Geometrical Aspects of Symmetrization

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1 Sets of finite perimeter

Symmetrization is one of the most powerful mathematical tools with several applications both in Analysis and Geometry. Probably the most remarkable application of Steiner symmetrization of sets is the De Giorgi proof (see [14], [25]) of the isoperimetric property of the sphere, while the spherical symmetrization of functions has several applications to PDEs and Calculus of Variations and to integral inequalities of Poincaré and Sobolev type (see for instance [23], [24], [19], [20]).

The two model functionals that we shall consider in the sequel are: the perimeter of a set E in $\mathbb{R}^n$ and the Dirichlet integral of a scalar function u. It is well known that on replacing E or u by its Steiner symmetral or its spherical symmetrization, respectively, both these quantities decrease. This fact is classical when E is a smooth open set and u is a C^1 function ([22], [21]). Moreover, on approximating a set of finite perimeter with smooth open sets or a Sobolev function by C^1 functions, these inequalities can be easily extended by lower semicontinuity to the general setting ([19], [25], [2], [4]). However, an approximation argument gives no information about the equality case. Thus, if one is interested in understanding when equality occurs, one has to carry on a deeper analysis, based on fine properties of sets of finite perimeter and Sobolev functions.

Let us start by recalling what the Steiner symmetrization of a measurable set E is. For simplicity, and without loss of generality, in the sequel we shall always consider the symmetrization of E in the vertical direction. To this aim, it is convenient to denote the points x in $\mathbb{R}^n$ also by (x', y), where $x' \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}$. Thus, given $x' \in \mathbb{R}^{n-1}$, we shall denote by $E_{x'}$ the corresponding one-dimensional section of E

$$E_{x'} = \{y \in \mathbb{R} : (x', y) \in E\} .$$

The *distribution function* μ of E is defined by setting for all $x' \in \mathbb{R}^{n-1}$

$$\mu(x') = \mathcal{L}^1(E_{x'}) .$$

Here and in the sequel we denote by $\mathcal{L}^k$ the Lebesgue measure in $\mathbb{R}^k$. Then, denoting the *essential projection* of E by $\pi(E)^+ = \{x' \in \mathbb{R}^{n-1} : \mu(x') > 0\}$, the *Steiner symmetral* of E with respect to the hyperplane $\{y = 0\}$ is the set

$$E^s = \{(x', y) : x' \in \pi(E)^+, |y| < \mu(x')/2\} .$$

Notice that by Fubini's theorem we get immediately that μ is a $\mathcal{L}^{n-1}$-measurable function in $\mathbb{R}^{n-1}$, hence E^s is a measurable set in $\mathbb{R}^n$ and $\mathcal{L}^n(E) = \mathcal{L}^n(E^s)$. Moreover, it is not hard to see that the diameter of E decreases under Steiner symmetrization, i.e., $\mathrm{diam}(E^s) \le \mathrm{diam}(E)$, an inequality which in turn implies ([1, Proposition 2.52]) the well known *isodiametric inequality*

$$\mathcal{L}^n(E) \le \omega_n \left(\frac{\mathrm{diam}(E)}{2}\right)^n ,$$

where ω_n denotes the measure of the unit ball in $\mathbb{R}^n$.

Denoting by $P(E)$ the *perimeter* of a measurable set in $\mathbb{R}^n$, the following result states that the perimeter too decreases under Steiner symmetrization.

Theorem 1.1. *Let $E \subset \mathbb{R}^n$ be a measurable set. Then,*

$$P(E^s) \le P(E) . \tag{1.1}$$

As we said before, inequality (1.1) is classic when E is a smooth set and can be proved by a simple approximation argument in the general case of a set of finite perimeter. However, following [9], we shall give here a different proof of Theorem 1.1, which has the advantage of providing valuable information in the case when (1.1) reduces to an equality.

Let us now recall the definition of perimeter. If E is a measurable set in $\mathbb{R}^n$ and $\Omega \subset \mathbb{R}^n$ is an open set, we say that E is a set of *finite perimeter* in Ω if the distributional derivative of the characteristic function of E, $D\chi_E$, is a vector-valued Radon measure in Ω, with finite total variation $|D\chi_E|(\Omega)$. Thus, denoting by $(D_1\chi_E, \dots, D_n\chi_E)$ the components of $D\chi_E$, we have that for all $i = 1, \dots, n$ and all test functions $\varphi \in C^1_0(\Omega)$

$$\int_\Omega \chi_E(x) \frac{\partial \varphi}{\partial x_i}(x)\, dx = -\int_\Omega \varphi(x)\, dD_i\chi_E(x) . \tag{1.2}$$

From this formula it follows that the total variation of $D\chi_E$ in Ω can be expressed as

$$\begin{aligned} |D\chi_E|(\Omega) &= \sup\left\{ \sum_{i=1}^n \int_\Omega \psi_i(x)\, dD_i\chi_E : \ \psi \in C^1_0(\Omega; \mathbb{R}^n),\ \|\psi\|_\infty \le 1 \right\} \qquad (1.3) \\ &= \sup\left\{ \int_E \mathrm{div}\psi(x)\, dx : \ \psi \in C^1_0(\Omega; \mathbb{R}^n),\ \|\psi\|_\infty \le 1 \right\} . \end{aligned}$$

Notice that, if E is a smooth bounded open set, equation (1.2) reduces to

$$\int_{E\cap\Omega} \frac{\partial\varphi}{\partial x_i}(x)\,dx = -\int_{\partial E\cap\Omega} \varphi(x)\nu_i^E(x)\,d\mathcal{H}^{n-1}(x)\,,$$

where ν^E denotes the inner normal to the boundary of E. Here and in the sequel $\mathcal{H}^k$, $1 \le k \le n-1$, stands for the Hausdorff k-dimensional measure in $\mathbb{R}^n$. Thus, for a smooth set E,

$$D_i\chi_E = \nu_i^E \mathcal{H}^{n-1} \llcorner \partial E \qquad\qquad i = 1, \dots, n\,,$$
$$|D\chi_E|(\Omega) = \mathcal{H}^{n-1}(\partial E \cap \Omega)\,.$$

Last equation suggests to define the *perimeter of* E *in* Ω by setting $P(E;\Omega) = |D\chi_E|(\Omega)$. More generally, if $B \subset \Omega$ is any Borel subset of Ω, we set

$$P(E;B) = |D\chi_E|(B)\,.$$

If $\Omega = \mathbb{R}^n$, the perimeter of E in $\mathbb{R}^n$ will be denoted simply by $P(E)$.
Notice that the last supremum in (1.3) makes sense for any measurable set E. Indeed, if for some E that supremum is finite, then an application of Riesz's theorem on functionals on $C_0(\Omega;\mathbb{R}^n)$ yields that $D\chi_E$ is a Radon measure and (1.3) holds. Thus, we may set for any measurable set $E \subset \mathbb{R}^n$ and any open set Ω

$$P(E;\Omega) = \sup\left\{\int_E \mathrm{div}\psi(x)\,dx :\ \psi \in C_0^1(\Omega;\mathbb{R}^n),\ \|\psi\|_\infty \le 1\right\}. \tag{1.4}$$

Clearly E a set of finite perimeter according to the definition given above if and only if the right hand side of (1.4) is finite. Notice also that from (1.4) it follows that if E_h is a sequence of measurable sets converging locally in measure to E in Ω, i.e., such that $\chi_{E_h} \to \chi_E$ in $L^1_{\mathrm{loc}}(\Omega)$, then

$$P(E;\Omega) \le \liminf_{h\to\infty} P(E_h;\Omega)\,. \tag{1.5}$$

Another immediate consequence of the definition of perimeter is that $P(E;\Omega)$ does not change if we modify E by a set of zero Lebesgue measure. Moreover, it is straightforward to check that

$$P(E;\Omega\setminus\partial E) = |D\chi_E|(\Omega\setminus\partial E) = 0\,,$$

i.e., $D\chi_E$ is concentrated on the topological boundary of E.
Next example shows that in general $D\chi_E$ may be concentrated on a much smaller set. Let us denote by $B_r(x)$ the ball with center x and radius r and set $E = \cup_{i=1}^\infty B_{1/2^i}(q_i)$, where $\{q_i\}$ a dense sequence in $\mathbb{R}^n$. Then E is an open set of finite measure such that $\mathcal{L}^n(\partial E) = \infty$. However, E is a set of finite perimeter in $\mathbb{R}^n$. In fact, given $\psi \in C_0^1(\mathbb{R}^n;\mathbb{R}^n)$ with $\|\psi\|_\infty \le 1$, by

applying the classical divergence theorem to the Lipschitz open sets $E_k = \cup_{i=1}^k B_{1/2^i}(q_i)$, with $k \geq 1$, we have

$$\int_E \operatorname{div}\psi\, dx = \lim_{k\to\infty} \int_{E_k} \operatorname{div}\psi\, dx = -\lim_{k\to\infty} \int_{\partial E_k} \langle \psi, \nu^{E_k} \rangle \, d\mathcal{H}^{n-1}$$

$$\leq \lim_{k\to\infty} \mathcal{H}^{n-1}(\partial E_k) \leq \sum_{i=1}^{\infty} \frac{n\omega_n}{2^{i(n-1)}} < \infty .$$

To identify the set of points where the measure "perimeter" $P(E;\cdot)$ is concentrated, we may use the Besicovitch derivation theorem (see [1, Theorem 2.22]), which guarantees that if E is a set of finite perimeter in $\mathbb{R}^n$, then for $|D\chi_E|$-a.e. point $x \in \operatorname{supp}|D\chi_E|$ (the support of the total variation of $D\chi_E$) there exists the derivative of $D\chi_E$ with respect to $|D\chi_E|$,

$$\lim_{r\to 0} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))} = \nu^E(x) , \tag{1.6}$$

and that

$$|\nu^E(x)| = 1 . \tag{1.7}$$

The set of points where (1.6) and (1.7) hold is called the *reduced boundary of E* and denoted by $\partial^* E$. If $x \in \partial^* E$, $\nu^E(x)$ is called the *generalized inner normal to E at x*. Since from Besicovitch theorem we have that $D\chi_E = \nu^E |D\chi_E| \llcorner \partial^* E$, formula (1.2) can be written as

$$\int_E \frac{\partial \varphi}{\partial x_i}\, dx = -\int_{\partial^* E} \varphi \nu_i^E \, d|D\chi_E| \qquad \text{for all } \varphi \in C_0^1(\mathbb{R}^n) \text{ and } i = 1, \ldots, n . \tag{1.8}$$

The following theorem ([13] or [1, Theorem 3.59]) describes the structure of the reduced boundary of a set of finite perimeter.

Theorem 1.2 (De Giorgi). *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter in $\mathbb{R}^n$, $n \geq 2$. Then*

(i)
$$\partial^* E = \bigcup_{h=1}^{\infty} K_h \cup N_0 ,$$

where each K_h is a compact subset of a C^1 manifold M_h and $\mathcal{H}^{n-1}(N_0) = 0$;

(ii)
$$|D\chi_E| = \mathcal{H}^{n-1} \llcorner \partial^* E ;$$

for $\mathcal{H}^{n-1}$-a.e. $x \in K_h$, $\nu^E(x)$ is orthogonal to the tangent plane to M_h at x.

