6

Natural Boundaries II: Algebraic Groups

6.1 Introduction

In this chapter, we use the analysis of the previous section to prove that the zeta functions of the classical groups $\mathrm{GO}_{2l+1}, \mathrm{GSp}_{2l}$ or GO_{2l}^+ of types B_l for $l \geq 2$, C_l for $l \geq 3$ and D_l for $l \geq 4$ have natural boundaries. These results were announced in [18]. We recall the definition of the local factors and the formula in terms of the root system established in [36] and [21].

Let G be one of the classical reductive groups GL_{l+1}, GO_{2l+1}, GSp_{2l} or GO_{2l}^+. For any field K, $G(K)$ will denote the appropriate subgroup of $\mathrm{GL}_n(K)$. Hey [35] and Tamagawa [56] proved that when $G = \mathrm{GL}_{l+1}$, the zeta function of G is something very classical, namely $Z_G(s) = \zeta(s) \dots \zeta(s-l)$, and hence has meromorphic continuation to the whole complex plane. Emboldened by the case of GL_{l+1}, the following definition of the zeta function of the classical group G had been proposed:

Definition 6.1. *1. For each prime p, let μ_G denote the Haar measure on $G(\mathbb{Q}_p)$ normalised such that $\mu_G(G(\mathbb{Z}_p)) = 1$. Define the* local *or* p-adic zeta function *of G to be*

$$Z_{G,p}(s) = \int_{G_p^+} |\det(g)|_p^s \, \mu_G(g) \ ,$$

where $G_p^+ = G(\mathbb{Q}_p) \cap \mathrm{M}_n(\mathbb{Z}_p)$, the set of matrices whose entries are all p-adic integers, and $|\cdot|_p$ denotes the p-adic valuation.

2. Define the global zeta function of G *to be*

$$Z_G(s) = \prod_{p \ prime} Z_{G,p}(s) \ .$$

Given any algebraic group G defined over a number field K and some K-rational representation $\rho : G \to \mathrm{GL}_n$ we can define in a similar manner an associated zeta function. In this paper we restrict ourselves to the above case of the classical groups, i.e. $\mathbb{Q}$-split reductive algebraic groups of type A_l, B_l, C_l, and D_l and their natural representations. In [18] we consider the exceptional types and the effect of changing the representation.

We describe now the formula in terms of the root system for the local zeta functions.

Let T denote the diagonal matrices of $G(\mathbb{Q}_p)$, namely a maximal split torus for $G(\mathbb{Q}_p)$. Let $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ be a basis for the root system $\Phi \subset \mathrm{Hom}(T, \mathbb{Q}_p)$ of $G(\mathbb{Q}_p)$ and let ϖ be the dominant weight of the contragredient (irreducible) representation $\rho^* = {}^{\mathrm{T}}\rho^{-1}$ of the natural representation ρ that we are taking for $G(\mathbb{Q}_p)$. Let m denote the order of the centre of the derived group $[G(\mathbb{C}), G(\mathbb{C})]$. Note that in particular m divides n. Let $\alpha_0 = \det^{n/m}\big|_T$. Then there exist integers $c_i > 0$ for $1 \le i \le l$ such that

$$\varpi^m = \alpha_0^{-1} \cdot \prod_{i=1}^{l} \alpha_i^{c_i} \ .$$

The second set of numerical data we need for our formula are the positive integers $b_1, \ldots, b_l$ which express the sum of the positive roots in terms of the primitive roots:

$$\prod_{\alpha \in \Phi^+} \alpha = \prod_{i=1}^{l} \alpha_i^{b_i} \ .$$

We can now write down our formula for the zeta function. Let W denote the finite Weyl group of Φ and $\lambda(w)$ the length of an element w of the Weyl group in terms of the fundamental reflections in the hyperplanes defined by the primitive roots.

Define two polynomials $P_G(X,Y)$, $Q_G(X,Y) \in \mathbb{Z}[X,Y]$ by

$$P_G(X,Y) = \sum_{w \in W} X^{-\lambda(w)} \prod_{\alpha_j \in w(\Phi^-)} X^{b_j} Y^{c_j} \ ,$$

$$Q_G(X,Y) = (1 - Y^m) \prod_{j=1}^{l} \left(1 - X^{b_j} Y^{c_j}\right) \ .$$

Then for each prime p,

$$Z_{G,p}(s) = \frac{P_G(p, p^{-(n/m)s})}{Q_G(p, p^{-(n/m)s})} \ .$$

It was proved in [36] and [21] that these polynomials satisfy a functional equation

$$\frac{P_G(X^{-1}, Y^{-1})}{Q_G(X^{-1}, Y^{-1})} = (-1)^{l+1} X^{\mathrm{card}(\Phi^+)} Y^m \frac{P_G(X,Y)}{Q_G(X,Y)} \ .$$

We are interested in the global behaviour of the zeta function defined as an Euler product of all these local factors. The denominator is always well behaved since it is just built out of the Riemann zeta function $\zeta_p(s)$. The interest lies in the numerator.

