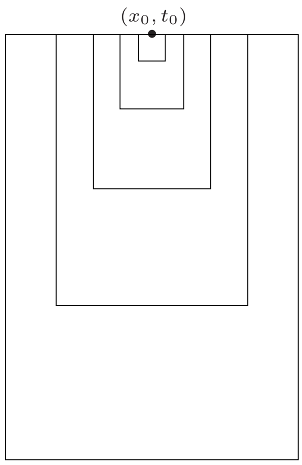
3

The Geometric Setting and an Alternative

We go back to equation

$$u_t - \operatorname{div} |\nabla u|^{p-2}\nabla u = 0 \tag{3.1}$$

and focus on the degenerate case $p > 2$. Results on the continuity of solutions at a point consist basically in constructing a sequence of nested and shrinking cylinders with vertex at that point, and in showing that the essential oscillation of the solution in those cylinders converges to zero as the cylinders shrink to the point.



This iterative procedure is based on energy and logarithmic estimates and works well with the standard parabolic cylinders if these estimates are homogeneous. The idea goes back to the work of De Giorgi, Moser and the Russian school (cf. [10], [42] and [37]), as explained in the introduction.

For degenerate or singular equations, the energy and logarithmic estimates are not homogeneous, as we have seen in the previous chapter. They involve integral norms corresponding to different powers, namely the powers 2 and p. To go about this difficulty, the equation has to be analyzed in a geometry dictated by its own degenerate structure. This amounts to rescale the standard parabolic cylinders by a factor that depends on the oscillation of the solution. This procedure of *intrinsic scaling*, which can be seen as an accommodation of the degeneracy, allows for the restoration of the homogeneity in the energy estimates, when written over the rescaled cylinders. We can say heuristically that the equation behaves in its own geometry like the heat equation. Let us make this idea precise.

3.1 A Geometry for the Equation

The standard parabolic cylinders

$$(x_0, t_0) + Q(R^2, R)$$

reflect the natural homogeneity between the space and time variables for the heat equation. Indeed, if $u(x,t)$ is a solution, then $u(\varepsilon x, \varepsilon^2 t)$, $\varepsilon \in \mathbb{R}$, is also a solution, *i.e.*, the equation remains invariant through a similarity transformation of the space-time variables that leaves constant the ratio $|x|^2/t$.

When dealing with the degenerate PDE (3.1), one might think, at first sight, that the adequate cylinders to perform the iterative method described above were cylinders of the form $Q(R^p, R)$, that correspond to the similarity scaling $|x|^p/t$ of the equation. But a more careful analysis shows that this is not to be expected. Indeed, it would work for the homogeneous equation

$$(u^{p-1})_t - \operatorname{div} |\nabla u|^{p-2} \nabla u = 0$$

but not for the inhomogeneous equation (3.1). By analogy, and in order to gain some hindsight on how to proceed, we recast (3.1) in the form

$$\left(\frac{u}{c}\right)^{2-p} (u^{p-1})_t - \operatorname{div} |\nabla u|^{p-2} \nabla u = 0,$$

for an appropriate constant c. This shows that the homogeneity can be recovered at the expense of a scaling factor, that depends on the solution itself and, modulo a constant, looks like u^{2-p}. The following is a sophisticated and rigorous way of implementing this heuristic reasoning.

Consider $0 < R < 1$, sufficiently small so that $Q(R^2, R) \subset \Omega_T$, and define the essential oscillation of the solution u within this cylinder

$$\omega := \operatorname*{ess\,osc}_{Q(R^2,R)} u = \mu^+ - \mu^-,$$

where

$$\mu^+ := \operatorname*{ess\,sup}_{Q(R^2,R)} u \qquad \text{and} \qquad \mu^- := \operatorname*{ess\,inf}_{Q(R^2,R)} u.$$

Then construct the rescaled cylinder

$$Q(a_0R^p, R) = K_R(0) \times (-a_0R^p, 0)\,, \quad \text{with} \quad a_0 = \left(\frac{\omega}{2^\lambda}\right)^{2-p}, \tag{3.2}$$

where $\lambda > 1$ is to be fixed later depending only on the data (see (4.15)). Note that for $p = 2$, *i.e.*, in the non-degenerate case, these are the standard parabolic cylinders reflecting the natural homogeneity between the space and time variables.

