian, and is denoted by $\mathfrak{a}$. All such subalgebras are conjugate under $K$. Let $\operatorname{ad}(X)$ denote the derivation $Y \mapsto [X,Y]$ of $\mathfrak{g}$. Then the Killing form $B$, given by

$$B(X,Y) = \operatorname{tr}(\operatorname{ad}(X)\operatorname{ad}(Y)) \qquad \forall X, Y \in \mathfrak{g},$$

gives rise to an inner product on $\mathfrak{a}$:

$$(X,Y)_B = -B(X, \theta Y) \qquad \forall X, Y \in \mathfrak{g},$$

which gives rise to a dual inner product, denoted in the same way, on $\mathfrak{a}^*$, which in turn extends to a bilinear form on $\mathfrak{a}_{\mathbb{C}}$, also denoted in the same way.

The third step in the description and construction of the various special subalgebras of $\mathfrak{g}$ and corresponding subgroups of $G$ is to decompose $\mathfrak{g}$ as a direct sum of *root spaces* $\mathfrak{g}_\alpha$ and a subalgebra $\mathfrak{g}_0$. Simultaneously diagonalise the operators $\operatorname{ad}(H)$, for $H$ in $\mathfrak{a}$. For $\alpha$ in the real dual $\mathfrak{a}^*$ of $\mathfrak{a}$ (that is, $\mathfrak{a}^* = \operatorname{Hom}_{\mathbb{R}}(\mathfrak{a}, \mathbb{R})$), define

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : [H,X] = \alpha(H)X \quad \forall H \in \mathfrak{a}\}.$$

For most $\alpha$ in $\mathfrak{a}^*$, $\mathfrak{g}_\alpha = \{0\}$, but when $\alpha = 0$, then $\mathfrak{a} \subseteq \mathfrak{g}_0$, so $\mathfrak{g}_0 \neq \{0\}$. There are finitely many nonzero $\alpha$ in $\mathfrak{a}^*$ for which $\mathfrak{g}_\alpha \neq \{0\}$; these $\alpha$ are called the *real roots* of $(\mathfrak{g}, \mathfrak{a})$, and the set thereof is written $\Sigma$. This set is a *root system*, a highly symmetric subset of $\mathfrak{a}^*$. Because $\mathfrak{g}_0$ is $\theta$-stable,

$$\mathfrak{g}_0 = (\mathfrak{g}_0 \cap \mathfrak{k}) \oplus (\mathfrak{g}_0 \cap \mathfrak{p}) = \mathfrak{m} \oplus \mathfrak{a},$$

say, where $\mathfrak{m}$ is the subalgebra of $\mathfrak{k}$ of elements which commute with $\mathfrak{a}$. Using the fact that $\mathrm{ad}(H)$ is a derivation of $\mathfrak{g}$ for each $H$ in $\mathfrak{a}$, it is easy to check that

$$[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}. \tag{1.1}$$

In particular, $\mathfrak{g}_0$ is a subalgebra, and $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ commute when $\mathfrak{g}_{\alpha+\beta} = \{0\}$. Clearly

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Sigma}^{\oplus} \mathfrak{g}_\alpha.$$

Now order the roots. The hyperplanes $\{H \in \mathfrak{a} : \alpha(H) = 0\}$, for $\alpha$ in $\Sigma$, divide $\mathfrak{a}$ into finitely many connected open cones, known as Weyl chambers. Pick one of these (arbitrarily) and fix it; it is called the positive Weyl chamber, and written $\mathfrak{a}^+$. A root $\alpha$ is now said to be *positive* or *negative* as $\alpha(H) > 0$ or $\alpha(H) < 0$ for all $H$ in $\mathfrak{a}^+$. Write $\Sigma^+$ for the set of positive roots; then $\Sigma = \Sigma^+ \cup -(\Sigma^+)$. For some roots $\alpha$ and real numbers $t$, $t\alpha$ is also a root; the possibilities are that $t = \pm 1$ (this always happens), $t = \pm 1/2$ or $t = \pm 2$ (these last four possibilities may or may not occur). If $(1/2)\alpha$ is not a root, then $\alpha$ is said to be *indivisible*; denote by $\Sigma_0^+$ the set of indivisible positive roots.

We can now define some more important subalgebras: let

$$\mathfrak{n} = \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha \qquad \text{and} \qquad \overline{\mathfrak{n}} = \sum_{\alpha \in \Sigma^+} \mathfrak{g}_{-\alpha}\,;$$

it is easy to deduce from formula (1.1) that $\mathfrak{n}$ and $\overline{\mathfrak{n}}$ are *nilpotent* subalgebras of $\mathfrak{g}$. Define $\rho$ by the formula

$$\rho(H) = \frac{1}{2}\,\mathrm{tr}(\mathrm{ad}(H)|_{\mathfrak{n}}) \qquad \forall H \in \mathfrak{a};$$

then $\rho = (1/2)\sum_{\alpha \in \Sigma^+} \dim(\mathfrak{g}_\alpha)\,\alpha$. We now have the ingredients for two more decompositions of $\mathfrak{g}$: the Iwasawa decomposition and the Bruhat decomposition, written

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n} \qquad \text{and} \qquad \mathfrak{g} = \overline{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

The proof of the second (Bruhat) decomposition is immediate. For the first (Iwasawa) decomposition, note that if $X \in \mathfrak{g}_\alpha$, then $\theta X \in \mathfrak{g}_{-\alpha}$, so that, if $X \in \overline{\mathfrak{n}}$, then

$$X = (X + \theta X) - \theta X \in \mathfrak{k} \oplus \mathfrak{n}.$$

### 1.2 Decompositions of Semisimple Lie Groups

At the group level, there are similar decompositions (usually known as factorisations in undergraduate linear algebra courses). Let $K$, $A$, $N$ and $\overline{N}$ denote the connected subgroups of $G$ with Lie algebras $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}$ and $\overline{\mathfrak{n}}$, and let $A^+$ and $\overline{A}^+$ be the subsemigroup $\exp(\mathfrak{a}^+)$ of $A$ and its closure. Let $M$ and $M'$ be the centraliser and normaliser of $\mathfrak{a}$ in $K$. Then both $M$ and $M'$ have $\mathfrak{m}$ as their Lie algebra. The group $M'$ is never connected, while $M$ is connected in some examples and is not in others. However, $M'/M$ is always finite. In fact, the adjoint action Ad of $M'$ on $\mathfrak{a}$ induces an isomorphism of $M'/M$ with a finite group of orthogonal transformations of $\mathfrak{a}$, generated by reflections. This is the Weyl group, $W(\mathfrak{g}, \mathfrak{a})$. It acts simply transitively on the space of Weyl chambers, that is, every Weyl chamber is the image of $\mathfrak{a}^+$ under a unique element of the Weyl group. By duality, this group also acts on $\mathfrak{a}^*$, and permutes the roots amongst themselves. Take a representative $s_w$ in $M'$ of each $w$ in the Weyl group.

At the group level, there are three important decompositions:

$$G = K\overline{A}^+K, \tag{1.2}$$

$$G = KAN, \tag{1.3}$$

$$G = \bigsqcup_{w\in W} MANs_wMAN \tag{1.4}$$

(this last formula involves a disjoint union). The Cartan decomposition (1.2) arises from the "polar decomposition" $G = K\exp(\mathfrak{p})$, in which the map $(k, X) \mapsto k\exp(X)$ is a diffeomorphism from $K \times \mathfrak{p}$ onto $G$; every element of $\mathfrak{p}$ is conjugate to an element of $\overline{\mathfrak{a}}^+$ by an element of $K$. In the Iwasawa decomposition (1.3), the map $(k, a, n) \mapsto kan$ is a diffeomorphism from $K\times A\times N$ onto $G$. In the Bruhat decomposition (1.4), each of the sets $MANs_wMAN$ is a submanifold of $G$, and the $|W|$ submanifolds are pairwise disjoint. There is a unique *longest element* $\overline{w}$ of the Weyl group, which maps $\mathfrak{a}^+$ to $-\mathfrak{a}^+$; the corresponding submanifold of $G$ is open and its complement is a union of submanifolds of lower dimension. More precisely,

$$\begin{aligned}
G &= \bigsqcup_{w\in W} s_{\overline{w}}MANs_wMAN \\
&= \bigsqcup_{w\in W} s_{\overline{w}}s_ws_w^{-1}NAMs_wMAN \\
&= \bigsqcup_{w\in W} s_{\overline{w}w}\overline{N}_wMAN,
\end{aligned}$$

where $\overline{N}_w = s_w^{-1}NS_w \cap \overline{N}$; each $\overline{N}_w$ is a Lie subgroup of $\overline{N}$, of lower dimension unless $w = \overline{w}$, and the map $(\overline{n}, m, a, n) \mapsto \overline{n}man$ is a diffeomorphism from $\overline{N}_w \times M \times A \times N$ onto $\overline{N}_wMAN$.

For many purposes it is sufficient to think of the Bruhat decomposition in the following way: the map $(\overline{n}, m, a, n) \mapsto \overline{n}MAN$ of $\overline{N} \times M \times A \times N$ to $G$ is a diffeomorphism of $\overline{N}MAN$ onto an open dense subset of $G$ whose complement is a finite union of lower dimensional submanifolds. In particular, $\overline{N}MAN$ is of full measure in $G/MAN$, equipped with any of the natural measures. I will use the abusive notation $G \simeq \overline{N}MAN$ to indicate this sort of "quasi-decomposition".

There are integral formulae associated with these group decompositions. In particular, we will use the formula

$$\begin{aligned} \int_G u(x)\,dx = C \int_K \int_{\overline{\mathfrak{a}}^+} \int_K & u(k_1 \exp(H) k_2) \\ & \prod_{\alpha \in \Sigma} \sinh(\alpha(H))^{\dim(\mathfrak{g}_\alpha)}\,dk_1\,dH\,dk_2, \end{aligned} \tag{1.5}$$

which relates the Haar measure on $G$ with the Haar measure $dk$ on $K$ and a weighted variant of Lebesgue measure $dH$ on $\mathfrak{a}^+$. For the formulae for the Iwasawa and Bruhat decompositions, see [60, Propositions I.5.1 and I.5.21].

### 1.3 Parabolic Subgroups

The subgroup $MAN$, often written $P$, is known as a *minimal parabolic subgroup.* Any subgroup $P_1$ of $G$ containing $MAN$ is known as a parabolic subgroup; such a group may be decomposed in the form

$$P_1 = M_1 A_1 N_1,$$

where $M_1 \supseteq M$, $A_1 \subseteq A$, and $N_1 \subseteq N$. The group $M_1$ is a semisimple subgroup of $G$, and has its own Iwasawa and Bruhat decompositions:

$$M_1 = K^1 A^1 N^1 \qquad \text{and} \qquad M_1 \simeq \overline{N}^1 M^1 A^1 N^1.$$

In these formulae, $K^1 \subseteq K$, $A^1 \subseteq A$, $N^1 \subseteq N$, $M^1 \supseteq M$, and $\overline{N}^1 \subseteq \overline{N}$; moreover, $\overline{N}^1 = \Theta N^1$. If $\mathfrak{a}_1$, $\mathfrak{a}^1$, $\mathfrak{n}_1$ and $\mathfrak{n}^1$ denote the subalgebras of $\mathfrak{a}$ and $\mathfrak{n}$ corresponding to $A_1$, $A^1$, $N_1$ and $N^1$, then $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}^1$ and $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}^1$. To each parabolic subgroup $P_1$, we associate $\rho_1$ on $\mathfrak{a}_1$, defined similarly to $\rho$; more precisely,

$$\rho_1(H) = \frac{1}{2}\operatorname{tr}(\operatorname{ad}(H)|_{\mathfrak{n}_1}) \qquad \forall H \in \mathfrak{a}_1$$

The point of this is mainly that the set of all subgroups $P_1$ of $G$ containing $P$ is well understood: it is a finite lattice with a well determined structure.

We conclude our discussion of the structure of $G$ with one more definition. A parabolic subgroup $P_1$ of $G$ is called *cuspidal* if $M_1$ has a compact Cartan subgroup, that is, if there is a compact abelian subgroup of $K_1$ which cannot be extended to a larger abelian subgroup of $M_1$. Since $M$ is compact,

$P$ is automatically cuspidal. It is a deep theorem of Harish-Chandra that the semisimple groups which have discrete series representations, that is, irreducible unitary representations which are subrepresentations of the regular representation, are precisely those with compact Cartan subgroups.

### 1.4 Spaces of Homogeneous Functions on $G$

For this section, fix a parabolic subgroup $M_1A_1N_1$ of $G$. Take an irreducible unitary representation $\mu$ of $M_1$ and $\lambda$ in the complexification $\mathfrak{a}_{1\mathbb{C}}^*$ of $\mathfrak{a}_1^*$ (that is, $\mathfrak{a}_{1\mathbb{C}}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a}_1, \mathbb{C})$). Let $\mathcal{H}_\mu$ denote the Hilbert space on which the representation $\mu$ acts. Consider the vector space $\mathcal{V}^{\mu,\lambda}$ of all smooth (infinitely differentiable) $\mathcal{H}_\mu$-valued functions $\xi$ on $G$ with the property that

$$\xi(xman) = e^{(i\lambda-\rho_1)(\log a)}\mu(m)^{-1}\xi(x),$$

for all $x$ in $G$, all $m$ in $M_1$, all $a$ in $A_1$ and all $n$ in $N_1$. These functions may also be viewed as functions on $G/N_1$, since $\xi(xn) = \xi(x)$ for all $x$ in $G$ and $n$ in $N_1$, or as sections of a vector bundle over $G/P_1$. I shall take the naive viewpoint that they are functions on $G$, even though there are often good geometric reasons for using vector bundle terminology. Write $\pi^{\mu,\lambda}$ for the left translation representation on $\mathcal{V}^{\mu,\lambda}$:

$$[\pi^{\mu,\lambda}(y)\xi](x) = \xi(y^{-1}x) \qquad \forall x, y \in G.$$

The inner product on $\mathcal{H}_\mu$ induces a pairing $\mathcal{V}^{\mu,\lambda'} \times \mathcal{V}^{\mu,\lambda} \to \mathcal{V}^{1,\lambda'-\overline{\lambda}+i\rho_1}$: indeed,

$$\begin{aligned}
&\langle \xi(xman), \eta(xman)\rangle \\
&\qquad = \langle e^{(i\lambda'-\rho_1)(\log a)}\mu(m)^{-1}\xi(x),\, e^{(i\lambda-\rho_1)(\log a)}\mu(m)^{-1}\eta(x)\rangle \\
&\qquad = e^{(i\lambda'-i\overline{\lambda}-2\rho_1)(\log a)}\langle \xi(x), \eta(x)\rangle,
\end{aligned}$$

so the complex-valued function $x \mapsto \langle \xi(x), \eta(x)\rangle$ indeed satisfies the covariance condition characterising $\mathcal{V}^{1,\lambda'-\overline{\lambda}+i\rho_1}$.

**Lemma 1.1.** *There is a $G$-invariant positive linear functional $I_{P_1}$ on $\mathcal{V}^{1,i\rho_1}$, which is unique up to a constant. It may be defined as (a constant multiple of) the Haar measure on $K$,*

$$\xi \mapsto \int_K \xi(k)\, dk,$$

*or as (a constant multiple of) the Haar measure on $\overline{N}_1$,*

$$\xi \mapsto \int_{\overline{N}_1} \xi(\overline{n})\, d\overline{n}.$$

*Proof.* This can be proved by fairly explicit means, involving calculation of Jacobians, to show that there is a constant $c$ such that

$$\int_K \xi(k)\, dk = c \int_{\overline{N}} \xi(\overline{n})\, d\overline{n},$$

and then deducing that this expression is $K$-invariant and $\overline{N}$-invariant, and so invariant under the group generated by $K$ and $\overline{N}$, which is $G$ itself.

Alternatively, this may be proved by using the fact that the "modular function" of $M_1A_1N_1$ is given by $man \mapsto e^{-2\rho_1(\log a)}$. See, for instance, [71, pp. 137–141].

Normalise $I_{P_1}$ so that

$$I_{P_1}(\xi) = \int_K \xi(k)\, dk.$$

An immediate corollary of this lemma is that the spaces $\mathcal{V}^{\mu,\lambda}$ and $\mathcal{V}^{\mu,\overline{\lambda}}$ are in duality: the map

$$\langle \xi, \eta\rangle \mapsto I_{P_1}\langle \xi(\cdot), \eta(\cdot)\rangle$$

is well defined. Further, it is easy to check that

$$\langle \pi^{\mu,\lambda}(y)\xi,\, \pi^{\mu,\overline{\lambda}}(y)\eta\rangle = \langle \xi, \eta\rangle \quad \forall y \in G.$$

In particular, if $\lambda$ is real, then the duality on $\mathcal{V}^{\mu,\lambda} \times \mathcal{V}^{\mu,\lambda}$ gives an inner product on $\mathcal{V}^{\mu,\lambda}$ relative to which $\pi^{\mu,\lambda}$ acts unitarily. In this case, $\mathcal{V}^{\mu,\lambda}$ may be completed to obtain a Hilbert space $\mathcal{H}^{\mu,\lambda}$ on which $\pi^{\mu,\lambda}$ acts unitarily.

