# 3

# $\mathbb{R}$-Trees and 0-Hyperbolic Spaces

## 3.1 Geodesic and Geodesically Linear Metric Spaces

We follow closely the development in [39] in this section and leave some of the more straightforward proofs to the reader.

**Definition 3.1.** *A* segment *in a metric space* $(X, d)$ *is the image of an isometry* $\alpha : [a, b] \to X$*. The end points of the segment are* $\alpha(a)$ *and* $\alpha(b)$*.*

**Definition 3.2.** *A metric space* $(X, d)$ *is* geodesic *if for all* $x, y \in X$*, there is a segment in* $X$ *with endpoints* $\{x, y\}$*, and* $(X, d)$ *is* geodesically linear *if, for all* $x, y \in X$*, there is a* **unique** *segment in* $X$ *with endpoints* $\{x, y\}$*.*

*Example 3.3.* Euclidean space $\mathbb{R}^d$ is geodesically linear. The closed annulus $\{z \in \mathbb{R}^2 : 1 \leqslant |z| \leqslant 2\}$ is not geodesic in the metric inherited from $\mathbb{R}^2$, but it is geodesic in the metric defined by taking the infimum of the Euclidean lengths of piecewise-linear paths between two points. The closed annulus is not geodesically linear in this latter metric: for example, a pair of points of the form $z$ and $-z$ are the endpoints of two segments – see Figure 3.1. The open annulus $\{z \in \mathbb{R}^2 : 1 < |z| < 2\}$ is not geodesic in the metric defined by taking the infimum over all piecewise-linear paths between two points: for example, there is no segment that has a pair of points of the form $z$ and $-z$ as its endpoints.

**Lemma 3.4.** *Consider a metric space* $(X, d)$*. Let* $\sigma$ *be a segment in* $X$ *with endpoints* $x$ *and* $z$*, and let* $\tau$ *be a segment in* $X$ *with endpoints* $y$ *and* $z$*.*

*(a) Suppose that* $d(u, v) = d(u, z) + d(z, v)$ *for all* $u \in \sigma$ *and* $v \in \tau$*. Then* $\sigma \cup \tau$ *is a segment with endpoints* $x$ *and* $y$*.*
*(b) Suppose that* $\sigma \cap \tau = \{z\}$ *and* $\sigma \cup \tau$ *is a segment. Then* $\sigma \cup \tau$ *has endpoints* $x$ *and* $y$*.*

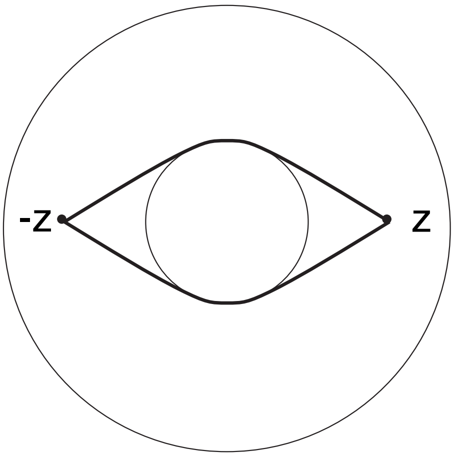


**Fig. 3.1.** Two geodesics with the same endpoints in the intrinsic path length metric on the annulus

**Lemma 3.5.** *Let $(X, d)$ be a geodesic metric space such that if two segments of $(X, d)$ intersect in a single point, which is an endpoint of both, then their union is a segment. Then $(X, d)$ is a geodesically linear metric space.*

*Proof.* Let $\sigma, \tau$ be segments, both with endpoints $u, v$. Fix $w \in \sigma$, and define $w'$ to be the point of $\tau$ such that $d(u, w) = d(u, w')$ (so that $d(v, w) = d(v, w')$). We have to show $w = w'$.

Let $\rho$ be a segment with endpoints $w, w'$. Now $\sigma = \sigma_1 \cup \sigma_2$, where $\sigma_1$ is a segment with endpoints $u, w$, and $\sigma_2$ is a segment with endpoints $w, v$ – see Figure 3.2.

We claim that either $\sigma_1 \cap \rho = \{w\}$ or $\sigma_2 \cap \rho = \{w\}$. This is so because if $x \in \sigma_1 \cap \rho$ and $y \in \sigma_2 \cap \rho$, then $d(x, y) = d(x, w) + d(w, y)$, and either $d(w, y) = d(w, x) + d(x, y)$ or $d(w, x) = d(w, y) + d(x, y)$, depending on how $x, y$ are situated in the segment $\rho$. It follows that either $x = w$ or $y = w$, establishing the claim.

Now, if $\sigma_1 \cap \rho = \{w\}$, then, by assumption, $\sigma_1 \cup \rho$ is a segment, and by Lemma 3.4(b) its endpoints are $u, w'$. Since $w \in \sigma_1 \cup \rho$, $d(u, w') = d(u, w) + d(w, w')$, so $w = w'$. Similarly, if $\sigma_2 \cap \rho = \{w\}$ then $w = w'$. □

**Lemma 3.6.** *Consider a geodesically linear metric space $(X, d)$.*

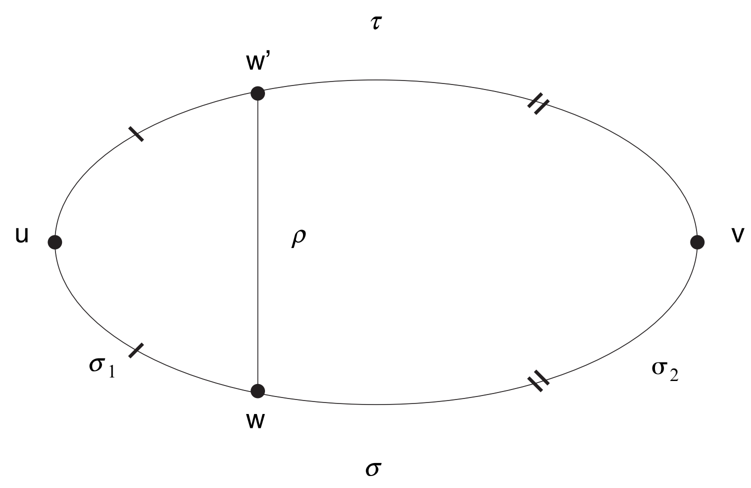


**Fig. 3.2.** Construction in the proof of Lemma 3.5

*(i) Given points $x, y, z \in X$, write $\sigma$ for the segment with endpoints $x, y$. Then $z \in \sigma$ if and only if $d(x, y) = d(x, z) + d(z, y)$.*
*(ii) The intersection of two segments in $X$ is also a segment if it is non-empty.*
*(iii) Given $x, y \in X$, there is a unique isometry $\alpha : [0, d(x, y)] \to X$ such that $\alpha(0) = x$ and $\alpha(d(x, y)) = y$. Write $[x, y]$ for the resulting segment. If $u, v \in [x, y]$, then $[u, v] \subseteq [x, y]$.*

## 3.2 0-Hyperbolic Spaces

**Definition 3.7.** *For $x, y, v$ in a metric space $(X, d)$, set*

$$(x \cdot y)_v := \frac{1}{2}(d(x, v) + d(y, v) - d(x, y))$$

*– see Figure 3.3.*

*Remark 3.8.* For $x, y, v, t \in X$,

$$0 \leqslant (x \cdot y)_v \leqslant d(x, v) \wedge d(y, v)$$

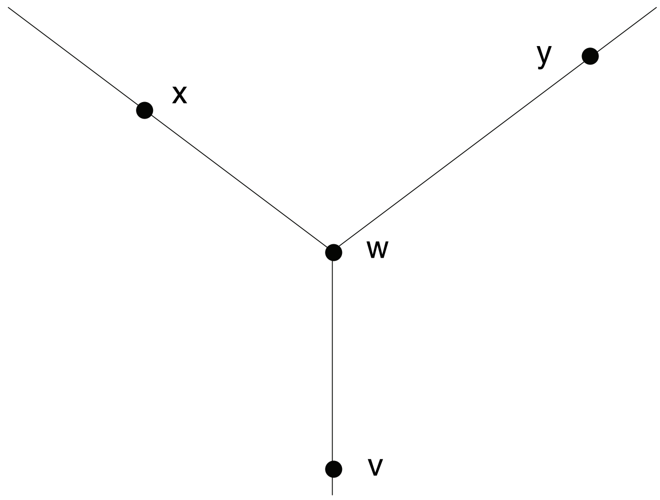


**Fig. 3.3.** $(x \cdot y)_v = d(w, v)$ in this tree

and

$$(x \cdot y)_t = d(t, v) + (x \cdot y)_v - (x \cdot t)_v - (y \cdot t)_t.$$

**Definition 3.9.** *A metric space $(X, d)$ is* 0-hyperbolic with respect to $v$ *if for all $x, y, z \in X$*

$$(x \cdot y)_v \geqslant (x \cdot z)_v \wedge (y \cdot z)_v$$

*– see Figure 3.4.*

**Lemma 3.10.** *If the metric space $(X, d)$ is 0-hyperbolic with respect to some point of $X$, then $(X, d)$ is 0-hyperbolic with respect to all points of $X$.*

*Remark 3.11.* In light of Lemma 3.10, we will refer to a metric space that is 0-hyperbolic with respect to one, and hence all, of its points as simply being 0-hyperbolic. Note that any subspace of a 0-hyperbolic metric space is also 0-hyperbolic.