From this theorem it is clear that for a set of finite perimeter the reduced boundary plays the same role of the topological boundary for smooth sets. In particular, the integration by parts formula (1.8) becomes

$$\int_E \frac{\partial \varphi}{\partial x_i}\, dx = -\int_{\partial^* E} \varphi \nu_i^E \, d\mathcal{H}^{n-1} \qquad \text{for all } \varphi \in C_0^1(\mathbb{R}^n) \text{ and } i = 1, \ldots, n\,, \tag{1.9}$$

an equation very similar to the one we have when E is a smooth open set.

In the one-dimensional case sets of finite perimeter are completely characterized by the following result (see [1, Proposition 3.52]).

Proposition 1.1. *Let $E \subset \mathbb{R}$ be a measurable set. Then E has finite perimeter in $\mathbb{R}$ if and only if there exist $-\infty \le a_1 < b_1 < a_2 < \cdots < b_{N-1} < a_N < b_N \le +\infty$, such that E is equivalent to $\bigcup_{i=1}^{N} (a_i, b_i)$. Moreover, if Ω is an open set in $\mathbb{R}$,*

$$P(E; \Omega) = \#\{i : \; a_i \in \Omega\} + \#\{i : \; b_i \in \Omega\}\,.$$

Notice that from this characterization we have that if $E \subset \mathbb{R}$ is a set of finite perimeter with finite measure, then $P(E) \ge 2$. Moreover, $P(E) = 2$ if and only if E is equivalent to a bounded interval. Notice also that Proposition 1.1 yields immediately Theorem 1.1 and the characterization of the equality case in (1.1).

If we translate Theorem 1.2 in the language of Geometric Measure theory, then assertion (i) says that the reduced boundary $\partial^* E$ of a set of finite perimeter E in $\mathbb{R}^n$ is a countably $\mathcal{H}^{n-1}$-rectifiable set (see [1, Definition 2.57]), while (iii) states that for $\mathcal{H}^{n-1}$-a.e. $x \in \partial^* E$ the approximate tangent plane to $\partial^* E$ at x (see [1, Section 2.11]) is orthogonal to $\nu^E(x)$. Therefore, from the coarea formula for rectifiable sets ([1, Remark 2.94]), we get that if $g : \mathbb{R}^n \to [0, +\infty]$ is a Borel function, then

$$\int_{\partial^* E} g(x) |\nu_n(x)|\, d\mathcal{H}^{n-1}(x) = \int_{\mathbb{R}^{n-1}} dx' \int_{(\partial^* E)_{x'}} g(x', y)\, d\mathcal{H}^0(y)\,, \tag{1.10}$$

where $\mathcal{H}^0$ denotes the counting measure.
Setting $V = \{x \in \partial E^* : \; \nu_n^E(x) = 0\}$ and $g(x) = \chi_V(x)$, from (1.10) we get that

$$\int_{\mathbb{R}^{n-1}} dx' \int_{(\partial^* E)_{x'}} \chi_V(x', y)\, d\mathcal{H}^0(y) = \int_{\partial^* E} \chi_V(x) |\nu_n(x)|\, d\mathcal{H}^{n-1}(x) = 0\,.$$

Therefore, if E is a set of finite perimeter, then $V_{x'} = \emptyset$ for $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$, i.e.,

$$\text{for } \mathcal{L}^{n-1}\text{-a.e. } x' \in \mathbb{R}^{n-1}, \quad \nu_n^E(x', y) \neq 0 \quad \text{for all } y \text{ such that } (x', y) \in \partial^* E\,. \tag{1.11}$$

Let Ω be an open subset of $\mathbb{R}^n$ and $u \in L^1(\Omega)$. We say that u is a *function of bounded variation* (shortly, a *BV-function*) in Ω, if the distributional derivative Du is a vector-valued Radon measure in Ω with finite total variation. Thus, denoting by $D_i u$, $i = 1, \ldots, n$, the components of Du we have that

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i}\, dx = -\int_\Omega \varphi \, dD_i u \qquad \text{for all } \varphi \in C_0^1(\Omega)\,. \tag{1.12}$$

The space of functions of bounded variation in Ω will be denoted by $BV(\Omega)$. Notice that if $u \in BV(\Omega)$, then, as in (1.3), we have

$$\begin{aligned} |Du|(\Omega) &= \sup\Big\{ \sum_{i=1}^n \int_\Omega \psi_i(x)\, dD_i u : \ \psi \in C_0^1(\Omega; \mathbb{R}^n),\ \|\psi\|_\infty \le 1 \Big\} \\ &= \sup\Big\{ \int_\Omega u \mathrm{div}\psi(x)\, dx : \ \psi \in C_0^1(\Omega; \mathbb{R}^n),\ \|\psi\|_\infty \le 1 \Big\}. \end{aligned}$$

Moreover, it is clear that if E is a measurable set such that $\mathcal{L}^n(E \cap \Omega) < \infty$, then $\chi_E \in BV(\Omega)$ if and only if E has finite perimeter in Ω.

In the sequel, we shall denote by $D^a u$ the absolutely continuous part of Du with respect to Lebesgue measure $\mathcal{L}^n$. The singular part of Du will be denoted by $D^s u$. Moreover, we shall use the symbol ∇u to denote the density of $D^a u$ with respect to $\mathcal{L}^n$. Therefore,

$$Du = \nabla u \mathcal{L}^n + D^s u\,.$$

Notice also that a function $u \in BV(\Omega)$ belongs to $W^{1,1}(\Omega)$ if and only if Du is absolutely continuous with respect to $\mathcal{L}^n$, i.e., $|Du|(B) = 0$ for all Borel sets $B \subset \Omega$ such that $\mathcal{L}^n(B) = 0$. In this case, the density of Du with respect to $\mathcal{L}^n$ reduces to the usual weak gradient ∇u of a Sobolev function.

Next result is an essential tool for studying the behavior of Steiner symmetrization with respect to perimeter.

Lemma 1.1. *Let E a set of finite perimeter in $\mathbb{R}^n$ with finite measure. Then $\mu \in BV(\mathbb{R}^{n-1})$ and for any bounded Borel function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$*

$$\int_{\mathbb{R}^{n-1}} \varphi(x')\, dD_i \mu(x') = \int_{\partial^* E} \varphi(x') \nu_i^E(x)\, d\mathcal{H}^{n-1}(x), \qquad i = 1, \dots, n-1\,. \tag{1.13}$$

Moreover, for any Borel set $B \subset \mathbb{R}^{n-1}$,

$$|D\mu|(B) \le P(E; B \times \mathbb{R})\,. \tag{1.14}$$

Proof. Let us fix $\varphi \in C_0^1(\mathbb{R}^{n-1})$ and a sequence $\{\psi_j\}$ of $C_0^1(\mathbb{R})$ functions, such that $0 \le \psi_j(y) \le 1$ for all $y \in \mathbb{R}$ and $j \in \mathbb{N}$, with $\lim_{j\to\infty} \psi_j(y) = 1$ for all y. For any $i \in \{1, \dots, n-1\}$, from Fubini's theorem and formula (1.9), we get immediately

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} \frac{\partial \varphi}{\partial x_i}(x')\mu(x')\, dx' &= \int_{\mathbb{R}^{n-1}} dx' \int_{\mathbb{R}} \frac{\partial \varphi}{\partial x_i}(x')\chi_E(x', y)\, dy \\ &= \lim_{j\to\infty} \int_E \frac{\partial \varphi}{\partial x_i}(x')\psi_j(y)\, dx' dy \qquad (1.15) \\ &= -\lim_{j\to\infty} \int_{\partial^* E} \varphi(x')\psi_j(y)\nu_i^E(x)\, d\mathcal{H}^{n-1} \\ &= -\int_{\partial^* E} \varphi(x')\nu_i^E(x)\, d\mathcal{H}^{n-1}\,. \end{aligned}$$

This proves that the distributional derivatives of μ are real measures with bounded variation. Therefore, since $\mathcal{L}^n(E) < \infty$, hence $\mu \in L^1(\mathbb{R}^{n-1})$, we have that $\mu \in BV(\mathbb{R}^{n-1})$ and thus, by applying (1.12) to μ, from (1.15) we get in particular that (1.13) holds with $\varphi \in C_0^1(\mathbb{R}^{n-1})$. The case of a bounded Borel function φ then follows easily by approximation (see [9, Lemma 3.1]).

Finally, when B is an open set of $\mathbb{R}^{n-1}$, (1.14) follows immediately from (1.13) and (ii) of Theorem 1.2. Again, the general case of a Borel set $B \subset \mathbb{R}^{n-1}$, follows by approximation. □

Next result provides a first estimate of the perimeter of E^s. Notice that in the statement below we have to assume that E^s is a set of finite perimeter, a fact that will be proved later.

Lemma 1.2. *Let E be any set of finite perimeter in $\mathbb{R}^n$ with finite measure. If E^s is a set of finite perimeter, then*

$$P(E^s; B \times \mathbb{R}) \leq P(E; B \times \mathbb{R}) + |D_n \chi_{E^s}|(B \times \mathbb{R}) \tag{1.16}$$

for every Borel set $B \subset \mathbb{R}^{n-1}$.