We will assume, without loss of generality, that

$$R < \frac{\omega}{2^\lambda}. \tag{3.3}$$

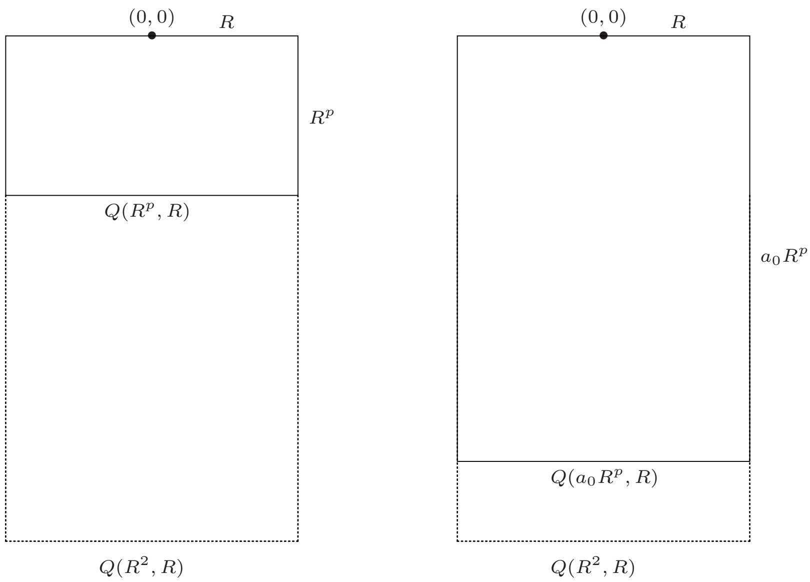
Indeed, if this does not hold, we have $\omega \le 2^\lambda R$ and there is nothing to prove since the oscillation is then comparable to the radius. Now, (3.3) implies the inclusion

$$Q(a_0R^p, R) \subset Q(R^2, R)$$

and the relation

$$\operatorname*{ess\,osc}_{Q(a_0R^p,R)} u \le \omega \tag{3.4}$$

which will be the starting point of an iteration process that leads to the main results. The schematics below give an idea of the stretching procedure, commonly referred to as *accommodation of the degeneracy* (the pictures are distorted on purpose in the t-direction).



Note that we had to consider the cylinder $Q(R^2, R)$ and assume (3.3), so that (3.4) would hold for the rescaled cylinder $Q(a_0 R^p, R)$. This is in general not true for a given cylinder, since its dimensions would have to be intrinsically defined in terms of the essential oscillation of the function within it.

We now consider subcylinders of $Q(a_0 R^p, R)$ of the form

$$(0, t^*) + Q(\theta R^p, R) \ , \quad \text{with} \quad \theta = \left(\frac{\omega}{2}\right)^{2-p} \tag{3.5}$$

that are contained in $Q(a_0 R^p, R)$ provided

$$\left(2^{p-2} - 2^{\lambda(p-2)}\right) \frac{R^p}{\omega^{p-2}} < t^* < 0. \tag{3.6}$$

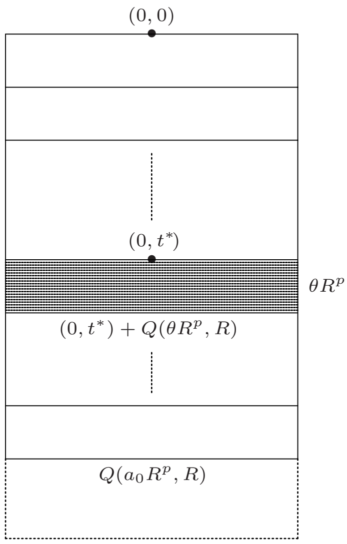
Once λ is chosen, we may redefine it, putting

$$\lambda^* = \frac{[p-1]}{p-2}\,[\lambda] + 1 > \lambda,$$

and assume that

$$N_0 = \frac{a_0}{\theta} = \left(\frac{\frac{\omega}{2^\lambda}}{\frac{\omega}{2}}\right)^{2-p} = 2^{\,(\lambda-1)(p-2)} \tag{3.7}$$

is an integer. Thus, we consider $Q(a_0 R^p, R)$ as being divided in subcylinders, all alike and congruent with $Q(\theta R^p, R)$:



The proof of the Hölder continuity of a weak solution u now follows from the analysis of two complementary cases. We briefly describe them in the following way: in the first case we assume that there is a cylinder of the type $(0,t^*)+Q(\theta R^p,R)$ where u is essentially away from its infimum. We show that going down to a smaller cylinder the oscillation decreases by a small factor that we can exhibit. If that cylinder can not be found then u is essentially away from its supremum in all cylinders of that type and we can compound this information to reach the same conclusion as in the previous case. We state this in a precise way.