We record now the results of our analysis of the polynomial $P_G(X,Y)$ for the classical groups and in particular that for large enough l the polynomials for $G = \mathrm{GO}_{2l+1}, \mathrm{GSp}_{2l}$ and GO_{2l}^+ satisfy, after some factorisation, the conditions of Corollary 5.9 in the previous chapter and hence have a natural boundary. We tabulate first the combinatorial data for the four examples (Table 6.1):

Table 6.1. Combinatorial data for algebraic groups

		m	b_i	c_i
GL_{l+1}	A_l	$l+1$	$i(l-i+1)$	$l-i+1$
GO_{2l+1}	B_l	1	$i(2l-i)$	1
GSp_{2l}	C_l	2	$\begin{cases} i(2l-i+1) & \text{if } i<l \\ l(l+1)/2 & \text{if } i=l \end{cases}$	$\begin{cases} 2 & \text{if } i<l \\ 1 & \text{if } i=l \end{cases}$
GO_{2l}^+	D_l	2	$\begin{cases} i(2l-i-1) & \text{if } i<l-1 \\ l(l-1)/2 & \text{if } i\geq l-1 \end{cases}$	$\begin{cases} 2 & \text{if } i<l-1 \\ 1 & \text{if } i\geq l-1 \end{cases}$

Let $P_G(s) = \prod P_G(p, p^{-s})$ and α_{P_G} be the abscissa of convergence of $P_G(s)$.

To satisfy the conditions of Corollary 5.10 it suffices to know what the ghosts of $P_G(X,Y)$ look like. The following descriptions were announced in [16] and proved in [18]. For convenience, we set $b_0 = 0$.

Proposition 6.2. *1. The ghost polynomial $\widetilde{P}_G(X,Y)$ associated to $G = \mathrm{GO}_{2l+1}$ is*

$$\prod_{i=0}^{l-1}(1+X^{b_i}Y)\,.$$

Hence $Z_{\mathrm{GO}_{2l+1}}(s)$ has a friendly ghost.

2. The ghost polynomial $\widetilde{P}_G(X,Y)$ associated to GSp_{2l} is

$$\prod_{i=0}^{l-1}(1+X^{b_i/2}Y)\prod_{i=0}^{l-2}(1+X^{b_i/2+1}Y)\,.$$

Hence $Z_{\mathrm{GSp}_{2l}}(s)$ has a friendly ghost.

3. The ghost polynomial $\widetilde{P}_G(X,Y)$ associated to GO_{2l}^+ or D_l and its natural representation is

$$\prod_{i=0}^{l-2}(1+X^{b_i/2}Y)^2\ .$$

Hence $Z_{\mathrm{GO}^+_{2l}}(s)$ *has a friendly ghost.*

Corollary 6.3. *If* $G=\mathrm{GO}_{2l+1}$, GSp_{2l} *or* GO^+_{2l} *then the inverse of the gradients of the Newton polygon of* $P_G(X,Y)$ *are all integers.*

Proof. This follows since the gradients are the same as the gradients of the Newton polygon of the ghost. □

Corollary 6.4. *The abscissa of convergence* α_{P_G} *of* $P_G(s)=\prod P_G(p,p^{-s})$ *for* $G=\mathrm{GO}_{2l+1}$, GSp_{2l} *or* GO^+_{2l} *is* b_l.

Proof.
1. If $G=\mathrm{GO}_{2l+1}$ then b_{l-1} is the maximal inverse gradient in the Newton polygon. Hence $\alpha_{P_G}=b_{l-1}+1=b_l$.
2. If $G=\mathrm{GSp}_{2l}$ then $b_{l-1}/2$ is the maximal inverse gradient in the Newton polygon. Hence $\alpha_{P_G}=b_{l-1}/2+1=b_l$.
3. If $G=\mathrm{GO}^+_{2l}$ then $b_{l-2}/2$ is the maximal inverse gradient in the Newton polygon. Hence $\alpha_{P_G}=b_{l-2}/2+1=b_l$. □

Note that in each case there is a term $X^{b_l-1}Y$ appearing in both $P_G(X,Y)$ and its ghost. In fact there is another way to see why b_l is the abscissa of convergence without passing to the ghost although the analysis below was essential in determining the ghost.

We know that $\alpha_{P_G}=\max\{\frac{1+n_k}{k}:k=1,\ldots,r\}$. We shall need to analyse the root system and the combinatorial data to ascertain the value of α_{P_G}.

Choose a subset of simple roots $\Pi_0\subseteq\Pi$. Let Φ_0 be the sub-root system that Π_0 generates. Notice that in the expression for $P_G(X,Y)$ we can realise the monomial term $X^{-\lambda(w)}\prod_{\alpha_j\in\Pi_0}X^{b_j}Y^{c_j}$ where w is a Weyl element such that $\Pi_0=\{w^{-1}\alpha_j\}\subset\Phi^-$. For each choice of Π_0, such elements w exist since we can take $w=w_0$ to be the unique element of W_0, the Weyl group of Φ_0, that sends all positive roots Φ_0^+ to negative roots Φ_0^-. To calculate the abscissa of convergence α_{P_G} we are going to be interested in choosing a w which is of minimal length since

$$\alpha_{P_G}=\max\left\{\frac{1-\lambda(w)+\sum_{\alpha_j\in\Pi_0}b_j}{\sum_{\alpha_j\in\Pi_0}c_j}:\Pi_0\subseteq\Pi,w\in W\text{ s.t. }w^{-1}\Pi_0\subset\Phi^-\right\}$$

The following lemma tells us that for any choice of a subset of simple roots Π_0, w_0 is the most efficient way to realise the corresponding monomial term:

Lemma 6.5. *Let* Π_0 *be a subset of the simple roots* Π *and let* Φ_0 *be the sub-root system of* Φ *generated by* Π_0. *Then the length of the shortest element* $w\in W$ *with the property that* $w(\Pi_0)\subset\Phi^-$ *but* $w(\Pi\setminus\Pi_0)\subset\Phi^+$ *is* $\mathrm{card}(\Phi_0^+)=\lambda(w_0)$.