**Lemma 3.12.** *The metric space $(X, d)$ is 0-hyperbolic if and only if*

$$d(x, y) + d(z, t) \leqslant \max\{d(x, z) + d(y, t),\ d(y, z) + d(x, t)\}$$

*for all $x, y, z, t \in X$,*

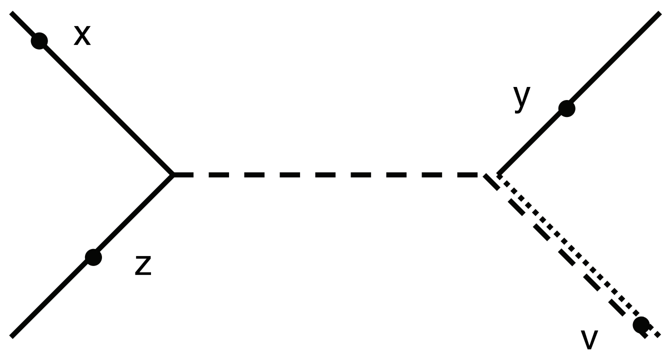


**Fig. 3.4.** The 0-hyperbolicity condition holds for this tree. Here $(x \cdot y)_v$ and $(y \cdot z)_v$ are both given by the length of the dotted segment, and $(x \cdot z)_v$ is the length of the dashed segment. Note that $(x \cdot y)_v \geqslant (x \cdot z)_v \wedge (y \cdot z)_v$, with similar inequalities when $x, y, z$ are permuted.

*Remark 3.13.* The set of inequalities in Lemma 3.12 is usually called the *four-point condition* – see Figure 3.5.

*Example 3.14.* Write $C(\mathbb{R}_+)$ for the space of continuous functions from $\mathbb{R}_+$ into $\mathbb{R}$. For $e \in C(\mathbb{R}_+)$, put $\zeta(e) := \inf\{t > 0 : e(t) = 0\}$ and write

$$U := \left\{ e \in C(\mathbb{R}_+) : \begin{array}{r} e(0) = 0,\, \zeta(e) < \infty, \\ e(t) > 0 \text{ for } 0 < t < \zeta(e), \\ \text{and } e(t) = 0 \text{ for } t \geqslant \zeta(e) \end{array} \right\}$$

for the space of positive excursion paths. Set $U^\ell := \{e \in U : \zeta(e) = \ell\}$.

We associate each $e \in U^\ell$ with a compact metric space as follows. Define an equivalence relation $\sim_e$ on $[0, \ell]$ by letting

$$u_1 \sim_e u_2, \qquad \text{iff} \qquad e(u_1) = \inf_{u \in [u_1 \wedge u_2, u_1 \vee u_2]} e(u) = e(u_2).$$

Consider the following semi-metric on $[0, \ell]$

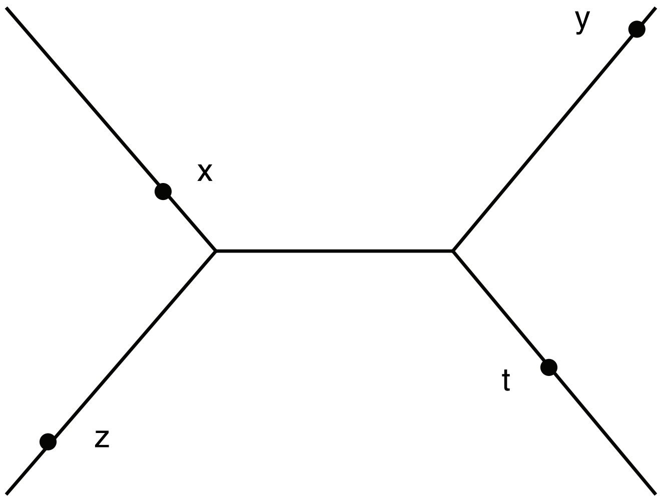


**Fig. 3.5.** The four-point condition holds on a tree: $d(x,z) + d(y,t) \leqslant d(x,y) + d(z,t) = d(x,t) + d(y,z)$

$$d_{T_e}(u_1, u_2) := e(u_1) - 2 \inf_{u \in [u_1 \wedge u_2, u_1 \vee u_2]} e(u) + e(u_2),$$

that becomes a true metric on the quotient space $T_e := [0,\ell]\big|_{\sim_e}$ – see Figure 3.6.

It is straightforward to check that the quotient map from $[0,\ell]$ onto $T_e$ is continuous with respect to $d_{T_e}$. Thus, $(T_e, d_{T_e})$ is path-connected and compact as the continuous image of a metric space with these properties. In particular, $(T_e, d_{T_e})$ is complete. It is not difficult to check that $(T_e, d_{T_e})$ satisfies the four-point condition, and, hence, is 0-hyperbolic.

## 3.3 ℝ-Trees

### 3.3.1 Definition, Examples, and Elementary Properties

**Definition 3.15.** *An* ℝ-tree *is a metric space* $(X,d)$ *with the following properties.*

*Axiom (a) The space* $(X,d)$ *is geodesic.*

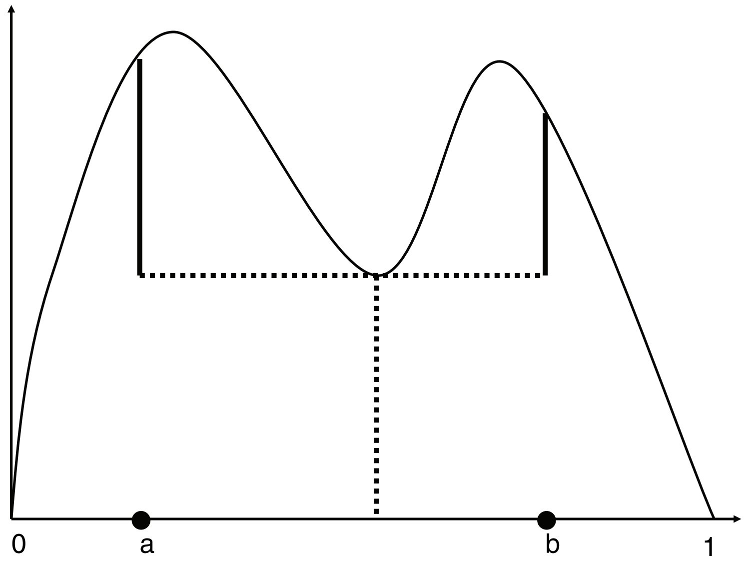


**Fig. 3.6.** An excursion path on $[0,1]$ determines a distance between the points $a$ and $b$

*Axiom (b) If two segments of $(X,d)$ intersect in a single point, which is an endpoint of both, then their union is a segment.*

*Example 3.16.* Finite trees with edge lengths (sometimes called weighted trees) are examples of ℝ-trees. To be a little more precise, we don't think of such a tree as just being its finite set of vertices with a collection of distances between them, but regard the edges connecting the vertices as also being part of the metric space.

*Example 3.17.* Take $X$ to be the plane $\mathbb{R}^2$ equipped with the metric

$$d((x_1,x_2),(y_1,y_2)) := \begin{cases} |x_2 - y_2|, & \text{if } x_1 = y_1, \\ |x_1 - y_1| + |x_2| + |y_2|, & \text{if } x_1 \neq y_1. \end{cases}$$

That is, we think of the plane as being something like the skeleton of a fish, in which the horizontal axis is the spine and vertical lines are the ribs. In order to compute the distance between two points on different ribs, we use the length of the path that goes from the first point to the spine, then along the spine to the rib containing the second point, and then along that second rib – see Figure 3.7.

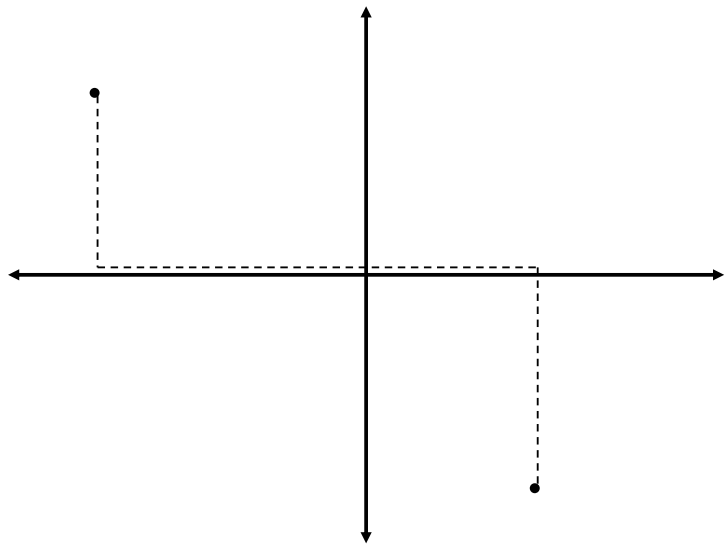

**Fig. 3.7.** The distance between two points of $\mathbb{R}^2$ in the metric of Example 3.17 is the (Euclidean) length of the dashed path

*Example 3.18.* Consider the collection $\mathcal{T}$ of bounded subsets of $\mathbb{R}$ that contain their supremum. We can think of the elements of $\mathcal{T}$ as being arrayed in a tree–like structure in the following way. Using genealogical terminology, write $h(B) := \sup B$ for the real–valued *generation* to which $B \in \mathcal{T}$ belongs and $B|t := (B \cap ]-\infty, t]) \cup \{t\} \in \mathcal{T}$ for $t \leqslant h(B)$ for the *ancestor* of $B$ in generation $t$. For $A, B \in \mathcal{T}$ the generation of the *most recent common ancestor* of $A$ and $B$ is $\tau(A,B) := \sup\{t \leqslant h(A) \wedge h(B) : A|t = B|t\}$. That is, $\tau(A,B)$ is the generation at which the lineages of $A$ and $B$ diverge. There is a natural genealogical distance on $\mathcal{T}$ given by

$$D(A,B) := [h(A) - \tau(A,B)] + [h(B) - \tau(A,B)].$$

See Figure 3.8.