Proof. Since $\mu \in BV(\mathbb{R}^{n-1})$, by a well known property of BV functions (see [1, Theorem 3.9], we may find a sequence $\{\mu_j\}$ of nonnegative functions from $C_0^1(\mathbb{R}^{n-1})$ such that $\mu_j \to \mu$ in $L^1(\mathbb{R}^{n-1})$, $\mu_j(x') \to \mu(x')$ for $\mathcal{L}^{n-1}$-a.e. x' in $\mathbb{R}^{n-1}$, $|D\mu_j|(\mathbb{R}^{n-1}) \to |D\mu|(\mathbb{R}^{n-1})$ and $|D\mu_j| \to |D\mu|$ weakly* in the sense of measures. Then, setting

$$E_j^s = \{(x', y) \in \mathbb{R}^{n-1} \times \mathbb{R} : \ \mu_j(x') > 0,\ |y| < \mu_j(x')/2\}\,,$$

we easily get that $\chi_{E_j^s}(x) \to \chi_{E^s}(x)$ in $L^1(\mathbb{R}^n)$. Fix an open set $U \subset \mathbb{R}^{n-1}$ and $\psi \in C_0^1(U \times \mathbb{R}, \mathbb{R}^n)$. Then, Fubini's theorem and a standard differentiation of integrals yield

$$\begin{aligned}
\int_{U \times \mathbb{R}} \chi_{E_j^s} \mathrm{div}\psi\, dx &= \int_U dx' \int_{-\mu_j(x')/2}^{\mu_j(x')/2} \sum_{i=1}^{n-1} \frac{\partial \psi_i}{\partial x_i}\, dy + \int_{\Omega \times \mathbb{R}} \chi_{E_j^s} \frac{\partial \psi_n}{\partial y}\, dx \\
&= -\frac{1}{2} \int_{\pi(\mathrm{supp}\psi)} \sum_{i=1}^{n-1} \left[\psi_i\Big(x', \frac{\mu_j(x')}{2}\Big) - \psi_i\Big(x', -\frac{\mu_j(x')}{2}\Big) \right] \frac{\partial \mu_j}{\partial x_i}\, dx' \\
&\quad + \int_{\Omega \times \mathbb{R}} \chi_{E_j^s} \frac{\partial \psi_n}{\partial y}\, dx\,,
\end{aligned}$$

where $\pi : \mathbb{R}^n \to \mathbb{R}^{n-1}$ denotes the projection over the first $n-1$ components. Thus

$$\begin{aligned}
&\int_{U \times \mathbb{R}} \chi_{E_j^s} \mathrm{div}\psi\, dx \leq \\
&\quad \leq \int_{\pi(\mathrm{supp}\psi)} \sqrt{\sum_{i=1}^{n-1} \left[\frac{1}{2}\Big(\psi_i\Big(x', \frac{\mu_j(x')}{2}\Big) - \psi_i\Big(x', -\frac{\mu_j(x')}{2}\Big)\Big) \right]^2} |\nabla \mu_j|\, dx' + \\
&\quad\quad + \int_{U \times \mathbb{R}} \chi_{E_j^s} \frac{\partial \psi_n}{\partial y}\, dx.
\end{aligned} \tag{1.17}$$

If $\|\psi\|_\infty \leq 1$, from (1.17) we get

$$\int_{U\times\mathbb{R}} \chi_{E_j^s} \mathrm{div}\psi\, dx \leq \int_{\pi(\mathrm{supp}\psi)} |\nabla \mu_j|\, dx' + \int_{U\times\mathbb{R}} \chi_{E_j^s} \frac{\partial \psi_n}{\partial y}\, dx\,. \tag{1.18}$$

Since $\chi_{E_j^s} \to \chi_{E^s}$ in $\mathcal{L}^1(\mathbb{R}^{n-1})$ and $\pi(\mathrm{supp}\psi)$ is a compact subset of U, recalling that $|D\mu_j| \to |D\mu|$ weakly* in the sense of measure and taking the lim sup in (1.18) as $j \to \infty$, we get

$$\begin{aligned}\int_{U\times\mathbb{R}} \chi_{E^s} \mathrm{div}\psi\, dx &\leq |D\mu|(\pi(\mathrm{supp}\psi)) + \int_{U\times\mathbb{R}} \chi_{E^s} \frac{\partial \psi_n}{\partial y}\, dx \\ &\leq |D\mu|(U) + |D_n\chi_{E^s}|(U\times\mathbb{R}) \\ &\leq P(E; U\times\mathbb{R}) + |D_n\chi_{E^s}|(U\times\mathbb{R})\,,\end{aligned} \tag{1.19}$$

where the last inequality follows from (1.14). Inequality (1.19) implies that (1.16) holds whenever B is an open set, and hence also when B is any Borel set. □

Remark 1.1. Notice that the argument used in the proof of Lemma 1.2 above yields that if E is a bounded set of finite perimeter, then E^s is a set of finite perimeter too. In fact, in this case, by applying (1.18) with $U = \mathbb{R}^{n-1}$ and $\|\psi\|_\infty \leq 1$ we get

$$\begin{aligned}&\int_{\mathbb{R}^n} \chi_{E_j^s} \mathrm{div}\psi\, dx \leq \\ &\leq \int_{\mathbb{R}^{n-1}} |\nabla\mu_j|\, dx' + \int_{\mathbb{R}^{n-1}} \Big[\psi_n(x', \mu_j(x')/2) - \psi_n(x', -\mu_j(x')/2)\Big]\, dx'\,.\end{aligned}$$

Hence, passing to the limit as $j \to \infty$, we get, from (1.14) and the assumption that E is bounded,

$$\begin{aligned}\int_{\mathbb{R}^n} \chi_{E^s} \mathrm{div}\psi\, dx &\leq |D\mu|(\mathbb{R}^{n-1}) + \int_{\mathbb{R}^{n-1}} \Big[\psi_n(x', \mu(x')/2) - \psi_n(x', -\mu(x')/2)\Big]\, dx' \\ &\leq P(E) + 2\mathcal{L}^{n-1}(\pi(E)^+) < \infty\,.\end{aligned}$$

Next result, due to Vol'pert ([26], [1, Theorem 3.108]), states that for $\mathcal{L}^{n-1}$-a.e. x' the section $E_{x'}$ is equivalent to a finite union of open intervals whose endpoints belong to the corresponding section $(\partial^* E)_{x'}$ of the reduced boundary.

Theorem 1.3. *Let E be a set of finite perimeter in $\mathbb{R}^n$. Then, for $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$,*

(i) $\qquad E_{x'}$ *has finite perimeter in* $\mathbb{R}$;

(ii) $\qquad \partial^* E_{x'} = (\partial^* E)_{x'}\,;$

(iii) $\nu_n^E(x',y) \neq 0$ *for all* y *such that* $(x',y) \in \partial^* E$;

(iv) $\chi_E(x',\cdot)$ *coincides* $\mathcal{L}^1$*-a.e. with a function* $g_{x'}$ *such that for all* $y \in \partial^* E_{x'}$

$$\begin{cases} \lim\limits_{z\to y^+} g_{x'}(z) = 1, & \lim\limits_{z\to y^-} g_{x'}(z) = 0 \quad \text{if } \nu_n^E(x',y) > 0\,, \\ \lim\limits_{z\to y^+} g_{x'}(z) = 0, & \lim\limits_{z\to y^-} g_{x'}(z) = 1 \quad \text{if } \nu_n^E(x',y) < 0\,. \end{cases}$$

The meaning of (i) and (ii) is clear. Property (iii) states that the section $(\partial^* E)_{x'}$ of the reduced boundary contains no vertical parts. As we have observed in (1.11), this is a consequence of the coarea formula (1.10). Finally, (iv) states that the normal $\nu^E(x)$ at a point $x \in \partial^* E$ has a positive vertical component if and only if $E_{x'}$ lies locally above x.

Notice also that from (ii) it follows that $(\partial^* E)_{x'} = \emptyset$ for $\mathcal{L}^{n-1}$-a.e. $x' \notin \pi(E)^+$ and that there exists a Borel set $G_E \subset \pi(E)^+$ such that

$$\begin{gathered} \text{the conclusions (i)-(iv) of Theorem 1.3 hold for every} \\ x' \in G_E, \qquad \mathcal{L}^{n-1}(\pi(E)^+ \setminus G_E) = 0. \end{gathered} \tag{1.20}$$

Let us now give a useful representation formula for the absolutely continuous part of the gradient of μ.

Lemma 1.3. *Let* $E \subset \mathbb{R}^n$ *be a set of finite perimeter with finite measure. Then, for* $\mathcal{L}^{n-1}$*-a.e.* $x' \in \pi(E)^+$,

$$\frac{\partial \mu}{\partial x_i}(x') = \sum_{y \in \partial^* E_{x'}} \frac{\nu_i^E(x',y)}{|\nu_n^E(x',y)|}, \qquad i = 1, \dots, n-1\,. \tag{1.21}$$

Proof. Let G_E be a Borel set satisfying (1.20) and g any function in $C_0(\mathbb{R}^{n-1})$. Set $\varphi(x') = g(x')\chi_{G_E}(x')$. From (1.13) and (1.10), recalling also (iii) and (ii) of Theorem 1.3, we have

$$\begin{aligned} \int_{G_E} g(x')\, dD_i\mu &= \int_{\partial^* E} g(x')\chi_{G_E}(x')\nu_i^E(x)\, d\mathcal{H}^{n-1}(x) = \\ &= \int_{\partial^* E} g(x')\chi_{G_E}(x') \frac{\nu_i^E(x)}{|\nu_n^E(x)|} |\nu_n^E(x)|\, d\mathcal{H}^{n-1}(x) \\ &= \int_{G_E} g(x') \sum_{y \in \partial^* E_{x'}} \frac{\nu_i^E(x',y)}{|\nu_n^E(x',y)|}\, dx'\,. \end{aligned}$$

Thus from this equality we get that

$$D_i\mu \llcorner G_E = \left(\sum_{y \in \partial^* E_{x'}} \frac{\nu_i^E(x',y)}{|\nu_n^E(x',y)|} \right) \mathcal{L}^{n-1} \llcorner G_E\,.$$

Hence the assertion follows, since by (1.20) $\mathcal{L}^{n-1}(\pi(E)^+ \setminus G_E) = 0$. $\square$

Remark 1.2. If E^s is a set of finite perimeter, since E and E^s have the same distribution function μ, we may apply Lemma 1.3 thus getting

$$\frac{\partial\mu}{\partial x_i}(x') = 2\frac{\nu_i^{E^s}(x', \frac{1}{2}\mu(x'))}{|\nu_n^{E^s}(x', \frac{1}{2}\mu(x'))|} \qquad \text{for } \mathcal{L}^{n-1}\text{-a.e. } x' \in \pi(E)^+ . \tag{1.22}$$

2 Steiner Symmetrization of Sets of Finite Perimeter

Let us start by proving the following version of Theorem 1.1.

Theorem 1.1 (Local version) *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter, $n \geq 2$. Then E^s is also of finite perimeter and for every Borel set $B \subset \mathbb{R}^{n-1}$,*

$$P(E^s; B) \leq P(E; B) . \tag{2.1}$$

Proof. Let $E \subset \mathbb{R}^n$ be a set of finite perimeter. If $\mathcal{L}^n(E) = \infty$, by the isoperimetric inequality (3.6) below, $\mathbb{R}^n \setminus E$ has finite measure, hence $\mathcal{L}^1(\mathbb{R} \setminus E_{x'}) < \infty$ for $\mathcal{L}^{n-1}$-a.e. $x' \in \mathbb{R}^{n-1}$, $E^s = \mathbb{R}^n$ and the assertion follows trivially.

Thus we may assume that E has finite measure. For the moment, let us assume also that E^s is a set of finite perimeter (we shall prove this fact later). Let us set $G = G_E \cap G_{E^s}$, where G_E and G_{E^s} are defined as in (1.20). To prove inequality (2.1) it is enough to assume $B \subset G$ or $B \subset \mathbb{R}^{n-1} \setminus G$.