Proof. Let w_0 be the element which maps Φ_0^+ to Φ_0^-. Certainly then $w_0(\Pi_0) \subset \Phi^-$. The length of this element is $\mathrm{card}(\Phi_0^+)$ in terms of the natural generators w_α where $\alpha \in \Pi_0$. We can't get any shorter than this using all generators w_α where $\alpha \in \Pi$ since the length is still the number of positive roots sent to negative roots in Φ which is at least $\mathrm{card}(\Phi_0^+)$. But notice that we have now shown that it is exactly that number hence $\Pi \setminus \Pi_0$ must be sent to positive roots since $(\Pi \setminus \Pi_0) \cap \Phi_0 = \varnothing$. But now the length of any element sending Π_0 to negative roots must be at least $\mathrm{card}(\Phi_0^+)$ since the length is the number of positive roots in Φ^+ sent to negative roots and if Π_0 gets sent to negative roots then so does Φ_0^+. This completes the proof of the lemma. □

Lemma 6.6. $\alpha_{P_G} = b_l$.

Proof. First note that $\alpha_{P_G} \geq b_l$ since we can take $\Pi_0 = \{\alpha_l\}$ and $w_0 = w_l$ the reflection in α_l which is a word of length 1. Next note that $\mathrm{card}(\Phi_0^+) \geq \mathrm{card}(\Pi_0)$. An analysis of the combinatorial data will confirm that $b_l - 1 = (b_l - 1)/c_l = \max\{\, b_i - 1/c_i : i = 1, \ldots, l \,\}$. The easiest way to check this is to note that for example in the case C_l we have $(2l - i + 1)/2 = \sum_{j=l-i}^{l-1} j$. Then we can use the fact that for any positive integers $x_1, \ldots, x_r, y_1, \ldots, y_r$ we have $\frac{x_1 + \ldots + x_r}{y_1 + \ldots + y_r} \leq \max \frac{x_i}{y_i}$ to deduce that for $w \in W$ such that $w^{-1}\Pi_0 \subset \Phi^-$,

$$\begin{aligned} \frac{1 - \lambda(w) + \sum_{\alpha_j \in \Pi_0} b_j}{\sum_{\alpha_j \in \Pi_0} c_j} &\leq \frac{1 + \sum_{\alpha_j \in \Pi_0} (b_j - 1)}{\sum_{\alpha_j \in \Pi_0} c_j} \\ &\leq \left(\sum_{\alpha_j \in \Pi_0} c_j \right)^{-1} + \max\{\, b_i - 1/c_i : \alpha_i \in \Pi_0 \,\} \,. \end{aligned}$$

Therefore $\alpha_{P_G} \leq 1 + (b_l - 1) = b_l$. This completes the lemma. □

We now put $\beta_P = b_l - 1 = \max\{\, \frac{n_k}{k} : k \in I \,\}$ where $I = \{\, k : \frac{1+n_k}{k} = \alpha_P \,\}$. The three examples B_l, C_l and D_l are perfect to illustrate the application of Hypotheses 1 and 2 (p. 134) of the previous chapter. For B_l with $l \geq 2$, we will find that the two hypotheses are satisfied and that β_P is a natural boundary. For C_l with $l \geq 3$, we will find that Hypothesis 2 actually fails, but because $P(X, Y)$ has a factor of the form $(1 + X^{\beta_P} Y)$ and hence the first candidate natural boundary can be passed. We then show that if $P(X, Y) = (1 + X^{\beta_P} Y) P_1(X, Y)$ then $P_1(X, Y)$ will give us a natural boundary. For D_l with $l \geq 4$, we will find that Hypothesis 1 fails. Again this is due to a factor of the form $(1 + X^{\beta_P} Y)$. Once this is removed we find that both Hypotheses 1 and 2 are satisfied and β_P is in fact a natural boundary.

6.2 $G = \mathrm{GO}_{2l+1}$ of Type B_l

Proposition 6.7. *If $G = \mathrm{GO}_{2l+1}$ of type B_l and $l \geq 2$, then $P_G(s)$ has a natural boundary at $\beta_P = b_{l-1} = b_l - 1 = l^2 - 1$.*

Proof. We make the change of variable $U = X^{\beta_P}Y$ and $V = X^{-1}$ so that $P(X, Y) = F(U, V)$. Then $A(U) = F(0, U) = 1 + U$. This follows because for all $\Pi_0 \subseteq \Pi, w \in W$ such that $w^{-1}\Pi_0 \subset \Phi^-$ except for the case $\Pi_0 = \{\alpha_l\}$ and $w_0 = w_l$ we have:

$$\frac{-\lambda(w) + \sum_{\alpha_j \in \Pi_0} b_j}{\sum_{\alpha_j \in \Pi_0} c_j} < \beta_P = \frac{b_l - 1}{c_l}$$

We set $\omega = -1$, the unique root of $A(U)$. Clearly Hypothesis 1 is satisfied since $A(U)$ does not have a multiple root at ω.

To check Hypothesis 2 we need to determine

$$B_1(U) = \frac{\partial}{\partial V} F(V, U)\bigg|_{V=0} = \sum_{\beta_P k - i = 1} a_{i,k} U^k .$$