In some cases, when $\lambda$ is not real, it is possible to find an inner product on $\mathcal{V}^{\mu,\lambda}$ relative to which $\pi^{\mu,\lambda}$ acts unitarily. The unitary representations which arise by completing $\mathcal{V}^{\mu,\lambda}$ relative to this inner product are known as *complementary series* representations. It is also possible to work with other completions of $\mathcal{V}^{\mu,\lambda}$. For example, if $1 \le p \le \infty$, and $\mathrm{Im}(\lambda) = (2/p - 1)\rho_1$, then

$$\begin{aligned}
\|\xi(xman)\|^p_{\mathcal{H}_\mu} &= \left\|\mu(m)e^{(i\lambda-\rho_1)(\log a)}\xi(x)\right\|^p_{\mathcal{H}_\mu} \\
&= e^{-p(\mathrm{Im}\,\lambda+\rho_1)(\log a)}\, \|\xi(x)\|^p_{\mathcal{H}_\mu} \\
&= e^{-2\rho_1(\log a)}\, \|\xi(x)\|^p_{\mathcal{H}_\mu}
\end{aligned}$$

for all $x$ in $G$, all $m$ in $M$, all $a$ in $A$ and all $n$ in $N$, so $\|\xi(\cdot)\|^p_{\mathcal{H}_\mu} \in \mathcal{V}^{1,i\rho_1}$. In this case, $\pi^{\mu,\lambda}$ acts isometrically on the completion of $\mathcal{V}^{\mu,\lambda}$ in the $L^p$-norm $\left(I_{P_1}(\|\cdot\|^p_{\mathcal{H}_\mu})\right)^{1/p}$, and to all intents and purposes we are dealing with a representation on a $\mathcal{H}_\mu$-valued $L^p$-space.

It is a notable fact that for the case where $G = \mathrm{SO}(1,n)$, the representations $\pi^{\mu,\lambda}$ may be completed to obtain unitary representations in Sobolev spaces, as well as isometric representations on $L^p$-spaces, and the Sobolev spaces and the $L^p$-spaces are linked as in the Sobolev embedding theorem: the degree of differentiation involved is such that the Sobolev space is either included in the $L^p$-space (when $p > 2$) or the $L^p$-space is included in the Sobolev space (when $p < 2$). To understand the corresponding result for other families of semisimple Lie groups, such as $\mathrm{SU}(1, n)$, is an important problem, to be discussed in the last lecture.

It is known that the representations $\pi^{\mu,\lambda}$ on $\mathcal{V}^{\mu,\lambda}$ are mostly irreducible—for a given $\mu$, the set of $\lambda$ in $\mathfrak{a}_{1\mathbb{C}}^*$ for which $\pi^{\mu,\lambda}$ is reducible is a countable union of hyperplanes in $\mathfrak{a}_{1\mathbb{C}}^*$. Here, reducible means that there are nontrivial closed (in the $C^\infty$-topology) $G$-invariant subspaces of $\mathcal{V}^{\mu,\lambda}$.

### 1.5 The Plancherel Formula

The Plancherel formula for semisimple Lie groups was proved by Harish-Chandra [53, 54, 55], following previous work by various people for various special cases. The representations involved are the representations $\pi^{\mu,\lambda}$, where $\mu$ is a discrete series representation of $M_1$ (written $\mu \in \widehat{M_1}_d$), and $\lambda$ in $\mathfrak{a}_{1\mathbb{C}}^*$ is real. Such representations are sometimes called *unitary principal series* representations—in this nomenclature, the *principal series* is the collection of all the $\pi^{\mu,\lambda}$ without the restriction on $\lambda$. Other authors call the smaller collection of representations the unitary principal series, and the larger collections of representations is then known as the *analytic continuation* of the principal series. All the cuspidal parabolic subgroups are involved.

A bit more notation is needed to state the Plancherel theorem: for $u$ in $C_c^\infty(G)$, the operator $\pi^{\mu,\lambda}(u)$ is given by the formula

$$\pi^{\mu,\lambda}(u) = \int_G u(y)\, \pi^{\mu,\lambda}(y)\, dy,$$

which is to be interpreted as an operator-valued integral. Let $\mathcal{P}$ be a set of nonconjugate cuspidal parabolic subgroups of $G$, and $\mathbf{c}$ be the more or less explicitly determined function known as the Harish-Chandra $\mathbf{c}$-function.

**Theorem 1.1.** *Suppose that $u \in C_c^\infty(G)$. Then the operators $\pi^{\mu,\lambda}(u)$ are trace-class for all $\mu$ in $\widehat{M_1}_d$ and $\lambda$ in $\mathfrak{a}_{1\mathbb{C}}^*$, and the $L^2(G)$-norm of $u$ is given by*

$$\|u\|_2^2 = \sum_{P_1 \in \mathcal{P}} \sum_{\mu \in \widehat{M_1}_d} c_{P_1} \int_{\mathfrak{a}_1^*} \operatorname{tr}\left(\pi^{\mu,\lambda}(u)^* \pi^{\mu,\lambda}(u)\right) \left|\mathbf{c}(P_1, \mu, \lambda)\right|^{-2} d\lambda.$$

Fortunately, a simpler formula is available for most of the analysis in the following lectures. If the function $u$ is $K$-invariant, on the left or the right or both, then $\pi^{\mu,\lambda}(u) = 0$ unless $P_1$ is the minimal parabolic and $\mu$ is the trivial representation 1 of $M$. This reduces substantially the number of terms in the Plancherel formula. Further, the operators $\pi^{1,\lambda}(u)$ are rank one operators, so that the Hilbert–Schmidt norm and the operator norm $\|\cdot\|$ (and all the other Schatten $p$-norms) coincide. Thus

$$\|u\|_2^2 = c_G \int_{\mathfrak{a}^*} \left\|\pi^{\mu,\lambda}(u)\right\|^2 \left|\mathbf{c}(\lambda)\right|^{-2} \, d\lambda.$$

Let $1^\lambda$ denote the function in $\mathcal{V}^{1,\lambda}$ which is identically equal to 1 on $K$. When $u$ is $K$-bi-invariant, the formula simplifies further: $\pi^{1,\lambda}(u)$ is a multiple of the projection onto $\mathbb{C}1^\lambda$. This multiple, denoted $\widetilde{u}(\lambda)$, is given by

$$\widetilde{u}(\lambda) = \int_G u(x)\, \varphi_\lambda(x)\, dx$$

where

$$\varphi_\lambda(x) = \langle \pi^{1,\lambda}(x)1^\lambda, 1^{\overline{\lambda}} \rangle \qquad \forall x \in G.$$

For $K$-bi-invariant functions $u$, the Plancherel formula becomes

$$\|u\|_2^2 = c_G \int_{\mathfrak{a}^*} \left|\widetilde{u}(\lambda)\right|^2 \left|\mathbf{c}(\lambda)\right|^{-2} \, d\lambda; \tag{1.6}$$

the corresponding inversion formula is

$$u(x) = c_G \int_{\mathfrak{a}^*} \widetilde{u}(\lambda)\, \overline{\varphi}_\lambda(x)\ \left|\mathbf{c}(\lambda)\right|^{-2} \, d\lambda \qquad \forall x \in G. \tag{1.7}$$

There are a number of integral formulae for $\varphi_\lambda$, which may be found in [60, Chapter IV]. One of these is the following: for any $x$ in $G$, denote by $A(x)$ the unique element of $\mathfrak{a}$ such that $x \in N \exp A(x)\, K$. For any $\lambda$ in $\mathfrak{a}^*_{\mathbb{C}}$, the spherical function $\varphi_\lambda$ is given by

$$\varphi_\lambda(x) = \int_K \exp\big((i\lambda + \rho)(A(kx))\big)\, dk \qquad \forall x \in G.$$

It is worth pointing out that when $G$ is complex, then there are explicit formulae for $\varphi_\lambda$, in terms of elementary functions, and for several other cases where $\dim(\mathfrak{a})$ is small, there are formulae in terms of hypergeometric functions or other less elementary functions (see, for instance, [64]). Perhaps the most important technique for understanding spherical functions in a fairly general context is M. Flensted-Jensen's method [48] of reducing the case of a normal real form to the complex case. It is possible to work with spherical functions fairly effectively: one can formulate conjectures using the complex case as a guide, and prove many of these for some or all general semisimple groups.

## 2 The Equations of Mathematical Physics on Symmetric Spaces

Let $G$ be a semisimple Lie group with a maximal compact subgroup $K$; then the quotient space $X = G/K$ is, in a natural way, a negatively curved Riemannian manifold. In particular, when $G = \mathrm{SO}(1,n)$, $\mathrm{SU}(1,n)$, or $\mathrm{Sp}(1,n)$, the manifold $X$ is a real, complex, or quaternionic hyperbolic space. The Laplace–Beltrami operator on $X$ is a natural second order elliptic differential operator. I prefer to deal with the positive operator $\varDelta$, equal to minus the Laplace–Beltrami operator. The $L^2$ spectrum of $\varDelta$ is the interval $[b,\infty)$, where $b = (\rho,\rho)_B$. It is natural to study not only $\varDelta$, but also $\varDelta - b$, which is still a positive operator and from some geometric points of view is more canonical than $\varDelta$. We shall consider $\varDelta - \theta b$, where $\theta \in [0,1]$.

This lecture deals with the equations of mathematical physics on $X$, that is, with the solutions of the equations

$$
\begin{aligned}
\frac{\partial}{\partial t} u_1(x,t) + (\varDelta - \theta b) u_1(x,t) = 0 \\
\frac{\partial^2}{\partial t^2} u_2(x,t) - (\varDelta - \theta b) u_2(x,t) = 0 \\
\frac{\partial^2}{\partial t^2} u_3(x,t) + (\varDelta - \theta b) u_3(x,t) = 0 \\
\frac{\partial}{\partial t} u_4(x,t) + i(\varDelta - \theta b) u_4(x,t) = 0
\end{aligned}
$$

for all $(x,t)$ in $X \times \mathbb{R}^+$, with boundary conditions $u_k(\cdot,0) = f$ for all $k$ in $\{1,2,3,4\}$, $u_2(\cdot,t) \to 0$ (in some sense, depending on $f$) as $t \to \infty$, and $\partial u_3/\partial t(\cdot,0) = i(\varDelta - \theta b)^{1/2} f$. These equations are the heat equation, Laplace's equation, the wave equation, and the Schrödinger equation. In the Euclidean case, these equations can be solved using the Fourier transform. The same is true in this case.

### 2.1 Spherical Analysis on Symmetric Spaces

Any function $f$ on $X$ gives rise canonically to a $K$-right-invariant function on $G$, also denoted by $f$. The key to the Fourier transform approach to these equations is that, for any $f$ in $C_c^\infty(X)$,

$$
\pi^{1,\lambda}(\varDelta f) = ((\lambda,\lambda)_B + (\rho,\rho)_B)\pi^{1,\lambda}(f) \qquad \forall \lambda \in \mathfrak{a}^*_{\mathbb{C}},
$$

that is, $\varDelta$ is a *Fourier multiplier*. Because $\varDelta$ corresponds to a positive operator on $L^2(X)$, it is possible to use spectral theory to define $m(\varDelta)$ for Borel measurable functions on $[b,\infty)$. For bounded $m$, the operator $m(\varDelta)$ is defined by

$$
m(\varDelta) f = \int_b^\infty m(\zeta)\, dP_\zeta f \qquad \forall f \in L^2(X),
$$

where $\{P_\zeta\}$ is the spectral resolution of the identity such that

$$\Delta f = \int_b^\infty \zeta \, dP_\zeta f \qquad \forall f \in \operatorname{Dom}(\Delta).$$

Define the quadratic function $Q_\theta$ by the formula

$$Q_\theta(\lambda) = (\lambda, \lambda)_B + (1-\theta)(\rho, \rho)_B \qquad \forall \lambda \in \mathfrak{a}_{\mathbb{C}}^*.$$

By spectral theory,

$$\pi^{1,\lambda}((\Delta - \theta b) f) = Q_\theta(\lambda) \, \pi^{1,\lambda}(f)$$

and

$$\pi^{1,\lambda}(m(\Delta - \theta b) f) = m(Q_\theta(\lambda)) \, \pi^{1,\lambda}(f).$$

At least formally, the solutions of the equations of mathematical physics on $L^2(X)$ are given by

$$\begin{aligned}
u_1(\cdot, t) &= e^{-t(\Delta - \theta b)} f \\
u_2(\cdot, t) &= e^{-t(\Delta - \theta b)^{1/2}} f \\
u_3(\cdot, t) &= e^{it(\Delta - \theta b)^{1/2}} f \\
u_4(\cdot, t) &= e^{it(\Delta - \theta b)} f
\end{aligned}$$

for all $t$ in $\mathbb{R}^+$. These solutions may also be expressed in terms of convolutions with kernels:

$$u_1(xK, t) = \mathcal{H}_{t,\theta} f(xK) = f \star h_{t,\theta}(x)$$

$$u_2(xK, t) = \mathcal{L}_{t,\theta} f(xK) = f \star l_{t,\theta}(x)$$

$$u_3(xK, t) = \mathcal{W}_{t,\theta} f(xK) = f \star w_{t,\theta}(x)$$

$$u_4(xK, t) = \mathcal{S}_{t,\theta} f(xK) = f \star s_{t,\theta}(x),$$

for all $x$ in $G$ and $t$ in $\mathbb{R}^+$; here, at least formally, the kernels are the $K$-bi-invariant objects on $G$ such that

$$\begin{aligned}
\widetilde{h}_{t,\theta} &= e^{-tQ_\theta} \\
\widetilde{l}_{t,\theta} &= e^{-tQ_\theta^{1/2}} \\
\widetilde{w}_{t,\theta} &= e^{itQ_\theta^{1/2}} \\
\widetilde{s}_{t,\theta} &= e^{itQ_\theta}.
\end{aligned}$$

These formulae should be compared with the results in the Euclidean case obtained by classical Fourier analysis. For example, for the heat equation in $\mathbb{R}^n$,

$$u(x,t) = f \star g_t$$
$$\mathcal{F}u(\xi,t) = (\mathcal{F}f)(\xi)\, e^{-t|\xi|^2}$$

for all $x$ and $\xi$ in $\mathbb{R}^n$ and $t$ in $\mathbb{R}^+$, where $g_t$ is the appropriately normalised Gaussian kernel on $\mathbb{R}^n$ and $\mathcal{F}$ denotes the spatial Fourier transform.

In order to obtain useful information from these formulae for the solutions to these equations, we need to have information about the kernels $h_{t,\theta}$, $l_{t,\theta}$, $w_{t,\theta}$, and $s_{t,\theta}$. This information can be of several types, for instance, pointwise estimates, $L^p$ estimates, or parametrix expressions (the latter means an expression of the kernel as a sum of distributions). It is also important to understand the regularity properties of $\Delta$ itself.

### 2.2 Harmonic Analysis on Semisimple Groups and Symmetric Spaces

This section outlines some of the features of harmonic analysis on noncompact semisimple Lie groups and symmetric spaces. In particular, we describe the spherical Fourier transformation, and the Plancherel measure and the **c**-function. We also prove a Hausdorff–Young type theorem about the Fourier transform of an $L^p(G)$-function for $p$ in $(1,2)$, and a partial converse.

We first discuss the spherical Fourier transform of an $L^1(G)$-function. Let $\mathbf{W}_1$ be the interior of the convex hull in $\mathfrak{a}^*$ of the images of $\rho$ under the Weyl group $W$ of $(\mathfrak{g},\mathfrak{a})$. For $\delta$ in $(0,1)$, denote by $\mathbf{W}_\delta$ and $\mathbf{T}_\delta$ the dilate of $\mathbf{W}_1$ by $\delta$ and the tube over the polygon $\mathbf{W}_\delta$, that is, $\mathbf{T}_\delta = \mathfrak{a}^* + i\delta\,\mathbf{W}_1$; $\overline{\mathbf{W}}_\delta$ and $\overline{\mathbf{T}}_\delta$ denote the closures of these sets in $\mathfrak{a}^*$ and $\mathfrak{a}^*_{\mathbb{C}}$ respectively.

If $\lambda = \lambda_0 - i\rho$, where $\lambda_0$ lies in $\mathfrak{a}^*$, then the formula (1.2) defining the spherical function $\varphi_\lambda$ becomes

$$\varphi_\lambda(x) = \int_K \exp\bigl(i\lambda_0(A(kx))\bigr)\,dk \qquad \forall x \in G,$$

which implies immediately that $\varphi_\lambda$ is bounded. The spherical functions are invariant under the Weyl group action on $\mathfrak{a}^*_{\mathbb{C}}$, that is, $\varphi_\lambda = \varphi_{w\lambda}$ for all $w$ in $W$. Hence $\varphi_\lambda$ is bounded whenever $\lambda$ lies in $\mathfrak{a}^* - iw\rho$, for any $w$ in $W$, and now a straightforward interpolation argument implies that $\varphi_\lambda$ is bounded whenever $\lambda$ lies in $\mathbf{T}_1$. A full proof of this is in Helgason [60, IV.8].

Further, the map $\lambda \mapsto \varphi_\lambda$ from $\mathfrak{a}^*_{\mathbb{C}}$ to $C(G)$, endowed with the topology of uniform convergence on compact sets, is holomorphic and so, in particular, continuous. It follows that if $f$ is in $L^1(G)$, then $\widetilde{f}$ extends to a continuous function in $\overline{\mathbf{T}}_1$, holomorphic in $\mathbf{T}_1$. If $f$ is a distribution on $G$ which convolves $L^1(G)$ into itself, then $f$ is a bounded measure on $G$, and similarly, $\widetilde{f}$ also extends continuously to $\overline{\mathbf{T}}_1$, and holomorphically to $\mathbf{T}_1$.