For a constant $\nu_0 \in (0,1)$, that will be determined depending only on the data, either

The First Alternative:

there is a cylinder of the type $(0,t^*)+Q(\theta R^p,R)$ for which

$$\frac{\left|\left\{(x,t)\in(0,t^*)+Q(\theta R^p,R)\ :\ u(x,t)<\mu^-+\frac{\omega}{2}\right\}\right|}{|Q(\theta R^p,R)|}\le\nu_0 \tag{3.8}$$

or this does not hold. Then, since $\mu^+-\frac{\omega}{2}=\mu^-+\frac{\omega}{2}$, it holds

The Second Alternative:

for every cylinder of the type $(0,t^*)+Q(\theta R^p,R)$

$$\frac{\left|\left\{(x,t)\in(0,t^*)+Q(\theta R^p,R)\ :\ u(x,t)>\mu^+-\frac{\omega}{2}\right\}\right|}{|Q(\theta R^p,R)|}<1-\nu_0. \tag{3.9}$$

3.2 The First Alternative

We start the analysis assuming the first alternative holds.

Lemma 3.1. *Assume (3.3) is in force. There exists a constant $\nu_0\in(0,1)$, depending only on the data, such that if (3.8) holds for some t^* as in (3.6) then*

$$u(x,t)>\mu^-+\frac{\omega}{4}\,,\quad\text{a.e. in }\ (0,t^*)+Q\left(\theta\left(\frac{R}{2}\right)^p,\frac{R}{2}\right).$$

Proof. Take the cylinder for which (3.8) holds and assume, by translation, that $t^*=0$. Let

$$R_n=\frac{R}{2}+\frac{R}{2^{n+1}}\,,\qquad n=0,1,\dots,$$

and construct the family of nested and shrinking cylinders $Q(\theta R_n^p,R_n)$. Consider piecewise smooth cutoff functions $0\le\zeta_n\le1$, defined in these cylinders, and satisfying the following set of assumptions:

$$\zeta_n = 1 \quad \text{in} \quad Q\left(\theta R_{n+1}^p, R_{n+1}\right); \qquad \zeta_n = 0 \quad \text{on} \quad \partial_p Q\left(\theta R_n^p, R_n\right);$$

$$|\nabla \zeta_n| \le \frac{2^{n+1}}{R}; \qquad 0 \le (\zeta_n)_t \le \frac{2^{p(n+1)}}{\theta R^p}.$$

Observe that the family of cylinders starts with $Q(\theta R^p, R)$ and converges to $Q\left(\theta\left(\frac{R}{2}\right)^p, \frac{R}{2}\right)$ and that the bounds on the gradient and the time derivative of ζ_n are strictly related to the dimensions of the cylinders.

Write the energy inequality (2.6) over the cylinders $Q\left(\theta R_n^p, R_n\right)$, for the functions $(u-k_n)_-$, with

$$k_n = \mu^- + \frac{\omega}{4} + \frac{\omega}{2^{n+2}}, \quad n = 0, 1, \ldots,$$

and $\zeta = \zeta_n$. They read, taking into account that ζ_n vanishes on $\partial_p Q\left(\theta R_n^p, R_n\right)$,

$$\begin{aligned}
&\sup_{-\theta R_n^p < t < 0} \int_{K_{R_n}\times\{t\}} (u-k_n)_-^2 \zeta_n^p \, dx + \int_{-\theta R_n^p}^0 \int_{K_{R_n}} |\nabla (u-k_n)_- \zeta_n|^p \, dx\, dt \\
&\le C \int_{-\theta R_n^p}^0 \int_{K_{R_n}} (u-k_n)_-^p |\nabla \zeta_n|^p \, dx\, dt + p \int_{-\theta R_n^p}^0 \int_{K_{R_n}} (u-k_n)_-^2 \zeta_n^{p-1} (\zeta_n)_t \, dx\, dt \\
&\le C \, \frac{2^{p(n+1)}}{R^p} \left\{ \int_{-\theta R_n^p}^0 \int_{K_{R_n}} (u-k_n)_-^p \, dx\, dt + \frac{1}{\theta} \int_{-\theta R_n^p}^0 \int_{K_{R_n}} (u-k_n)_-^2 \, dx\, dt \right\}.
\end{aligned}$$