We claim that for $i = \beta_P k - 1$, $a_{i,k} \neq 0$ if and only if $k = 1$. For $k = 1$ we are required to show there is a monomial of the form $X^{b_{l-1}-1}Y$. This can be realised by taking $\Pi_0 = \{\alpha_{l-1}\}$ and $w_0 = w_{l-1}$ the reflection defined by the root α_{l-1}. For $k > 1$, for each choice of Π_0 with k elements and a corresponding w such that $w^{-1}\Pi_0 \subset \Phi^-$,

$$\begin{aligned} -\lambda(w) + \sum_{\alpha_j \in \Pi_0} b_j &\leq \sum_{\alpha_j \in \Pi_0} (b_j - 1) \\ &< (k-1)(b_{l-1} - 1) + b_l - 1 \text{ if } \Pi_0 \neq \{\alpha_{l-1}, \alpha_l\} \\ &\leq (k-1)(\beta_P - 1) + \beta_P \leq i \end{aligned}$$

since the b_i are a strictly increasing sequence. For $\Pi_0 = \{\alpha_{l-1}, \alpha_l\}$ we just have to use the stronger inequality that if $w^{-1}\Pi_0 \subset \Phi^-$ then $\lambda(w) \geq 3$. Hence we have shown that for each $k > 1$, there are no monomials of the form $X^{\beta_P k - 1}Y^k$. Hence $B_1(U) = a_{\beta_P - 1, 1}U$ and

$$-\frac{B_1(\omega)}{\omega A'(\omega)} = -a_{\beta_P - 1, 1}.$$

Since $a_{\beta_P - 1, 1} > 0$, Hypothesis 2 is satisfied. Therefore we can apply Theorem 5.13 to deduce that $P_G(s)$ has a natural boundary at $\beta_P = b_{l-1} = b_l - 1 = l^2 - 1$. □

Corollary 6.8. *If $G = \mathrm{GO}_{2l+1}$ of type B_l and $l \geq 2$ then $Z_G(s)$ has abscissa of convergence at $\alpha_G = b_l + 1$ and a natural boundary at $\beta_P = b_{l-1} = b_l - 1 = l^2 - 1$.*

Proof. We just have to add that $Q_G(s)^{-1} = \prod Q_G(p, p^{-s})^{-1}$ is a meromorphic function with abscissa of convergence at $\alpha_G = b_l + 1$. □

6.3 $G = \mathrm{GSp}_{2l}$ of Type C_l or $G = \mathrm{GO}_{2l}^+$ of Type D_l

In these two examples, there is an initial problem with performing the analysis of the previous example because $(1 + X^{\beta_P}Y)$ is a factor of $P_G(X, Y)$. This means that after the substitution $U = X^{\beta_P}Y$ and $V = X^{-1}$, $P(X, Y) = F(U, V) = (1 + U)F_1(U, V)$ and hence for all n

$$B_n(-1) = \frac{1}{n!}\frac{\partial}{\partial V^n}F(V, U)\bigg|_{V=0,U=-1} = 0\,.$$

Hence Hypothesis 2 is never satisfied. This is what we would expect since if $(1 + X^{\beta_P}Y)$ is a factor then the potential natural boundary it might cause at $s = \beta_P$ can be passed by multiplying by the meromorphic function $\prod(1 + p^{\beta_P}p^{-s})^{-1}$. Note that in the case of $G = \mathrm{GO}_{2l}^+$ of type D_l, even Hypothesis 1 fails since $A(U) = F(0, U) = 1 + 2U + U^2$. In this case once the factor $(1 + X^{\beta_P}Y)$ is removed the remaining term $F_1(U, V)$ still has the property that $U = -1$ is a zero of $F_1(U, 0)$. We will find that $s = \beta_P$ will now produce a natural boundary. In the case $G = \mathrm{GSp}_{2l}$ of type C_l we will have to move a little further to the left to find our natural boundary.

The polynomial $P_G(X, Y)$ actually has a number of other natural factors, not only $(1 + X^{\beta_P}Y)$. This fact was announced in [16]. Its proof is technical and has been consigned to Appendix B:

Theorem 6.9. *If $G = \mathrm{GSp}_{2l}$ of type C_l or $G = \mathrm{GO}_{2l}^+$ of type D_l then $P_G(X, Y)$ has a factor of the form*

$$(1 + Y)\prod_{i=1}^{r}(1 + X^{b_i/2}Y)\,,$$

where $r = l - 1$ for $G = \mathrm{GSp}_{2l}$ and $r = l - 2$ for $G = \mathrm{GO}_{2l}^+$.

Corollary 6.10. *1. If $G = \mathrm{GSp}_{2l}$ then*

$$P_G(X, Y) = (1 + Y)\prod_{i=1}^{l-1}(1 + X^{b_i/2}Y)R_G(X, Y)\,,$$

where $R_G(X, Y)$ has ghost polynomial

$$\widetilde{R_G}(X, Y) = (1 + XY)\prod_{i=1}^{l-2}(1 + X^{b_i/2+1}Y)\,.$$

2. If $G = \mathrm{GO}_{2l}^+$ then

$$P_G(X, Y) = (1 + Y)\prod_{i=1}^{l-2}(1 + X^{b_i/2}Y)R_G(X, Y)\,,$$

where $R_G(X,Y)$ has ghost polynomial

$$\widetilde{R_G}(X,Y) = (1+Y)\prod_{i=1}^{l-2}(1+X^{b_i/2}Y)\,.$$

In Appendix B we give a description of the polynomials $R_G(X,Y)$ in terms of the root system.

6.3.1 $G = \mathrm{GSp}_{2l}$ of Type C_l

Let us recall the structure of the root system C_l and its corresponding Weyl group. Let $\mathbf{e}_i$ be the standard basis for the l-dimensional vector space $\mathbb{R}^l$, where we assume $l \geq 3$.

$C_l^+ = \{\, 2\mathbf{e}_i, \mathbf{e}_i \pm \mathbf{e}_j : 1 \leq i < j \leq l \,\}$ with simple roots $\alpha_1 = \mathbf{e}_1 - \mathbf{e}_2, \ldots, \alpha_{l-1} = \mathbf{e}_{l-1} - \mathbf{e}_l, \alpha_l = 2\mathbf{e}_l$. $W(C_l)$ is a semi-direct product of the symmetric group on $\mathbf{e}_i$ and the group $(\mathbb{Z}/2\mathbb{Z})^l$ operating by $\mathbf{e}_i \mapsto (\pm 1)_i \mathbf{e}_i$.