We now discuss the Plancherel formula. Recall (1.6): for $K$-bi-invariant functions $u$,

$$\|u\|_2^2 = c_G \int_{\mathfrak{a}^*} \bigl|\widetilde{u}(\lambda)\bigr|^2\,|\mathbf{c}(\lambda)|^{-2}\,d\lambda.$$

The Gindikin–Karpelevič formula for $\mathbf{c}$ states that

$$|\mathbf{c}(\lambda)|^{-2} = \prod_{\alpha\in\Sigma_0^+} |\mathbf{c}_\alpha((\alpha,\lambda)_B)|^{-2} \qquad \forall\lambda\in\mathfrak{a}^*,$$

where each "Plancherel factor" $|\mathbf{c}_\alpha(\cdot)|^{-2}$, which is given by an explicit formula involving several $\Gamma$-functions, extends to an analytic function in a neighbourhood of the real axis and satisfies

$$|\mathbf{c}_\alpha(z)|^{-2} \sim |z|^2\,(1+|z|)^{d_\alpha-2} \qquad \forall z\in\mathbb{R}, \tag{2.1}$$

where $d_\alpha = \dim(\mathfrak{g}_\alpha) + \dim(\mathfrak{g}_{2\alpha})$. This and other useful results about the $\mathbf{c}$-function may be found in Helgason's book [60, IV.6]. It follows easily that there exists a positive constant $C$ such that

$$|\mathbf{c}(\lambda)|^{-2} \le C\,|\lambda|^{\nu-\ell}\,(1+|\lambda|)^{n-\nu} \qquad \forall\lambda\in\mathfrak{a}^*.$$

We shall use a modified version $\mu$ of the Plancherel measure as well as an auxiliary function $\Upsilon$ on $\mathfrak{a}^*$ defined by the rule

$$d\mu(\lambda)/d\lambda = \Upsilon(\lambda) = \prod_{\alpha\in\Sigma_0^+} (1+|(\alpha,\lambda)_B|)^{d_\alpha}.$$

The estimate (2.1) implies that

$$\|f\|_2 \le C\Big(\int_{\mathfrak{a}^*} \big|\widetilde{f}(\lambda)\big|^2\,d\mu(\lambda)\Big)^{1/2} \qquad \forall f\in L^2(K\backslash X).$$

The modified Plancherel measure is invariant under the action of the Weyl group $W$ (like the Plancherel measure), and moreover it is quasi-invariant under translations, in the sense that for any measurable subset $S$ of $\mathfrak{a}^*$,

$$\mu(S+\lambda) \le \Upsilon(\lambda)\,\mu(S) \qquad \forall\lambda\in\mathfrak{a}^*.$$

We now describe a version of the Hausdorff–Young inequality valid for semisimple Lie groups. Write $\delta(p)$ for $2/p-1$.

**Theorem 2.1.** *Equip $\mathfrak{a}^*$ with the modified Plancherel measure $\mu$. Suppose that $1<p<2$, and that $f$ lies in $L^p(G)$. Then $\widetilde{f}$ may be extended to a measurable function on the tube $\overline{\mathbf{T}}_{\delta(p)}$, holomorphic in $\mathbf{T}_{\delta(p)}$, such that $\lambda_0\mapsto\widetilde{f}(\cdot+i\lambda_0)$ is continuous from $\overline{\mathbf{W}}_{\delta(p)}$ to $L^{p'}(\mathfrak{a}^*)$, and such that*

$$\Big(\int_{\mathfrak{a}^*} \big|\widetilde{f}(\lambda+i\lambda_0)\big|^{p'}\,d\mu(\lambda)\Big)^{1/p'} \le C\,\|f\|_p \qquad \forall f\in L^p(G)\quad \forall\lambda_0\in\overline{\mathbf{W}}_{\delta(p)}.$$

*Further, for any closed subtube $\mathbf{T}$ of $\mathbf{T}_{\delta(p)}$, there exists a constant $C$ such that*

$$\big|\widetilde{f}(\lambda)\big| \le C\,\Upsilon(\lambda)^{-1/p'}\,\|f\|_p \qquad \forall f\in L^p(G)\quad \forall\lambda\in\mathbf{T}.$$

*Proof.* See [36].

It follows from this that if $1 < p < 2$ and $\lambda$ is in $\mathbf{T}_{\delta(p)}$, then $\varphi_\lambda$ is in $L^{p'}(G)$, and for every closed subtube $\mathbf{T}$ of $\mathbf{T}_{\delta(p)}$, there exists a constant $C$ such that $\|\varphi_\lambda\|_{p'} \leq C\Upsilon(\lambda)^{-1/p'}$. This is a little sharper than the standard result, that $\|\varphi_\lambda\|_{p'} \leq C$, which is based on the pointwise inequality $|\varphi_{\lambda_1+i\lambda_2}| \leq \varphi_{i\lambda_2}$, trivially true when $\lambda_1$ and $\lambda_2$ lie in $\mathfrak{a}^*$.

Using only the fact that the spherical functions $\varphi_\lambda$ are in $L^{p'}(G)$ when $\lambda$ is in $\mathbf{T}_{\delta(p)}$ and $1 \leq p < 2$ (which may be proved by interpolation, as above, or by careful estimates on the spherical functions), J.-L. Clerc and E.M. Stein [23, Theorem 1] showed that if $1 \leq p < 2$, then the spherical Fourier transform of an $L^p$-function extends to a holomorphic function in $\mathbf{T}_{\delta(p)}$, bounded in closed subtubes thereof, and that if $f$ convolves $L^p(G)$ into itself, then its spherical Fourier transform extends to a bounded holomorphic function in $\mathbf{T}_{\delta(p)}$.

Another consequence of Theorem 2.1 is the $K$-bi-invariant version of the Kunze–Stein phenomenon: if $1 \leq p < 2$ and $k$ is in $L^p(K\backslash X)$, then the maps $f \mapsto f * k$ and $f \mapsto k * f$ are bounded on $L^2(G)$. Indeed, without loss of generality we may assume that $k \geq 0$, and then Herz' *principe de majoration* [63] shows that it suffices to have $\widetilde{k}(0)$ bounded. Some of our computations require this result, and others the stronger result that the maps $f \mapsto f*k$ and $f \mapsto k*f$ are bounded on $L^2(G)$, if $k$ is in $L^p(G)$. This stronger result is known as the Kunze–Stein phenomenon. The Kunze–Stein phenomenon (which is discussed in more detail in Section 3.3) and the generalisation of Young's inequality to locally compact groups have the following consequences.

**Theorem 2.2.** *Suppose that $1 \leq r \leq \infty$. For a function $k$ in $L^r(G)$, denote by $\mathcal{K}$ and $\mathcal{K}'$ the operators $f \mapsto k * f$ and $f \mapsto f * k$ from $S(G)$ to $C(G)$. Then $\mathcal{K}$ and $\mathcal{K}'$ are bounded from $L^p(G)$ to $L^q(G)$, with a corresponding operator norm inequality, provided that one of the following conditions holds:*

*(i) if $r = 1$, and $1 \leq p = q \leq \infty$;*
*(ii) if $1 < r \leq 2$, $q \geq r$, $p \leq r'$, $0 \leq 1/p - 1/q \leq 1/r'$, and $(p, q) \neq (r, r)$ or $(r', r')$;*
*(iii) if $2 < r < \infty$, $q \geq r$, $p \leq r'$, $0 \leq 1/p - 1/q \leq 1/r'$, and $(p, q) \neq (r, r')$;*
*(iv) if $r = \infty$, $p = 1$, and $q = \infty$.*

*Consequently, if $k$ is in $L^r(G)$ for all $r$ in $(2, \infty]$, $\mathcal{K}$ and $\mathcal{K}'$ are bounded from $L^p(G)$ to $L^q(G)$ when $1 \leq p < 2 < q \leq \infty$.*

*Proof.* We give the details of this proof because it is short and indicative of the differences between harmonic analysis on symmetric spaces and on Euclidean spaces. Without loss of generality, we may restrict our attention to the operator $\mathcal{K}$, because $G$ is unimodular, so the mapping $f \mapsto \check{f}$, where $\check{f}(x) = f(x^{-1})$ for all $x$ in $G$, which has the property that $(f*k)\check{} = \check{k}*\check{f}$, acts isometrically on each of the spaces $L^s(G)$. Thus $\mathcal{K}$ is bounded from $L^p(G)$ to $L^q(G)$ if and only if $\mathcal{K}'$ is.

For the cases where $r = 1$ and $r = \infty$, the result is standard. If $1 < r < 2$, then it suffices to prove that $L^r(G) * L^p(G) \subseteq L^r(G)$ when $1 \le p < r$ and $L^r(G) * L^p(G) \subseteq L^p(G)$ when $r < p \le 2$. For then duality arguments establish that $L^r(G) * L^{r'}(G) \subseteq L^p(G)$ when $r' < p \le \infty$ and $L^r(G) * L^p(G) \subseteq L^p(G)$ when $2 \le p < r'$, and interpolation arguments establish the boundedness in the set claimed. The first inclusion follows by multilinear interpolation between the inclusions $L^1(G) * L^1(G) \subseteq L^1(G)$ and $L^2(G) * L^s(G) \subseteq L^2(G)$, for any $s$ in $[1, 2)$. The second inclusion follows by multilinear interpolation between the inclusions $L^1(G) * L^s(G) \subseteq L^s(G)$ and $L^t(G) * L^2(G) \subseteq L^2(G)$, for any $s$ and $t$ in $[1, 2)$.

When $r = 2$, the result is easy: one first reformulates the Kunze–Stein phenomenon to show that $L^2(G) * L^2(G) \subseteq L^s(G)$ when $2 < s \le \infty$, then applies duality and interpolation.

When $2 < r < \infty$, the result follows by multilinear interpolation between the results when $r = 2$ and when $r = \infty$.

The final consequence is proved by combining the results of (iii) and (iv).

We shall deal with functions which belong to $L^r(G)$ for all $r$ in $(2, \infty]$; the convolution properties thereof are described in the theorem just proved. As a corollary of Theorem 2.1, we may establish a criterion for a function on $\mathfrak{a}^*$ to be the spherical Fourier transform of such a function. A technical definition is necessary: for any small positive $\epsilon$, let $H^\infty(\mathbf{T}_\epsilon)$ denote the space of all bounded holomorphic functions in $\mathbf{T}_\epsilon$, with the supremum norm; clearly when $\delta < \epsilon$, $H^\infty(\mathbf{T}_\epsilon)$ may be injected into $H^\infty(\mathbf{T}_\delta)$ by restricting $H^\infty(\mathbf{T}_\epsilon)$ functions to $\mathbf{T}_\delta$. The inductive limit space $\bigcup_{\epsilon > 0} H^\infty(\mathbf{T}_\epsilon)$ is denoted by $A(\mathfrak{a}^*)$. An element of the dual space of $A(\mathfrak{a}^*)$ will be called, somewhat abusively, an analytic functional on $\mathfrak{a}^*$.

**Corollary 2.1.** *Suppose that $T$ is an analytic functional on $\mathfrak{a}^*$. Then there exists a $K$-bi-invariant function on $G$, $k$ say, which belongs to $L^r(G)$ for all $r$ in $(2, \infty]$, such that*

$$\int_G k(x)\, f(x)\, dx = T(\widetilde{f}) \qquad \forall f \in S(G).$$

*If $\mathfrak{a}^*$ is endowed with the Plancherel measure and if*

$$T(\widetilde{f}) = \int_{\mathfrak{a}^*} \widetilde{f}(\lambda)\, t(\lambda)\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda \qquad \forall f \in S(G),$$

*where $t$ is Weyl group invariant and lies in $L^1(\mathfrak{a}^*) \cap L^2(\mathfrak{a}^*)$, then $\widetilde{k} = t$.*

*Proof.* This is an immediate consequence of Theorem 2.1. Fix $s$ in $[1, 2)$. If $f$ is in $L^s(G)$ then $\widetilde{f}$ is in $A(\mathfrak{a}^*)$, and so composition with $T$ provides a linear functional on $L^s(G)$. Thus there exists $k$ in $L^{s'}(G)$ such that

$$\int_G k(x)\, f(x)\, dx = T(\widetilde{f}) \qquad \forall f \in L^s(G).$$

Since this works for arbitrary $s$, $k$ has the properties claimed. The second part of the corollary follows from the Plancherel formula.

It is clear that if $T$ is an analytic functional on $\mathfrak{a}^*$, then for any $h$ belonging to $H^\infty(\mathbf{T}_\epsilon)$ for some positive $\epsilon$, $hT$, defined by the rule $hT(g) = T(hg)$ for all $g$ in $A(\mathfrak{a}^*)$, is also an analytic functional on $\mathfrak{a}^*$.

Now we consider the question whether Theorem 2.1 has a converse: if $\widetilde{f}$ extends to a holomorphic function in $\mathbf{T}_{\delta(p)}$, is it necessarily true that $f$ must lie in $L^p(G)$? It is too much to expect that we will be able to prove the converse in the semisimple case, when even in Euclidean Fourier analysis this is impossible. However, we can prove a result which is useful.

To find an estimate for $\|f\|_p$, it is tempting to try to interpolate between estimates for $\|f\|_1$ and $\|f\|_2$. Unfortunately, the straightforward interpolation argument gives

$$\|f\|_p \leq \|f\|_1^{\delta(p)} \, \|f\|_2^{1-\delta(p)} \,,$$

which is useless unless $\widetilde{f}$ is holomorphic in $\mathbf{T}_1$, for otherwise $f$ could not be in $L^1(G)$. To obviate this problem, the obvious estimate is replaced by an estimate

$$\|f\|_p \leq \|f\varphi_{ic\rho}\|_1^{\delta(p)} \, \|f\varphi_{ic'\rho}\|_2^{1-\delta(p)} \,,$$

where the spherical functions $\varphi_{ic\rho}$ and $\varphi_{ic'\rho}$ are chosen so that the first factor lies in $L^1(G)$. A technique of L. Vretare [101] enables us to compute the second factor in terms of an $L^2$-norm of $\widetilde{f}$.

**Theorem 2.3.** *Suppose* $1 < p < 2$, *that* $f$ *is a measurable* $K$*-bi-invariant function on* $G$, *and that* $f\varphi_{i\delta(p)\rho}$ *lies in* $L^1(G)$. *Then the spherical Fourier transform of* $f$ *extends holomorphically into the tube* $\mathbf{T}_{\delta(p)}$, *and continuously to* $\overline{\mathbf{T}}_\delta(p)$. *If moreover* $N < \infty$, *where*

$$N = \Big( \int_{\mathfrak{a}^*} \big|\widetilde{f}(\lambda + i\delta(p)\rho)\big|^2 d\mu(\lambda) \Big)^{1/2},$$

*then* $f$ *lies in* $L^p(G)$, *and*

$$\|f\|_p \leq C \left\|f\varphi_{i\delta(p)\rho}\right\|_1^{\delta(p)} N^{1-\delta(p)}.$$

Vretare [102] also proved a form of inverse Hausdorff–Young theorem for semisimple groups.

### 2.3 Regularity of the Laplace–Beltrami Operator

The following result, taken from [36], but based on much previous work, encapsulates the various Sobolev-type regularity theorems for the Laplace–Beltrami operator. In the following, we denote by $n$ the dimension of $X$, by $\ell$ its real rank, that is, the (real) dimension of $A$, and by $\nu$ the *pseudo-dimension* or *dimension at infinity* $2\left|\Sigma_0^+\right|+\ell$, where $\left|\Sigma_0^+\right|$ is the cardinality of the set of the indivisible positive roots. The pseudo-dimension $\nu$ may very well be strictly larger than the dimension $n$, as, for instance, in the case of $\mathrm{SL}(p,\mathbb{R})$.

If $1 \leq p, q \leq \infty$, we denote by $\left|\!\left|\!\left| T \right|\!\right|\!\right|_{p;q}$ the norm of the linear operator $T$ from $L^p(X)$ to $L^q(X)$. In the case where $p = q$ we shall simply write $\left|\!\left|\!\left| T \right|\!\right|\!\right|_p$. By $C$ we denote a constant which may not be the same at different occurrences. The expression

$$A(t) \sim B(t) \qquad \forall t \in \mathbf{D},$$

where $\mathbf{D}$ is some subset of the domains of $A$ and of $B$, means that there exist (positive) constants $C$ and $C'$ such that

$$CA(t) \leq B(t) \leq C'A(t) \qquad \forall t \in \mathbf{D};$$

$C$ and $C'$ may depend on any quantifiers written *before* the displayed formula.

Finally, $p_\theta$ denotes $2/\left[1+(1-\theta)^{1/2}\right]$, and for $\alpha$ in $\mathbb{C}$ with positive real part, $\mathbf{I}_\alpha$ denotes the interval $[2, 2n/(n-2\operatorname{Re}\alpha)]$ if $0 \leq \operatorname{Re}\alpha < n/2$, the interval $[2,\infty)$ if $\operatorname{Re}\alpha = n/2$, and the interval $[2,\infty]$ if $\operatorname{Re}\alpha > n/2$, while $\mathbf{P}_\alpha$ denotes the set of all $(p,q)$ satisfying the following conditions:

(i) if $\alpha = 0$ then $1 \leq p = q \leq \infty$;
(ii) if $\operatorname{Re}\alpha = 0$ and $\operatorname{Im}\alpha \neq 0$, then $1 < p = q < \infty$;
(iii) if $0 < \operatorname{Re}\alpha < n$, then $1 \leq p \leq q \leq \infty$ and either $1/p - 1/q < \operatorname{Re}\alpha/n$ or $1/p - 1/q = \operatorname{Re}\alpha/n$, $p > 1$, and $q < \infty$;
(iv) if $\operatorname{Re}\alpha = n$, then $1 \leq p \leq q \leq \infty$ and either $1/p - 1/q < \operatorname{Re}\alpha/n$ or $\operatorname{Im}\alpha \neq 0$, $p = 1$, and $q = \infty$;
(v) if $\operatorname{Re}\alpha > n$, then $1 \leq p \leq q \leq \infty$.