In the first case, using Theorem 1.2 (ii), Theorem 1.3 (iii), coarea formula (1.10) and formulas (1.22) and (1.21), we get easily

$$\begin{aligned} P(E^s; B \times \mathbb{R}) &= \int_{\partial^* E^s \cap (B\times\mathbb{R})} \frac{1}{|\nu_n^{E^s}|} |\nu_n^{E^s}| \, d\mathcal{H}^{n-1} = \int_B \sum_{y \in \partial^* E^s_{x'}} \frac{1}{|\nu_n^{E^s}(x', y)|} \, dx' \\ &= 2\int_B \frac{1}{|\nu_n^{E^s}(x', \frac{1}{2}\mu(x'))|} \, dx' \qquad (2.2) \\ &= 2\int_B \sqrt{1 + \sum_{i=1}^{n-1} \left(\frac{|\nu_i^{E^s}(x', \frac{1}{2}\mu(x'))|}{|\nu_n^{E^s}(x', \frac{1}{2}\mu(x'))|} \right)^2} \, dx' \\ &= \int_B \sqrt{4 + \sum_{i=1}^{n-1} \left(\frac{\partial\mu}{\partial x_i}(x') \right)^2} \, dx' \\ &= \int_B \sqrt{4 + \sum_{i=1}^{n-1} \Bigg(\sum_{y \in \partial^* E_{x'}} \frac{\nu_i^E(x', y)}{|\nu_n^E(x', y)|} \Bigg)^2} \, dx' . \end{aligned}$$

Notice that, since E has finite measure, for a.e. $x' \in \mathbb{R}^{n-1}$, $\mathcal{L}^1(E_{x'}) < \infty$ and thus $P(E_{x'}) \geq 2$. Hence from the equality above, using the discrete Minkowski inequality, we get

$$P(E^s; B \times \mathbb{R}) = \int_B \sqrt{4 + \sum_{i=1}^{n-1} \left(\sum_{y \in \partial^* E_{x'}} \frac{\nu_i^E(x', y)}{|\nu_n^E(x', y)|} \right)^2} \, dx' \tag{2.3}$$

$$\leq \int_B \sqrt{\left(\#\{y : y \in \partial^* E_{x'}\} \right)^2 + \sum_{i=1}^{n-1} \left(\sum_{y \in \partial^* E_{x'}} \frac{\nu_i^E(x', y)}{|\nu_n^E(x', y)|} \right)^2} \, dx'$$

$$\leq \int_B \sum_{y \in \partial^* E_{x'}} \sqrt{1 + \sum_{i=1}^{n-1} \left(\frac{\nu_i^E(x', y)}{|\nu_n^E(x', y)|} \right)^2} \, dx'$$

$$= \int_B \sum_{y \in \partial^* E_{x'}} \frac{1}{|\nu_n^E(x', y)|} \, dx' = P(E; B \times \mathbb{R}) \,,$$

where the last two equalities, as in (2.2), are a consequence of the coarea formula and of the assumption $B \subset G_E$.
When $B \subset \mathbb{R}^{n-1} \setminus G$, we use (1.6), Theorem 1.2 (ii), coarea formula again, Theorem 1.3 (ii) and the fact that $\mathcal{L}^{n-1}(\pi(E)^+ \cap B) = 0$, thus getting

$$|D_n \chi_{E^s}|(B \times \mathbb{R}) = \int_{\partial^* E^s \cap (B \times \mathbb{R})} |\nu_n^{E^s}| d\mathcal{H}^{n-1} = \int_B \#\{y \in \partial^* E^s_{x'}\} dx'$$

$$= \int_{B \setminus \pi(E)^+} \#\{y \in \partial^* E^s_{x'}\} dx' = 0 \,,$$

where the last equality is a consequence of the fact that $E^s_{x'} = \emptyset$ for all $x' \notin \pi(E)^+$. Then (2.1) immediately follows from (1.16).
Let us now prove now that E^s is a set of finite perimeter. If E is bounded, this property follows from what we have already observed in Remark 1.1. If E is not bounded, we may always find a sequence of smooth bounded open sets E_h such that $\mathcal{L}^n(E \Delta E_h) \to 0$ and $P(E_h) \to P(E)$ as $h \to \infty$ (see [1, Theorem 3.42]). Notice that, by Fubini's theorem,

$$\mathcal{L}^n(E^s \Delta (E_h)^s) = \int_{\mathbb{R}^{n-1}} |\mathcal{L}^1(E^s_{x'}) - \mathcal{L}^1((E_h)^s_{x'})| \, dx'$$

$$= \int_{\mathbb{R}^{n-1}} |\mathcal{L}^1(E_{x'}) - \mathcal{L}^1((E_h)_{x'})| \, dx'$$

$$\leq \int_{\mathbb{R}^{n-1}} |\mathcal{L}^1(E_{x'} \Delta (E_h)_{x'})| \, dx' = \mathcal{L}^n(E \Delta E_h) \,.$$

Therefore, from the lower semicontinuity of perimeters with respect to convergence in measure (1.5) and from what we have proved above we get

$$P(E^s) \leq \liminf_{h \to \infty} P((E_h)^s) \leq \lim_{h \to \infty} P(E_h) = P(E)$$

and thus E^s has finite perimeter. □

The result we have just proved was more or less already known in the literature though with a different proof (see for instance [25]). The interesting point of the above proof is that it provides almost immediately some non trivial information about the case when equality holds in (1.1), as shown by the next result.

Theorem 2.1. *Let E be a set of finite perimeter in $\mathbb{R}^n$, with $n \geq 2$, such that equality holds in (1.1). Then, either E is equivalent to $\mathbb{R}^n$ or $\mathcal{L}^n(E) < \infty$ and for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi(E)^+$*

$$E_{x'} \text{ is equivalent to a segment } (y_1(x'), y_2(x')), \tag{2.4}$$

$$(\nu_1^E, \ldots, \nu_{n-1}^E, \nu_n^E)(x', y_1(x')) = (\nu_1^E, \ldots, \nu_{n-1}^E, -\nu_n^E)(x', y_2(x')). \tag{2.5}$$

Proof. If $\mathcal{L}^n(E) = \infty$, as we have already observed in the previous proof, $E^s = \mathbb{R}^n$. Then, since $P(E) = P(E^s) = 0$, it follows that also E is equivalent to $\mathbb{R}^n$.

If $\mathcal{L}^n(E) < \infty$, from the assumption $P(E) = P(E^s)$ and from inequality (2.1) it follows that $P(E^s; B \times \mathbb{R}) = P(E; B \times \mathbb{R})$ for all Borel sets $B \subset \mathbb{R}^{n-1}$. By applying this equality with $B = G$, where G is the set introduced in the proof above, it follows that both inequalities in (2.3) are indeed equalities.

In particular, since the first inequality holds as an equality, we get

$$\#\{y : y \in \partial^* E_{x'}\} = 2 \qquad \text{for } \mathcal{L}^{n-1}\text{-a.e. } x' \in G.$$

Hence (2.4) follows, recalling that, by (1.20), $\mathcal{L}^{n-1}(\pi(E)^+ \setminus G) = 0$.

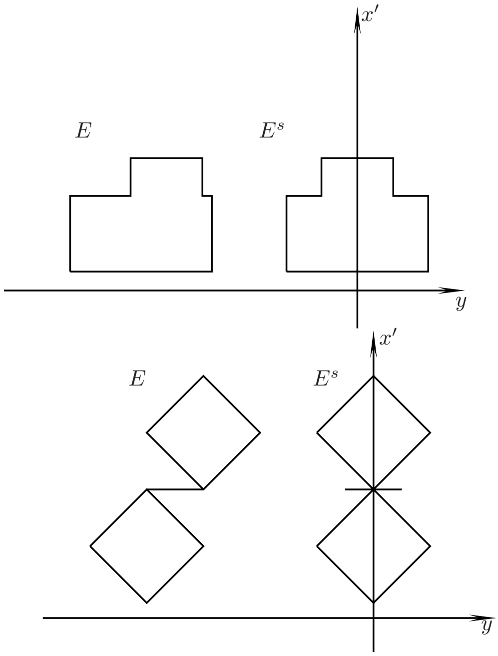
The fact that also the second inequality in (2.3) is an equality implies that

$$\frac{\nu_i^E(x', y_1(x'))}{|\nu_n^E(x', y_1(x'))|} = \frac{\nu_i^E(x', y_2(x'))}{|\nu_n^E(x', y_2(x'))|}$$

$$\text{for } i = 1, \ldots, n-1 \text{ and for } \mathcal{L}^{n-1}\text{-a.e. } x' \in G.$$

From this equation, since $|\nu^E| = 1$, we have that $\nu_i^E(x', y_1(x')) = \nu_i^E(x', y_2(x'))$ and $|\nu_n^E(x', y_1(x'))| = |\nu_n^E(x', y_2(x'))|$ for $\mathcal{L}^{n-1}$-a.e. $x' \in G$. Then, equality $\nu_n^E(x', y_1(x')) = -\nu_n^E(x', y_2(x')$ is an easy consequence of assertion (iv) of Theorem 1.3. Hence, (2.5) follows. □

As we have just seen, Theorem 2.1 states that if E has the same perimeter of its Steiner symmetral E^s, then almost every section of E in the y direction is a segment and the two normals at the endpoints of the segment are symmetric. However, this is not enough to conclude that E coincides with E^s (up to a transaltion), as it is clear by looking at the picture below.



Thus, in order to deduce from the equality $P(E) = P(E^s)$ that E and E^s coincide, up to a translation in the y direction, we need to make some assumption on the set E or on E^s. To this aim let us start by assuming that, given an open set $U \subset \mathbb{R}^{n-1}$,

$$(H_1) \qquad \mathcal{H}^{n-1}(\{x \in \partial^* E^s : \nu_n^{E^s}(x) = 0\} \cap (U \times \mathbb{R})\} = 0\,,$$

i.e., the (reduced) boundary of E^s has no flat parts parallel to the y direction. Notice that this assumption rules out the example shown on the upper part of the picture. Moreover, as we shall see in a moment, (H_1) holds in an open set U if and only if the distribution function is a $W^{1,1}$ function in U. To this aim, let us recall the following well known result concerning the graph of a BV function (see, for instance, [18, Ch. 4, Sec. 1.5, Th. 1, and Ch. 4, Sec. 2.4, Th. 4]).

Theorem 2.2. *Let $U \subset \mathbb{R}^{n-1}$ be a bounded open set and $u \in L^1(U)$. Then the subgraph of U,*

$$\mathcal{S}_u = \{(x', y) \in U \times \mathbb{R} : y < u(x')\}\,,$$

is a set of finite perimeter in $U \times \mathbb{R}$ *if and only if* $u \in BV(U)$. *Moreover, in this case,*

$$P(\mathcal{S}_u; B \times \mathbb{R}) = \int_B \sqrt{1 + |\nabla u|^2} dx' + |D^s u|(B) \tag{2.6}$$

for every Borel set $B \subset U$.

Notice that if E is a bounded set of finite perimeter, since $\mu \in BV(\mathbb{R}^{n-1})$ by Lemma 1.1, and

$$E^s = \{(x', y) \in \mathbb{R}^{n-1} \times \mathbb{R} : \, -\mu(x')/2 < y < \mu(x')/2\}, \tag{2.7}$$

from Theorem 2.2 we get immediately that E^s is a set of finite perimeter, being the intersection of the two sets of finite perimeter $\{(x', y) : y > -\mu(x')/2\}$ and $\{(x', y) : y < \mu(x')/2\}$.