We shall write $w = \pi_w \sigma_w$ where π_w is the permutation and σ_w is the sign change.

Let w_l be the element sending α_l to $-\alpha_l$. The element w_l is the sign change $\mathbf{e}_i \mapsto \mathbf{e}_i$ for $i = 1, \ldots, l-1$ and $\mathbf{e}_l \mapsto -\mathbf{e}_l$. Let Φ_{k+1} be the sub-root system generated by $\{\, \alpha_{l-k}, \ldots, \alpha_l \,\}$ and $w_{\Phi_{k+1}}$ be the element sending Φ_{k+1}^+ to Φ_{k+1}^-. For $G = \mathrm{GSp}_{2l}$ we prove in Appendix B that for $k = 1, \ldots, l$,

$$\begin{aligned} P_G(X,Y) &= (1+X^{b_{k-1}/2}Y)\left(\sum_{w\in W(k)} X^{-\lambda(w)} \prod_{\alpha_j \in w(\Phi^-)} X^{b_j}Y^{c_j}\right) \\ &= (1+X^{b_{k-1}/2}Y)P_k(X,Y)\,, \end{aligned}$$

where

$$\begin{aligned} W(k) = &\left\{ \begin{matrix} w = \pi_w\sigma_w : w^{-1}(\alpha_{k-1}) \text{ and } (w_{\Phi_{l-k+1}} w w(k))^{-1}(\alpha_{k-1}) \\ \text{have the same sign and } (\sigma_{w^{-1}})_k = 1 \end{matrix} \right\} \\ &\cup \left\{ \begin{matrix} w = \pi_w\sigma_w : w^{-1}(\alpha_{k-1}) \text{ and } (w_{\Phi_{l-k+1}} w w(k))^{-1}(\alpha_{k-1}) \\ \text{have opposite signs and } (\sigma_{w^{-1}})_k = -1 \end{matrix} \right\} \\ = &\; W(k)^+ \cup W(k)^-\,, \end{aligned}$$

where for each $w \in W$, $w(k)$ denotes the permutation of $\mathbf{e}_{\pi_{w^{-1}}(i)}$ for $i = k, \ldots, l$ which alters the order. Here it suffices to know the following: for $G = \mathrm{GSp}_{2l}$,

$$\begin{aligned} &P_G(X,Y) \\ &= (1+X^{b_{l-1}/2}Y)(1+X^{b_{l-2}/2}Y)P(X,Y) \\ &= (1+X^{b_{l-1}/2}Y)(1+X^{b_{l-2}/2}Y) \\ &\quad \times \left(\sum_{w\in W(l)\cap W(l-1)} X^{-\lambda(w)} \prod_{\alpha_j\in w(\Phi^-)} X^{b_j}Y^{c_j} \right)\,. \end{aligned}$$

The ghost of $P(X,Y)$ indicates that the first candidate natural boundary is at $\beta = b_{l-2}/2 + 1$. We see that Hypotheses 1 and 2 apply now to $P(X,Y)$. We make our change of variable $U = X^\beta Y$ and $V = X^{-1}$ so that $P(X,Y) = F(U,V)$. The corresponding polynomial $A(U) = 1 + U$ hence this satisfies Hypothesis 1 for the unique root $\omega = -1$.

To check Hypothesis 2 we need to determine

$$B_1(U) = \frac{\partial}{\partial V} F(V,U)\bigg|_{V=0} = \sum_{\beta k - i = 1} a_{i,k} U^k \, .$$

We claim that for $i = \beta k - 1$, $a_{i,k} \neq 0$ if and only if $k = 1$. For $k = 1$, we are required to show that there is a monomial of the form $X^{b_{l-2}/2} Y$ in $P(X,Y)$. Now $X^{b_{l-2}/2} Y = X^{b_l - 3} Y$. This can be realised by taking $w = w_l w_{l-1} w_{l-2}$, where w_i is the reflection defined by the root α_i. Now

$$\begin{aligned} w^{-1} &: \mathbf{e}_l \mapsto -\mathbf{e}_{l-2} \\ &: \mathbf{e}_{l-1} \mapsto \mathbf{e}_l \\ &: \mathbf{e}_{l-2} \mapsto \mathbf{e}_{l-1} \end{aligned}$$

We show that $w \in W(l)$. Now $(w_l w_{l-1} w_{l-2})^{-1}(\alpha_{l-1}) = \mathbf{e}_l + \mathbf{e}_{l-2} \in \Phi^+$ and $(w_l w_l w_{l-1} w_{l-2})^{-1}(\alpha_{l-1}) = \mathbf{e}_l - \mathbf{e}_{l-2} \in \Phi^-$. Since $(\sigma_{w^{-1}})_l = -1$ this implies that $w \in W(l)$.

Next we need that $w \in W(l-1)$. We have that $(w_l w_{l-1} w_{l-2})^{-1}(\alpha_{l-2}) = \alpha_{l-1} \in \Phi^+$. Now $w(l-1)$ is defined as the permutation which swaps $\mathbf{e}_{l-2} = \mathbf{e}_{\pi_{w^{-1}}(l)}$ and $\mathbf{e}_l = \mathbf{e}_{\pi_{w^{-1}}(l-1)}$ and w_{Φ_2} sends $\mathbf{e}_l$ to $-\mathbf{e}_l$ and $\mathbf{e}_{l-1}$ to $-\mathbf{e}_{l-1}$. Hence

$$w(l-1)^{-1} w^{-1} w_{\Phi_2}^{-1}(\alpha_{l-2}) = \mathbf{e}_{l-1} + \mathbf{e}_{l-2} \, .$$

Since $(\sigma_{w^{-1}})_{l-1} = 1$ this implies that $w \in W(l-1)$.