Next, observe that either $(u-k_n)_- = 0$ or

$$(u-k_n)_- = (\mu^- - u) + \frac{\omega}{4} + \frac{\omega}{2^{n+2}} \le \frac{\omega}{2},$$

and thus, since $2-p<0$,

$$\begin{aligned}
(u-k_n)_-^2 &= (u-k_n)_-^{2-p} (u-k_n)_-^p \\
&\ge \left(\frac{\omega}{2}\right)^{2-p} (u-k_n)_-^p \\
&= \theta\, (u-k_n)_-^p,
\end{aligned}$$

recalling that $\theta = \left(\frac{\omega}{2}\right)^{2-p}$. We obtain, homogenizing the powers in the integral norms,

$$\begin{aligned}
&\theta \sup_{-\theta R_n^p < t < 0} \int_{K_{R_n}\times\{t\}} (u-k_n)_-^p \zeta_n^p \, dx + \int_{-\theta R_n^p}^0 \int_{K_{R_n}} |\nabla (u-k_n)_- \zeta_n|^p \, dx\, dt \\
&\quad \le C \, \frac{2^{p(n+1)}}{R^p} \left\{ \left(\frac{\omega}{2}\right)^p + \frac{1}{\theta}\left(\frac{\omega}{2}\right)^2 \right\} \int_{-\theta R_n^p}^0 \int_{K_{R_n}} \chi_{\{(u-k_n)_- > 0\}} \, dx\, dt,
\end{aligned}$$

where χ_E denotes the characteristic function of the set E. Finally, divide throughout by θ to get

$$\sup_{-\theta R_n^p < t < 0} \int_{K_{R_n}\times\{t\}} (u-k_n)_-^p \zeta_n^p \, dx + \frac{1}{\theta}\int_{-\theta R_n^p}^0 \int_{K_{R_n}} |\nabla (u-k_n)_- \zeta_n|^p \, dx\, dt$$
$$\leq C\, \frac{2^{p(n+1)}}{R^p} \left(\frac{\omega}{2}\right)^p \frac{1}{\theta}\int_{-\theta R_n^p}^0 \int_{K_{R_n}} \chi_{\{(u-k_n)_->0\}} \, dx\, dt. \tag{3.10}$$

The next step, in which the intrinsic geometric framework is crucial, is to perform a change in the time variable, putting $\bar{t} = t/\theta$, and to define

$$\overline{u}(\cdot,\bar{t}) := u(\cdot,t) \ , \qquad \overline{\zeta_n}(\cdot,\bar{t}) := \zeta_n(\cdot,t).$$

We obtain the simplified inequality

$$\left\|(\overline{u}-k_n)_- \ \overline{\zeta_n}\right\|^p_{V^p(Q(R_n^p,R_n))} \leq C\frac{2^{pn}}{R^p}\left(\frac{\omega}{2}\right)^p \int_{-R_n^p}^0\int_{K_{R_n}} \chi_{\{(\overline{u}-k_n)_->0\}}\, dx\, d\bar{t}, \tag{3.11}$$

which reveals the appropriate functional framework.

To conclude, define, for each n,

$$A_n = \int_{-R_n^p}^0 \int_{K_{R_n}} \chi_{\{(\overline{u}-k_n)_->0\}} \ dx \ d\bar{t}$$

and observe that

$$\begin{aligned}
\frac{1}{2^{p(n+2)}} \left(\frac{\omega}{2}\right)^p A_{n+1} &= |k_n - k_{n+1}|^p \ A_{n+1}\\
&\leq \|(\overline{u}-k_n)_-\|^p_{p,Q(R^p_{n+1},R_{n+1})}\\
&\leq \left\|(\overline{u}-k_n)_-\overline{\zeta_n}\right\|^p_{p,Q(R_n^p,R_n)}\\
&\leq C\left\|(\overline{u}-k_n)_- \ \overline{\zeta_n}\right\|^p_{V^p(Q(R_n^p,R_n))} A_n^{\frac{p}{d+p}}\\
&\leq C\frac{2^{pn}}{R^p}\left(\frac{\omega}{2}\right)^p A_n^{1+\frac{p}{d+p}}.
\end{aligned} \tag{3.12}$$

[The first two inequalities follow from the definition of A_n and the fact that $k_{n+1} < k_n$; the third inequality is a consequence of Theorem 2.11 and the last one follows from (3.11).] Next, define the numbers

$$X_n = \frac{A_n}{|Q(R_n^p, R_n)|},$$

divide (3.12) by $\left|Q(R^p_{n+1},R_{n+1})\right|$ and obtain the recursive relation

$$X_{n+1} \leq C\, 4^{pn} \ X_n^{1+\frac{p}{d+p}},$$

for a constant C depending only upon d and p. By Lemma 2.9 on fast geometric convergence, if

$$X_0 \leq C^{-\frac{d+p}{p}} 4^{-\frac{(d+p)^2}{p}} =: \nu_0 \tag{3.13}$$

then

$$X_n \longrightarrow 0. \tag{3.14}$$

But (3.13) is precisely our hypothesis (3.8), for the indicated choice of ν_0, and from (3.14) we immediately obtain, returning to the original variables,

$$\left|\left\{(x,t) \in Q\left(\theta(\tfrac{R}{2})^p, \tfrac{R}{2}\right) \ : \ u(x,t) \le \mu^- + \frac{\omega}{4}\right\}\right| = 0.$$