It is not difficult to show that the metric space $(\mathcal{T}, D)$ is a $\mathbb{R}$-tree. For example, the segment with end-points $A$ and $B$ is the set $\{A|t : \tau(A,B) \leqslant t \leqslant h(A)\} \cup \{B|t : \tau(A,B) \leqslant t \leqslant h(B)\}$.

The metric space $(\mathcal{T}, D)$ is essentially "the" real tree of [47, 137] (the latter space has as its points the bounded subsets of $\mathbb{R}$ that contain their infimum and the corresponding metric is such that the map from $(\mathcal{T}, D)$ into this latter space given by $A \mapsto -A$ is an isometry). With a slight abuse of

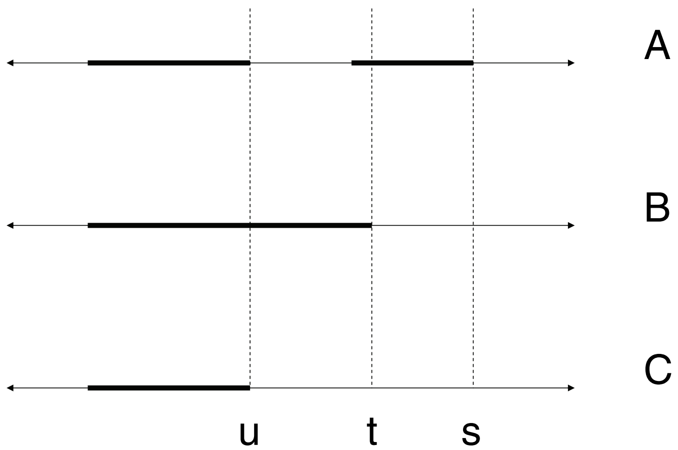


**Fig. 3.8.** The set $C$ is the most recent common ancestor of the sets $A, B \subset \mathbb{R}$ thought of as points of "the" real tree of Example 3.18. The distance $D(A, B)$ is $[s - u] + [t - u]$.

nomenclature, we will refer here to $(\mathcal{T}, D)$ as the real tree. Note that $(\mathcal{T}, D)$ is huge: for example, the removal of any point shatters $\mathcal{T}$ into uncountably many connected components.

*Example 3.19.* We will see in Example 3.37 that the compact 0-hyperbolic metric space $(T_e, d_{T_e})$ of Example 3.14 that arises from an excursion path $e \in U$ is a $\mathbb{R}$-tree.

The following result is a consequence of Axioms (a) and (b) and Lemma 3.5.

**Lemma 3.20.** *An $\mathbb{R}$-tree is geodesically linear. Moreover, if $(X, d)$ is a $\mathbb{R}$-tree and $x, y, z \in X$ then $[x, y] \cap [x, z] = [x, w]$ for some unique $w \in X$.*

*Remark 3.21.* It follows from Lemma 3.4, Lemma 3.6 and Lemma 3.20 that Axioms (a) and (b) together imply following condition that is stronger than Axiom (b):

Axiom (b') If $(X, d)$ is a $\mathbb{R}$-tree, $x, y, z \in X$ and $[x, y] \cap [x, z] = \{x\}$, then $[x, y] \cup [x, z] = [y, z]$

**Lemma 3.22.** *Let $x, y, z$ be points of a $\mathbb{R}$-tree $(X, d)$, and write $w$ for the unique point such that $[x, y] \cap [x, z] = [x, w]$.*

*(i) The points $x, y, z, w$ and the segments connecting them form a Y shape, with $x, y, z$ at the tips of the Y and $w$ at the center. More precisely, $[y, w] \cap [w, z] = \{w\}$, $[y, z] = [y, w] \cup [w, z]$ and $[x, y] \cap [w, z] = \{w\}$.*
*(ii) If $y' \in [x, y]$ and $z' \in [x, z]$, then*

$$d(y', z') = \begin{cases} |d(x, y') - d(x, z')|, & \text{if } d(x, y') \wedge d(x, z') \leqslant d(x, w), \\ d(x, y') + d(x, z') - 2d(x, w), & \text{otherwise.} \end{cases}$$

*(iii) The "centroid" $w$ depends only on the set $\{x, y, z\}$, not on the order in which the elements are written.*

*Proof.* (i) Since $y, w \in [x, y]$, we have $[y, w] \subseteq [x, y]$. Similarly, $[w, z] \subseteq [x, z]$. So, if $u \in [y, w] \cap [w, z]$, then $u \in [x, y] \cap [x, z] = [x, w]$. Hence $u \in [x, w] \cap [y, w] = \{w\}$ (because $w \in [x, y]$). Thus, $[y, w] \cap [w, z] = \{w\}$, and $[y, z] = [y, w] \cup [w, z]$ by Axiom (b').

Now, since $w \in [x, y]$, we have $[x, y] = [x, w] \cup [w, y]$, so $[x, y] \cap [w, z] = ([x, w] \cap [w, z]) \cup ([y, w] \cap [w, z])$, and both intersections are equal to $\{w\}$ ($w \in [x, z]$).
(ii) If $d(x, y') \leqslant d(x, w)$ then $y', z' \in [x, z]$, and so $d(y', z') = |d(x, y') - d(x, z')|$. Similarly, if $d(x, z') \leqslant d(x, w)$, then $y', z' \in [x, y]$, and once again $d(y', z') = |d(x, y') - d(x, z')|$.

If $d(x, y') > d(x, w)$ and $d(x, z') > d(x, w)$, then $y' \in [y, w]$ and $z' \in [z, w]$. Hence, by part (i),

$$\begin{aligned} d(y', z') &= d(y', w) + d(w, z') \\ &= (d(x, y') - d(x, w)) + (d(x, z') - d(x, w) \\ &= d(x, y') + d(x, z') - 2d(x, w). \end{aligned}$$

(iii) We have by part (i) that

$$\begin{aligned} [y, x] \cap [y, z] &= [y, x] \cap ([y, w] \cup [w, z]) \\ &= [y, w] \cup ([y, x] \cap [w, z]) \\ &= [y, w] \cup ([y, w] \cap [w, z]) \cup ([w, x] \cap [w, z]) \end{aligned}$$

Now $[y, w] \cap [w, z] = \{w\}$ by part (1) and $[w, x] \cap [w, z] = \{w\}$ since $w \in [x, z]$. Hence, $[y, x] \cap [y, z] = [y, w]$. Similarly, $[z, x] \cap [z, y] = [z, w]$, and part (iii) follows. □

**Definition 3.23.** *In the notation of Lemma 3.22, write $Y(x, y, z) := w$ for the* centroid *of $\{x, y, z\}$.*

*Remark 3.24.* Note that we have

$$[x, y] \cap [w, z] = [x, z] \cap [w, y] = [y, z] \cap [w, x] = \{w\}.$$

Also, $d(x, w) = (y \cdot z)_x$, $d(y, w) = (x \cdot z)_y$, and $d(z, w) = (x \cdot y)_z$. In Figure 3.3, $Y(x, y, v) = w$.

**Corollary 3.25.** *Consider a $\mathbb{R}$-tree $(X, d)$ and points $x_0, x_1, \ldots, x_n \in X$. The segment $[x_0, x_n]$ is a subset of $\bigcup_{i=1}^{n}[x_{i-1}, x_i]$.*

*Proof.* If $n = 2$, then, by Lemma 3.22,

$$[x_0, x_2] = [x_0, Y(x_0, x_1, x_2)] \cup [Y(x_0, x_1, x_2), x_2] \subseteq [x_0, x_1] \cup [x_1, x_2].$$

If $n > 2$, then $[x_0, x_n] \subseteq [x_0, x_{n-1}] \cup [x_{n-1}, x_n]$ by the case $n = 2$, and the result follows by induction on $n$. □

**Lemma 3.26.** *Consider a $\mathbb{R}$-tree $(X, d)$. Let $\alpha : [a, b] \to X$ be a continuous map. If $x = \alpha(a)$ and $y = \alpha(b)$, then $[x, y]$ is a subset of the image of $\alpha$.*

*Proof.* Let $A$ denote the image of $\alpha$. Since $A$ is a closed subset of $X$ (being compact as the image of a compact interval by a continuous map), it is enough to show that every point of $[x, y]$ is within distance $\epsilon$ of $A$, for all $\epsilon > 0$.

Given $\epsilon > 0$, the collection $\{\alpha^{-1}(B(x, \epsilon/2)) : x \in A\}$ is an open covering of the compact metric space $[a, b]$, so there is a number $\delta > 0$ such that any two points of $[a, b]$ that are distance less than $\delta$ apart belong to some common set in the cover.