Proposition 2.1. *Let* E *be any set of finite perimeter in* $\mathbb{R}^n$, $n \geq 2$, *with finite measure. Let* U *be an open subset of* $\mathbb{R}^{n-1}$. *Then the following conditions are equivalent:*

(i) $\mathcal{H}^{n-1}\big(\{x \in \partial^* E^s : \nu_n^{E^s}(x) = 0\} \cap (U \times \mathbb{R})\big) = 0$,
(ii) $P(E^s; B \times \mathbb{R}) = 0$ *for every Borel set* $B \subset U$ *such that* $\mathcal{L}^{n-1}(B) = 0$,
(iii) $\mu \in W^{1,1}(U)$.

Proof. Let us assume that (i) holds and fix a Borel set $B \subset U$ such that $\mathcal{L}^{n-1}(B) = 0$. Using coarea formula (1.10) we get

$$\begin{aligned} P(E^s; B \times \mathbb{R}) &= \mathcal{H}^{n-1}(\partial^* E^s \cap (B \times \mathbb{R})) \\ &= \mathcal{H}^{n-1}(\{x \in \partial^* E^s : \nu_n^{E^s}(x) \neq 0\} \cap (B \times \mathbb{R})) \\ &= \int_{\partial^* E^s} \frac{1}{|\nu_n^{E^s}(x)|} \chi_{\{\nu_n^{E^s} \neq 0\} \cap (B \times \mathbb{R})}(x) |\nu_n^{E^s}(x)| \, d\mathcal{H}^{n-1} \\ &= \int_B dx' \int_{(\partial^* E^s)_{x'}} \frac{\chi_{\{\nu_n^{E^s} \neq 0\}}(x', y)}{|\nu_n^{E^s}(x', y)|} \, d\mathcal{H}^0(y) = 0, \end{aligned}$$

hence (ii) follows.
If (ii) holds and B is a null set in U, by applying (1.14) with E replaced by E^s we get $|D\mu|(B) = 0$. Thus, $D\mu$ is absolutely continuous with respect to $\mathcal{L}^{n-1}$, hence $\mu \in W^{1,1}(U)$.
Notice that, if E_1, E_2 are two sets of finite perimeter and B is an open set, then (see [1, Proposition 3.38]) $P(E_1 \cap E_2; B) \leq P(E_1; B) + P(E_2; B)$ and, by approximation, the same inequality holds also when B is a Borel set. Therefore, recalling (2.7) and (2.6) we get that, if (iii) holds, for any Borel set B in U

$$P(E^s; B \times \mathbb{R}) \leq 2P(\mathcal{S}_{\mu/2}; B \times \mathbb{R}) = \int_B \sqrt{4 + |\nabla \mu|^2} \, dx'. \tag{2.8}$$

Set $B_0 = \pi(\partial^* E^s) \setminus G_{E^s}$, where $G_{E^s} \subset \pi(E)^+$ is a Borel set satisfying (1.20) with E replaced by E^s. Since by Theorem 1.3 $(\partial^* E^s)_{x'} = \emptyset$ for $\mathcal{L}^{n-1}$-a.e.

$x' \notin \pi(E)^+$, we have $\mathcal{L}^{n-1}(B_0) = \mathcal{L}^{n-1}(\pi(\partial^* E^s) \setminus \pi(E)^+) + \mathcal{L}^{n-1}(\pi(E)^+ \setminus G_{E^s}) = 0$. Therefore, from (2.8) we get that $P(E^s; (B_0 \cap U) \times \mathbb{R}) = 0$, i.e. $\mathcal{H}^{n-1}\left((\partial^* E^s \setminus (G_{E^s} \times \mathbb{R})) \cap (U \times \mathbb{R})\right) = 0$. Then, (i) follows since by definition $\{x \in \partial^* E^s : \nu_n^{E^s}(x) = 0\} \subset \partial^* E^s \setminus (G_{E^s} \times \mathbb{R})$. □

It may seem strange that assumption (H_1) is made on the Steiner symmetral E^s. Alternatively, we could make a similar assumption on E by requiring that

$$(H_1') \qquad \mathcal{H}^{n-1}(\{x \in \partial^* E : \nu_n^E(x) = 0\} \cap (U \times \mathbb{R})\} = 0\,.$$

Actually, it is not difficult to show that (H_1') implies (H_1), while the converse is false in general, as one can see by simple examples. In fact, if (H_1') holds, arguing exactly as in the proof of the implication '(i)⇒(ii)' in Proposition 2.1 we get that $P(E; B \times \mathbb{R}) = 0$ for any Borel set $B \subset U$ with zero measure. Then (2.1) implies that the same property holds also for E^s and thus, by Proposition 2.1, we get that E^s satisfies (H_1). Notice also that when $P(E) = P(E^s)$, then by (2.1) we have that $P(E; B \times \mathbb{R}) = P(E^s; B \times \mathbb{R})$ for any Borel set $B \subset \mathbb{R}^{n-1}$. Thus one immediately gets that in this case the two conditions (H_1), (H_1') are equivalent.

Let us now comment on the example on the lower part of the picture above. It is clear that in that case things go wrong, in the sense that E and E^s are not equal, because even though the set E is connected in a strict topological sense it is 'essentially disconnected'. Therefore, to deal with similar examples one could device to use a suitable notion of connectedness set up in the context of sets of finite perimeter (see, for instance, [1, Example 4.18]). However, we will not follow this path. Instead, we will use the information provided by Proposition 2.1.

If the distribution function μ is of class $W^{1,1}(U)$, then for $\mathcal{H}^{n-2}$-a.e. $x' \in U$ we can define its *precise representative* $\widetilde{\mu}(x')$ (see [15] or [27]) as the unique value such that

$$\lim_{r \to 0} \fint_{B_r^{n-1}(x')} |\mu(y) - \widetilde{\mu}(x')|\, dx' = 0\,, \tag{2.9}$$

where by $B_r^{n-1}(x')$ we have denoted the $(n-1)$-dimensional ball with centre x' and radius r. Then, in order to rule out a situation like the one on the bottom of the picture above, we make the assumption

$$(H_2) \qquad \widetilde{\mu}(x') > 0 \qquad \text{for } \mathcal{H}^{n-2}\text{-a.e. } x' \in U\,.$$

Next result, proved in [9], shows that the two examples in the picture are indeed the only cases where the equality $P(E) = P(E^s)$ does not imply that the two sets are equal. As for Theorem 1.1, we state the result in a local form.

Theorem 2.3. *Let E be a set of finite perimeter $\mathbb{R}^n$, with $n \geq 2$, such that*

$$P(E^s) = P(E)\,. \tag{2.10}$$

Let us assume that (H_1) and (H_2) hold in some open set $U \subset \mathbb{R}^{n-1}$. Then, for every connected open subset U_α of U, $E \cap (U_\alpha \times \mathbb{R})$ is equivalent to $E^s \cap (U_\alpha \times \mathbb{R})$, up to a translation in the y direction.
In particular, if (H_1) and (H_2) hold in a connected open set U such that $\mathcal{L}^{n-1}(\pi(E)^+ \setminus U) = 0$, then E is equivalent to E^s, up to a translation in the y direction.

As far as I know, this result was known in the literature only for convex sets, where it can be proved with a simple argument. In fact, let us assume that E is an open convex set such that $P(E) = P(E^s) < \infty$. Then, $\pi(E)$ is also an open convex set and there exist two functions $y_1, y_2 : \pi(E) \to \mathbb{R}$, y_1 convex, and y_2 concave, such that

$$E = \{(x', y) : x' \in \pi(E),\, y_1(x') < y < y_2(x')\}\,.$$

Let us now fix an open set $U \subset\subset \pi(E)$. From assumption (2.10) and from (2.1) we have that $P(E^s; U \times \mathbb{R}) = P(E; U \times \mathbb{R})$. Since y_1 and y_2 are Lipschitz continuous in U, we can write this equality as

$$2\int_U \sqrt{1 + \frac{|\nabla(y_2 - y_1)|^2}{4}}\, dx' = \int_U \sqrt{1 + |\nabla y_1|^2}\, dx' + \int_U \sqrt{1 + |\nabla y_1|^2}\, dx'\,.$$

From this equality, the strict convexity of the function $t \mapsto \sqrt{1+t^2}$ and the arbitrariness of U, we get that $\nabla y_2 = -\nabla y_1$ in $\pi(E)$ and thus $y_2 = -y_1 + const.$. This shows that E coincides with E^s, up to a translation in y direction.

The proof of Theorem 2.3, for which we refer to [9], uses delicate tools from Geometric Measure theory. However, in the special case considered below it can be greatly simplified.

Proof of Theorem 2.3 in a Special Case. Let us assume that E is an open set, that $\pi(E)$ is connected and that E is bounded in y direction.
Notice that since E is open, then μ is a lower semicontinuous function and for any open set $U \subset\subset \pi(E)$ there exists a constant $c(U) > 0$ such that

$$\mu(x') \geq c(U) \qquad \text{for all } x' \in U\,. \tag{2.11}$$

Moreover, since E is bounded in the y direction, the function

$$x' \in \pi(E) \mapsto m(x') = \int_{E_{x'}} y\, dy$$

is bounded in $\pi(E)$. Then, the same arguments used in the proofs of Lemmas 1.1 and 1.3 yield that $m \in BV_{\mathrm{loc}}(\pi(E))$ and that for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi(E)$, $i = 1, \ldots, n-1$,

$$\frac{\partial m}{\partial x_i}(x') = \sum_{y \in \partial^* E_{x'}} \frac{y\, \nu_i^E(x', y)}{|\nu_n^E(x', y)|}\,, \tag{2.12}$$

where we have denoted by $\partial m/\partial x_i$ the absolutely continuous part of the derivative $D_i m$.