Finally $\{\, \alpha_j : \alpha_j \in w_l w_{l-1} w_{l-2}(\Phi^-) \,\} = \{\alpha_l\}$ and $\lambda(w_l w_{l-1} w_{l-2}) = 3$.

Consider any monomial term $X^r Y^{2j+\varepsilon}$ where $2j + \varepsilon > 1$ and $\varepsilon = 0$ or 1. Then

$$r = -\lambda(w) + \sum_{\alpha_i \in \Pi'} b_i \, ,$$

where w is an element of $W(l)$ such that w^{-1} sends Π' (a subset of the simple roots of size $j+\varepsilon$) to negative roots. Now $\beta = b_{l-2}/2 + 1 = b_{l-1}/2 - 1 = b_l - 2$ and b_i is strictly increasing for $i \leq l-1$. Suppose first that $j \geq 2$ then since $\lambda(w) \geq j + \varepsilon$,

$$\begin{aligned} r &= -\lambda(w) + \sum_{\alpha_i \in \Pi'} b_i \\ &\leq (j-1) b_{l-2} + b_{l-1} + \varepsilon b_l - \lambda(w) \\ &\leq 2j\beta + \varepsilon b_l - (\varepsilon + 1) - (j-1) \\ &< (2j+\varepsilon)\beta - 1 \, . \end{aligned}$$

Suppose that $j = 1$. Then (except if $\Pi' = \{\alpha_{l-1}\}$ or $\{\alpha_{l-1}, \alpha_l\}$)

$$\begin{aligned} r &= -\lambda(w) + \sum_{\alpha_i \in \Pi'} b_i \\ &\le b_{l-2} + \varepsilon b_l - \lambda(w) \\ &\le (2j+\varepsilon)\beta - 2\,. \end{aligned}$$

This finishes the cases except for $\Pi' = \{\alpha_{l-1}\}$ or $\{\alpha_{l-1}, \alpha_l\}$.

If $w \in W(l) \cap W(l-1)$ and $\Pi' = \{\alpha_{l-1}\}$, then we are required to show that

$$\begin{aligned} 1 &< 2\beta - r \\ &= (b_{l-1} - 2) - (b_{l-1} - \lambda(w))\,, \end{aligned}$$

i.e. that $\lambda(w) \ge 4$. In this case $(\sigma_{w^{-1}})_l = 1$ and $w^{-1}(\alpha_{l-1}) \in \Phi^-$ hence $w^{-1}w_l^{-1}(\alpha_{l-1}) = w^{-1}(\mathbf{e}_{l-1} + \mathbf{e}_l) \in \Phi^-$. This in turn implies that $(\sigma_{w^{-1}})_{l-1} = -1$ and $\pi_{w^{-1}}(l-1) < \pi_{w^{-1}}(l)$ So we have already found three positive roots ($\mathbf{e}_{l-1} + \mathbf{e}_l, \mathbf{e}_{l-1} - \mathbf{e}_l$ and $2\mathbf{e}_{l-1}$) that are sent to negative roots by w^{-1}. We just have to demonstrate a fourth such root to guarantee $\lambda(w) \ge 4$. Now since $(\sigma_{w^{-1}})_{l-1} = -1$ and $w \in W(l-1)$ we get that $w^{-1}(\alpha_{l-2})$ and $(w_{\Phi_2} w w(l-1))^{-1}(\alpha_{l-2}) = w^{-1}(ww(l-1)w^{-1}w_{\Phi_2})(\alpha_{l-2})$ have opposite signs. So we just need to know that $(ww(l-1)w^{-1}w_{\Phi_2})(\alpha_{l-2}) \ne \mathbf{e}_{l-1} \pm \mathbf{e}_l$ but is a positive root. Now $(ww(l-1)w^{-1}w_{\Phi_2})(\mathbf{e}_{l-2}) = \mathbf{e}_{l-2}$ whilst $(ww(l-1)$ $w^{-1}w_{\Phi_2})(\mathbf{e}_{l-1}) = -(\sigma_w)_{\pi_{w^{-1}}(l)}\mathbf{e}_l$ which confirms both these facts. Hence we have a fourth positive root (either α_{l-2} or $(ww(l-1)w^{-1}w_{\Phi_2})(\alpha_{l-2})$) sent to a negative root by w^{-1}. This confirms that $\lambda(w) \ge 4$.

We show that if $w \in W(l) \cap W(l-1)$ then $\Pi' \ne \{\alpha_{l-1}, \alpha_l\}$. Suppose otherwise. In this case $(\sigma_{w^{-1}})_l = -1$ and $w^{-1}(\alpha_{l-1}) \in \Phi^-$ hence (1) $(\sigma_{w^{-1}})_{l-1} = -1$ and (2) $w^{-1}w_l^{-1}(\alpha_{l-1}) = w^{-1}(\mathbf{e}_{l-1} + \mathbf{e}_l) \in \Phi^+$ since $w \in W(l)$. But

$$w^{-1}(\mathbf{e}_{l-1} + \mathbf{e}_l) = -\mathbf{e}_{\pi_{w^{-1}}(l-1)} - \mathbf{e}_{\pi_{w^{-1}}(l)} \in \Phi^-\,.$$

Hence we have a contradiction.