**Theorem 2.4.** *Suppose that* $0 \leq \theta < 1$, $1 \leq p \leq \infty$, *and* $\operatorname{Re}\alpha \geq 0$. *The operator* $(\Delta - \theta b)^{-\alpha/2}$ *is bounded on* $L^p(X)$ *if and only if one of the following conditions holds:*

*(i)* $\alpha = 0$;
*(ii)* $\operatorname{Re}\alpha = 0$, $\alpha \neq 0$, $1 < p < \infty$, *and* $p_\theta \leq p \leq p'_\theta$;
*(iii)* $\operatorname{Re}\alpha > 0$ *and* $p_\theta < p < p'_\theta$.

*Suppose that* $0 \leq \theta < 1$, $\operatorname{Re}\alpha > 0$, *and* $1 \leq p < q \leq \infty$. *Then the operator* $(\Delta - \theta b)^{-\alpha/2}$ *is bounded from* $L^p(X)$ *to* $L^q(X)$ *if and only if the following conditions both hold:*

*(iv) either* $0 < \operatorname{Re}\alpha < (\ell+1)/p'_\theta$, $p \leq p'_\theta$, *and* $q \geq p_\theta$ *or* $\operatorname{Re}\alpha \geq (\ell+1)/p'_\theta$, $p < p'_\theta$, *and* $q > p_\theta$;
*(v)* $(p,q)$ *is in* $\mathbf{P}_\alpha$.

*Suppose that* $\operatorname{Re}\alpha \geq 0$ *and* $1 \leq p \leq q \leq \infty$. *Then* $(\Delta - b)^{-\alpha/2}$ *is bounded from* $L^p(X)$ *to* $L^q(X)$ *if and only if one of the following conditions holds:*

*(vi)* $p = q = 2$ *and* $\operatorname{Re}\alpha = 0$;
*(vii)* $p = 2 < q$, $0 < \operatorname{Re}\alpha < \nu/2$, *and* $q$ *is in* $\mathbf{I}_\alpha$;

*(viii)* $p < 2 = q$, $0 < \operatorname{Re}\alpha < \nu/2$, *and* $p'$ *is in* $\mathbf{I}_\alpha$;
*(ix)* $p < 2 < q$, $\operatorname{Re}\alpha - \nu$ *is not in* $2\mathbb{N}$ *and* $1/p - 1/q < \operatorname{Re}\alpha/n$.
*(x)* $p < 2 < q$, $\operatorname{Re}\alpha - \nu$ *is not in* $2\mathbb{N}$, $1/p - 1/q = \operatorname{Re}\alpha/n$, *and if* $p = 1$ *or* $q = \infty$ *then both*
*(xi)* $\operatorname{Re}\alpha = n$ *and* $\operatorname{Im}\alpha \neq 0$.

### 2.4 Approaches to the Heat Equation

Quite a bit is known about the heat kernel $h_{t,\theta}$. For complex Lie groups, an explicit expression is available [57]. For other Lie groups, less explicit but nevertheless useful pointwise formulae are known. In particular, P. Sawyer [91, 92] estimated the heat kernels in a number of special cases. In the general case, the best estimates are due to J.-Ph. Anker [3], Anker and L. Ji [4, 5], and Anker and P. Ostellari [6]. Those who study these questions often speak of having problems "at the walls"; this is, for example, the major problem with the pointwise estimates for the spherical functions. To a large extent, these problems are unimportant, since, for instance, it is possible to show that the heat kernel is very small near the walls so that the precise behaviour there is irrelevant. By the inversion theorem for the spherical Fourier transform (1.7),

$$h_{t,\theta}(x) = c_G \int_{\mathfrak{a}^*} e^{-tQ_\theta(\lambda)} \varphi_\lambda(x) \left|\mathbf{c}(\lambda)\right|^{-2} d\lambda.$$

To estimate $h_{t,\theta}$, one needs to know about the functions $\varphi_\lambda$. By the Cartan decomposition (1.2), it suffices to consider $x$ in $A$. The "difficulties at the walls" are two-fold: for $H$ in $\mathfrak{a}$, $\varphi_\lambda(\exp H)$ is hard to handle when $\lambda$ is close to a wall of a Weyl chamber in $\mathfrak{a}^*$ or when $H$ is close to a wall of a Weyl chamber in $\mathfrak{a}$. Even in the complex case, obtaining estimates "close to the walls" is tricky, especially if one wants estimates which are uniform in both $\lambda$ and $H$. Some progress on this problem has been made recently by Cowling and A. Nevo [42], based on an idea of H. Gunawan [52].

The other approach to the problem is to try to obtain other sorts of estimates for the kernels. For the heat equation, this has been quite effective. In the next section, we summarise some of the results of [36, 37], and extend one of these a little.

### 2.5 Estimates for the Heat and Laplace Equations

Putting together the facts of the previous section, it is relatively easy to obtain estimates for the operators arising in the heat and Laplace equations.

**Theorem 2.5.** *Let* $(\mathcal{H}_t)_{t>0}$ *be the heat semigroup. Then the following hold:*

*(i) for all* $p$ *in* $[1,\infty]$,

$$|||\mathcal{H}_t|||_p = \exp\left(-(1 - \delta(p)^2)bt\right) \qquad \forall t \in \mathbb{R}^+;$$

*(ii) for all $p, q$ such that $1 \leq p \leq q \leq \infty$,*

$$\|\mathcal{H}_t\|_{p;q} \sim t^{-n(1/p-1/q)/2} \qquad \forall t \in (0,1];$$

*(iii) for all $p, q$ such that either $1 \leq p < q = 2$ or $2 = p < q \leq \infty$,*

$$\|\mathcal{H}_t\|_{p;q} \sim t^{-\nu/4} \exp(-bt) \qquad \forall t \in [1,\infty);$$

*(iv) for all $p, q$ such that $1 \leq p < 2 < q \leq \infty$,*

$$\|\mathcal{H}_t\|_{p;q} \sim t^{-\nu/2} \exp(-bt) \qquad \forall t \in [1,\infty);$$

*(v) for all $p, q$ such that $1 \leq p < q < 2$,*

$$\|\mathcal{H}_t\|_{p;q} \sim t^{-\ell/2q'} \exp\big(-(1-\delta(q)^2)bt\big) \qquad \forall t \in [1,\infty);$$

*(vi) for all $p, q$ such that $2 < p < q \leq \infty$,*

$$\|\mathcal{H}_t\|_{p;q} \sim t^{-\ell/2p} \exp\big(-(1-\delta(p)^2)bt\big) \qquad \forall t \in [1,\infty).$$

**Theorem 2.6.** *Suppose that $0 \leq \theta \leq 1$ and $1 \leq p, q \leq \infty$. The following hold:*

*(i) if $t > 0$, then $\mathcal{L}_{t,\theta}$ is bounded from $L^p(X)$ to $L^q(X)$ only if $p \leq q$, $p \leq p'_\theta$, and $q \geq p_\theta$;*

*(ii) if $p_\theta \leq p \leq p'_\theta$, then*

$$\|\mathcal{L}_{t,\theta}\|_{p;p} = \exp\big(-[(\frac{4}{pp'} - \theta)b]^{1/2} t\big) \qquad \forall t \in \mathbb{R}^+;$$

*(iii) if $p \leq q$, $p \leq p'_\theta$ and $q \geq p_\theta$, then*

$$\|\mathcal{L}_{t,\theta}\|_{p;q} \sim t^{-n(1/p-1/q)} \qquad \forall t \in (0,1];$$

*(iv) if $p < q = 2$ or $2 = p < q$, then*

$$\|\mathcal{L}_{t,\theta}\|_{p;q} \sim t^{-\nu/4} \exp\big(-[(1-\theta)b]^{1/2} t\big) \qquad \forall t \in [1,\infty);$$

*(v) if $p < 2 < q$, then*

$$\|\mathcal{L}_{t,\theta}\|_{p;q} \sim t^{-\nu/2} \exp\big(-[(1-\theta)b]^{1/2} t\big) \qquad \forall t \in [1,\infty);$$

*(vi) if $p < q < 2$ and $q > p_\theta$, then*

$$\|\mathcal{L}_{t,\theta}\|_{p;q} \sim t^{-\ell/2q'} \exp\big(-[(\frac{4}{qq'} - \theta)b]^{1/2} t\big) \qquad \forall t \in [1,\infty);$$

*(vii) if $p < q < 2$ and $q > p_\theta$, then*

$$\|\mathcal{L}_{t,\theta}\|_{p;q} \sim t^{-\ell/2q'} \exp\big(-[(\frac{4}{qq'} - \theta)b]^{1/2} t\big) \qquad \forall t \in [1,\infty);$$

*(viii) if* $p < q = p_\theta$*, then*

$$\left\| \mathcal{L}_{t,\theta} \right\|_{p;q} \sim t^{-(\ell+1)/q'} \qquad \forall t \in [1, \infty);$$

*(ix) if* $2 < p < q$ *and* $p < p'_\theta$*, then*

$$\left\| \mathcal{L}_{t,\theta} \right\|_{p;q} \sim t^{-\ell/2p} \exp\bigl(-[(\frac{4}{pp'} - \theta)b]^{1/2} t\bigr) \qquad \forall t \in [1, \infty);$$

*(x) if* $p'_\theta = p < q$*, then*

$$\left\| \mathcal{L}_{t,\theta} \right\|_{p;q} \sim t^{-(\ell+1)/p} \qquad \forall t \in [1, \infty).$$

It is worth pointing out that the significant difference in the behaviour of the solutions of the heat and Laplace equations is due to the fact that the function $\exp(-tQ_\theta(\cdot))$ extends to an entire function in $\mathfrak{a}^*_{\mathbb{C}}$ while the function $\exp(-tQ_\theta(\cdot)^{1/2})$ extends into a tube $\mathbf{T}_\delta$, where $\delta = (1 - \theta)^{1/2}$, but into no larger tube. This shows clearly that harmonic analysis on a noncompact symmetric space involves phenomena with no Euclidean analogue.

### 2.6 Approaches to the Wave and Schrödinger Equations

Dealing with the wave equation is tricky. There is a method, due originally to Hadamard, for obtaining a parametrix for the fundamental solutions, but for large values of the time parameter this is not very easy to deal with except in the complex case and a few other relatively simple special cases.

Despite this, considerable progress has been made, and there are many important papers on this topic, starting, perhaps, with work of Helgason [61]. T. Branson, G. Ólafsson, and H. Schlichtkrull, in various combinations (see [13] and the references cited there), have studied the heat equation by analytic methods, while O.A. Chalykh and A.P. Veselov [19] used algebraic methods.

S. Giulini, S. Meda and I [38] have given a parametrix expression for the wave operator and used this to obtain $L^p$-$L^q$ mapping estimates for the complexified Poisson semigroup, and we have also [39] obtained $L^p$-$L^q$ estimates for the operator with convolution kernel $w^\alpha_\theta$, defined by the condition

$$\widetilde{w}^\alpha_\theta(\lambda) = Q_\theta(\lambda)^{-\alpha/2} \exp(iQ_\theta(\lambda)^{1/2}),$$

using a representation of this operator originating in T.P. Schonbek [94] in the Euclidean case. There is interest in obtaining similar inequalities where the exponential is replaced by $\exp(itQ_\theta(\lambda)^{1/2})$, and $t$ is allowed to vary. Such inequalities, which belong to the family known as Strichartz estimates, are now a standard tool in hyperbolic partial differential equations.

Last but not least, let us consider the Schrödinger operator. Here, not very much is known except in the usual special cases (complex groups and real rank one groups). Note, however, that the spherical Fourier transform $\widetilde{s}_{t,\theta}$ extends to an entire function, but this grows exponentially in any tube $T_\delta$ when $\delta > 0$; this suggests that, as far as restrictions on the indices are concerned, the results should resemble those for the heat equation case more than those for Laplace's equation. Note also that the rapid oscillation of $\widetilde{s}_{t,\theta}(\lambda)$ as $\lambda \to \infty$ in $\mathfrak{a}^*$ implies that $\widetilde{s}_{t,\theta}(\lambda)$ defines an analytic functional, so that the kernel is in $L^{2+\epsilon}(G)$ for all positive $\epsilon$. In a "discrete symmetric space" (see later for an indication of what this might mean), some results have been obtained by A.G. Setti [95].

### 2.7 Further Results

Much more is known about harmonic analysis on semisimple groups than is outlined here. In the area of spherical Fourier analysis, it is more than appropriate to mention the book of R. Gangolli and V.S. Varadarajan [50], which presents a complete picture of the Harish-Chandra viewpoint. Much of the theory of spherical functions may be viewed as statements about certain special functions, and generalised. T. Koornwinder [72] presents a pleasant account of this interface between group theory and special functions. As mentioned above, in the complex case, there are explicit formulae for the spherical functions. For some other groups, there are ways of getting some control of the spherical functions.

For the purposes of harmonic analysis, there are a number of important characterisations of the image under the spherical Fourier transformation of spaces on the semisimple group $G$. In particular, there is a family of "Schwartz spaces" on $G$, whose images were characterised by various authors, as well as a Paley–Wiener theorem characterising the compactly supported functions. The original Paley–Wiener results are due to Helgason [58] and Gangolli [49], and the Schwartz space results are due to P.C. Trombi and V.S. Varadarajan [100]. The proofs have been simplified since then. See [2, 24, 60] for more in this direction.

A lot of effort has been put into determining conditions on a distribution or on its Fourier transform which imply that it convolves $L^p(G)$ into itself. The major contributions here include [1, 77, 78, 96]. In the more general setting of a Riemannian manifold, there are many results on the functional calculus for the Laplace–Beltrami operator. Arguably, the key technique here has been the use of the finite propagation speed of the solutions to the heat equation, pioneered by J. Cheeger, M. Gromov, and especially M.E. Taylor [20, 99].

## 3 The Vanishing of Matrix Coefficients

Semisimple groups differ from other locally compact groups in the sense that their unitary representations may be characterised by the rate of decay of their matrix coefficients. In this lecture, I make this statement more precise, describing different ways in which this decay can be quantified.

### 3.1 Some Examples in Representation Theory

Suppose that $G$ is a Lie group. A *unitary representation* $(\pi, \mathcal{H}_\pi)$ of $G$ is a Hilbert space $\mathcal{H}_\pi$ and a homomorphism $\pi$ from $G$ into $\mathcal{U}(\mathcal{H}_\pi)$, the group of unitary operators on $\mathcal{H}_\pi$. We always suppose that $\pi$ is continuous when $\mathcal{U}(\mathcal{H}_\pi)$ is equipped with the strong operator topology. We usually abuse notation a little and talk of "the representation $\pi$", the Hilbert space being implicit.

Recall that the representation $\pi$ is said to be *reducible* if there are nontrivial $G$-invariant closed subspaces of $\mathcal{H}_\pi$, and *irreducible* otherwise. A vector $\xi$ in $\mathcal{H}_\pi$ is said to be *smooth* if the $\mathcal{H}_\pi$-valued function $x \mapsto \pi(x)\xi$ on $G$ is smooth, or equivalently if all the $\mathbb{C}$-valued functions $x \mapsto \langle \pi(x)\xi, \eta\rangle$ (where $\eta$ varies over $\mathcal{H}_\pi$) are smooth. Similar definitions may be made when the Hilbert space $\mathcal{H}_\pi$ is replaced by a Banach space.

For unitary representations, as distinct from Banach space representation, there is a reasonably satisfactory theory of decompositions into irreducible representations. An arbitrary unitary representation can be written as a *direct integral* (a generalisation of a direct sum) of irreducible representations. Many Lie groups, including semisimple Lie groups and real algebraic groups (groups of matrices defined by algebraic equation in the entries), have the property that this direct integral decomposition is essentially unique. On the other hand, many groups, such as noncommutative free groups, do not have unique direct integral decompositions; this makes analysis on these groups harder. To give an indication of the sorts of decomposition which appear for "good" groups, we give two examples.

For the group $\mathbb{R}^n$, the irreducible representations are the characters $\chi_y \colon x \mapsto \exp(-2\pi i y \cdot x)$, where $y$ varies over $\mathbb{R}^n$. Given a positive Borel measure $\nu$ on $\mathbb{R}^n$ with support $S_\nu$, form the usual Hilbert space $L^2(S_\nu, \nu)$ of complex-valued functions on $S_\nu$, and define the representation $\pi_\nu$ on $L^2(S_\nu, \nu)$ by the formula

$$[\pi_\nu(x)\xi](y) = \chi_y(x)\, \xi(y) \qquad \forall y \in S_\nu$$

for all $\xi$ in $L^2(S_\nu, \nu)$ and all $x$ in $\mathbb{R}^n$. Any unitary representation of $\mathbb{R}^n$ is unitarily equivalent to a direct sum of representations $\pi_\nu$, with possibly different $\nu$'s.

The *$ax + b$ group* $Q$ is defined to be the group of all matrices $M_{a,b}$ of the form

$$M_{a,b} = \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix},$$

where $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$, equipped with the obvious topology.