Choose a partition of $[a, b]$, say $a = t_0 < \cdots < t_n = b$, so that for $1 \leqslant i \leqslant n$ we have $t_i - t_{i-1} < \delta$, and, therefore, $d(\alpha(t_{i-1}), \alpha(t_i)) < \epsilon$. Then all points of $[\alpha(t_{i-1}), \alpha(t_i)]$ are at distance less than $\epsilon$ from $\{\alpha(t_{i-1}), \alpha(t_i)\} \subseteq A$ for $1 \leqslant i \leqslant n$. Finally, $[x, y] \subseteq \bigcup_{i=1}^{n}[\alpha(t_{i-1}), \alpha(t_i)]$, by Corollary 3.25. □

**Definition 3.27.** *For points $x_0, x_1, \ldots, x_n$ in a $\mathbb{R}$-tree $(X, d)$, write $[x_0, x_n] = [x_0, x_1, \ldots, x_n]$ to mean that, if $\alpha : [0, d(x_0, x_n)] \to X$ is the unique isometry with $\alpha(0) = x_0$ and $\alpha(d(x_0, x_n)) = x_n$, then $x_i = \alpha(a_i)$, for some $a_0, a_1, a_2, \ldots, a_n$ with $0 = a_0 \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_n = d(x_0, x_n)$.*

**Lemma 3.28.** *Consider a $\mathbb{R}$-tree $(X, d)$. If $x_0, \ldots, x_n \in X$, $x_i \neq x_{i+1}$ for $1 \leqslant i \leqslant n-2$ and $[x_{i-1}, x_i] \cap [x_i, x_{i+1}] = \{x_i\}$ for $1 \leqslant i \leqslant n-1$, then $[x_0, x_n] = [x_0, x_1, \ldots, x_n]$.*

*Proof.* There is nothing to prove if $n \leqslant 2$. Suppose $n = 3$. We can assume $x_0 \neq x_1$ and $x_2 \neq x_3$, otherwise there is again nothing to prove. Let $w = Y(x_0, x_2, x_3)$.

Now $w \in [x_0, x_2]$ and $x_1 \in [x_0, x_2]$, so $[x_2, w] \cap [x_2, x_1] = [x_2, v]$, where $v$ is either $w$ or $x_1$, depending on which is closer to $x_2$. But $[x_2, w] \cap [x_2, x_1] \subseteq [x_2, x_3] \cap [x_2, x_1] = \{x_2\}$, so $v = x_2$.

Since $x_1 \neq x_2$, we conclude that $w = x_2$. Hence $[x_0, x_2] \cap [x_2, x_3] = \{x_2\}$, which implies $[x_0, x_3] = [x_0, x_2, x_3] = [x_0, x_1, x_2, x_3]$.

Now suppose $n > 3$. By induction,

$$[x_0, x_{n-1}] = [x_0, x_1, \ldots, x_{n-2}, x_{n-1}] = [x_0, x_{n-2}, x_{n-1}].$$

By the $n = 3$ case,

$$[x_0, x_n] = [x_0, x_{n-2}, x_{n-1}, x_n] = [x_0, x_1, \ldots, x_{n-2}, x_{n-1}, x_n]$$

as required. □

### 3.3.2 ℝ-Trees are 0-Hyperbolic

**Lemma 3.29.** *A* $\mathbb{R}$*-tree* $(X, d)$ *is* 0*-hyperbolic.*

*Proof.* Fix $v \in X$. We have to show

$$(x \cdot y)_v \geqslant (x \cdot z)_v \wedge (y \cdot z)_v$$
$$(x \cdot z)_v \geqslant (x \cdot y)_v \wedge (y \cdot z)_v$$
$$(y \cdot z)_v \geqslant (x \cdot y)_v \wedge (x \cdot z)_v$$

for all $x, y, z$. Note that if this is so, then one of $(x \cdot y)_v, (x \cdot z)_v, (y \cdot z)_v$ is at least as great as the other two, which are equal.

Let $q = Y(x, v, y)$, $r = Y(y, v, z)$, and $s = Y(z, v, x)$. We have $(x \cdot y)_v = d(v, q)$, $(y \cdot z)_v = d(v, s)$, and $(z \cdot x)_v = d(v, r)$. We may assume without loss of generality that

$$d(v, q) \leqslant d(v, r) \leqslant d(v, s),$$

in which case have to show that $q = r$ – see Figure 3.9.

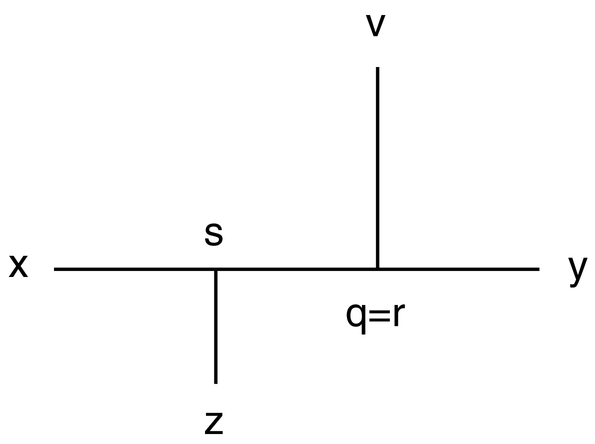


**Fig. 3.9.** The configuration demonstrated in the proof of Lemma 3.29

Now $r, s \in [v, z]$ by definition, and $d(v, r) \leqslant d(v, s)$, so that $[v, s] = [v, r, s]$. Also, by definition of $s$, $[v, x] = [v, s, x] = [v, r, s, x]$. Hence $r \in [v, x] \cap [v, y] = [v, q]$. Since $d(v, q) \leqslant d(v, r)$, we have $q = r$, as required. □

*Remark 3.30.* Because any subspace of a 0-hyperbolic space is still 0-hyperbolic, we can't expect that the converse to Lemma 3.29 holds. However, we will see in Theorem 3.38 that any 0-hyperbolic space is isometric to a subspace of a ℝ-tree.

### 3.3.3 Centroids in a 0-Hyperbolic Space

**Definition 3.31.** *A set* $\{a, b, c\} \subset \mathbb{R}$ *is called an* isosceles triple *if*

$$a \geqslant b \wedge c,\ b \geqslant c \wedge a,\ \text{and}\ c \geqslant a \wedge b.$$

*(This means that at least two of* $a, b, c$ *are equal, and not greater than the third.)*

*Remark 3.32.* The metric space $(X, d)$ is 0-hyperbolic if and only if $(x \cdot y)_v, (x \cdot z)_v, (y \cdot z)_v$ is an isosceles triple for all $x, y, z, v \in X$.

**Lemma 3.33.** *(i) If* $\{a, b, c\}$ *is any triple then*

$$\{a \wedge b, b \wedge c, c \wedge a\}$$

*is an isosceles triple.*
*(ii) If* $\{a, b, c\}$ *and* $\{d, e, f\}$ *are isosceles triples then so is*

$$\{a \wedge d, b \wedge e, c \wedge f\}.$$

**Lemma 3.34.** *Consider a* 0*-hyperbolic metric space* $(X, d)$*. Let* $\sigma, \tau$ *be segments in* $X$ *with endpoints* $v, x$ *and* $v, y$ *respectively. Write* $x \cdot y := (x \cdot y)_v$.

*(i) If* $x' \in \sigma$*, then* $x' \in \tau$ *if and only if* $d(v, x') \leqslant x \cdot y$.
*(ii) If* $w$ *is the point of* $\sigma$ *at distance* $x \cdot y$ *from* $v$*, then* $\sigma \cap \tau$ *is a segment with endpoints* $v$ *and* $w$.

*Proof.* If $d(x', v) > d(y, v)$ then $x' \notin \tau$, and $d(x', v) > x \cdot y$, so we can assume that $d(x', v) < d(y, v)$. Let $y'$ be the point in $\tau$ such that $d(v, x') = d(v, y')$. Define

$$\alpha = x \cdot y,\ \beta = x' \cdot y,\ \gamma = x \cdot x',\ \alpha' = x' \cdot y'.$$

Since $x' \in \sigma$ and $y' \in \tau$, we have $\gamma = d(v, x') = d(v, y') = y \cdot y'$. Hence, $(\alpha, \beta, \gamma)$ and $(\alpha', \beta, \gamma)$ are isosceles triples. We have to show that $x' \in \tau$ if and only if $\alpha \geqslant \gamma$. The two cases $\alpha < \gamma$ and $\alpha \geqslant \gamma$ are illustrated in Figure 3.10 and Figure 3.11 respectively.