□

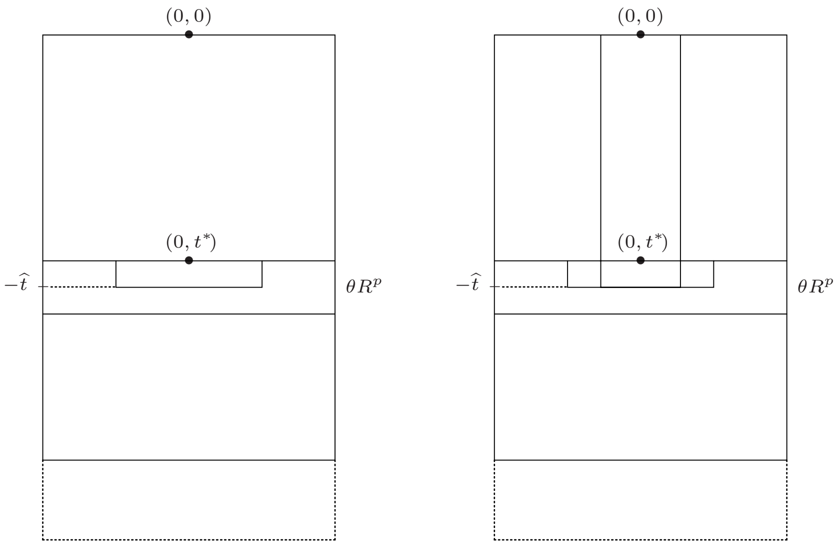
Remark 3.2. The constant ν_0, that appears in the formulation of the alternative, is now fixed by (3.13). Note that indeed $\nu_0 \in (0,1)$.

3.3 The Role of the Logarithmic Estimates: Expansion in Time

Our next aim is to show that the conclusion of Lemma 3.1 holds in a full cylinder $Q(\tau, \rho)$. The idea is to use the fact that at the time level

$$-\widehat{t} := t^* - \theta\left(\tfrac{R}{2}\right)^p \tag{3.15}$$

the function $u(x, -\widehat{t}\,)$ is strictly above the level $\mu^- + \frac{\omega}{4}$ in the cube $K_{\frac{R}{2}}$, and look at this time level as an initial condition to make the conclusion hold up to $t = 0$ in a smaller cylinder, as sketched in the following diagram:



As an intermediate step we need the following lemma, in which the use of the logarithmic estimates is crucial.

Lemma 3.3. *Assume (3.8) holds for some t^* as in (3.6) and that (3.3) is in force. Given $\nu_* \in (0,1)$, there exists $s_* \in \mathbb{N}$, depending only on the data, such that*

$$\left|\left\{x \in K_{\frac{R}{4}} \ :\ u(x,t) < \mu^- + \frac{\omega}{2^{s_*}}\right\}\right| \leq \nu_* \left|K_{\frac{R}{4}}\right| \ , \quad \forall t \in (-\widehat{t}, 0).$$

Proof. We use the logarithmic estimate (2.8) applied to the function $(u-k)_-$ in the cylinder $Q(\widehat{t}, \frac{R}{2})$, with the choices

$$k = \mu^- + \frac{\omega}{4} \qquad \text{and} \qquad c = \frac{\omega}{2^{n+2}},$$

where $n \in \mathbb{N}$ will be chosen later. In this cylinder, we have

$$k - u \leq H^-_{u,k} = \operatorname*{ess\,sup}_{Q(\widehat{t}, \frac{R}{2})} \left|\left(u - \mu^- - \frac{\omega}{4}\right)_-\right| \leq \frac{\omega}{4}. \tag{3.16}$$

If $H^-_{u,k} \leq \frac{\omega}{8}$, the result is trivial for the choice $s^* = 3$. Assuming $H^-_{u,k} > \frac{\omega}{8}$, recall from section 2.3 that the logarithmic function $\psi^-(u)$ is defined in the whole of $Q(\widehat{t}, \frac{R}{2})$ and it is given by

$$\psi^-_{\{H^-_{u,k}, k, \frac{\omega}{2^{n+2}}\}}(u) = \begin{cases} \ln\left\{\dfrac{H^-_{u,k}}{H^-_{u,k} + u - k + \frac{\omega}{2^{n+2}}}\right\} & \text{if } u < k - \frac{\omega}{2^{n+2}} \\ 0 & \text{if } u \geq k - \frac{\omega}{2^{n+2}}. \end{cases}$$