This completes the analysis and confirms that $B_1(U) = a_{\beta-1,1}U$ where $a_{\beta-1,1} \ge 1$ (in fact it is possible to show that $a_{\beta-1,1} = 1$). Hence

$$-\frac{B_1(-1)}{(-1)A'(-1)} = -a_{\beta-1,1}$$

and so $\Re\left(-\frac{B_\gamma(\omega)}{\omega A'(\omega)}\right) < 0$, confirming Hypothesis 2. Therefore we can apply Theorem 5.13 to deduce that $P_G(s)$ has a natural boundary at $\beta_P = b_{l-2}/2 + 1 = l(l+1)/2 - 2$.

Corollary 6.11. *If $G = \mathrm{GSp}_{2l}$ of type C_l then $Z_G(s)$ has abscissa of convergence at $\alpha_G = b_l + 1$ and a natural boundary at $\beta_P = b_{l-2}/2 + 1 = l(l+1)/2 - 2$.*

Proof. We just have to add that $Q_G(s)^{-1} = \prod Q_G(p, p^{-s})^{-1}$ is a meromorphic function with abscissa of convergence at $\alpha_G = b_l + 1$. □

Note that had we not factored out $(1+X^{b_{l-2}/2}Y)$ as well to define $P(X,Y)$ we would have got that $B_1(-1) = 0$. In the next example we only have to remove one factor.

6.3.2 $G = \mathrm{GO}^+_{2l}$ of Type D_l

We turn now to proving that Hypotheses 1 and 2 hold for D_l if $l \geq 4$. We recall the structure of the root system in this case. $D_l^+ = \{\mathbf{e}_i \pm \mathbf{e}_j : 1 \leq i < j \leq l\}$ with simple roots $\alpha_1 = \mathbf{e}_1 - \mathbf{e}_2, \ldots, \alpha_{l-1} = \mathbf{e}_{l-1} - \mathbf{e}_l$, $\alpha_l = \mathbf{e}_{l-1} + \mathbf{e}_l$. $W(D_l)$ is a semi-direct product of the symmetric group on $\mathbf{e}_i$ and the group $(\mathbb{Z}/2\mathbb{Z})^{l-1}$ operating by $\mathbf{e}_i \mapsto (\pm 1)_i \mathbf{e}_i$ with $\prod_i (\pm 1)_i = 1$. Again we write an element of w as $\pi_w \sigma_w$.

In a similar fashion to the case of GSp_{2l} we prove for GO^+_{2l} in Appendix B that $k = 1, \ldots, l-1$

$$
\begin{aligned}
P_G(X,Y) &= (1 + X^{b_{k-1}/2}Y)\left(\sum_{w \in W(k)} X^{-\lambda(w)} \prod_{\alpha_j \in w(\Phi^-)} X^{b_j} Y^{c_j}\right) \\
&= (1 + X^{b_{k-1}/2}Y) P_k(X,Y) \,,
\end{aligned}
$$

where

$$
\begin{aligned}
W(k) = &\left\{ \begin{array}{c} w = \pi_w \sigma_w : w^{-1}(\alpha_{k-1}) \text{ and } (w_{\Phi_{l-k+1}} w w(k))^{-1}(\alpha_{k-1}) \\ \text{have the same sign and } (\sigma_{w^{-1}})_k = 1 \end{array} \right\} \\
&\cup \left\{ \begin{array}{c} w = \pi_w \sigma_w : w^{-1}(\alpha_{k-1}) \text{ and } (w_{\Phi_{l-k+1}} w w(k))^{-1}(\alpha_{k-1}) \\ \text{have opposite signs and } (\sigma_{w^{-1}})_k = -1 \end{array} \right\} \\
= &\; W(k)^+ \cup W(k)^- \,,
\end{aligned}
$$

where for each $w \in W$, $w(k)$ denotes the permutation of $\mathbf{e}_{\pi_{w^{-1}}(i)}$ for $i = k, \ldots, l$ which alters the order. In this case we only need to know that

$$
\begin{aligned}
P_G(X,Y) &= (1 + X^{b_{l-2}/2}Y) P(X,Y) \\
&= (1 + X^{b_{l-2}/2}Y)\left(\sum_{w \in W(l-1)} X^{-\lambda(w)} \prod_{\alpha_j \in w(\Phi^-)} X^{b_j} Y^{c_j}\right) .
\end{aligned}
$$

The ghost of $P(X,Y)$ indicates that the first candidate natural boundary is at $\beta = b_{l-2}/2$. We see that Hypotheses 1 and 2 apply now to $P(X,Y)$. We make our change of variable $U = X^\beta Y$ and $V = X^{-1}$ so that $P(X,Y) = F(U,V)$. The corresponding polynomial $A(U) = 1 + U$ hence this satisfies Hypothesis 1 for the unique root $\omega = -1$.

To check Hypothesis 2 we need to determine

$$B_1(U) = \frac{\partial}{\partial V} F(V,U)\bigg|_{V=0} = \sum_{\beta k - i = 1} b_{i,k} U^k \ .$$

We claim that for $i = \beta k - 1$, $a_{i,k} \neq 0$ if and only if $k = 1$.

For $k = 1$, we are required to show that there is a monomial of the form $X^{b_{l-2}/2-1}Y$ in $P(X,Y)$. If we rewrite $X^{b_{l-2}/2-1}Y = X^{b_l-2}Y = X^{b_{l-1}-2}Y$ we see that we are looking for an element $w \in W(l-1)$ of length two with either $w^{-1}(\alpha_l)$ or $w^{-1}(\alpha_{l-1}) \in \Phi^-$. If we choose either $w = w_{l-1}w_{l-2}$ or $w = w_l w_{l-2}$ then we can satisfy these criterion.