Now,

$$\beta = x' \cdot y \leqslant d(v, x') = x \cdot x' = \gamma.$$

Also,

$$\alpha' = d(v, x') - \frac{1}{2} d(x', y') = \gamma - \frac{1}{2} d(x', y') \leqslant \gamma$$

and

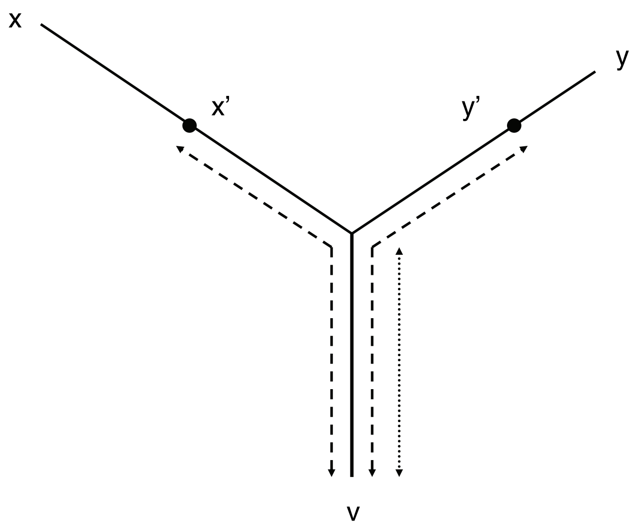


**Fig. 3.10.** First case of the construction in the proof of Lemma 3.34. Here $\gamma$ is either of the two equal dashed lengths and $\alpha = \beta = \alpha'$ is the dotted length. As claimed, $\alpha < \gamma$ and $x' \notin \tau$.

$$x' \in \tau \Leftrightarrow x' = y' \Leftrightarrow d(x', y') = 0 \Leftrightarrow \alpha' = \gamma.$$

Moreover, $\alpha' = \gamma$ if and only if $\beta = \gamma$, because $(\alpha', \beta, \gamma)$ is an isosceles triple and $\alpha', \beta \leqslant \gamma$. Since $(\alpha, \beta, \gamma)$ is also an isosceles triple, the equality $\beta = \gamma$ is equivalent to the inequality $\alpha \geqslant \gamma$. This proves part (i). Part (ii) of the lemma follows immediately. □

**Lemma 3.35.** *Consider a* 0*-hyperbolic metric space* $(X, d)$*. Let* $\sigma, \tau$ *be segments in* $X$ *with endpoints* $v, x$ *and* $v, y$ *respectively. Set* $x \cdot y := (x \cdot y)_v$*. Write* $w$ *for the point of* $\sigma$ *at distance* $x \cdot y$ *from* $v$ *(so that* $w$ *is an endpoint of* $\sigma \cap \tau$ *by Lemma 3.34). Consider two points* $x' \in \sigma$*,* $y' \in \tau$*, and suppose* $d(x', v) \geqslant x \cdot y$ *and* $d(y', v) \geqslant x \cdot y$*. Then*

$$d(x', y') = d(x', w) + d(y', w).$$

*Proof.* The conclusion is clear if $d(x', v) = x \cdot y$ (when $x' = w$) or $d(y', v) = x \cdot y$ (when $y' = w$), so we assume that $d(x', v) > x \cdot y$ and $d(y', v) > x \cdot y$. As in the proof of Lemma 3.34, we put

$$\alpha = x \cdot y,\ \beta = x' \cdot y,\ \gamma = x \cdot x',\ \alpha' = x' \cdot y',$$

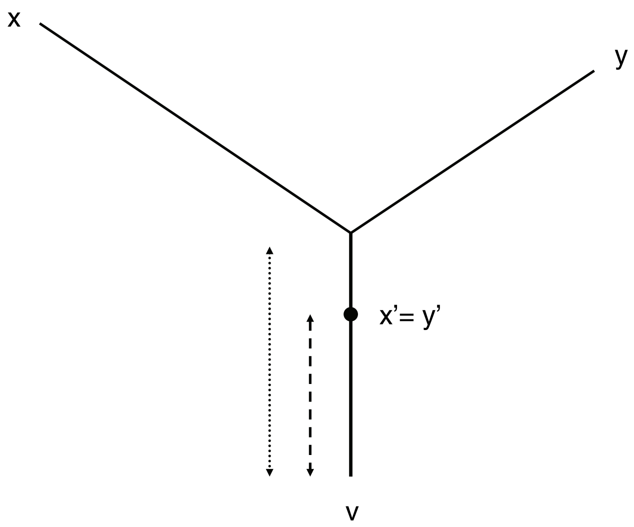


**Fig. 3.11.** Second case of the construction in the proof of Lemma 3.34. Here $\gamma = \beta = \alpha'$ is the dashed length and $\alpha$ is the dotted length. As claimed, $\alpha \geqslant \gamma$ and $x' \in \tau$.

and we also put $\gamma' = y \cdot y'$, so that $\gamma = d(v, x')$ and $\gamma' = d(v, y')$. Thus, $\alpha < \gamma$. Hence, $\alpha = \beta$ since $(\alpha, \beta, \gamma)$ is an isosceles triple. Also, $\alpha < \gamma'$, so that $\beta < \gamma'$. Hence, $\alpha = \alpha' = \beta$ because $(\alpha', \beta, \gamma)$ is an isosceles triple.

By definition of $\alpha'$,

$$\begin{aligned} d(x', y') &= d(v, x') + d(v, y') - 2\alpha' \\ &= d(v, x') + d(v, y') - 2\alpha. \end{aligned}$$

Since $w \in \sigma \cap \tau$, $\alpha = d(v, w) < d(v, x'), d(v, y')$ and $\sigma, \tau$ are segments, it follows that

$$d(x', w) = d(v, x') - \alpha$$

and

$$d(y', w) = d(v, y') - \alpha,$$

and the lemma follows on adding these equations. □

### 3.3.4 An Alternative Characterization of ℝ-Trees

**Lemma 3.36.** *Consider a* $0$*-hyperbolic metric space* $(X, d)$*. Suppose that there is a point* $v \in X$ *such that for every* $x \in X$ *there is a segment with endpoints* $v, x$*. Then* $(X, d)$ *is a* ℝ*-tree.*

*Proof.* Take $x, y \in X$ and let $\sigma, \tau$ be segments with endpoints $v, x$ and $v, y$ respectively.

By Lemma 3.34, if $w$ is the point of $\sigma \cap \tau$ at distance $(x \cdot y)_v$ from $v$, then $\sigma$ is the union $(\sigma \cap \tau) \cup \sigma_1$, where

$$\sigma_1 := \{u \in \sigma \, : \, d(v, u) \geqslant (x \cdot y)_v\}$$

is a segment with endpoints $w, x$. Similarly, $\tau$ is the union $(\sigma \cap \tau) \cup \tau_1$, where

$$\tau_1 := \{u \in \tau \, : \, d(v, u) \geqslant (x \cdot y)_v\}$$

is a segment with endpoints $w, y$.

By Lemma 3.35 and Lemma 3.4, $\sigma_1 \cup \tau_1$ is a segment with endpoints $x, y$. Thus, $(X, d)$ is geodesic.

Note that by Lemma 3.34, $\sigma \cap \tau$ is a segment with endpoints $v, w$. Also, by Lemma 3.34, if $\sigma \cap \tau = \{w\}$ then $(x \cdot y)_v = 0$ and $\sigma_1 = \sigma$, $\tau_1 = \tau$. Hence, $\sigma \cup \tau$ is a segment. Now, by Lemma 3.10, we may replace $v$ in this argument by any other point of $X$. Hence, $(X, d)$ satisfies the axioms for a ℝ-tree. □

*Example 3.37.* We noted in Example 3.14 that the compact metric space $(T_e, d_{T_e})$ that arises from an excursion path $e \in U$ is 0-hyperbolic. We can use Lemma 3.36 to show that $(T_e, d_{T_e})$ is a ℝ-tree. Suppose that $e \in U^\ell$. Take $x \in T_e$ and write $t$ for a point in $[0, \ell]$ such that $x$ is the image of $t$ under the quotient map from $[0, \ell]$ onto $T_e$. Write $v \in T_e$ for the image of $0 \in [0, \ell]$ under the quotient map from $[0, \ell]$ onto $T_e$. Note that $v$ is also the image of $\ell \in [0, \ell]$. For $h \in [0, e(t)]$, set $\lambda_h := \sup\{s \in [0, t] : e(s) = h\}$. Then the image of the set $\{\lambda_h : h \in [0, e(t)]\} \subseteq [0, \ell]$ under the quotient map is a segment in $T_e$ that has endpoints $v$ and $x$.

### 3.3.5 Embedding 0-Hyperbolic Spaces in ℝ-Trees

**Theorem 3.38.** *Let* $(X, d)$ *be a* $0$*-hyperbolic metric space. There exists a* ℝ*-tree* $(X', d')$ *and an isometry* $\phi : X \to X'$*.*

*Proof.* Fix $v \in X$. Write $x \cdot y := (x \cdot y)_v$ for $x, y \in X$. Let

$$Y = \{(x, m) : x \in X, m \in \mathbb{R} \text{ and } 0 \leqslant m \leqslant d(v, x)\}.$$

Define, for $(x, m), (y, n) \in Y$,

$$(x, m) \sim (y, n) \text{ if and only if } x \cdot y \geqslant m = n.$$

This is an equivalence relation on $Y$. Let $X' = Y/\sim$, and let $\langle x, m\rangle$ denote the equivalence class of $(x, m)$. We define the metric by

$$d'(\langle x, m\rangle, \langle y, n\rangle) = m + n - 2[m \wedge n \wedge (x \cdot y)].$$

The construction is illustrated in Figure 3.12.