From (3.16), we estimate

$$\psi^-(u) \leq n \ln 2 \qquad \text{since} \qquad \frac{H^-_{u,k}}{H^-_{u,k} + u - k + \frac{\omega}{2^{n+2}}} \leq \frac{\frac{\omega}{4}}{\frac{\omega}{2^{n+2}}} = 2^n \tag{3.17}$$

and

$$\left|\left(\psi^-\right)'(u)\right|^{2-p} = \left(H^-_{u,k} + u - k + c\right)^{p-2} \leq \left(\frac{\omega}{2}\right)^{p-2}. \tag{3.18}$$

Now observe that, as a consequence of Lemma 3.1, we have $u(x, -\widehat{t}\,) > k$ in the cube $K_{\frac{R}{2}}$, which implies that

$$\left[\psi^-(u)\right](x, -\widehat{t}) = 0 \ , \quad x \in K_{\frac{R}{2}}.$$

Choosing a piecewise smooth cutoff function $0 < \zeta(x) \leq 1$, defined in $K_{\frac{R}{2}}$ and such that

$$\zeta = 1 \quad \text{in } K_{\frac{R}{4}} \qquad \text{and} \qquad |\nabla \zeta| \leq \frac{8}{R},$$

inequality (2.8) reads

$$\sup_{-\hat{t}<t<0} \int_{K_{\frac{R}{2}}\times\{t\}} \big[\psi^-(u)\big]^2 \zeta^p \, dx \\ \le C \int_{-\hat{t}}^{0} \int_{K_{\frac{R}{2}}} \psi^-(u) \, \Big|(\psi^-)'(u)\Big|^{2-p} |\nabla\zeta|^p \; dx \, dt. \qquad (3.19)$$

The right hand side is estimated above, using (3.17) and (3.18), by

$$C\, n(\ln 2) \left(\frac{\omega}{2}\right)^{p-2} \left(\frac{8}{R}\right)^p \hat{t} \, \Big|K_{\frac{R}{2}}\Big| \le C\, n\, 2^{\lambda(p-2)} \, \Big|K_{\frac{R}{4}}\Big|,$$

since, by (3.15),

$$\hat{t} \le a_0 R^p = \left(\frac{\omega}{2^\lambda}\right)^{2-p} R^p.$$

We estimate below the left hand side of (3.19) by integrating over the smaller set

$$S = \left\{ x \in K_{\frac{R}{4}} \; : \; u(x,t) < \mu^- + \frac{\omega}{2^{n+2}} \right\} \subset K_{\frac{R}{2}},$$

and observing that in S, $\zeta = 1$ and

$$\frac{H^-_{u,k}}{H^-_{u,k} + u - k + \frac{\omega}{2^{n+2}}}$$

is a decreasing function of $H^-_{u,k}$ because $u - k + \frac{\omega}{2^{n+2}} < 0$. Thus, from (3.16),

$$\frac{H^-_{u,k}}{H^-_{u,k} + u - k + \frac{\omega}{2^{n+2}}} \ge \frac{\frac{\omega}{4}}{\frac{\omega}{4} + u - k + \frac{\omega}{2^{n+2}}} = \frac{\frac{\omega}{4}}{u - \mu^- + \frac{\omega}{2^{n+2}}} > \frac{\frac{\omega}{4}}{\frac{\omega}{2^{n+1}}} = 2^{n-1}$$

since $u - \mu^- < \dfrac{\omega}{2^{n+2}}$ in S. Therefore,

$$\big[\psi^-(u)\big]^2 \ge \big[\ln\big(2^{n-1}\big)\big]^2 = (n-1)^2 (\ln 2)^2 \quad \text{in } S.$$

Combining these estimates in (3.19), we get

$$\left|\left\{ x \in K_{\frac{R}{4}} \; : \; u(x,t) < \mu^- + \frac{\omega}{2^{n+2}} \right\}\right| \le C \, \frac{n}{(n-1)^2} \, 2^{\lambda(p-2)} \, \Big|K_{\frac{R}{4}}\Big|,$$

for all $t \in (-\hat{t}, 0)$, and to prove the lemma we choose

$$s_* = n + 2 \qquad \text{with} \qquad n > 1 + \frac{2C}{\nu_*} \, 2^{\lambda(p-2)}.$$

□

3.4 Reduction of the Oscillation

We now state the main result in the context of the first alternative.