By Proposition 2.1 we have that (H_1) implies that the distribution function μ is a Sobolev function. The same assumption implies also that $m \in W^{1,1}_{\mathrm{loc}}(\pi(E))$. In fact, the argument used to prove (1.14) shows that if $B \subset \pi(E)$ is a Borel set, then $|Dm|(B) \leq MP(E; B \times \mathbb{R})$, where M is a constant such that $E \subset \mathbb{R}^{n-1} \times (-M, M)$. Therefore, if $\mathcal{L}^{n-1}(B) = 0$, from (2.1) and Proposition 2.1 we have that $P(E; B \times \mathbb{R}) = P(E^s; B \times \mathbb{R}) = 0$, hence Dm is absolutely continuous with respect to $\mathcal{L}^{n-1}$.
Let us now denote, for any $x' \in \pi(E)$ by $b(x')$ the baricenter of the section $E_{x'}$, i.e.,

$$b(x') = \frac{\displaystyle\int_{E_{x'}} y\,dy}{\mu(x')} .$$

From (2.11) and Proposition 2.1 we have that b too belongs to the space $W^{1,1}_{\mathrm{loc}}(\pi(E))$. Thus, to prove the assertion, since $\pi(E)$ is a connected open set, it is enough to show that $\nabla b \equiv 0$, hence b is constant on $\pi(E)$. To this aim, let us evaluate the partial derivatives of b, using the representation formulas (2.12) and (1.21). We have, for any $i = 1, \dots, n-1$ and for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi(E)$,

$$\frac{\partial b}{\partial x_i}(x') = \frac{1}{\mu(x')}\left(\sum_{y \in \partial^* E_{x'}} \frac{y\,\nu_i^E(x', y)}{|\nu_n^E(x', y)|} - \frac{\displaystyle\int_{E_{x'}} y\,dy}{\mu(x')} \sum_{y \in \partial^* E_{x'}} \frac{\nu_i^E(x', y)}{|\nu_n^E(x', y)|} \right) . \quad (2.13)$$

Since for $\mathcal{L}^{n-1}$-a.e. $x' \in \pi(E)$ (2.4) and (2.5) hold, the right hand side of (2.13) is equal to

$$\frac{1}{\mu(x')}\left[\big(y_2(x') + y_1(x')\big) \frac{\nu_i^E(x', y_2(x'))}{|\nu_n^E(x', y_2(x'))|} - \frac{1}{2} \frac{y_2^2(x') - y_1^2(x')}{y_2(x') - y_1(x')} \frac{2\nu_i^E(x', y_2(x'))}{|\nu_n^E(x', y_2(x'))|} \right] = 0.$$

Hence, the assertion follows. □

3 The Pòlya–Szegö Inequality

We are going to present the classical Pòlya–Szegö inequality for the spherical rearrangement of a Sobolev function u and discuss what can be said about the function u when the equality holds. In order to simplify the exposition, we shall assume that u is a nonnegative measurable function from $\mathbb{R}^n$, with compact support. However, most of the results presented here, like Theorem 3.1, still hold with no restrictions on the support or on the sign of u.

Given u, we set, for any $t \geq 0$,

$$\mu_u(t) = \mathcal{L}^n(\{x \in \mathbb{R}^n : u(x) > t\}) .$$

The function μ_u is called the *distribution function* of u. Clearly μ_u is a decreasing, right-continuous function such that

$$\mu_u(0)=\mathcal{L}^n(\mathrm{supp}u), \quad \mu_u(\mathrm{esssup}\, u)=0, \quad \mu_u(t-)=\mathcal{L}^n(\{u \geq t\}) \quad \text{for all } t > 0\,. \tag{3.1}$$

Notice that from the last equality we have that when $t > 0$

$$\mu_u \text{ is continuous in } t \text{ iff } \mathcal{L}^n(\{u = t\}) = 0\,.$$

Let us now introduce the *decreasing rearrangement* of u, that is the function $u^* : [0, +\infty) \to [0, +\infty)$ defined, for any $s \geq 0$, by setting

$$u^*(s) = \sup\{t \geq 0 : \mu_u(t) > s\}\,.$$

Clearly, u^* is a decreasing, right-continuous function. The following elementary properties of u^* are easily checked:

(j) $u^*(\mu_u(t)) \leq t \leq u^*(\mu_u(t)-)$ for all $0 \leq t < \mathrm{esssup}\, u$;
(jj) $\mu_u(u^*(s)) \leq s \leq \mu_u(u^*(s)-)$ for all $0 \leq s < \mathcal{L}^n(\mathrm{supp}u)$;
(jjj) $\mathcal{L}^1(\{s : u^*(s) > t\} = \mu_u(t)$ for all $t \geq 0$.

Notice that (jjj) states that the functions u and u^* are *equi-distributed*, i.e., $\mu_u = \mu_{u^*}$. Let us now define the *spherical symmetric rearrangement* of u, that is the function $u^\star : \mathbb{R}^n \to [0, +\infty)$, such that for all $x \in \mathbb{R}^n$

$$u^\star(x) = u^*(\omega_n |x|^n)\,. \tag{3.2}$$

By definition and by (jjj) we have

$$\mathcal{L}^n(\{u^\star > t\}) = \mathcal{L}^n(\{u > t\}) \qquad \text{for all } t \geq 0\,,$$

i.e., $\mu_u = \mu_{u^\star}$. Thus also u and $u^\star$ are equi-distributed. As a simple consequence of this equality and Fubini's theorem we have, for all $p \geq 1$,

$$\int_{\mathbb{R}^n} |u^\star(x)|^p\, dx = \int_{\mathbb{R}^n} |u(x)|^p\, dx\,,$$

and, letting $p \to +\infty$, $\mathrm{esssup}\, u = \mathrm{esssup}\, u^\star$.

If u is a smooth function, in general its symmetric rearrangement will be no longer smooth (actually the best we may expect from Theorem 3.1 below is that $u^\star$ is Lipschitz continuous). However, the symmetric rearrangement behaves nicely on Sobolev functions, as shown by the next result.

Theorem 3.1 (Pòlya–Szegö Inequality). *Let $u \in W^{1,p}(\mathbb{R}^n)$, $p \geq 1$, be a nonnegative function with compact support. Then $u^\star \in W^{1,p}(\mathbb{R}^n)$ and*

$$\int_{\mathbb{R}^n} |\nabla u^\star(x)|^p\, dx \leq \int_{\mathbb{R}^n} |\nabla u(x)|^p\, dx\,. \tag{3.3}$$

The proof of this result relies upon two main ingredients, the isoperimetric inequality and the coarea formula for BV functions. Let us start by recalling the latter.

Let u be a $BV(\Omega)$ function. Then, for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$, the set $\{u > t\}$ has finite perimeter in Ω. Moreover, for any Borel function $g : \Omega \to [0, +\infty]$, the following formula holds (see [1, Theorem 3.40]).

$$\int_\Omega g(x)\, d|Du| = \int_{-\infty}^{\infty} dt \int_{\partial^*\{u>t\}} g(x)\, d\mathcal{H}^{n-1} . \tag{3.4}$$

In the special case $u \in W^{1,1}(\Omega)$, it can be shown that for $\mathcal{L}^1$-a.e. t the reduced boundary $\partial^*\{u > t\}$ coincides, modulo a set of $\mathcal{H}^{n-1}$-measure zero, with the level set $\{\widetilde{u} = t\}$, where $\widetilde{u}$ denotes the *precise representative* of u, which is defined $\mathcal{H}^{n-1}$-a.e. in Ω as in (2.9). Therefore, if $u \in W^{1,1}(\Omega)$, (3.4) becomes

$$\int_\Omega g(x)|\nabla u(x)|\, dx = \int_{-\infty}^{\infty} dt \int_{\{\tilde{u}=t\}} g(x)\, d\mathcal{H}^{n-1}(x) . \tag{3.5}$$

The *isoperimetric inequality* states that if E be a set of finite perimeter, then

$$\Big(\min\{\mathcal{L}^n(E), \mathcal{L}^n(\mathbb{R}^n \setminus E)\}\Big)^{\frac{n-1}{n}} \leq \frac{1}{n\omega_n^{1/n}} P(E) . \tag{3.6}$$

Moreover, the equality holds if and only if E is (equivalent to) a ball.

Next lemma shows that if u is a Sobolev function, then the same is also true for $u^\star$.

Lemma 3.1. *Let u be a nonnegative function with compact support from the space $W^{1,1}(\mathbb{R}^n)$. Then $u^\star$ belongs to $W^{1,1}(\mathbb{R}^n)$ and*

$$\int_{\mathbb{R}^n} |\nabla u^\star|\, dx \leq \int_{\mathbb{R}^n} |\nabla u|\, dx . \tag{3.7}$$

Proof. Let us first prove that for any $0 < a < b$, the function u^* is absolutely continuous in (a, b) and

$$\int_a^b |u^{*\prime}(s)|\, ds \leq \frac{1}{n\omega_n^{1/n} a^{\frac{n-1}{n}}} \int_{\{u^*(b)<u<u^*(a)\}} |\nabla u|\, dx . \tag{3.8}$$

To this aim, we start by observing that from the third equality in (3.1) and from inequality (jj) we have

$$\mathcal{L}^n(\{x \in \mathbb{R}^n : u^*(b) < u(x) < u^*(a)\}) = \mu_u(u^*(b)) - \mu_u(u^*(a)-) \leq b - a . \tag{3.9}$$

Let us denote by $\omega : [0, \infty) \to [0, +\infty)$ the modulus of continuity of the integral of $|\nabla u|$, i.e., a continuous function, vanishing at zero and such that for any set of finite measure E

$$\int_E |\nabla u|\, dx \le \omega(\mathcal{L}^n(E))\,.$$

Using the coarea formula (3.5), the isoperimetric inequality (3.6) and (jj) again, we obtain the following estimate for the integral of $|\nabla u|$ between two level sets,

$$\begin{aligned}\int_{\{u^*(b)<u<u^*(a)\}} |\nabla u|\, dx &= \int_{u^*(b)}^{u^*(a)} P(\{u>t\})\, dt \\ &\ge n\omega_n^{1/n} \int_{u^*(b)}^{u^*(a)} \Big(\mathcal{L}^n(\{u>t\}\Big)^{\frac{n-1}{n}} dt \\ &\ge n\omega_n^{1/n} [\mu_u(u^*(a)-)]^{\frac{n-1}{n}} (u^*(a)-u^*(b)) \quad (3.10) \\ &\ge n\omega_n^{1/n} a^{\frac{n-1}{n}} (u^*(a)-u^*(b))\,.\end{aligned}$$

Let us now take a finite number of pairwise disjoint intervals $(a_i, b_i) \subset (a, b)$, $i = 1, \dots, N$. By applying (3.9) and (3.10) to each interval (a_i, b_i), we get

$$\begin{aligned}\sum_{i=1}^N |u^*(b_i) - u^*(a_i)| &\le \frac{1}{n\omega_n^{1/n} a^{\frac{n-1}{n}}} \sum_{i=1}^N \int_{\{u^*(b_i)<u<u^*(a_i)\}} |\nabla u|\, dx \\ &\le \frac{1}{n\omega_n^{1/n} a^{\frac{n-1}{n}}} \omega\Big(\sum_{i=1}^N (b_i - a_i)\Big)\,. \qquad (3.11)\end{aligned}$$

From this inequality it follows immediately that u is absolutely continuous in (a, b), since the left hand side is smaller than a given $\varepsilon > 0$ as soon as the sum of the lengths of the intervals (a_i, b_i) is sufficiently small. Moreover, by taking the supremum of the left hand side of (3.11) over all possible partitions of the interval (a, b), from the first inequality in (3.11) we get immediately (3.8).