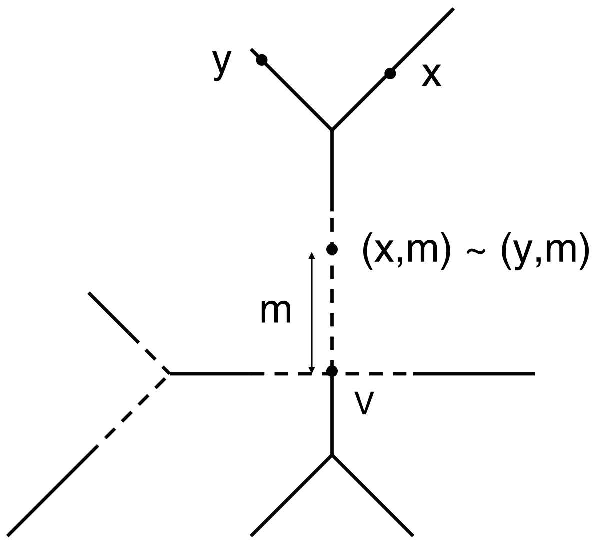


**Fig. 3.12.** The embedding of Theorem 3.38. Solid lines represent points that are in $X$, while dashed lines represent points that are added to form $X'$.

It follows by assumption that $d'$ is well defined. Note that

$$d'(\langle x, m\rangle, \langle x, n\rangle) = |m - n|$$

and $\langle x, 0\rangle = \langle v, 0\rangle$ for all $x \in X$, so $d'(\langle x, m\rangle, \langle v, 0\rangle) = m$. Clearly $d'$ is symmetric, and it is easy to see that $d'(\langle x, m\rangle, \langle y, n\rangle) = 0$ if and only if $\langle x, m\rangle = \langle y, n\rangle$. Also, in $X'$,

$$(\langle x, m\rangle \cdot \langle y, n\rangle)_{\langle v,0\rangle} = m \wedge n \wedge (x \cdot y).$$

If $\langle x, m\rangle, \langle y, n\rangle$ and $\langle z, p\rangle$ are three points of $X'$, then

$$\{m \wedge n, n \wedge p, p \wedge m\}$$

is an isosceles triple by Lemma 3.33(1). Hence, by Lemma 3.33(2), so is $\{m \wedge n \wedge (x \cdot y), n \wedge p \wedge (y \cdot z), p \wedge m \wedge (z \cdot x)\}$. It follows that $(X', d')$ is a 0-hyperbolic metric space.

If $\langle x, m\rangle \in X'$, then the mapping $\alpha : [0, m] \to X'$ given by $\alpha(n) = \langle x, n\rangle$ is an isometry, so the image of $\alpha$ is a segment with endpoints $\langle v, 0\rangle$ and $\langle x, m\rangle$. It now follows from Lemma 3.36 that $(X', d')$ is a $\mathbb{R}$-tree. Further, the mapping $\phi : X \to X'$ defined by $\phi(x) = \langle x, d(v, x)\rangle$ is easily seen to be an isometry. □

### 3.3.6 Yet another Characterization of $\mathbb{R}$-Trees

**Lemma 3.39.** *Let $(X, d)$ be a $\mathbb{R}$-tree. Fix $v \in X$.*

*(i) For $x, y \in X\backslash\{v\}$, $[v, x] \cap [v, y] \neq \{v\}$ if and only if $x, y$ are in the same path component of $X\backslash\{v\}$.*
*(ii) The space $X\backslash\{v\}$ is locally path connected, the components of $X\backslash\{v\}$ coincide with its path components, and they are open sets in $X$.*

*Proof.* (i) Suppose that $[v, x] \cap [v, y] \neq \{v\}$. It can't be that $v \in [x, y]$, because that would imply $[x, v] \cap [v, y] = \{v\}$. Thus, $[x, y] \subseteq X\backslash\{v\}$ and $x, y$ are in the same path component of $X\backslash\{v\}$. Conversely, if $\alpha : [a, b] \to X\backslash\{v\}$ is a continuous map, with $x = \alpha(a)$, $y = \alpha(b)$, then $[a, b]$ is a subset of the image of $\alpha$ by Lemma 3.26, so $v \notin [x, y]$, and $[v, x] \cap [v, y] \neq \{v\}$ by Axiom (b') for a $\mathbb{R}$-tree.
(ii) For $x \in X\backslash\{v\}$, the set $U := \{y \in X : d(x, y) < d(x, v)\}$ is an open set in $X$, $U \subseteq X\backslash\{v\}$, $x \in U$, and $U$ is path connected. For if $y, z \in U$, then $[x, y] \cup [x, z] \subseteq U$, and so $[y, z] \subseteq U$ by Corollary 3.25. Thus, $X\backslash\{v\}$ is locally path connected. It follows that the path components of $X\backslash\{v\}$ are both open and closed, and (ii) follows easily. □

**Theorem 3.40.** *A metric space $(X, d)$ is a $\mathbb{R}$-tree if and only if it is connected and 0-hyperbolic.*

*Proof.* An $\mathbb{R}$-tree is geodesic, so it is path connected. Hence, it is connected. Therefore, it is 0-hyperbolic by Lemma 3.29.

Conversely, assume that a metric space $(X, d)$ is connected and 0-hyperbolic. By Theorem 3.38 there is an embedding of $(X, d)$ in a $\mathbb{R}$-tree $(X', d')$. Let $x, y \in X$, suppose $v \in X'\backslash X$ and $v \in [x, y]$. Then $[v, x] \cap [v, y] = \{v\}$ and so by Lemma 3.39, $x, y$ are in different components of $X\backslash\{v\}$.

Let $C$ be the component of $X\backslash\{v\}$ containing $x$. By Lemma 3.39, $C$ is open and closed, so $X \cap C$ is open and closed in $X$. Since $x \in X \cap C$, $y \notin X \cap C$, this contradicts the connectedness of $X$. Thus, $[x, y] \subseteq X$ and $(X, d)$ is geodesic. It follows that $(X, d)$ is a $\mathbb{R}$-tree by Lemma 3.36. □

*Example 3.41.* Let $\mathcal{P}$ denote the collection of partitions of the positive integers $\mathbb{N}$. There is a natural partial order $\leqslant$ on $\mathcal{P}$ defined by $P \leqslant Q$ if every block of $Q$ is a subset of some block of $P$ (that is, the blocks of $P$ are unions of blocks of $Q$). Thus, the partition $\{\{1\},\{2\},\ldots\}$ consisting of singletons is the unique largest element of $\mathcal{P}$, while the partition $\{\{1,2,\ldots\}\}$ consisting of a single block is the unique smallest element. Consider a function $\Pi : \mathbb{R}_+ \mapsto \mathcal{P}$ that is non-increasing in this partial order. Suppose that $\Pi(0) = \{\{1\},\{2\},\ldots\}$ and $\Pi(t) = \{\{1,2,\ldots\}\}$ for all $t$ sufficiently large. Suppose also that if $\Pi$ is right-continuous in the sense that if $i$ and $j$ don't belong to the same block of $\Pi(t)$ for some $t \in \mathbb{R}_+$, then they don't belong to the same block of $\Pi(u)$ for $u > t$ sufficiently close to $t$.

Let $T$ denote the set consisting of points of the form $(t, B)$, where $t \in \mathbb{R}_+$ and $B \in \Pi(t)$. Given two point $(s, A), (t, B) \in T$, set

$$\begin{aligned} &m((s,A),(t,B)) \\ &\quad := \inf\{u > s \wedge t : A \text{ and } B \text{ subsets of a common block of } \Pi(u)\}, \end{aligned}$$

and put

$$d((s,A),(t,B)) := [m((s,A),(t,B)) - s] + [m((s,A),(t,B)) - t].$$

It is not difficult to check that $d$ is a metric that satisfies the four point condition and that the space $T$ is connected. Hence, $(T, d)$ is a $\mathbb{R}$-tree by Theorem 3.40. The analogue of this construction with $\mathbb{N}$ replaced by $\{1, 2, 3, 4\}$ is shown in Figure 3.13.

Moreover, if we let $\bar{T}$ denote the completion of $T$ with respect to the metric $d$, then $\bar{T}$ is also a $\mathbb{R}$-tree. It is straightforward to check that $\bar{T}$ is compact if and only if $\Pi(t)$ has finitely many blocks for all $t > 0$.

Write $\delta$ for the restriction of $d$ to the positive integers $\mathbb{N}$, so that

$$\delta(i,j) = 2 \inf\{t > 0 : i \text{ and } j \text{ belong to the same block of } \Pi(t)\}.$$

The completion $\mathbb{S}$ of $\mathbb{N}$ with respect to $\delta$ is isometric to the closure of $\mathbb{N}$ in $\bar{T}$, and $\mathbb{S}$ is compact if and only if $\Pi(t)$ has finitely many blocks for all $t > 0$. Note that $\delta$ is an *ultrametric* , that is, $\delta(x,y) \leqslant \delta(x,z) \vee \delta(z,y)$ for $x, y, z \in \mathbb{S}$. This implies that at least two of the distances are equal and are no smaller than the third. Hence, all triangles are *isosceles*. When $\mathbb{S}$ is compact, the open balls for the metric $\delta$ coincide with the closed balls and are obtained by taking the closure of the blocks of $\Pi(t)$ for $t > 0$. In particular, $\mathbb{S}$ is *totally disconnected*.

The correspondence between coalescing partitions, tree structures and ultrametrics is a familiar idea in the physics literature – see, for example, [109].