The set $N$ of matrices $M_{1,b}$, with $b$ in $\mathbb{R}$, is a closed normal subgroup of $Q$, and the quotient group $Q/N$ is isomorphic to the multiplicative group $\mathbb{R}^+$. By identifying $x$ in $\mathbb{R}$ with the vector $\begin{pmatrix} x \\ 1 \end{pmatrix}$ in $\mathbb{R}^2$, we obtain an action of $Q$ on $\mathbb{R}$, given by

$$M_{a,b} \circ x = a^2 x + ab \qquad \forall x \in \mathbb{R} \quad \forall M_{a,b} \in Q.$$

Using this action, we define the representation $\sigma$ of $Q$ on $L^2(\mathbb{R})$ by the formula

$$\sigma(M_{a,b})\xi(x) = a^{-1}\xi(M_{a,b}^{-1} \circ x) \qquad \forall x \in \mathbb{R}.$$

Under the representation $\sigma$ of $Q$, the Hilbert space $L^2(\mathbb{R})$ breaks up into two irreducible subspaces, $L^2(\mathbb{R})_+$ and $L^2(\mathbb{R})_-$, containing those functions whose Fourier transforms are supported in $[0, +\infty)$ and in $(-\infty, 0]$ respectively. The restrictions of $\sigma$ to these two subspaces are denoted $\sigma_+$ and $\sigma_-$. Any unitary representation $\pi$ of the group $Q$ decomposes as a sum $\pi_1 \oplus \pi_0$, where $\pi_1$ is trivial on $N$ and hence is essentially a representation of the quotient group $Q/N$, and $\pi_0$ is a direct sum of copies of the representations $\sigma_+$ and $\sigma_-$.

Unitary representations of semisimple groups have been described, albeit incompletely and briefly, in the first lecture. Much more is known—see the references there.

Associated to a unitary representation $\pi$ of $G$, there are *matrix coefficients*. For $\xi$ and $\eta$ in $\mathcal{H}$, $x \mapsto \langle \pi(x)\xi, \eta \rangle$ is a bounded continuous function on $G$, written $\langle \pi(\cdot)\xi, \eta \rangle$. If $\xi$ and $\eta$ run over an orthogonal basis of $\mathcal{H}_\pi$, then we obtain a matrix of functions corresponding to the representation of $\pi(\cdot)$ as a matrix in this basis. Thus the collection of all matrix coefficients contains complete information about $\pi$; for many purposes, however, it is easier to deal with spaces of functions on $G$ rather than representations. As we may decompose a unitary representation $\pi$ of $G$, so we may decompose the matrix coefficients of $\pi$ into sums of matrix coefficients of "smaller" representations. In the case of $\mathbb{R}^n$, this decomposition writes a function on $\mathbb{R}^n$ as a sum or an integral of characters—this is just Fourier analysis under a different guise.

When $G = \mathbb{R}^n$, the matrix coefficients of the irreducible representations of $G$ are (multiples of) characters. These are constant in absolute value, and in particular do not vanish at infinity or belong to any $L^p$ space with finite $p$. However, the regular representation $\lambda$ of $G$ on $L^2(G)$ has matrix coefficients which decay at infinity: if $\xi, \eta \in L^2(G)$, then

$$\begin{aligned} \langle \lambda(x)\xi, \eta \rangle &= \int_G \lambda(x)\xi(y)\,\overline{\eta}(y)\,dy \\ &= \int_{\mathbb{R}^n} \xi(y-x)\,\overline{\eta}(y)\,dy \quad \forall x \in \mathbb{R}^n, \end{aligned}$$

and $\langle \lambda(\cdot)\xi, \eta \rangle$ has compact support if $\xi$ and $\eta$ do, and is in $C_0(G)$ in general. It is easy to show that, by choosing $\xi$ and $\eta$ appropriately, it is possible to make $\langle \lambda(\cdot)\xi, \eta \rangle$ decay arbitrarily slowly.

In the $ax + b$ case, all the matrix coefficients of a representation $\pi_1$ which is trivial on $N$ are constant on cosets of $N$ in $Q$, and do not vanish at infinity, no matter how $\pi_1$ behaves on the quotient group $Q/N$. On the other hand, the representations $\sigma_+$ and $\sigma_-$ have the property that all their matrix coefficients vanish at infinity; as in the $\mathbb{R}^n$ case, this decay may be arbitrarily slow.

### 3.2 Matrix Coefficients of Representations of Semisimple Groups

All semisimple Lie groups are "almost direct products" of "simple factors"; unitary representations break up as "outer tensor products" of representations of the various factors, and the matrix coefficients decompose similarly. There is therefore little loss of generality in restricting attention to simple Lie groups, that is, those whose Lie algebra is simple, for the rest of this lecture.

An important notion in the study of representations of a semisimple Lie group is $K$-finiteness. A vector $\xi$ in $\mathcal{H}_\pi$ is said to be $K$-finite if $\{\pi(k)\xi : k \in K\}$ spans a finite dimensional subspace. The set of $K$-finite vectors is a dense subspace of $\mathcal{H}_\pi$.

The first key fact about a unitary representation $\pi$ of a simple Lie group $G$ is that it decomposes into two pieces, $\pi_1$ and $\pi_0$. The representation $\pi_1$ is a multiple of the trivial representation, and the associated matrix coefficients are constants, while all the matrix coefficients of $\pi_0$ vanish at infinity in $G$. Several of the proofs of this involve looking at subgroups $R$ of $G$ similar to the group $Q$ described above, and "lifting" to $G$ the decomposition from $R$. The difficulty of the proof is in showing that $G$ acts trivially on the vectors where the normal subgroup $N$ of $R$ acts trivially.

The remarkable fact is that we can often say more than this: for most representations of interest, there is control on the rate of decay. There are two ways to quantify the rate of decay of matrix coefficients: uniform estimates and $L^{p+}$ estimates.

Recall, from Lecture 1, the Cartan decomposition: every $x$ in $G$ may be written in the form

$$x = k_1 a k_2,$$

where $k_1, k_2 \in K$ and $a \in \overline{A}^+$, the closure of $\exp(\mathfrak{a}^+)$ in $A$. A uniform estimate for a matrix coefficient $u$ is an estimate of the form

$$|u(k_1 a k_2)| \leq C\,\phi(a) \qquad \forall k_1, k_2 \in K \quad \forall a \in \overline{A}^+,$$

for some function $\phi$ in $C_0(\overline{A}^+)$. An $L^{p+}$ estimate for $u$ is the statement that, for any positive $\epsilon$, the function $u$ is in $L^{p+\epsilon}(G)$, and $\|u\|_{p+\epsilon}$, the $L^{p+\epsilon}(G)$ norm of $u$, may be estimated. Matrix coefficients of unitary representations are always bounded, so that if $u \in L^p(G)$, then $u \in L^q(G)$ for all $q$ in $[p, \infty]$. Thus the set of $q$ such that $u \in L^q(G)$ is an interval containing $\infty$.

For irreducible representations of $G$, the $K$-finite matrix coefficients are solutions of differential equations on $A$. These are $\ell$-dimensional generalisations of the hypergeometric differential equation, and an extension of the analysis of differential equations with regular singular points to $A$ leads to proofs of the existence of asymptotic expressions for matrix coefficients (first carried out by Harish-Chandra, but published in an improved and simplified version by W. Casselman and D. Miličić [18]). In particular, it can be shown that, for all $K$-finite matrix coefficients of an irreducible unitary representation,

$$\langle \pi(\exp(H))\xi, \eta \rangle \sim \sum_{\gamma \in I} C(\xi, \eta, \gamma) \wp_\gamma(H) e^{-(\rho+\gamma)(H)}$$

as $H \to \infty$ in $\mathfrak{a}^+$, keeping away from the walls of $a^+$, for some finite subset $I$ of $\mathfrak{a}^*_{\mathbb{C}}$ with the property that $\operatorname{Re}\gamma(H) \geq 0$ for all $H$ in $\mathfrak{a}^+$ and $\gamma$ in $I$, and some polynomials $\wp_\gamma$ of bounded degree; both the set $I$ of "leading terms" and the polynomials $\wp_\gamma$ are independent of $\xi$ and $\eta$.

From this fact, it appears that the best sort of uniform estimate to consider is one of the form

$$|u(k_1 \exp(H) k_2)| \leq C\, \wp(H) e^{-\gamma(H)} \qquad \forall k_1, k_2 \in A \quad \forall H \in \overline{\mathfrak{a}}^+,$$

where $\gamma \in \mathfrak{a}^+$ and $\wp$ is a polynomial. It can be shown that such estimates hold for $K$-finite matrix coefficients of irreducible representations.

Let us now formulate a conjecture.

**Conjecture 3.1.** *Suppose that $\pi$ is a unitary representation of a simple Lie group $G$, that $\gamma \in \mathfrak{a}^*$, and that $0 < \gamma(H) \leq \rho(H)$ for all $H$ in $a^+$. Then*

$$|\langle \pi(k_1 \exp(H) k_2)\xi, \eta \rangle| \leq C(\xi, \eta)\, \wp_\pi(H) e^{-\gamma(H)} \qquad \forall k_1, k_2 \in A \quad \forall H \in \overline{\mathfrak{a}}^+$$

*if and only if a similar inequality, with the same $\gamma$, holds for each of the irreducible representations involved in the decomposition of $\pi$.*

Note that some of the representations involved in the decomposition of $\pi$ may admit uniform estimates with much more rapid decay rates. I know of no proof of this conjecture in general, but it is certainly true in a few simple cases.

By using $L^{p+}$ estimates, we may prove a version of Conjecture 3.1 for the case where $\alpha = (1/m)\rho$, for some positive integer $m$. To do this, we have to find a connection between uniform estimates and $L^{p+}$ estimates.

Suppose that $0 < t \leq 1$, that $\wp$ is a positive polynomial on $\overline{\mathfrak{a}}^+$, and that

$$\phi(\exp(H)) = \wp(H) e^{-t\rho(H)} \qquad \forall H \in \overline{\mathfrak{a}}^+.$$

If $pt \geq 2$, and

$$|u(k_1 a k_2)| \leq \phi(a) \qquad \forall k_1, k_2 \in K \quad \forall a \in \overline{A}^+,$$

then $u$ satisfies $L^{p+}$ estimates. Indeed, from (1.5),

$$\begin{aligned}
\|u\|_{p+\epsilon} &= \Big(C \int_K \int_{\overline{\mathfrak{a}}^+} \int_K |u(k_1 \exp(H) k_2)|^{p+\epsilon} \\
&\qquad\qquad \prod_{\alpha \in \Sigma} \sinh(\alpha(H))^{\dim(\mathfrak{g}_\alpha)}\, dk_1\, dH\, dk_2\Big)^{1/(p+\epsilon)} \\
&\leq \Big(C \int_{\overline{\mathfrak{a}}^+} \left|\wp(H) e^{-t\rho(H)}\right|^{p+\epsilon} e^{2\rho(H)}\, dH\Big)^{1/(p+\epsilon)} \\
&= \Big(C \int_{\overline{\mathfrak{a}}^+} |\wp(H)|^{p+\epsilon} \exp(-t(p+\epsilon)\rho(H) + 2\rho(H))\, dH\Big)^{1/(p+\epsilon)} \\
&\leq \Big(C \int_{\overline{\mathfrak{a}}^+} |\wp(H)|^{p+\epsilon} \exp(-t\epsilon\rho(H))\, dH\Big)^{1/(p+\epsilon)} \\
&= C_{G,p,\epsilon} < \infty.
\end{aligned}$$

More generally, if $\alpha \in \mathbf{T}_t$, and

$$|u(k_1 \exp(H) k_2)| \leq \wp(H) e^{-\alpha(H)} \qquad \forall H \in \overline{\mathfrak{a}}^+,$$

for some polynomial $\wp$, then $u$ satisfies $L^{2/t\,+}$ estimates. This shows that uniform estimates imply $L^{p+}$ estimates.

Conversely, good $L^{p+}$ estimates imply uniform estimates. More precisely, if we know that $\langle \pi(\cdot)\xi, \eta\rangle \in L^{p+\epsilon}(G)$ for all positive $\epsilon$ and all $K$-finite smooth vectors $\xi$ and $\eta$ in $\mathcal{H}_\pi$, then we argue that the function $\langle \pi(\cdot)\xi, \eta\rangle$ and lots of its derivatives lie in $L^{p+\epsilon}(G)$ for all positive $\epsilon$; using an argument involving the Sobolev embedding theorem and the exponential growth of $G$, we may then show that

$$|\langle \pi(k_1 \exp(H) k_2)\xi, \eta\rangle| \leq C(\epsilon, \xi, \eta) e^{-(2/p+\epsilon)\rho(H)} \qquad \forall k_1, k_2 \in K \quad \forall H \in \overline{\mathfrak{a}}^+.$$

The details of this may be found in [30].

Observe that the uniform estimates are "nicer" than the $L^{p+}$ estimates because they contain more information: decay can be faster in some directions than others. However, they have the weakness that they are not translation-invariant: if

$$|u(k_1 a k_2)| \leq \phi(a) \qquad \forall k_1, k_2 \in K \quad \forall a \in \overline{A}^+,$$

and $v(x) = u(xy)$ for some $y$ and all $x$ in $G$, it need not follow that

$$|v(k_1 a k_2)| \leq \phi(a) \qquad \forall k_1, k_2 \in K \quad \forall a \in \overline{A}^+.$$

On the other hand, spaces of matrix coefficients are translation-invariant: indeed

$$\langle \pi(\cdot y)\xi, \eta\rangle = \langle \pi(\cdot)(\pi(y)\xi), \eta\rangle.$$

### 3.3 The Kunze–Stein Phenomenon

Possibly the most important decay estimate for matrix coefficients of simple (or semisimple) Lie groups is the Kunze–Stein phenomenon. This says that $L^{2+}$ estimates hold for the matrix coefficients of the regular representation $\lambda$ of $G$ on $L^2(G)$. More precisely, for all positive $\epsilon$, there exists a constant $C_\epsilon$ such that, if $\xi$ and $\eta$ are in $L^2(G)$, then $\langle \lambda(\cdot)\xi, \eta\rangle \in L^{2+\epsilon}(G)$ and

$$\|\langle \lambda(\cdot)\xi, \eta\rangle\|_{2+\epsilon} \leq C_\epsilon \,\|\xi\|_2 \,\|\eta\|_2 \,.$$

This result was first observed by R.A. Kunze and Stein for the case where $G = \mathrm{SL}(2,\mathbb{R})$, then extended to a number of other simple Lie groups by Kunze and Stein, and by others. Inspired by Kunze and Stein, C.S. Herz [63] and then P. Eymard and N. Lohoué [46] made inroads into the general case. The first general proof, which uses a simplified version of the argument of Kunze and Stein, may be found in [29]. An important corollary of a little functional analysis combined with the Kunze–Stein phenomenon is that, if $\pi$ is any representation of a simple Lie group $G$ and, for some positive integer $m$, $\langle \pi(\cdot)\xi, \eta\rangle \in L^{2m+\epsilon}(G)$ for all positive $\epsilon$ and all $\xi$ and $\eta$ in a dense subspace of $\mathcal{H}_\pi$, then $\langle \pi(\cdot)\xi, \eta\rangle \in L^{2m+\epsilon}(G)$ for all positive $\epsilon$ and all $\xi$ and $\eta$ in $\mathcal{H}_\pi$; further there exists a constant $C(G,\epsilon,m)$ such that

$$\|\langle \pi(\cdot)\xi, \eta\rangle\|_{2m+\epsilon} \leq C(G,\epsilon,m)\;\|\xi\|\,\|\eta\| \qquad \forall \xi, \eta \in \mathcal{H}_\pi.$$

The functional analysis serves to show that the $m$-fold tensor product $\pi\otimes^m$ of $\pi$ is "weakly contained in the regular representation"; the Kunze–Stein phenomenon then gives $L^{2+\epsilon/m}$ estimates for $\langle \pi(\cdot)\xi, \eta\rangle^m$. See [35] for more details.

It may be conjectured that, if $q \geq 2$ and $\langle \pi(\cdot)\xi, \eta\rangle \in L^{q+\epsilon}(G)$ for all positive $\epsilon$ and all $\xi$ and $\eta$ in a dense subspace of $\mathcal{H}_\pi$, then $\langle \pi(\cdot)\xi, \eta\rangle \in L^{q+\epsilon}$ for all $\xi$ and $\eta$ in $\mathcal{H}$, and $L^{q+}$ estimates hold. This is certainly correct for the cases when $G = \mathrm{SO}(1,n)$ or $\mathrm{SU}(1,n)$, but the proof for these groups does not generalise. No such result can hold if $q < 2$: see [30] for the argument.

In any case, these ideas, together with the links between uniform estimates and $L^{p+}$ estimates, establish the claim earlier that Conjecture 3.1 holds when $\alpha = (1/m)\rho$, for some positive integer $m$.

Recently, sharper versions of the Kunze–Stein phenomenon have been discovered, at least for groups of real rank one. In particular, the Kunze–Stein estimates are dual to the convolution estimate $L^p(G) * L^2(G) \subseteq L^2(G)$; by interpolation with the obvious result $L^1(G) * L^1(G) \subseteq L^1(G)$, this implies that $L^r(G) * L^s(G) \subseteq L^s(G)$ when $1 \leq r < s \leq 2$. By using Lorentz spaces $L^{p,q}(G)$, it is possible to formulate versions of the Kunze–Stein convolution theorem such as $L^{p,1}(G) * L^p(G) \subseteq L^p(G)$ for the case where $p < 2$ (see [32]) and $L^{2,1}(G) * L^{2,1}(G) \subseteq L^{2,\infty}(G)$ (see [66]). The study of related operators, such as maximal operators, has also begun; the major result here is that of J.-O. Strömberg [97].

### 3.4 Property $T$

Certain Lie groups (and many more locally compact groups) have property $T$. This is a property with several equivalent formulations, one of which is that the trivial representation is isolated in the unitary dual $\widehat{G}$ of $G$, that is, the set of all equivalence classes of irreducible unitary representations of $G$, equipped with a natural topology.