Proposition 3.4. *Assume (3.8) holds for some t^* as in (3.6) and that (3.3) is in force. There exists $s_1 \in \mathbb{N}$, depending only on the data, such that*

$$u(x,t) > \mu^- + \frac{\omega}{2^{s_1+1}} , \quad \text{a.e. in } Q\left(\widehat{t}, \frac{R}{8}\right).$$

Proof. Consider the cylinder for which (3.8) holds, let

$$R_n = \frac{R}{8} + \frac{R}{2^{n+3}} , \quad n = 0, 1, \ldots$$

and construct the family of nested and shrinking cylinders $Q(\widehat{t}, R_n)$, where $\widehat{t}$ is given by (3.15). Take piecewise smooth cutoff functions $0 < \zeta_n(x) \le 1$, independent of t, defined in K_{R_n} and satisfying

$$\zeta_n = 1 \quad \text{in } K_{R_{n+1}} ; \qquad |\nabla \zeta_n| \le \frac{2^{n+4}}{R}.$$

Write the local energy inequalities (2.6) for the functions $(u-k_n)_-$, in the cylinders $Q\left(\widehat{t}, R_n\right)$, with

$$k_n = \mu^- + \frac{\omega}{2^{s_1+1}} + \frac{\omega}{2^{s_1+1+n}} , \quad n = 0, 1, \ldots,$$

s_1 to be chosen, and $\zeta = \zeta_n$. Observing that, due to Lemma 3.1, we have

$$u(x, -\widehat{t}) > \mu^- + \frac{\omega}{4} \ge k_n \quad \text{in } K_{\frac{R}{2}} \supset K_{R_n},$$

which implies that

$$(u-k_n)_-(x, -\widehat{t}) = 0 \quad \text{in } K_{R_n}, \quad n = 0, 1, \ldots,$$

the estimates read

$$\begin{aligned} \sup_{-\hat{t}<t<0} \int_{K_{R_n}\times\{t\}} (u-k_n)_-^2 \zeta_n^p \, dx &+ \int_{-\hat{t}}^0 \int_{K_{R_n}} |\nabla (u-k_n)_- \zeta_n|^p \, dx\, dt \\ &\le C \int_{-\hat{t}}^0 \int_{K_{R_n}} (u-k_n)_-^p |\nabla \zeta_n|^p \, dx\, dt \\ &\le C\, \frac{2^{p(n+4)}}{R^p} \int_{-\hat{t}}^0 \int_{K_{R_n}} (u-k_n)_-^p \, dx\, dt. \end{aligned} \tag{3.20}$$

From (3.15), we estimate

$$\widehat{t} \le a_0 R^p = \left(\frac{\omega}{2^\lambda}\right)^{2-p} R^p,$$

where a_0 is defined in (3.2). From this,

$$\begin{aligned}(u-k_n)_-^2 &\geq \left(\frac{\omega}{2^{s_1}}\right)^{2-p} (u-k_n)_-^p \\ &\geq \left(\frac{2^{s_1}}{2^{\lambda}}\right)^{p-2} \frac{\widehat{t}}{R^p} (u-k_n)_-^p \\ &\geq \frac{\widehat{t}}{\left(\frac{R}{2}\right)^p} (u-k_n)_-^p ,\end{aligned}$$

provided $s_1 > \lambda + \frac{p}{p-2}$. Dividing now by $\dfrac{\widehat{t}}{(\frac{R}{2})^p}$ throughout (3.20) gives

$$\begin{aligned}\sup_{-\widehat{t}<t<0} \int_{K_{R_n}\times\{t\}} (u-k_n)_-^p \zeta_n^p \, dx + \frac{\left(\frac{R}{2}\right)^p}{\widehat{t}} \int_{-\widehat{t}}^0 \int_{K_{R_n}} |\nabla(u-k_n)_-\zeta_n|^p \, dx\, dt \\ \leq C \, \frac{2^{pn}}{\widehat{t}} \int_{-\widehat{t}}^0 \int_{K_{R_n}} (u-k_n)_-^p \, dx\, dt.\end{aligned}$$

The change of the time variable $\bar{t} = t\frac{(\frac{R}{2})^p}{\widehat{t}}$, along with defining the new function

$$\overline{u}(\cdot,\bar{t}) := u(\cdot,t),$$

leads to the simplified inequality

$$\|(\overline{u}-k_n)_- \, \zeta_n\|^p_{V^p\left(Q\left((\frac{R}{2})^p,R_n\right)\right)} \leq C \, \frac{2^{pn}}{(\frac{R}{2})^p} \left(\frac{\omega}{2^{s_1}}\right)^p \int_{-(\frac{R}{2})^p}^0 \int_{K_{R_n}} \chi_{\{\overline{u}<k_n\}} \, dx\, d\bar{t}.$$

Define, for each n,

$$A_n = \int_{-(\frac{R}{2})^p}^0 \int_{K_{R_n}} \chi_{\{(\overline{u}-k_n)_->0\}} \, dx\, d\bar{t}.$$