Notice that from (3.8) it follows that $u^\star$ is in $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\})$. To prove the assertion, we fix $\sigma > 1$ and estimate the integral of $|\nabla u|$ in the annuli $A_{k,\sigma} = \{x \in \mathbb{R}^n : \omega_n^{-1/n}\sigma^{k/n} < |x| < \omega_n^{-1/n}\sigma^{(k+1)/n}\}$, for $k \in \mathbb{Z}$. Using (3.8) again, and recalling the definition (3.2), we get, for any $k \in \mathbb{Z}$,

$$\begin{aligned}\int_{A_{k,\sigma}} |\nabla u^\star|\, dx &= n\omega_n \int_{A_{k,\sigma}} |x|^{n-1} |u^{*\prime}(\omega_n |x|^n)|\, dx \\ &= n^2\omega_n^2 \int_{\omega_n^{-1/n}\sigma^{k/n}}^{\omega_n^{-1/n}\sigma^{(k+1)/n}} r^{2n-2} |u^{*\prime}(\omega_n r^n)|\, dr \\ &= n\omega_n^{1/n} \int_{\sigma^k}^{\sigma^{k+1}} s^{\frac{n-1}{n}} |u^{*\prime}(s)|\, ds \\ &\le \sigma^{\frac{n-1}{n}} \int_{\{u^*(\sigma^k)<u<u^*(\sigma^{k+1})\}} |\nabla u|\, dx\,.\end{aligned}$$

Then the assertion immediately follows by summing up both sides of this inequality over all $k \in \mathbb{Z}$ and then letting $\sigma \to 1^+$. □

Notice that the lemma we have just proved provides the Pólya–Szegö inequality for $p = 1$. However, for the general case $p \geq 1$ we present a different proof which has the advantage of giving better information when the inequality becomes an equality.

To this aim let us introduce a few quantities that will be useful later. If $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$, we set

$$\mathcal{D}_u^+ = \{x \in \mathbb{R}^n : \nabla u(x) \neq 0\}, \qquad \mathcal{D}_u^0 = \mathbb{R}^n \setminus \mathcal{D}_u^+ .$$

We can now give a representation formula for the derivative of μ_u. Notice that the formula stated in (3.12) uses the fact that the $|\nabla u^\star|$ is $\mathcal{H}^{n-1}$-a.e. constant on $\mathcal{L}^1$-a.e. level set $\{u^\star = t\}$.

Lemma 3.2. *Let $u \in W^{1,1}(\mathbb{R}^n)$ be a nonnegative function with compact support. Then, for $\mathcal{L}^1$-a.e. $t > 0$,*

$$\mu_u'(t) = -\frac{\mathcal{H}^{n-1}(\{u^\star = t\})}{|\nabla u^\star|_{|\{u^\star=t\}}} \leq -\int_{\{\tilde{u}=t\}} \frac{1}{|\nabla u|}\, d\mathcal{H}^{n-1} . \tag{3.12}$$

Proof. First of all let us evaluate $\mu_u(t)$ using the coarea formula (3.5). We get, for all $t \geq 0$,

$$\begin{aligned}
\mu_u(t) &= \mathcal{L}^n\left(\{u > t\} \cap \mathcal{D}_u^0\right) + \mathcal{L}^n\left(\{u > t\} \cap \mathcal{D}_u^+\right) \\
&= \mathcal{L}^n\left(\{u > t\} \cap \mathcal{D}_u^0\right) + \int_{\mathcal{D}_u^+} \chi_{\{u>t\}}(x)\, dx \qquad (3.13)\\
&= \mathcal{L}^n\left(\{u > t\} \cap \mathcal{D}_u^0\right) + \int_t^{+\infty} ds \int_{\{\tilde{u}=s\}} \frac{\chi_{\mathcal{D}_u^+}}{|\nabla u|}\, d\mathcal{H}^{n-1} \\
&= \mathcal{L}^n\left(\{u > t\} \cap \mathcal{D}_u^0\right) + \int_t^{+\infty} ds \int_{\{\tilde{u}=s\}} \frac{1}{|\nabla u|}\, d\mathcal{H}^{n-1} ,
\end{aligned}$$

where the last equality follows by observing that coarea formula (3.5) implies that $\mathcal{H}^{n-1}(\{\tilde{u} = t\} \cap \mathcal{D}_u^0) = 0$ for $\mathcal{L}^1$-a.e. $t \geq 0$. By applying (3.13) to $u^\star$, we get also that for all $t \geq 0$

$$\mu_u(t) = \mathcal{L}^n\left(\{u^\star > t\} \cap \mathcal{D}_{u^\star}^0\right) + \int_t^{+\infty} \frac{\mathcal{H}^{n-1}(\{u^\star = t\})}{|\nabla u^\star|_{|\{u^\star=t\}}} \tag{3.14}$$

Let us now recall a nice property of absolutely continuous functions (see for instance [10, Lemma 2.4]).

If g is an absolutely continuous function in a bounded open interval I and, for all $t \in \mathbb{R}$, we set $\phi_g(t) = \mathcal{L}^1(\{g > t\} \cap \mathcal{D}_g^0)$, then ϕ_g is a nondecreasing function such that $\phi_g'(t) = 0$ for $\mathcal{L}^1$-a.e. t.

By applying this result with $g = u^*$ and observing that $\mathcal{L}^n\left(\{u^\star > t\} \cap \mathcal{D}_{u^\star}^0\right) = \mathcal{L}^1\left(\{u^* > t\} \cap \mathcal{D}_{u^*}^0\right)$, for all $t > 0$, from (3.14) we get immediately the equality in (3.12). On the other hand, the inequality

$$\mu_u'(t) \leq -\int_{\{\tilde{u}=t\}} \frac{1}{|\nabla u|}\, d\mathcal{H}^{n-1}$$

follows immediately from (3.13). □

We are now ready to prove the Pólya–Szegö inequality (3.3).

Proof of Theorem 3.1. Let us fix a nonnegative function $u \in W^{1,p}(\mathbb{R}^n)$ with compact support and let us assume, without loss of generality, that u coincides with its precise representative $\widetilde{u}$. From Lemma 3.1 we know already that $u^\star$ belongs to the space $W^{1,1}(\mathbb{R}^n)$. Thus, using the coarea formula (3.5) with u replaced by $u^\star$ and recalling that $|\nabla u^\star|$ is constant on the level sets of $u^\star$, we get

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla u^\star|^p\, dx &= \int_0^{+\infty} dt \int_{\{u^\star=t\}} |\nabla u^\star|^{p-1}\, d\mathcal{H}^{n-1} \\ &= \int_0^{+\infty} \mathcal{H}^{n-1}(\{u^\star=t\}) |\nabla u^\star|^{p-1}_{|\{u^\star=t\}}\, dt\,.\end{aligned}$$

From this equation, using twice (3.12), the isoperimetric inequality (3.6), Hölder's inequality and coarea formula again, we get

$$\begin{aligned}\int_{\mathbb{R}^n} |\nabla u^\star|^p\, dx &= \int_0^{+\infty} \frac{\big[\mathcal{H}^{n-1}(\{u^\star=t\})\big]^p}{[-\mu_u'(t)]^{p-1}}\, dt \\ &\leq \int_0^{+\infty} \frac{\big[\mathcal{H}^{n-1}(\{u^\star=t\})\big]^p}{\left(\displaystyle\int_{\{u^\star=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|}\right)^{p-1}}\, dt \\ &\leq \int_0^{+\infty} \frac{\big[\mathcal{H}^{n-1}(\{u=t\})\big]^p}{\left(\displaystyle\int_{\{u=t\}} \frac{d\mathcal{H}^{n-1}}{|\nabla u|}\right)^{p-1}}\, dt \\ &\leq \int_0^{+\infty} dt \int_{\{u=t\}} |\nabla u|^{p-1}\, d\mathcal{H}^{n-1} = \int_{\mathbb{R}^n} |\nabla u|^p\, dx\,.\end{aligned} \tag{3.15}$$

Hence (3.3) follows. □

Let us now discuss the equality case in (3.3). First, notice that if this is the case, then all inequalities in (3.15) are in fact equalities. In particular, if the second inequality in (3.15) holds as an equality, then we can conclude that the set $\{u > t\}$ is (equivalent to) a ball for $\mathcal{L}^1$-a.e. $t \geq 0$. Moreover, if the equality holds in the third inequality (where we have used Hölder inequality), the conclusion is that for $\mathcal{L}^1$-a.e. $t \geq 0$, $|\nabla u|$ is $\mathcal{H}^{n-1}$-a.e. constant on the level set $\{u = t\}$.

These are the immediate consequences of the equality case. However, with some extra work, one can prove the following, more precise, result (see [5] or [11, Theorem 2.3]).

Proposition 3.1. *Let $u \in W^{1,p}(\mathbb{R}^n)$, $p \geq 1$, a nonnegative function with compact support such that*

$$\int_{\mathbb{R}^n} |\nabla u^\star|^p \, dx = \int_{\mathbb{R}^n} |\nabla u|^p \, dx \,. \tag{3.16}$$

Then there exist a function v, equivalent to u, i.e. such that $v(x) = u(x)$ for $\mathcal{L}^n$-a.e. $x \in \mathbb{R}^n$, and a family of open balls $\{U_t\}_{t\geq 0}$ such that:

(i) $\{v > t\} = U_t$ *for* $t \in [0, \operatorname{esssup} u)$*;*

(ii) $\{v = \operatorname{ess\,sup} u\} = \bigcap_{0\leq t<\operatorname{esssup} u} \overline{U}_t,$ *and is a closed ball (possibly a point);*

(iii) v *is lower semicontinuous in* $\{v < \operatorname{esssup} u\}$*;*

(iv) *if* $v(x) \in (0, \operatorname{esssup} u)$ *and* $\mathcal{L}^n(\{u = v(x)\}) = 0$*, then* $x \in \partial U_{v(x)}$*;*

(v) *for every* $t \in (0, \operatorname{esssup} u)$ *there exists at most one point* $x \in \partial U_t$ *such that* $v(x) \neq t$*;*

(vi) *the coarea formula (3.5) holds with* $\widetilde{u}$ *replaced by* v*;*

(vii) *for* $\mathcal{L}^1$*-a.e.* $t \in (0, \operatorname{esssup} u)$, $|\nabla v(x)| = |\nabla u^\star|_{|\{u^\star=t\}}$ *for* $\mathcal{H}^{n-1}$*-a.e.* $x \in \partial U_t$ *.*

This proposition contains all the information that we can extract from equality (3.16). However it is not true in general that (3.16) implies that u coincides with $u^\star$, up to a translation in x. This can be easily seen by considering any spherically symmetric nonnegative function w, such that $\mathcal{L}^n(\{w = t_0\}) > 0$ for some $t_0 \in (0, \operatorname{esssup} w)$ and another function u whose graph agrees with the graph of w where $u < t_0$ and with a slight translated of the graph of w where $u > t_0$. Then $u^\star = w$ and (3.16) holds, but u is not spherically symmetric. What goes wrong in this example is the fact that the gradient of w (and of u) vanishes in a set of positive measure. Thus, this example suggests to introduce the following assumption,

$$(H) \qquad \mathcal{L}^n(\{0 < u^\star < \operatorname{esssup} u\} \cap \mathcal{D}^0_{u^\star}) = 0 \,.$$

Notice that we are in a situation similar to the one we were in the previous lecture when dealing with the assumption (H_1). In fact, it can be proved (see for instance [10, Lemma 3.3]) that (H) is implied by the stronger assumption

$$(H') \qquad \mathcal{L}^n(\{0 < u < \operatorname{esssup} u\} \cap \mathcal{D}^0_{u}) = 0 \,.$$

Moreover, (H) is equivalent to the absolute continuity in $(0, +\infty)$ of the distribution function μ_u and the two conditions (H) and (H') are equivalent if (3.16) holds (see [10, Lemma 3.3] again).