This property was introduced by D.A. Kazhdan [70] who proved that $\mathrm{SL}(3,\mathbb{R})$, and many other simple Lie groups with $\dim(A) \geq 2$, have it. Shortly after, S.P. Wang [103] observed that Kazhdan's argument could be developed to prove that all simple Lie groups with $\dim(A) \geq 2$ have property $T$. At about the same times, B. Kostant [73, 74] established that $\mathrm{Sp}(1,n)$ and $F_{4,-20}$ have property $T$, while $\mathrm{SO}(1,n)$ and $\mathrm{SU}(1,n)$ do not. Since then property $T$ has appeared in a number of different applications of representation theory, including the proof (of J.M. Rosenblatt, D. Sullivan and G.A. Margulis [82, 88, 98]) that Lebesgue measure is the only finitely additive rotation-invariant position additive set function on the sphere $S^k$, for $k \geq 5$, and the construction by Margulis [81, 83] of "expanders", graphs with a very high degree of connectivity. The monograph of P. de la Harpe and A. Valette [56] presents a detailed account of these applications, and much more; for more recent applications, see also the monographs of P. Sarnak [89] and A. Lubotzky [79]. If $G$ is a simple Lie group with property $T$, then there exists $p_G$ in $(2,\infty)$ such that

$$\langle \pi(\cdot)\xi, \eta\rangle \in L^{p_G+\epsilon}(G) \qquad \forall \xi, \eta \in \mathcal{H}_\pi$$

for all unitary representations $\pi$ of $G$ with no trivial subrepresentations, and further

$$\left\| \langle \pi(\cdot)\xi, \eta\rangle \right\|_{p_G+\epsilon} \leq C_{G,\epsilon} \left\|\xi\right\| \left\|\eta\right\| \qquad \forall \xi, \eta \in \mathcal{H}_\pi .$$

In other words, there is uniform vanishing at infinity of all matrix coefficients which vanish at infinity. This can also be expressed with uniform estimates for smooth $K$-finite matrix coefficients.

Kazhdan's proof that $\mathrm{SL}(3,\mathbb{R})$ has property $T$ uses an argument like the argument already given to show that matrix coefficients decay at infinity. Indeed, $\mathrm{SL}(3,\mathbb{R})$ contains the subgroup $Q_1$, of all elements of the form

$$\begin{bmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{bmatrix},$$

where $a, b, c, d, x, y \in \mathbb{R}$, and $ad - bc = 1$. The subgroups $M_1$ and $N_1$ of $Q_1$ are defined by the conditions that $x = y = 0$ (for $M_1$) and $a = d = 1$ and $b = c = 0$ (for $N_1$).

Representation theory ("the Mackey machine") shows that any unitary representation $\pi$ of $Q_1$ splits into two: $\pi = \pi_1 \oplus \pi_0$, where $\pi_1$ is trivial on $N_1$ and $\pi_0|_{M_1}$ is a subrepresentation of the regular representation of $M_1$. A representation whose matrix coefficients vanish at infinity cannot have a $\pi_1$ component. Kazhdan used this analysis to deduce that $\pi$ cannot approach 1. In [30], it is shown that the matrix coefficients of $\pi$, restricted to $M_1$, satisfy $L^{2+\epsilon}$ estimates; this is then used to show that the matrix coefficients satisfy $L^{p_G+}$ estimates on $G$. Later, R.E. Howe [65], R. Scaramuzzi [93], J.-S. Li [75, 76] and H. Oh [85, 86] analysed the various possibilities more carefully, and found optimal values for $p_G$.

### 3.5 The Generalised Ramanujan–Selberg Property

Suppose that the simple Lie group $G$ acts on a probability space $\Omega$, preserving the measure. Then there is a unitary representation $\pi$ of $G$ on $L^2(\Omega)$ given by the formula

$$[\pi(x)\xi](\omega) = \xi(x^{-1}\omega) \qquad \forall\omega \in \Omega \quad \forall\xi \in L^2(\Omega).$$

The constant functions form a 1-dimensional $G$-invariant subspace of $L^2(\Omega)$; denote by $L^2(\Omega)_0$ its orthogonal complement. When $G$ acts ergodically on $\Omega$, there is no invariant vector in $L^2(\Omega)_0$ (this may be taken as the definition of ergodicity). It follows that all the matrix coefficients of the restriction $\pi_0$ of $\pi$ to $L^2(\Omega)_0$ vanish at infinity. If $G$ has property $T$, then these matrix coefficients satisfy a $L^{p_G+}$ estimate. We define an action of $G$ on a probability space $\Omega$ to be a $T$-action if there is some finite $q$ such that the restricted representation $\pi_0$ has matrix coefficients which satisfy a $L^{q+}$ estimate. Then every ergodic action of a group with property $T$ is a $T$-action.

If the action of $G$ on a probability space $\Omega$ is an $T$-action, then information about the representation of $G$ on $L^p(\Omega)$ comes from complex interpolation. Indeed, suppose that $p < 2$, and that $\xi \in L^p(\Omega)_0$ and $\eta \in L^{p'}(\Omega)_0$, that is, $\xi \in L^p(\Omega)$, $\eta \in L^{p'}(\Omega)$, and both have zero mean on $\Omega$. For a complex number $z$ with $\mathrm{Re}(z)$ in $[0, 1]$, define $\xi_z$ and $\eta_z$:

$$\xi_z = |\xi|^{p(1-z/2)-1}\,\xi - \int_\Omega |\xi|^{p(1-z/2)-1}\,\xi$$

and

$$\overline{\eta}_z = |\eta|^{p'z/2-1}\,\overline{\eta} - \int_\Omega |\eta|^{p'z/2-1}\,\overline{\eta}.$$

If $\mathrm{Re}(z) = 0$, then $\left||\xi|^{p(1-z/2)-1}\,\xi\right| = |\xi|^p$ and $\left||\eta|^{p'z/2-1}\,\overline{\eta}\right| = 1$, whence

$$\|\xi_z\|_1 \le 2\,\|\xi\|_p^p \qquad \text{and} \qquad \|\eta_z\|_\infty \le 2;$$

similarly if $\mathrm{Re}(z) = 1$, then

$$\|\xi_z\|_2 \le 2\,\|\xi\|_p^{p/2} \qquad \text{and} \qquad \|\eta_z\|_2 \le 2\,\|\eta\|_{p'}^{p'/2}.$$

Consider the analytic family of functions on $\{z \in \mathbb{C} : \mathrm{Re}(z) \in [0, 1]\}$ given by

$$z \mapsto \langle \pi(\cdot)\xi_z, \eta_z \rangle.$$

When $\mathrm{Re}(z) = 0$, these functions are bounded on $G$, while when $\mathrm{Re}(z) = 1$, these functions are coefficients of $\pi_0$, so satisfy $L^{q+}$ estimates. When $z = 2/p'$, we get $\langle \pi(\cdot)\xi, \eta \rangle$. By a standard complex interpolation argument, this function on $G$ satisfies $L^{2q/p'+}$ estimates.

There are some important examples of $T$-actions of groups which do not have property $T$. In particular, if $G = \mathrm{SL}(2,\mathbb{R})$ and $X = G/\Gamma$, where $\Gamma$ is a congruence subgroup, that is, for some $N$ in $\mathbb{Z}^+$,

$$\Gamma = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z}) : a - 1 \equiv d - 1 \equiv b \equiv c \equiv 0 \mod N \right\},$$

then the action of $G$ on $X$ is a $T$-action. Indeed, $\pi_0$ satisfies a $L^{4+}$ estimate. This is a reformulation of a celebrated result of A. Selberg (generalising a result of G. Roelcke for $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$ which states, in our language, that for this choice of $\Gamma$, the representation $\pi_0$ satisfies a $L^{2+}$ estimate). Selberg also conjectured that $\pi_0$ satisfies an $L^{2+}$ estimate. This result is usually phrased in terms of the first nonzero eigenvalue of the Laplace–Beltrami operator on the space $K\backslash G/\Gamma$, a quotient of the hyperbolic upper half plane; the representation theoretic version is due to I. Satake [90].

Similar results were discovered by M. Burger, J.-S. Li and Sarnak [16, 17], and formulated in terms of the "Ramanujan dual". It is now known that every action of a real simple algebraic group $G$ on the quotient space $G/\Gamma$ is a $T$-action, for any lattice $\Gamma$ in $G$ (arithmetic or not). As pointed out by M.E.B. Bekka, this follows from the Burger–Sarnak argument and work of A. Borel and H. Garland [12] (see [9] for more details). The Burger–Sarnak argument has been reworked by a number of people, including L. Clozel, Oh and E. Ullmo [26] and Cowling [33].

An observation by C.C. Moore [84] is relevant here; representations of simple Lie groups with finite centres which do not weakly contain the trivial representation automatically satisfy $L^{p+}$ estimates.

## 4 More General Semisimple Groups

In this lecture, I look at questions in graph theory and number theory. The initial motivation is to shed light on analytical problems, such as finding the behaviour of the eigenvalues of the Laplace–Beltrami operator on Riemannian manifolds, by studying this problem for the eigenvalues of graph Laplacians. It turns out that certain problems in discrete mathematics can also be attacked effectively using approaches and results from analysis.

### 4.1 Graph Theory and its Riemannian Connection

A *graph* $\mathcal{G}(V, E)$, usually written $\mathcal{G}$, is a set $V$ of *vertices* and a set $E$ of *edges*, that is, a symmetric subset of $V \times V$. A *path* in $\mathcal{G}$ from $v_0$ to $v_n$ of length $n$ is a list of vertices $[v_0, v_1, \dots, v_n]$ with the property that $(v_{i-1}, v_i) \in E$ when $i = 1, 2, \dots, n$; we consider $[v_0]$ to be a path of length 0. The graph is said to be *connected* if there is a path between any two vertices. The *distance* $d(v, w)$ between vertices $v$ and $w$ in a connected graph is the length of a shortest path between them. The *diameter* of a finite connected graph is the greatest distance between any pair of vertices. The *degree* of a vertex $v$, written $\deg v$, is the cardinality of the set of vertices at distance 1 from $v$; the degree of the graph $\mathcal{G}$ is the supremum of the degrees of the vertices. We shall deal with connected graphs of finite degree.

For a connected graph $\mathcal{G}$ of finite degree, the graph Laplacian $\Delta_{\mathcal{G}}$ is defined as a map on functions on $V$:

$$\Delta_{\mathcal{G}} f(v) = f(v) - \frac{1}{\deg v} \sum_{\substack{w \in V \\ d(v,w)=1}} f(w).$$

This operator is bounded on $L^2(V)$ and self-adjoint. It is a natural analogue of the Laplace–Beltrami operator $\Delta_M$ on a Riemannian manifold $M$ but, being bounded, is easier to analyse.

There is already an extensive theory of "approximation" of a Riemannian manifold $M$ and $\Delta_M$ by a graph $\mathcal{G}$ and $\Delta_{\mathcal{G}}$. The underlying philosophy is that properties of $\Delta_M$ which are "local" in the manifold and are reflected spectrally at infinity are lost when the manifold is discretised, but that properties of $\Delta_M$ which are "global" in the manifold and are reflected in the spectrum "near 0" will be seen in the properties of $\Delta_{\mathcal{G}_0}$. For some examples in this direction, see [27, 28, 69].

Interesting examples of graphs arise in the study of semisimple groups in two ways: as Cayley graphs of discrete groups and as "discrete symmetric spaces". We next consider Cayley graphs and then describe the $p$-adic numbers and discrete symmetric spaces.

### 4.2 Cayley Graphs

Suppose $X$ is a set of generators for a group $G$, closed under the taking of inverses. The Cayley graph of $(G, X)$ is the graph $\mathcal{G}(G, E)$, where $E$ is the subset of $G \times G$ defined by the condition that $(x, y) \in E$ if and only if $xy^{-1} \in X$ (or equivalently $yx^{-1} \in X$). The group $G$ acts simply transitively and isometrically on $\mathcal{G}(G, E)$ by left multiplication, so that Cayley graphs are homogeneous: all points "look alike". Cayley graphs are good for obtaining examples of graphs of small degree and small diameter but high cardinality (these are "expanders", which are important in discrete mathematics).

Suppose that $(u_n : n \in \mathbb{N})$ is a sequence of positive definite functions on $G$, normalised in the sense that $u_n(e) = 1$ for all $n$ (where $e$ denotes the identity of $G$). Then $u_n = \langle \pi_n(\cdot)\xi_n, \xi_n \rangle$, where $\|\xi_n\| = 1$. Suppose that $|u_n(x) - 1| < \epsilon_n$ for all $x$ in $X$. Then

$$\begin{aligned} \|\pi_n(x)\xi_n - \xi_n\|^2 &= \langle \pi_n(x)\xi_n - \xi_n,\, \pi_n(x)\xi_n - \xi_n \rangle \\ &= 2 - 2\operatorname{Re}\langle \pi_n(x)\xi_n, \xi_n \rangle, \end{aligned}$$

and so

$$\|\pi_n(x)\xi_n - \xi_n\| \leq (2\epsilon)^{1/2}.$$

It follows, by induction, that

$$\|\pi_n(x_1 \, \dots \, x_m)\xi_n - \xi_n\| \leq m(2\epsilon_n)^{1/2} \qquad \forall x_1, \dots, x_m \in X;$$

indeed

$$\begin{aligned} \|\pi_n(x_1 \dots x_m)\xi_n - \xi_n\| \leq{}& \|\pi_n(x_1 \, \dots \, x_{m-1})(\pi_n(x_m)\xi_n - \xi_n)\| \\ &+ \|\pi_n(x_1 \, \dots \, x_{m-1})\xi_n - \xi_n\| \, . \end{aligned}$$

Thus if $u_n(x) \to 1$ as $n \to \infty$ for all $x$ in $X$, $u_n \to 1$ as $n \to \infty$ locally uniformly on $G$.

Property $T$ may be expressed in the following form: if none of the unitary representations $\pi_n$ has a trivial subrepresentation, then the corresponding matrix coefficients $u_n$ cannot tend to 1 locally uniformly. It becomes possible to quantify property $T$, by finding numbers $\tau_G$ such that

$$\sup_{x \in X} |u(x) - 1| > \tau_G,$$

or perhaps (if $G$ is finitely generated)

$$\sum_{x \in X} |u(x) - 1| > \tau_G \qquad \text{or} \qquad \left( \sum_{x \in X} |u(x) - 1|^2 \right)^{1/2} > \tau_G,$$

for all normalised positive definite functions $u$ which are associated to unitary representations without trivial subrepresentations. (It might be argued, on the basis of results about the free group, and of its utility in formulae like inequality (4.1) that this final definition is the best). This quantification of property $T$ leads to estimates for a spectral gap for $\Delta_{\mathcal{G}}$, acting on matrix coefficients of unitary representations. For the normalised positive definite function $u$, equal to $\langle \pi(\cdot)\xi, \xi \rangle$,

$$|u(x)| = |\langle \pi(x)\xi, \xi \rangle| \leq \|\pi(x)\xi\| \, \|\xi\| \leq 1,$$

and for any complex number $z$ in the closed unit disc,

$$\mathrm{Re}(1 - z) \geq \frac{|1 - z|^2}{2}.$$

Thus

$$\Delta_{\mathcal{G}} u(e) = \frac{1}{\deg e} \mathrm{Re}\left( \sum_{x \in X} (1 - u(x)) \right) \geq \frac{1}{2|X|} \sum_{x \in X} |1 - u(x)|^2, \tag{4.1}$$

which is bounded away from 0 if $\tau_G$ is bounded away from 0 (using any of the above definitions).

### 4.3 An Example Involving Cayley Graphs

Let $G$ denote the group $\mathrm{SL}(3, \mathbb{Z})$ of all $3 \times 3$ integer matrices with determinant 1, and let $X$ denote the symmetric generating subset

$$\left\{ \begin{bmatrix} 1 & x_{12} & x_{13} \\ x_{21} & 1 & x_{23} \\ x_{31} & x_{32} & 1 \end{bmatrix} : x_{ij} \in \{0, \pm 1\}, \quad |x_{12}| + \cdots + |x_{32}| = 1 \right\}$$

(that is, all $x_{ij}$ but one are equal to 0). Burger [15] estimated $\tau_G$ for this group (using the first definition). Let $\pi$ be a unitary representation of $\mathrm{SL}(3, \mathbb{Z})$ in a Hilbert space $\mathcal{H}_\pi$, and $S$ a finite set of generators of $\mathrm{SL}(3, \mathbb{Z})$. For various examples of $S$ and $\pi$, he obtains an explicit positive $\varepsilon$ such that $\max_{\gamma \in S} \|\pi(\gamma)\xi - \xi\| \,/\, \|\xi\| \geq \varepsilon$, for all $\xi$ in the space of $\pi$. As he observes, these results give a partial solution to the problem of giving a quantitative version of Kazhdan's property (T) for $\mathrm{SL}(3, \mathbb{Z})$.

For any prime number $p$, let $G_p$ denote the finite group $\mathrm{SL}(3, \mathbb{F}_p)$, that is, the group of matrices of determinant 1 with entries in the finite field $\mathbb{F}_p$ with $p$ elements. This is a quotient group of $G$. Indeed, define the normal subgroup $\Gamma_p$ of $G$ by

$$\Gamma_p = \left\{ \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \in \mathrm{SL}(3, \mathbb{Z}) : x_{ij} \equiv \delta_{ij} \mod p \right\},$$

where $\delta_{ij}$ is the Kronecker delta; then $G_p$ is isomorphic to $G/\Gamma_p$. A unitary representation of $G/\Gamma_p$ with no trivial subrepresentation lifts canonically to a unitary representation of $G$ with no trivial subrepresentation, and so the estimates on the matrix coefficients of all unitary representations of $G$ imply, in particular, estimates for the matrix coefficients of these lifted representations. Thus we obtain estimates on the degree of isolation of the trivial representation of $G_p$ which are uniform in $p$. A number of estimates of this type have recently been summarised in the survey of P. Diaconis and L. Saloff-Coste [44].