## 3.4 ℝ–Trees without Leaves

### 3.4.1 Ends

**Definition 3.42.** *An* $\mathbb{R}$-tree without leaves *is a* $\mathbb{R}$*–trees* $(T, d)$ *that satisfies the following extra axioms.*

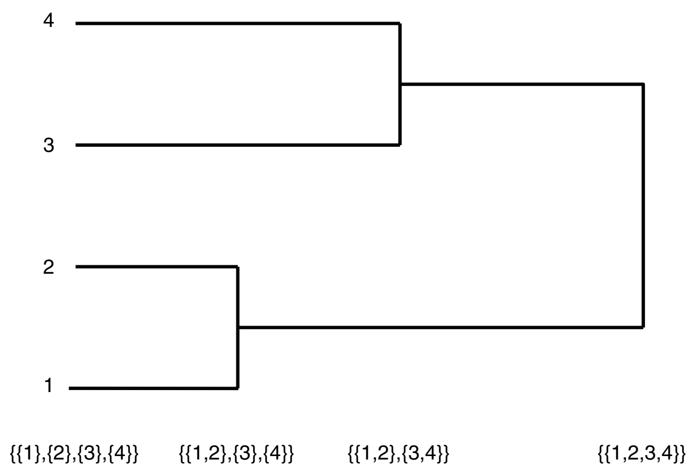


**Fig. 3.13.** The construction of a $\mathbb{R}$-tree from a non-increasing function taking values in the partitions of $\{1, 2, 3, 4\}$.

*Axiom (c) The metric space $(T, d)$ is* complete*.*
*Axiom (d) For each $x \in T$ there is at least one isometric embedding $\theta : \mathbb{R} \to T$ with $x \in \theta(\mathbb{R})$.*

*Example 3.43.* "The" real tree $(\mathcal{T}, D)$ of Example 3.18 satisfies Axioms (c) and (d).

We will suppose in this section that we are always working with a $\mathbb{R}$-tree $(T, d)$ that is without leaves.

**Definition 3.44.** *An* end *of $T$ is an equivalence class of isometric embeddings from $\mathbb{R}_+$ into $T$, where we regard two such embeddings $\phi$ and $\psi$ as being equivalent if there exist $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}_+$ such that $\alpha + \beta \geqslant 0$ and $\phi(t) = \psi(t + \alpha)$ for all $t \geqslant \beta$. Write $E$ for the set of ends of $T$.*

By Axiom (d), $E$ has at least 2 points. Fix a distinguished element † of $E$. For each $x \in T$ there is a unique isometric embedding $\kappa_x : \mathbb{R}_+ \to T$ such that $\kappa_x(0) = x$ and $\kappa_x$ is a representative of the equivalence class of †. Similarly, for each $\xi \in E_+ := E \backslash \{†\}$ there is at least one isometric embedding $\theta : \mathbb{R} \to T$ such that $t \mapsto \theta(t)$, $t \geqslant 0$, is a representative of the equivalence class of $\xi$

and $t \mapsto \theta(-t)$, $t \geqslant 0$, is a representative of the equivalence class of †. Denote the collection of all such embeddings by $\Theta_\xi$. If $\theta, \theta' \in \Theta_\xi$, then there exists $\gamma \in \mathbb{R}$ such that $\theta(t) = \theta'(t + \gamma)$ for all $t \in \mathbb{R}$. Thus, it is possible to select an embedding $\theta_\xi \in \Theta_\xi$ for each $\xi \in E_+$ in such a way that for any pair $\xi, \zeta \in E_+$ there exists $t_0$ (depending on $\xi, \zeta$) such that $\theta_\xi(t) = \theta_\zeta(t)$ for all $t \leqslant t_0$ (and $\theta_\xi(]t_0, \infty[) \cap \theta_\zeta(]t_0, \infty[) = \varnothing$). Extend $\theta_\xi$ to $\mathbb{R}^* := \mathbb{R} \cup \{\pm\infty\}$ by setting $\theta_\xi(-\infty) := \dagger$ and $\theta_\xi(+\infty) := \xi$.

*Example 3.45.* The ends of the real tree $(\mathcal{T}, D)$ of Example 3.18 can be identified with the collection consisting of the empty set and the elements of $\mathcal{E}_+$, where $\mathcal{E}_+$ consists of subsets $B \subset \mathbb{R}$ such that $-\infty < \inf B$ and $\sup B = +\infty$. If we choose † to be the empty set so that $\mathcal{E}_+$ plays the role of $E_+$, then we can define the isometric embedding $\theta_A$ for $A \in \mathcal{E}_+$ by $\theta_A(t) := (A \cap ]-\infty, t]) \cup \{t\} = A|t$, in the notation of Example 3.18.

The map $(t, \xi) \mapsto \theta_\xi(t)$ from $\mathbb{R} \times E_+$ (resp. $\mathbb{R}^* \times E_+$) into $T$ (resp. $T \cup E$) is surjective. Moreover, if $\eta \in T \cup E$ is in $\theta_\xi(\mathbb{R}^*) \cap \theta_\zeta(\mathbb{R}^*)$ for $\xi, \zeta \in E_+$, then $\theta_\xi^{-1}(\eta) = \theta_\zeta^{-1}(\eta)$. Denote this common value by $h(\eta)$, the *height* of $\eta$. In genealogical terminology, we think of $h(\eta)$ as the *generation* to which $\eta$ belongs. In particular, $h(\dagger) := -\infty$ and $h(\xi) = +\infty$ for $\xi \in E_+$. For the real tree $(\mathcal{T}, D)$ of Example 3.18 with corresponding isometric embeddings defined as above, $h(B)$ is just $\sup B$, with the usual convention that $\sup \varnothing := -\infty$ (in accord with the notation of Example 3.18).

Define a *partial order* $\leqslant$ on $T \cup E$ by declaring that $\eta \leqslant \rho$ if there exists $-\infty \leqslant s \leqslant t \leqslant +\infty$ and $\xi \in E_+$ such that $\eta = \theta_\xi(s)$ and $\rho = \theta_\xi(t)$. In genealogical terminology, $\eta \leqslant \rho$ corresponds to $\eta$ being an ancestor of $\rho$ (note that individuals are their own ancestors). In particular, † is the unique point that is an ancestor of everybody, while points of $E_+$ are characterized by being only ancestors of themselves. For the real tree $(\mathcal{T}, D)$ of Example 3.18, $A \leqslant B$ if and only if $A = (B \cap ]-\infty, \sup A]) \cup \{\sup A\}$. In particular, this partial order is **not** the usual inclusion partial order (for example, the singleton $\{0\}$ is an ancestor of the singleton $\{1\}$).

Each pair $\eta, \rho \in T \cup E$ has a well-defined *greatest common lower bound* $\eta \wedge \rho$ in this partial order, with $\eta \wedge \rho \in T$ unless $\eta = \rho \in E_+$, $\eta = \dagger$ or $\rho = \dagger$. In genealogical terminology, $\eta \wedge \rho$ is the *most recent common ancestor* of $\eta$ and $\rho$. For $x, y \in T$ we have

$$\begin{aligned} d(x, y) &= h(x) + h(y) - 2h(x \wedge y) \\ &= [h(x) - h(x \wedge y)] + [h(y) - h(x \wedge y)]. \end{aligned} \tag{3.1}$$

Therefore, $h(x) = d(x, y) - h(y) + 2h(x \wedge y) \leqslant d(x, y) + h(y)$ and, similarly, $h(y) \leqslant d(x, y) + h(x)$, so that

$$|h(x) - h(y)| \leqslant d(x, y), \tag{3.2}$$

with equality if $x, y \in T$ are comparable in the partial order (that is, if $x \leqslant y$ or $y \leqslant x$).

If $x, x' \in T$ are such that $h(x \wedge y) = h(x' \wedge y)$ for all $y \in T$, then, by (3.1), $d(x, x') = [h(x) - h(x \wedge x')] + [h(x') - h(x \wedge x')] = [h(x) - h(x \wedge x)] + [h(x') - h(x' \wedge x')] = 0$, so that $x = x'$. Slight elaborations of this argument show that if $\eta, \eta' \in T \cup E$ are such that $h(\eta \wedge y) = h(\eta' \wedge y)$ for all $y$ in some dense subset of $T$, then $\eta = \eta'$.

For $x, x', z \in T$ we have that if $h(x \wedge z) < h(x' \wedge z)$, then $x \wedge x' = x \wedge z$ and a similar conclusion holds with the roles of $x$ and $x'$ reversed; whereas if $h(x \wedge z) = h(x' \wedge z)$, then $x \wedge z = x' \wedge z \leqslant x \wedge x'$. Using (3.1) and (3.2) and checking the various cases we find that

$$|h(x \wedge z) - h(x' \wedge z)| \leqslant d(x \wedge z, x' \wedge z) \leqslant d(x, x'). \tag{3.3}$$

For $\eta \in T \cup E$ and $t \in \mathbb{R}^*$ with $t \leqslant h(\eta)$, let $\eta|t$ denote the unique $\rho \in T \cup E$ with $\rho \leqslant \eta$ and $h(\rho) = t$. Equivalently, if $\eta = \theta_\xi(u)$ for some $u \in \mathbb{R}^*$ and $\xi \in E_+$, then $\eta|t = \theta_\xi(t)$ for $t \leqslant u$. For the real tree of Example 3.18, this definition coincides with the one given in Example 3.18.

The metric space $(E_+, \delta)$, where

$$\delta(\xi, \zeta) := 2^{-h(\xi \wedge \zeta)},$$

is complete. Moreover, the metric $\delta$ is actually an *ultrametric*; that is, $\delta(\xi, \zeta) \leqslant \delta(\xi, \eta) \vee \delta(\eta, \zeta)$ for all $\xi, \zeta, \eta \in E_+$.