The following result was proved for the first time in the Sobolev setting by Brothers and Ziemer ([5]). It shows that when equality holds in (3.3), assumption (H) guarantees that u and $u^\star$ agree.

Theorem 3.2. *Let $u \in W^{1,p}(\mathbb{R}^n)$, $p > 1$, be a nonnegative function with compact support such that (3.16) and (H) hold. Then $u^\star = u$, up to a translation in x.*

Notice that the above result is in general false if $p = 1$, even in one dimension. To see this it is enough to take a function which is increasing in the interval $(-\infty, a)$ and decreasing in $(a, +\infty)$.

The starting point of the proof of Brothers and Ziemer is to observe, as we have done before, that the equality (3.1) yields that $\mathcal{L}^n$-a.e. set $\{u > t\}$ is a ball and that $|\nabla u|$ is constant on the corresponding boundary. Then the difficult part of the proof consists in exploiting assumption (H) to deduce that all these balls are concentric, i.e. u is spherically symmetric. To prove this, we shall not follow the original argument contained in ([5]), but a somewhat simpler one used in [11], which is in turn inspired to an alternative proof of Theorem 3.2 given in [17].

To this aim from now on we shall assume, without loss of generality, that u agrees with the representative v provided by Proposition 3.1 and that $U_0 = \{u > 0\}$ is a ball centered at the origin. Then, for all $0 < t < \operatorname{esssup} u$ we denote by R_t the radius of the ball U_t and set, for all $x \in U_0$,

$$\Phi(x) = \Big(\frac{\mu_u(u(x))}{\omega_n}\Big)^{1/n}. \tag{3.17}$$

To understand the role of the function Φ, observe that if $x \in U_0$ is a point such that $u(x) = t$, then $\mu_u(u(x)) = \mathcal{L}^n(U_t)$. Therefore, $\Phi(x)$ is equal to the radius of the ball U_t.

The following lemma is a crucial step toward the proof of Theorem 3.2.

Lemma 3.3. *Under the assumptions of Theorem 3.2, $\Phi \in W^{1,\infty}(U_0)$ and*

$$|\nabla\Phi(x)| = 1 \qquad \text{for } \mathcal{L}^n\text{-a.e. } x \in U_0 \setminus \{u = \operatorname{esssup} u\}. \tag{3.18}$$

Proof. We claim that $\mu_u \circ u \in W^{1,\infty}(U_0)$ and that

$$\nabla(\mu_u \circ u)(x) = -\mathcal{H}^{n-1}(\{u = u(x)\}) \frac{\nabla u(x)}{|\nabla u|_{|\{u=u(x)\}}} \chi_{\mathcal{D}_u^+}(x) \qquad \text{for } \mathcal{L}^n\text{-a.e. } x \in U_0. \tag{3.19}$$

From assumption (H), which is equivalent by (3.16) to (H'), using (3.13) and Proposition 3.1 (vii), we get that for all $0 \le t \le \operatorname{esssup} u$,

$$\mu_u(t) = \mathcal{L}^n(\{u = \operatorname{esssup} u\}) + \int_t^{+\infty} \frac{\mathcal{H}^{n-1}(\{u = s\})}{|\nabla u|_{|\{u=s\}}}\, ds\,.$$

For $\varepsilon > 0$, we set

$$\mu_{u,\varepsilon}(t) = \mathcal{L}^n(\{u = \operatorname{esssup} u\}) + \int_t^{+\infty} \frac{\mathcal{H}^{n-1}(\{u = s\})}{|\nabla u|_{|\{u=s\}} + \varepsilon}\, ds\,.$$

Clearly, $\mu_{u,\varepsilon}(t) \uparrow \mu_u(t)$ for every $t \geq 0$ as $\varepsilon \downarrow 0$. Moreover, $\mu_{u,\varepsilon}$ is Lipschitz continuous in $[0, +\infty)$, and

$$\mu'_{u,\varepsilon}(t) = -\frac{\mathcal{H}^{n-1}(\{u=t\})}{|\nabla u|_{|\{u=t\}} + \varepsilon} \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \geq 0\,,$$

whence

$$|\mu'_{u,\varepsilon}(t) - \mu'_u(t)| \leq \frac{\varepsilon \mathcal{H}^{n-1}(\{u=t\})}{|\nabla u|_{|\{u=t\}} \left(|\nabla u|_{|\{u=t\}} + \varepsilon\right)} \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \geq 0\,.$$

Thus, $\mu'_{u,\varepsilon}(t) \to \mu'_u(t)$ $\mathcal{L}^1$-a.e. in $[0, +\infty)$ as $\varepsilon \to 0$, since $|\nabla u|_{|\{u=t\}} \neq 0$ for $\mathcal{L}^1$-a.e. $t \geq 0$, in as much as $\dfrac{\mathcal{H}^{n-1}(\{u=t\})}{|\nabla u|_{|\{u=t\}}} \in L^1(0, \infty)$. This membership and the fact that

$$|\mu'_{u,\varepsilon}(t) - \mu'_u(t)| \leq \frac{\mathcal{H}^{n-1}(\{u=t\})}{|\nabla u|_{|\{u=t\}}} \qquad \text{for } \mathcal{L}^1\text{-a.e. } t \geq 0$$

entail that $\mu'_{u,\varepsilon} \to \mu'_u$ in $L^1(0, \infty)$. Hence, $\mu_{u,\varepsilon} \to \mu_u$ uniformly in $(0, \infty)$. Consequently, the functions $\mu_{u,\varepsilon} \circ u$ converge uniformly to $\mu_u \circ u$. Furthermore, by the chain rule for Sobolev functions (see e.g. [1, Theorem 3.96]),

$$\nabla(\mu_{u,\varepsilon} \circ u)(x) = \mu'_{u,\varepsilon}(u(x))\nabla u(x) = -\mathcal{H}^{n-1}(\{u = u(x)\}) \frac{\nabla u(x)}{|\nabla u|_{|\{u=u(x)\}} + \varepsilon}$$
$$\text{for } \mathcal{L}^n\text{-a.e. } x \in U_0\,.$$

The last expression clearly converges to the right-hand side of (3.19). Moreover, from Theorem 3.1, we have that $\mathcal{H}^{n-1}(\{u = t\}) = \mathcal{H}^{n-1}(\partial U_t) \leq n\omega_n R_0^{n-1}$ for $\mathcal{L}^1$-a.e. $t \in (0, \operatorname{ess\,sup} u)$. Thus,

$$|\nabla(\mu_{u,\varepsilon} \circ u)(x)| \leq n\omega_n R_0^{n-1} \qquad \text{for } \mathcal{L}^n\text{-a.e. } x \in U_0\,.$$

By dominated convergence, $\nabla(\mu_{u,\varepsilon} \circ u)$ converges to the right-hand side of (3.19) in $L^1(U_0)$. Hence, the claim follows.
To conclude the proof let us now observe that for all $t \in (0, \operatorname{esssup} u)$, $\mu_u(u(x)) \geq \mu_u(t) > 0$ for all $x \in U_0 \setminus \overline{U}_t$. Therefore, we can compute the derivatives of Φ in $U_0 \setminus \overline{U}_t$ by the usual chain rule formula for Sobolev functions, thus getting, from (3.19), that for $\mathcal{L}^n$-a.e. $x \in U_0 \setminus \overline{U}_t$

$$\begin{aligned} \nabla\Phi(x) &= -\frac{1}{n\omega_n^{1/n}} \left(\mu_u(u(x))\right)^{\frac{1-n}{n}} \mathcal{H}^{n-1}(\{u=u(x)\}) \frac{\nabla u(x)}{|\nabla u|_{|\{u=u(x)\}}} \chi_{\mathcal{D}_u^+}(x) \\ &= -\frac{\nabla u(x)}{|\nabla u|_{|\{u=u(x)\}}} \chi_{\mathcal{D}_u^+}(x). \end{aligned} \tag{3.20}$$

Then, (3.18) follows immediately from Proposition 3.1 (ii). In particular, this proves that Φ is a $W^{1,\infty}$ function in the open set $U_0 \setminus \{u = \operatorname{esssup} u\}$ (which is the difference of an open ball and a closed one). To conclude the proof it is enough to observe, that since $\mu_{u,\varepsilon} \circ u \in W^{1,\infty}(U_0)$, Φ has a continuous representative in U_0 which is constant on the closed ball $\{u = \operatorname{esssup} u\}$. □

Notice that Lemma 3.3 is not telling us that Φ is Lipschitz continuous in U_0. It just says that Φ coincides $\mathcal{L}^n$-a.e. with a Lipschitz continuous function, with Lipschitz constant less than or equal to one. Therefore, we may only conclude that there exists a set N_0, with $\mathcal{L}^n(N_0) = 0$ such that

$$|\Phi(x) - \Phi(y)| \leq |x - y| \qquad \text{for all } x, y \in U_0 \setminus N_0 \,. \tag{3.21}$$

However, this information is enough to achieve the proof of Theorem 3.2.

Proof of Theorem 3.2. From the coarea formula (3.5), recalling (iv), (v) and (vi) of Proposition 3.1, we get

$$\int_0^{+\infty} \mathcal{H}^{n-1}(\partial U_t \cap N_0)\, dt = \int_{N_0} |\nabla u|\, dx = 0 \,.$$

Therefore, there exists a set $I_0 \subset (0, +\infty)$, with $\mathcal{L}^1(I_0) = 0$, such that

$$\mathcal{H}^{n-1}(\partial U_t \cap N_0) = 0 \qquad \text{for all } t \in (0, +\infty) \setminus I_0 \,.$$

Let us now fix $0 < s < t < \operatorname{esssup} u$, with $s, t \notin I_0$. From Proposition 3.1 (v) we can find two sequences $\{x_h\} \subset \partial U_s \setminus N_0$ and $\{y_h\} \subset \partial U_t \setminus N_0$ such that

$$u(x_h) = s, \quad u(y_h) = t \quad \text{for all } h, \qquad |x_h - y_h| \to \operatorname{dist}(\partial U_s, \partial U_t) \,.$$

Since $\Phi(x_h) = R_s$ and $\Phi(y_h) = R_t$, from (3.21) we get that

$$|R_s - R_t| \leq \lim_{h \to +\infty} |x_h - y_h| = \operatorname{dist}(\partial U_s, \partial U_t)$$

Hence, U_s and U_t are concentric balls. From this one can easily conclude that U_s and U_t are indeed concentric for all $0 \leq s < t \leq \operatorname{esssup} u$, thus proving the assertion. □

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