### 4.4 The Field of $p$-adic Numbers

For a prime number $p$, the $p$-adic norm on the set of rational numbers $\mathbb{Q}$ is defined by

$$|0|_p = 0 \qquad \text{and} \qquad |x|_p = p^{-\alpha},$$

where $x = mp^\alpha/n$, $m$ and $n$ being integers with no factors of $p$. It is easy to check that $|x|_p = 0$ only if $x = 0$, that $|xy|_p = |x|_p|y|_p$, and that

$$|x + y|_p \le \max\{|x|_p, |y|_p\} \qquad \forall x, y \in \mathbb{Q}.$$

The completion of $\mathbb{Q}$ in the associated distance $d_p$, that is, $d_p(x, y) = |x-y|_p$, is a totally disconnected locally compact field, called the field of $p$-adic numbers, and written $\mathbb{Q}_p$. The algebraic operations of $\mathbb{Q}_p$ are those of formal series of the form

$$\sum_{n=N}^{\infty} a_n p^n,$$

where $N \in \mathbb{Z}$ and $a_n \in \{0, 1, 2, \ldots, p-1\}$, with "carrying", for instance,

$$\begin{aligned}(-1)p^k &= (p-1)p^k + (-1)p^{k+1} \\ &= (p-1)p^k + (p-1)p^{k+1} + (-1)p^{k+2} \\ &= (p-1)p^k + (p-1)p^{k+1} + (p-1)p^{k+2} + \ldots .\end{aligned}$$

The subset of $\mathbb{Q}_p$ of all series where $N \ge 0$ is an open and closed subring of $\mathbb{Q}_p$, known as the ring of $p$-adic integers, and written $\mathcal{O}_p$; this is the completion of $\mathbb{Z}$ in $\mathbb{Q}_p$. The field $\mathbb{Q}_p$ presents a few surprises to the uninitiated: for example, if $p \equiv 1 \mod 4$, then $\mathbb{Q}_p$ contains a square root of $-1$. However, $\mathbb{Q}_p$ does not contain very many new roots, and the algebraic completion of $\mathbb{Q}_p$ is of infinite degree over $\mathbb{Q}_p$.

Apart from $\mathbb{R}$ and $\mathbb{C}$, the real and complex numbers, the locally compact complete normed fields are "local fields", that is, they are totally disconnected. Every local field is either a finite algebraic extension of $\mathbb{Q}_p$ or a field of Laurent series in one variable over a finite field. Like $\mathbb{Q}_p$, these all have a compact open "ring of integers" $\mathcal{O}$. There is a unique translation-invariant measure on any local field which assigns measure 1 to $\mathcal{O}$.

### 4.5 Lattices in Vector Spaces over Local Fields

Let $V$ be the vector space $\mathbb{Q}_p^n$ over the local field $\mathbb{Q}_p$, with the standard basis $\{e_1, \ldots, e_n\}$. A *lattice* $L$ in $V$ is a subset of $V$ of the form

$$L = \{m_1 v_1 + m_2 v_2 + \cdots + m_n v_n : m_1, m_2, \ldots, m_n \in \mathcal{O}_p\},$$

where $\{v_1, \ldots, v_n\}$ is a basis for $V$. The set $\{v_1, \ldots, v_n\}$ is also called a basis for $L$ over $\mathcal{O}_p$, for obvious reasons. The standard lattice $L_0$ is the lattice with basis $\{e_1, \ldots, e_n\}$. All lattices are compact open subsets of $V$.

Given two lattices $L_1$ and $L_2$, it is possible to find a basis $\{v_1, \ldots, v_n\}$ of $L_1$ over $\mathcal{O}_p$ such that, for suitable integers $a_1, \ldots, a_n$,

$$L_2 = \{m_1 p^{a_1} v_1 + m_2 p^{a_2} v_2 + \cdots + m_n p^{a_n} v_n : m_1, m_2, \ldots, m_n \in \mathcal{O}_p\}. \tag{4.2}$$

The order of the numbers $a_i$ may depend on the basis chosen, but the numbers themselves do not. This result, known as the invariant factor theorem, may be found in many texts on algebra, such as C.W. Curtis and I. Reiner [43, pp. 150–153].

The group $\mathrm{GL}(n, \mathbb{Q}_p)$ acts on the vector space $V$ and hence on the space $\mathcal{L}$ of lattices in $V$. The stabiliser of the standard lattice $L_0$ is the compact subgroup $\mathrm{GL}(n, \mathcal{O}_p)$ of invertible $\mathcal{O}_p$-valued $n \times n$ matrices whose inverses are also $\mathcal{O}_p$-valued (equivalently, whose determinant has norm 1). Thus the space $\mathcal{L}$ may be identified with the coset space $\mathrm{GL}(n, \mathbb{Q}_p)/\mathrm{GL}(n, \mathcal{O}_p)$. The group $\mathrm{GL}(n, \mathbb{Q}_p)$ is not semisimple, and this is not quite the analogue of a Riemannian symmetric space.

One example of a discrete symmetric space may be obtained by restricting attention to the space $\mathcal{L}_1$ of lattices whose volume is equal to that of the standard lattice. It follows from the invariant factor theorem that $\mathcal{L}_1$ may be identified with the coset space $\mathrm{SL}(n, \mathbb{Q}_p)/\mathrm{SL}(n, \mathcal{O}_p)$.

The more standard example of a discrete symmetric space is a quotient space of $\mathcal{L}$. Define an equivalence relation $\sim$ on $\mathcal{L}$ by the stipulation that $L_1 \sim L_2$ if $L_1 = \lambda L_2$ for some $\lambda$ in $\mathbb{Q}_p$; the equivalence class of $L$ is written $[L]$. Define $d \colon \mathcal{L} \times \mathcal{L} \to \mathbb{N}$ by

$$d(L_1, L_2) = \max\{a_i : 1 \le i \le n\} - \min\{a_i : 1 \le i \le n\},$$

where $\{a_i : 1 \le i \le n\}$ is as in formula (4.2) above. It is simple to check that $d$ factors to a distance function on the space $[\mathcal{L}]$ of equivalence classes of lattices. We may identify $[\mathcal{L}]$ with the coset space $\mathrm{PGL}(n, \mathbb{Q}_p)/\mathrm{PGL}(n, \mathcal{O}_p)$, where $\mathrm{PGL}(n, \mathbb{Q}_p)$ is the quotient group $\mathrm{GL}(n, \mathbb{Q}_p)/Z$, $Z$ being its centre (that is, the group of nonzero diagonal matrices), and $\mathrm{PGL}(n, \mathcal{O}_p)$ is the image of $\mathrm{GL}(n, \mathcal{O}_p)$ in $\mathrm{PGL}(n, \mathbb{Q}_p)$. The key to this identification is the observation that the scalar matrix $\lambda I$ moves $L$ to $\lambda L$, and preserves the equivalence classes. The space $[\mathcal{L}]$ has the structure of a simplicial complex, in which the vertices are the equivalence classes $[L]$, and the edges are pairs $([L_1], [L_2])$ where $d(L_1, L_2) = 1$.

Similar constructions apply when $\mathbb{Q}_p$ is replaced by another local field.

Perhaps the moral of this is just that a discrete symmetric space is a well defined combinatorial object; it has "invariant difference operators" analogous to the "invariant differential operators" on a symmetric space, and various functions and function spaces on a symmetric space have discrete analogues which are easier to work with. In particular, the formulae for the spherical functions are easier to deal with, so that, for example, it should be easier to analyse the heat equation on a discrete symmetric space than on a normal symmetric space. For the rank one case, compare [36] and [40, 41].

Useful bibliography on discrete symmetric spaces includes [14, 87] (for the geometric and combinatorial structure), and [80] (for the spherical functions, Plancherel theorem, ...). For the rank one case, these are "trees"(that is, simply connected graphs), and analysis on these structures was developed by A. Figà-Talamanca and C. Nebbia [47]. More detailed analysis on trees may be found in, for instance, [40, 41].

### 4.6 Adèles

We conclude this outline of some of the generalisations of semisimple Lie groups with a brief discussion of the adèles and adèle groups.

The ring of adèles, $\mathbb{A}$, is the "restricted direct product" $\mathbb{R} \times \Pi_{p\in P} \mathbb{Q}_p$, where $P$ is the set of prime numbers. An adèle is a "vector" $(x_\infty, x_2, x_3, \ldots, x_p, \ldots)$, where $x_\infty \in \mathbb{R}$, and $x_p \in \mathbb{Q}_p$; further, $|x_p|_p > 1$ for only finitely many $p$ in $P$. The operations in the ring are componentwise addition, subtraction, and multiplication. For an adèle to be invertible, it is necessary and sufficient that no component be zero, and that $|x_p|_p \neq 1$ for only finitely many components.

The ring of adèles may be topologised by defining a basis of open sets at 0 to be all sets of the form $U_\infty \times U_2 \times U_3 \times \cdots \times U_p \times \cdots$, where $U_\infty$ is an open set containing 0, as is each $U_p$, and all but finitely many $U_p$ are equal to $\mathcal{O}_p$. The translates of these sets by $x$ then form a basis for the topology at $x$. Similarly $\mathbb{A}$ may be equipped with product measure.

It is possible to form groups such as $\mathrm{SL}(2, \mathbb{A})$. One may think either of matrices with "vector" entries, or equivalently as a "vector" of matrices:

$$\begin{bmatrix} (a_\infty, a_2, \cdots) & (b_\infty, b_2, \cdots) \\ (c_\infty, c_2, \cdots) & (d_\infty, d_2, \cdots) \end{bmatrix} \sim \left( \begin{bmatrix} a_\infty & b_\infty \\ c_\infty & d_\infty \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}, \ldots \right).$$

Because the operations are component-by-component, these are equivalent formulations.

The rational numbers may be injected diagonally into the adèles, that is, the rational number $r$ corresponds to the adèle $(r, r, r, \ldots)$. Then $\mathbb{Q}$ "is" a discrete subring of $\mathbb{A}$, and $\mathbb{A}/\mathbb{Q}$ is compact.

One of the major goals of number theorists is to understand the unitary representation $\lambda$ of $\mathrm{SL}(2, \mathbb{A})$ on the space $L^2(\mathrm{SL}(2, \mathbb{A})/\mathrm{SL}(2, \mathbb{Q}))$, and similar representations involving other groups, such as $\mathrm{SL}(n, \mathbb{A})$. For the $\mathrm{SL}(2)$ case, much information is contained in Gel'fand, Graev, and Pyateckii-Shapiro [51].

The group $\mathrm{SL}(2, \mathbb{A})$ has unitary representations, which are "restricted tensor products" of unitary representations of the factors, and quite a bit is known about how $\lambda$ decomposes into irreducible components. A number of important conjectures in number theory may be reformulated in terms of the harmonic analysis of $\mathrm{SL}(2, \mathbb{A})$.

An important recent result is concerned with the representation of $G(\mathbb{A})$ on $L^2(G(\mathbb{A})/G(\mathbb{Q}))$. The space $G(\mathbb{A})/G(\mathbb{Q})$ has finite volume, so the constant functions lie in the Hilbert space, and the representation contains a trivial subrepresentation. Clozel [25] has recently proved a conjecture of A. Lubotzky and R.J. Zimmer that all the other components of this representation are isolated away from the trivial representation. A consequence of this is that there exists $p$ such that each of the other components satisfies an $L^{p+}$ estimate, which in turn implies that the restriction of many unitary representations of $G(\mathbb{A})$ to $G(\mathbb{Q})$ are irreducible (see [10, 11]).

### 4.7 Further Results

The work of I. Cherednik [21, 22] offers another unification of the real and $p$-adic settings.

# 5 Carnot–Carathéodory Geometry and Group Representations

In this lecture, following [7], I construct some unitary and uniformly bounded representations of simple Lie groups of real rank one, using geometric methods.

## 5.1 A Decomposition for Real Rank One Groups

**Theorem 5.1.** *Suppose that $G$ is a real rank one simple Lie group, with an Iwasawa decomposition $KAN$. Then $G = KNK$, in the sense that every element $g$ of $G$ may be written (not uniquely) in the form $k_1 n k_2$, where $n \in N$ and $k_1, k_2 \in N$.*

*Proof.* Consider the action of the group $G$ on the associated symmetric space $X$ (which may be identified with $G/K$). It will suffice to show that any point $x$ in $X$ may be written in the form $kno$, where $o$ is the base point of $X$ (that is, the point stabilised by $K$). Suppose that the distance of $x$ from $o$ is $d$. As $n$ varies over the connected group $N$, the point $no$ varies over a subset $No$ of $X$ which contains $o$. Since this subset is connected and unbounded, there exists a point $no$ in $No$ whose distance from $o$ is $d$. Now $K$ acts transitively on all the spheres with centre $o$, so there exists $k$ in $K$ such that $kno = x$, as required.

## 5.2 The Conformal Group of the Sphere in $\mathbb{R}^n$

Stereographic projection from $\mathbb{R}^n$ to $S^n$ may be defined by the formula

$$\sigma(x) = \Big(1 + \frac{|x|^2}{4}\Big)^{-1}\Big(x, 1 - \frac{|x|^2}{4}\Big) \qquad \forall x \in \mathbb{R}^n,$$

where $(x,t)$ is shorthand for $(x_1, \dots, x_n, t)$. It is a conformal map, that is, its differential is a multiple $D_\sigma$ of an orthogonal map. Its Jacobian $J_\sigma$ is the $n^{\text{th}}$ power of this multiple, that is,

$$J_\sigma(x) = \Big(1 + \frac{|x|^2}{4}\Big)^{-n} \qquad \forall x \in \mathbb{R}^n.$$

It is well known that $G$, the conformal group of the sphere, that is, the group of all orientation-preserving conformal diffeomorphisms of $S^n$, may be identified with $\mathrm{SO}_0(1, n+1)$. Let $P$ denote the subgroup of $G$ of conformal maps which fix the north pole $b$. By conjugation with $\sigma$, we may identify $P$ with the group of all conformal diffeomorphisms of $\mathbb{R}^n$. This is the Euclidean motion group, which is the semidirect product of the group $\mathrm{SO}(n) \times \mathbb{R}^+$ of conformal linear maps of $\mathbb{R}^n$ (which are all products of rotations and dilations) and the group of translations of $\mathbb{R}^n$ (isomorphic to $\mathbb{R}^n$ itself). Then $P$ may be decomposed as $MAN$, where $MA$ is the subgroup of $P$ giving linear conformal maps of

$\mathbb{R}^n$ (that is, $\sigma(\mathrm{SO}(n) \times \mathbb{R}^n)\sigma^{-1}$ and $N = \sigma\{\tau_x : x \in \mathbb{R}^n\}\sigma^{-1}$, where $\tau_x$ denotes translation by $x$ on $\mathbb{R}^n$ (that is, $\tau_x y = x + y$). Write $K$ for $\mathrm{SO}(n)$. Then the groups $K$, $M$, $A$ and $N$ are those which arise in the Iwasawa and Bruhat decompositions of $G$, described in Lecture 1. We can also establish these decompositions geometrically: for instance, given $g$ in $G$, there exists a rotation $k$ of $S^n$ such that $gb = kb$. Then $k^{-1}gb = b$, so $k^{-1}g \in P$, and $k^{-1}g$ may be written in the form $man$, where $m \in M$, $a \in A$, and $n \in N$. Consequently, $g = (km)an$, and $km \in K$; thus we have shown that $G = KAN$.

**Theorem 5.2.** *Suppose that $\mathfrak{F}_z(S^n)$ and $\mathfrak{F}_z(\mathbb{R}^n)$ are function spaces on $S^n$ and $\mathbb{R}^n$, such that $T_z \colon \mathfrak{F}_z(S^n) \to \mathfrak{F}_z(\mathbb{R}^n)$ is an isomorphism, where*

$$T_z f = J_\sigma^{z/n+1/2} f \circ \sigma,$$

*and that translations and rotations act isometrically on $\mathfrak{F}(\mathbb{R}^n)$ and $\mathfrak{F}_z(S^n)$ respectively. Then $\pi_z \colon G \to \mathrm{End}(\mathfrak{F}_z(S^n))$, given by*

$$\pi_z(g)f(x) = J_{g^{-1}}^{z/n+1/2}(x) f(g^{-1}x)$$

*(where $J_g$ is the Jacobian of the conformal map $g$ on $S^n$), is a representation of $G$ on $\mathfrak{F}_z(S^n)$ by isomorphisms. If the maps $T_z$ are isometric, then so is $\pi_z$.*

*Proof.* Since $G = KNK$, it suffices to show that $N$ acts by isomorphisms (which are isometric if the map $T_z$ is isometric), since $K$ acts isometrically. Now

$$\begin{aligned}
\pi_z(\sigma\tau_x^{-1}\sigma^{-1})f(y) &= \Big(\frac{d\sigma\tau_x\sigma^{-1}y}{dy}\Big)^{z/n+1/2} f(\sigma\tau_x\sigma^{-1}y) \\
&= \Big(\frac{d\sigma\tau_x\sigma^{-1}y}{d\tau_x\sigma^{-1}y}\,\frac{d\tau_x\sigma^{-1}y}{d\sigma^{-1}y}\,\frac{d\sigma^{-1}y}{dy}\Big)^{z/n+1/2} f(\sigma\tau_x\sigma^{-1}y) \qquad (5.1)\\
&= (T_z^{-1}\tau_x^{-1}T_z f)(y),
\end{aligned}$$

by the chain rule and the definitions, so

$$\|\pi_z(n)f\| \le \|T_z^{-1}\|\,\|T_z\|\,\|f\|\,.$$

Clearly, if $T_z$ is isometric, then $\pi_z$ is isometric.