By a reasoning similar to the one leading to (3.12), we obtain

$$\begin{aligned}\frac{1}{2^{p(n+2)}} \left(\frac{\omega}{2^{s_1}}\right)^p A_{n+1} &= |k_n-k_{n+1}|^p \; A_{n+1} \\ &\leq \|(\overline{u}-k_n)_-\|^p_{p,Q\left((\frac{R}{2})^p,R_{n+1}\right)} \\ &\leq \|(\overline{u}-k_n)_-\zeta_n\|^p_{p,Q\left((\frac{R}{2})^p,R_n\right)} \\ &\leq C \; \|(\overline{u}-k_n)_- \, \zeta_n\|^p_{V^p\left(Q\left((\frac{R}{2})^p,R_n\right)\right)} \, A_n^{\frac{p}{d+p}} \\ &\leq C \, \frac{2^{pn}}{(\frac{R}{2})^p} \left(\frac{\omega}{2^{s_1}}\right)^p A_n^{1+\frac{p}{d+p}} .\end{aligned}$$

Next, define the numbers

$$X_n = \frac{A_n}{\left|Q\left((\frac{R}{2})^p, R_n\right)\right|},$$

and divide the previous inequality by $\left|Q\left((\frac{R}{2})^p, R_{n+1}\right)\right|$ to obtain the recursive relations

$$X_{n+1} \leq C\, 4^{pn}\; X_n^{1+\frac{p}{d+p}}.$$

By Lemma 2.9 on fast geometric convergence, if

$$X_0 \leq C^{-\frac{d+p}{p}} 4^{-\frac{(d+p)^2}{p}} \quad =: \nu_* \in (0,1) \tag{3.21}$$

then

$$X_n \longrightarrow 0. \tag{3.22}$$

Apply Lemma 3.3 with such a ν_* and conclude that there exists $s_* =: s_1$, depending only on the data, such that

$$\left|\left\{x \in K_{\frac{R}{4}} \;:\; u(x,t) < \mu^- + \frac{\omega}{2^{s_1}}\right\}\right| \leq \nu_* \left|K_{\frac{R}{4}}\right| \;, \quad \forall t \in (-\widehat{t}, 0),$$

which is exactly (3.21). Since (3.22) implies that $A_n \to 0$, we conclude that

$$\begin{aligned} &\left|\left\{(x,\bar{t}) \in Q\left(\left(\tfrac{R}{2}\right)^p, \frac{R}{8}\right) \;:\; \overline{u}(x,\bar{t}) \leq \mu^- + \frac{\omega}{2^{s_1+1}}\right\}\right| \\ &\quad = \left|\left\{(x,t) \in Q\left(\widehat{t}, \frac{R}{8}\right) \;:\; u(x,t) \leq \mu^- + \frac{\omega}{2^{s_1+1}}\right\}\right| = 0. \end{aligned}$$

□

We finally reach the conclusion of the first alternative, namely the reduction of the oscillation.

Corollary 3.5. *Assume (3.8) holds for some t^* as in (3.6) and that (3.3) is in force. There exists a constant $\sigma_0 \in (0,1)$, depending only on the data, such that*

$$\operatorname*{ess\,osc}_{Q\left(\theta(\frac{R}{8})^p, \frac{R}{8}\right)} u \;\leq\; \sigma_0\, \omega. \tag{3.23}$$

Proof. By Proposition 3.4, there exists $s_1 \in \mathbb{N}$ such that

$$\operatorname*{ess\,inf}_{Q(\widehat{t}, \frac{R}{8})} u \geq \mu^- + \frac{\omega}{2^{s_1+1}}$$

and thus

$$\begin{aligned} \operatorname*{ess\,osc}_{Q(\widehat{t},\frac{R}{8})} u &= \operatorname*{ess\,sup}_{Q(\widehat{t},\frac{R}{8})} u - \operatorname*{ess\,inf}_{Q(\widehat{t},\frac{R}{8})} u \\ &\le \mu^+ - \mu^- - \frac{\omega}{2^{s_1+1}} \\ &= \left(1 - \frac{1}{2^{s_1+1}}\right)\omega. \end{aligned}$$

Since $\theta\left(\frac{R}{8}\right)^p \le \widehat{t} = -t^* + \theta\left(\frac{R}{2}\right)^p$, with $t^* < 0$, we have

$$Q\left(\theta\left(\frac{R}{8}\right)^p, \frac{R}{8}\right) \subset Q\left(\widehat{t}, \frac{R}{8}\right),$$

and the corollary follows with $\sigma_0 = \left(1 - \frac{1}{2^{s_1+1}}\right)$. □