### 3.4.2 The Ends Compactification

Suppose in this subsection that the metric space $(E_+, \delta)$ is separable. For $t \in \mathbb{R}$ consider the set

$$T_t := \{x \in T : h(x) = t\} = \{\xi|t : \xi \in E_+\} \tag{3.4}$$

of points in $T$ that have height $t$. For each $x \in T_t$ the set $\{\zeta \in E_+ : \zeta|t = x\}$ is a ball in $E_+$ of diameter at most $2^{-t}$ and two such balls are disjoint. Thus, the separability of $E_+$ is equivalent to each of the sets $T_t$ being countable. In particular, separability of $E_+$ implies that $T$ is also separable, with countable dense set $\{\xi|t : \xi \in E_+,\, t \in \mathbb{Q}\}$, say.

We can, via a standard Stone–Čech-like procedure, embed $T \cup E$ in a compact metric space in such a way that for each $y \in T \cup E$ the map $x \mapsto h(x \wedge y)$ has a continuous extension to the compactification (as an extended real–valued function).

More specifically, let $S$ be a countable dense subset of $T$. Let $\pi$ be a strictly increasing, continuous function that maps $\mathbb{R}$ onto $]0, 1[$. Define an injective map $\Pi$ from $T$ into the compact, metrizable space $[0, 1]^S$ by $\Pi(x) := (\pi(h(x \wedge y)))_{y \in S}$. Identify $T$ with $\Pi(T)$ and write $\overline{T}$ for the closure of $T (= \Pi(T))$ in $[0, 1]^T$. In other words, a sequence $\{x_n\}_{n \in \mathbb{N}} \subset T$ converges to a point in $\overline{T}$ if $h(x_n \wedge y)$ converges (possibly to $-\infty$) for all $y \in S$, and two such sequences $\{x_n\}_{n \in \mathbb{N}}$ and $\{x'_n\}_{n \in \mathbb{N}}$ converge to the same point if and only if $\lim_n h(x_n \wedge y) = \lim_n h(x'_n \wedge y)$ for all $y \in S$.

We can identify distinct points in $T \cup E$ with distinct points in $\overline{T}$. If $\{x_n\}_{n\in\mathbb{N}} \subset T$ and $\xi \in E_+$ are such that for all $t \in \mathbb{R}$ we have $\xi|t \leqslant x_n$ for all sufficiently large $n$, then $\lim_n h(x_n \wedge y) = h(\xi \wedge y)$ for all $y \in S$. We leave the identification of † to the reader.

In fact, we have $\overline{T} = T \cup E$. To see this, suppose that $\{x_n\}_{n\in\mathbb{N}} \subset T$ converges to $x_\infty \in \overline{T}$. Put $h_\infty := \sup_{y\in S} \lim_n h(x_n \wedge y)$. Assume for the moment that $h_\infty \in \mathbb{R}$. We will show that $x_\infty \in T$ with $h(x_\infty) = h_\infty$. For all $k \in \mathbb{N}$ we can find $y_k \in S$ such that

$$h_\infty - \frac{1}{k} \leqslant \lim_n h(x_n \wedge y_k) \leqslant h(y_k) \leqslant h_\infty + \frac{1}{k}.$$

Observe that

$$\begin{aligned} d(y_k, y_\ell) &\leqslant \limsup_n \Big( d(y_k, x_n \wedge y_k) + d(x_n \wedge y_k, x_n \wedge y_\ell) \\ &\quad + d(x_n \wedge y_\ell, y_\ell) \Big) \\ &= \limsup_n \Big( [h(y_k) - h(x_n \wedge y_k)] + |h(x_n \wedge y_k) - h(x_n \wedge y_\ell)| \\ &\quad + [h(y_\ell) - h(x_n \wedge y_\ell)] \Big) \\ &\leqslant \frac{2}{k} + \left( \frac{1}{k} + \frac{1}{\ell} \right) + \frac{2}{\ell}. \end{aligned}$$

Therefore, $(y_k)_{k\in\mathbb{N}}$ is a $d$-Cauchy sequence and, by Axiom (c), this sequence converges to $y_\infty \in T$. Moreover, by (3.2) and (3.3), $\lim_n h(x_n \wedge y_\infty) = h(y_\infty) = h_\infty$.

We claim that $y_\infty = x_\infty$; that is, $\lim_n h(x_n \wedge z) = h(y_\infty \wedge z)$ for all $z \in S$. To see this, fix $z \in T$ and $\epsilon > 0$. If $n$ is sufficiently large, then

$$h(x_n \wedge z) \leqslant h(y_\infty) + \epsilon \tag{3.5}$$

and

$$h(y_\infty) - \epsilon \leqslant h(x_n \wedge y_\infty) \leqslant h(y_\infty). \tag{3.6}$$

If $h(y_\infty \wedge z) \leqslant h(y_\infty) - \epsilon$, then (3.6) implies that $y_\infty \wedge z = x_n \wedge z$. On the other hand, if $h(y_\infty \wedge z) \geqslant h(y_\infty) - \epsilon$, then (3.6) implies that

$$h(x_n \wedge z) \geqslant h(y_\infty) - \epsilon, \tag{3.7}$$

and so, by (3.5) and (3.6),

$$\begin{aligned} &|h(y_\infty \wedge z) - h(x_n, z)| \\ &\quad \leqslant [h(y_\infty) - (h(y_\infty) - \epsilon)] \vee [(h(y_\infty) + \epsilon) - (h(y_\infty) - \epsilon)] \\ &\quad = 2\epsilon. \end{aligned} \tag{3.8}$$

We leave the analogous arguments for $h_\infty = +\infty$ (in which case $x_\infty \in E_+$) and $h_\infty = -\infty$ (in which case $x_\infty = †$) to the reader.

We have just seen that the construction of $\overline{T}$ does not depend on $T$ (more precisely, any two such compactifications are homeomorphic). Moreover, a sequence $\{x_n\}_{n\in\mathbb{N}} \subset T \cup E$ converges to a limit in $T \cup E$ if and only if $\lim_n h(x_n \wedge y)$ exists for all $y \in T$, and two convergent sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{x'_n\}_{n\in\mathbb{N}}$ converge to the same limit if and only if $\lim_n h(x_n \wedge y) = \lim_n h(x'_n \wedge y)$ for all $y \in T$.

### 3.4.3 Examples of ℝ-Trees without Leaves

Fix a prime number $p$ and constants $r_-, r_+ \geqslant 1$. Let $\mathbb{Q}$ denote the rational numbers. Define an equivalence relation $\sim$ on $\mathbb{Q} \times \mathbb{R}$ as follows. Given $a, b \in \mathbb{Q}$ with $a \neq b$ write $a - b = p^{v(a,b)}(m/n)$ for some $v(a,b), m, n \in \mathbb{Z}$ with $m$ and $n$ not divisible by $p$. For $v(a,b) \geqslant 0$ put $w(a,b) = \sum_{i=0}^{v(a,b)} r_+^i$, and for $v(a,b) < 0$ put $w(a,b) := 1 - \sum_{i=0}^{-v(a,b)} r_-^i$. Set $w(a,a) := +\infty$. Given $(a,s), (b,t) \in \mathbb{Q} \times \mathbb{R}$ declare that $(a,s) \sim (b,t)$ if and only if $s = t \leqslant w(a,b)$. Note that

$$v(a,c) \geqslant v(a,b) \wedge v(b,c) \tag{3.9}$$

so that

$$w(a,c) \geqslant w(a,b) \wedge w(b,c) \tag{3.10}$$

and $\sim$ is certainly transitive (reflexivity and symmetry are obvious).

Let $T$ denote the collection of equivalence classes for this equivalence relation. Define a partial order $\leqslant$ on $T$ as follows. Suppose that $x, y \in T$ are equivalence classes with representatives $(a,s)$ and $(b,t)$. Say that $x \leqslant y$ if and only if $s \leqslant w(a,b) \wedge t$. It follows from (3.10) that $\leqslant$ is indeed a partial order. A pair $x, y \in T$ with representatives $(a,s)$ and $(b,t)$ has a unique greatest common lower bound $x \wedge y$ in this order given by the equivalence class of $(a, s \wedge t \wedge w(a,b))$, which is also the equivalence class of $(b, s \wedge t \wedge w(a,b))$.

For $x \in T$ with representative $(a,s)$, put $h(x) := s$. Define a metric $d$ on $T$ by setting $d(x,y) := h(x) + h(y) - 2h(x \wedge y)$. We leave it to the reader to check that $(T,d)$ is a ℝ–tree satisfying Axioms (a)–(d), and that the definitions of $x \leqslant y$, $x \wedge y$ and $h(x)$ fit into the general framework of Section 3.4, with the set $E_+$ corresponding to $\mathbb{Q} \times \mathbb{R}$–valued paths $s \mapsto (a(s), s)$ such that $s \leqslant w(a(s), a(t)) \wedge t$.

Note that there is a natural Abelian group structure on $E_+$: if $\xi$ and $\zeta$ correspond to paths $s \mapsto (a(s), s)$ and $s \mapsto (b(s), s)$, then define $\xi + \zeta$ to correspond to the path $s \mapsto (a(s) + b(s), s)$. We mention in passing that there is a bi–continuous group isomorphism between $E_+$ and the additive group of the $p$–adic integers $\mathbb{Q}_p$. (This map is, however, not an isometry if $E_+$ is equipped with the $\delta$ metric and $\mathbb{Q}_p$ is equipped with the usual $p$-adic metric.)