Here are some examples. First, if $\mathfrak{F}_z = L^p$, where $\mathrm{Re}(z) = n(1/p - 1/2)$, then $T_z$ is an isometry, by definition. In particular, if $\mathrm{Re}(z) = 0$, then $\pi_z$ acts unitarily on $L^2(S)$, giving us the "unitary class-one principal series" of representations, indicated by the heavy vertical line in the diagram at the end of this section.

Next, if $\mathfrak{F} = H^s$, where

$$s = -\mathrm{Re}(z) \in \Big(-\frac{n}{2}, \frac{n}{2}\Big),$$

then $T_z$ is an isomorphism. Here

$$\begin{aligned} H^s(\mathbb{R}^n) &= \{f \in \mathcal{S}'(\mathbb{R}^n) : |\cdot|^s \hat{f} \in L^2(\mathbb{R}^n)\} \\ H^s(S^n) &= \{f \in L^1(S^n) : \sum_k (1+k)^s f_k \in L^2(S^n)\}, \end{aligned}$$

where $\sum_k f_k$ is the decomposition of $f$ as a sum of spherical harmonics $f_k$ of degree $k$.

One proof of this uses the remarkable formula

$$|1 - \sigma(x) \cdot \sigma(y)| = J_\sigma(x)^{1/2n} J_\sigma(y)^{1/2n} |x - y| \qquad \forall x, y \in N. \tag{5.2}$$

Note that $|x - y|$ is the Euclidean distance between $x$ and $y$ in $\mathbb{R}^n$, and that $|1 - p \cdot q|$, henceforth written $d_{S^n}(p, q)$, is the distance between $p$ and $q$ in $S^n$ (the length of the chord joining them, not the geodesic distance in the sphere). This implies that, if $s \in (-n/2, n/2)$, then

$$\begin{aligned} &\int_{S^n} \int_{S^n} f(p)\, g(q)\, d_{S^n}(p,q)^{-2s-n}\, dp\, dq \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(x)\, g(y)\, J_\sigma(x)^{-s/n+1/2}\, J_\sigma(y)^{-s/n+1/2}\, |x-y|^{-2s-n}\, dx\, dy \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} T_{-s} f(x)\, T_{-s} g(y)\, |x-y|^{-2s-n}\, dx\, dy \end{aligned} \tag{5.3}$$

where we integrate using the Riemannian volume element on the sphere. By putting $f = \bar{g}$ and taking square roots, we deduce that

$$\|T_{-s} f\|_{H^s(\mathbb{R}^n)} = \|f\|_{\tilde{H}^s(S^n)} ,$$

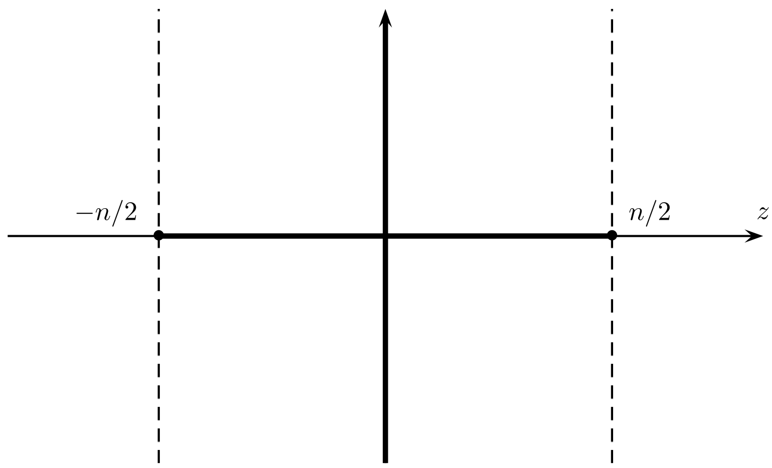
where the Hilbert space $\tilde{H}^s(S^n)$ is defined like $H^s(S^n)$, but with $(1 + k)^s$ replaced by a quotient of $\Gamma$ functions determined by the spherical harmonic decomposition of $d_{S^n}(\cdot, \cdot)$ (see, for instance, [67]). More precisely,

$$\begin{aligned} \|f\|_{H^s(S^n)} &= \Big\| \sum_k (1+k)^s f_k \Big\|_2 = \Big( \sum_k (1+k)^{2s} \|f_k\|_2^2 \Big)^{1/2} \\ \|f\|_{\tilde{H}^s(S^n)} &= C_s \Big( \sum_k \frac{\Gamma(k+s)}{\Gamma(k-s)} \|f_k\|_2^2 \Big)^{1/2}, \end{aligned}$$

where $C_s$ depends only on $n$ and $s$. Hence $\pi_{-s}$ acts unitarily on $\tilde{H}^s(S^n)$ when $s \in (-n/2, n/2)$, indicated by the heavy horizontal line in the diagram at the end of this section.

To show that $\pi_z$ acts uniformly boundedly on the Hilbert space $\tilde{H}^s(S^n)$ when $\mathrm{Re}(z) = -s$ and $s \in (-n/2, n/2)$, we use the fact that $T_{\mathrm{Re}\, z}$ is a unitary map from $\tilde{H}^s(S^n)$ to $H^s(\mathbb{R}^n)$. Now $T_z f = m_{i \,\mathrm{Im}\, z/n} T_{\mathrm{Re}\, z}$, where $m_{iy}$ denotes pointwise multiplication by the function $J_\sigma^{iy}$, so it suffices to show that the functions $m_{iy}$ multiply the spaces $H^s(\mathbb{R}^n)$ pointwise. This is a little tedious, but not hard (see [7]). Thus the region where the representations can be made uniformly bounded is between the dashed lines in the diagram at the end of this section.

It is easy to show that $\pi_z$, as defined in this section, is the same as $\pi^{1,\lambda}$, as defined in Lecture 1, where $\lambda = (2iz/n)\rho$.




Class-one Unitary Representations of $\mathrm{SO}_0(1, n+1)$

### 5.3 The Groups $\mathbf{SU}(1, n+1)$ and $\mathbf{Sp}(1, n+1)$

Denote by $\mathbb{F}$ either the complex numbers $\mathbb{C}$ or the quaternions $\mathbb{H}$; define the "$\mathbb{F}$-valued inner product" on the right vector space $\mathbb{F}^n$ by

$$x \cdot x' = \sum_{j=1}^{n} x'_j \bar{x}_j \qquad \forall x, x' \in \mathbb{F}^n.$$

We denote the projection of $\mathbb{F}$ onto the subspace of purely imaginary elements by $\Im$. The spheres $S^{2n+1}$ in $\mathbb{C}^n$ and $S^{4n+3}$ in $\mathbb{H}^{n+1}$ are Carnot–Carathéodory (generalised CR) manifolds. For $p$ in $S^{2n+1}$ or $S^{4n+3}$, we denote the subspace of the tangent space $T_p$ to the sphere at $p$ of vectors orthogonal to $p\mathbb{C}$ or $p\mathbb{H}$ by $U_p$, and endow $U_p$ with the restriction of the standard Riemannian metric. We denote either of these spheres, with its Carnot–Carathéodory structure (that is, a privileged nonintegrable subbundle of the tangent bundle, equipped with an inner product), by $S$.

A diffeomorphism $f\colon S \to S$ is said to be Carnot–Carathéodory contact if $f_*$ maps $U_p$ into $U_{f(p)}$ for all $p$, and Carnot–Carathéodory conformal if it is contact and in addition $f_*\big|_{U_p}$ is a multiple of an orthogonal map for all $p$. The groups $\mathrm{SU}(1, n+1)$ and $\mathrm{Sp}(1, n+1)$ may be identified with the conformal groups of $S^{2n+1}$ (in the complex case) and $S^{4n+3}$ (in the quaternionic case).

The analogue of the stereographic projection is the Cayley transform from a Heisenberg-type group $N$ to the sphere $S$. We define $N$ to be the set $\mathbb{F}^n \times \Im(\mathbb{F})$, equipped with the multiplication

$$(x,t)(x',t') = (x + x', t + t' + \frac{1}{2}\Im(x' \cdot x)) \qquad \forall (x,t),(x',t') \in \mathbb{F}^n \times \Im(\mathbb{F}).$$

The homogeneous dimension $Q$ of $N$ is defined to be $2n + 2$ in the complex case and $4n+6$ in the quaternionic case. The homogeneous dimension double-counts the dimension of the "missing directions". The Cayley transform is the map $\sigma \colon N \to S$, given by

$$\sigma(x,t) = \left(\left(1 + \frac{|x|^2}{4}\right)^2 + |z|^2\right)^{-1} \left(\left(1 + \frac{|x|^4}{4} - z\right)x, -1 + \frac{|x|^4}{16} + |z|^2 + 2z\right)$$

for all $(x,t)$ in $N$. It is a (nontrivial) exercise in calculus to show that $\sigma$ is Carnot–Carathéodory conformal when $N$ is given the left-invariant Carnot–Carathéodory structure which at the group identity $(0,0)$ is $\mathbb{F}^n$ (that is, $\{(x,0) : x \in \mathbb{F}^n\}$) with its standard inner product. The Jacobian of the transformation is given by the $Q^{\text{th}}$ power of the dilation factor, that is,

$$J_\sigma(x,t) = \left(\left(1 + \frac{|x|^2}{4}\right)^2 + |z|^2\right)^{-Q/2}.$$

The representations that we wish to investigate are given by the formula

$$\pi_z(g)f(x) = J_{g^{-1}}^{z/Q+1/2}(x) f(g^{-1}x).$$

We define the nonhomogeneous distance $d_S$ on $S$ by

$$d_S(p,q) = |1 - p \cdot q|,$$

and the nonhomogeneous distance $d_N$ on $N$ to be the left-invariant distance such that

$$d_N((0,0),(x,z)) = \left(\frac{|x|^4}{16} + |z|^2\right)^{1/4}.$$

Then, as proved geometrically in [7], the analogue of the remarkable formula (5.2) is

$$d_S(\sigma(x),\sigma(y)) = J_\sigma(x)^{1/2Q}\, J_\sigma(y)^{1/2Q}\, d_N(x,y).$$

We now define the map $T_z$ taking functions on $S$ to functions on $N$ much as before:

$$T_z f(x,t) = J_\sigma(x,t)^{z/Q+1/2} f(\sigma(x,t))$$

(compare with (5.1)), and a very similar calculation to (5.3) shows that

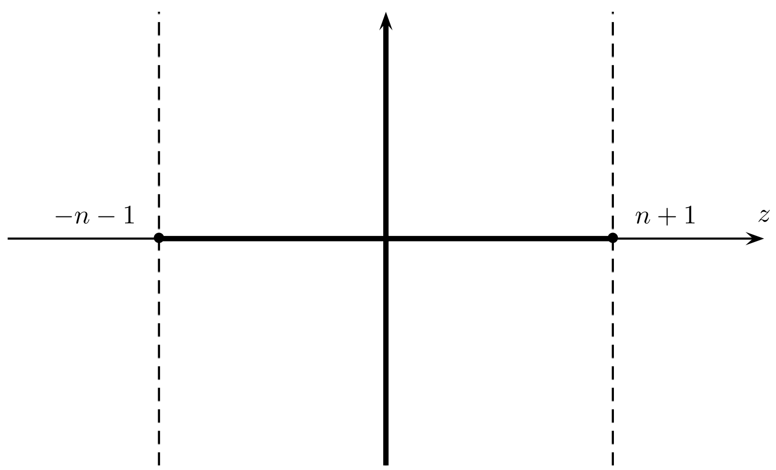
$$\|T_{-s}f\|_{\tilde{H}^2(N)} = \|f\|_{\tilde{H}^s(S)} ,$$

where $\tilde{H}^s(N)$ is the "Sobolev space" of functions $f$ such that

$$\Big(\int_N \int_N f(x)\,\bar{f}(y)\,d_N(x,y)^{-2s-Q}\,dx\,dy\Big)^{1/2} < \infty,$$

and $\tilde{H}^s(S)$ is defined similarly on $S$.

If $G = \mathrm{SU}(1, n+1)$, then $S = S^{2n+1}$ and $N$ is $\mathbb{C} \times (i\mathbb{R})$ (setwise). In this case, $\pi_z$ acts unitarily on $L^2(S)$ where $\mathrm{Re}(z) = 0$, giving us the unitary class-one principal series of representations (indicated by the heavy vertical line in the diagram below). Further, the two kernels $d_S^{-2s-Q}$ and $d_N^{-2s-Q}$ are positive definite and the "Sobolev spaces" are defined for all $s$ in $(-Q/2, Q/2)$ (the calculations may be found in [67] for $d_S$, and in [31] for $d_N$). This gives a construction of the class-one complementary series (indicated by the heavy horizontal line in the diagram below). Further, it can be shown that pointwise multiplication by purely imaginary powers of $J_\sigma$ is a bounded map on $\tilde{H}^s(N)$ for all $s$ in this range, and so when $\mathrm{Re}(z) \in (-Q/2, Q/2)$ the representations $\pi_z$ are uniformly bounded on $\tilde{H}^s(S)$, where $s = -\operatorname{Re} z$ (see [7]).




Class-one Unitary Representations of $\mathrm{SU}_0(1, n+1)$

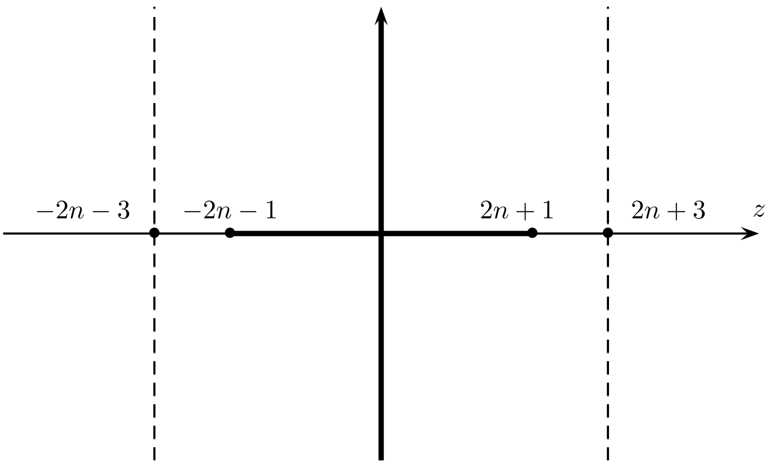
For $\mathrm{Sp}(1, n+1)$, the picture is different. The representations $\pi_z$ act by isometries on $L^p$ spaces when $-2n-3 \leq \mathrm{Re}(z) \leq 2n+3$, and we might hope that all the representations $\pi_x$ for $x$ in $(-2n-3, 2n+3)$ might be unitary on some Hilbert space, and all the representations $\pi_z$ for $z$ inside this strip might be made to act uniformly boundedly on some Hilbert space. When

$\mathrm{Re}(z) = 0$, we again obtain unitary representations on $L^2(S)$, the unitary class-one principal series, represented by the heavy vertical line in the diagram below.

However, $\tilde{H}^s(S)$ and $\tilde{H}^s(N)$ are only Hilbert spaces when the kernels $d_S(\cdot,\cdot)^{-2s-Q}$ and $d_N(\cdot,\cdot)^{-2s-Q}$ are positive (semi) definite, and this is only while

$$-2n - 1 \le s \le 2n + 1.$$

Nevertheless, at the expense of loosing the isometry of the representation it is possible to modify the spaces $\tilde{H}^s$, taking more standard Sobolev spaces $H^s(S)$ (spaces of functions with $s$ derivatives in $L^2(S)$—but only derivatives in the Carnot–Carathéodory directions) and show that the representations may still be made uniformly bounded. This is achieved in [7].




Class-one Unitary Representations of $\mathrm{Sp}_0(1, n+1)$

It is easy to show that $\pi_z$, as defined in this section, is the same as $\pi^{1,\lambda}$, as defined in Lecture 1, where $\lambda = (2iz/Q)\rho$; this applies for both the unitary and symplectic groups.

Cowling [30] showed that it is possible to make the representations inside the strip uniformly bounded, but this paper provides no control on the norms of the representations. Then Cowling and Haagerup [34] showed that it is possible to control the spherical functions associated to the representations $\pi_x$, where $-Q/2 < x < Q/2$. A.H. Dooley [45] has recently shown that, for the group $\mathrm{Sp}(1, n+1)$, it is possible to choose a "Sobolev type" norm on $N$ (this is called the "noncompact picture") so that $\|\pi_x\|^2 \le 2n - 1$ whenever $-Q/2 < x < Q/2$. This is the best possible constant. It has also been shown, working on the sphere (this is the “compact picture”), that the representations of $\mathrm{Sp}(1, n+1)$ have special properties which show that this group satisfies the Baum–Connes conjecture “with coefficients” (see [7, 68]). It would be nice to obtain a similar best possible result for the “picture changing” version, that is, to show that $\|T_s\| \|T_s^{-1}\| \leq (2n-1)^{1/2}$, as this would unify all the known results in this direction, and be best possible. This result should presumably be approached from a geometric point of view; more relevant information on the geometry of the spaces $N$ and $K/M$ is in the papers [7, 8]. There has also been much work on understanding these representations from a differential geometric point of view; see, for instance, [13], and other papers by these authors.

It may also be possible to extend some of the ideas here to more general semisimple groups: the Cayley transform treated here arises by composition of various natural mappings which appear when one considers the Iwasawa and Bruhat decompositions:

$$N \to NMA\bar{N}/MA\bar{N} \to G/MA\bar{N} = KA\bar{N}/MA\bar{N} \cong K/M.$$

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