Lemma 6.12. *1. If $w = w_{l-1}w_{l-2}$ then $\{\, \alpha_i \in \Pi : \alpha_i \in w(\Phi^-)\,\} = \{\alpha_{l-1}\}$ and $w \in W(l-1)^+$.*
2. If $w = w_l w_{l-2}$ then $\{\, \alpha_i \in \Pi : \alpha_i \in w(\Phi^-)\,\} = \{\alpha_l\}$ and $w \in W(l-1)^-$.

Proof. 1.

$$\begin{aligned} w_{l-2}w_{l-1} &: \mathbf{e}_{l-2} - \mathbf{e}_{l-1} \mapsto \mathbf{e}_{l-1} - \mathbf{e}_l \\ &: \mathbf{e}_{l-1} - \mathbf{e}_l \mapsto \mathbf{e}_l - \mathbf{e}_{l-2} \\ &: \mathbf{e}_{l-1} + \mathbf{e}_l \mapsto \mathbf{e}_l + \mathbf{e}_{l-2} \ . \end{aligned}$$

This is enough to check that $\{\, \alpha_i \in \Pi : \alpha_i \in w(\Phi^-)\,\} = \{\alpha_{l-1}\}$. The element $w(l-1)$ is the permutation of $\mathbf{e}_l = \mathbf{e}_{\pi_{w^{-1}}(l-1)}$ and $\mathbf{e}_{l-2} = \mathbf{e}_{\pi_{w^{-1}}(l)}$ whilst the element w_{Φ_2} maps $\mathbf{e}_{l-1}$ to $-\mathbf{e}_{l-1}$ and $\mathbf{e}_l$ to $-\mathbf{e}_l$. Hence

$$w(l-1)w^{-1}w_{\Phi_2} : \mathbf{e}_{l-2} - \mathbf{e}_{l-1} \mapsto \mathbf{e}_{l-1} + \mathbf{e}_{l-2} \in \Phi^+ \ .$$

Since $w_{l-2}w_{l-1}(\mathbf{e}_{l-2} - \mathbf{e}_{l-1}) \in \Phi^+$ and $(\sigma_{w^{-1}})_{l-1} = 1$ this confirms that $w \in W(l-1)^+$.

2.

$$\begin{aligned} w_{l-2}w_l &: \mathbf{e}_{l-2} - \mathbf{e}_{l-1} \mapsto \mathbf{e}_{l-1} + \mathbf{e}_l \\ &: \mathbf{e}_{l-1} - \mathbf{e}_l \mapsto -\mathbf{e}_l + \mathbf{e}_{l-2} \\ &: \mathbf{e}_{l-1} + \mathbf{e}_l \mapsto -\mathbf{e}_l - \mathbf{e}_{l-2} \ . \end{aligned}$$

From this we can deduce $\{\, \alpha_i \in \Pi : \alpha_i \in w(\Phi^-)\,\} = \{\alpha_l\}$. The element $w(l-1)$ is again the permutation of $\mathbf{e}_l = \mathbf{e}_{\pi_{w^{-1}}(l-1)}$ and $\mathbf{e}_{l-2} = \mathbf{e}_{\pi_{w^{-1}}(l)}$. Hence

$$w(l-1)w^{-1}w_{\Phi_2} : \mathbf{e}_{l-2} - \mathbf{e}_{l-1} \mapsto \mathbf{e}_{l-1} - \mathbf{e}_{l-2} \in \Phi^- \ .$$

Since $w_{l-2}w_l(\mathbf{e}_{l-2} - \mathbf{e}_{l-1}) \in \Phi^+$ and $(\sigma_{w^{-1}})_{l-1} = -1$ this confirms that $w \in W(l-1)^-$.

So $a_{\beta-1,1} \geq 2$ (and in fact it is possible to show that $a_{\beta-1,1} = 2$).

Now we need to show that we don't pick up any other terms.

Suppose we have a monomial term $X^r Y^{2j+\mathbf{e}_{l-1}+\mathbf{e}_l}$ corresponding to a $w \in W(l-1)$ where $\mathbf{e}_{l-1}$ (respectively, $\mathbf{e}_l$) $= 0$ or 1 according to whether α_{l-1} (respectively, α_l) $\in \{\, \alpha_i \in \Pi : \alpha_i \in w(\Phi^-) \,\} = \Pi'$ and Π' is a set of size $j + \mathbf{e}_{l-1} + \mathbf{e}_l$.
Firstly assume $j > 1$. Then using the fact that b_i is a strictly increasing sequence for $i \le l-2$ and $b_l = b_{l-1} = b_{l-2}/2 + 1$ we can deduce

$$\begin{aligned} r &= -\lambda(w) + \sum_{\alpha_i \in \Pi'} b_i \\ &\le (j-1)b_{l-3} + b_{l-2} + \mathbf{e}_{l-1}b_{l-1} + \mathbf{e}_l b_l - (j + \mathbf{e}_{l-1} + \mathbf{e}_l) \\ &< (2j + \mathbf{e}_{l-1} + \mathbf{e}_l)\beta - j \\ &< (2j + \mathbf{e}_{l-1} + \mathbf{e}_l)\beta - 1 \,. \end{aligned}$$

So we are left with the cases that $\Pi' = \{\alpha_{l-2}\}$, $\{\alpha_{l-2}, \alpha_{l-1}\}$, $\{\alpha_{l-2}, \alpha_l\}$ or $\{\alpha_{l-2}, \alpha_{l-1}, \alpha_l\}$. It suffices to show that $\lambda(w) > |\Pi'|$. Recall that $\lambda(w)$ is the number of positive roots sent to negative roots by w. Hence it suffices to show at least one positive root outside of Π' which gets sent to a negative root.
In the case that $\Pi' = \{\alpha_{l-2}\}$ we just have to demonstrate that $\lambda(w) > 1$. Now there is a unique element w of length one with $w^{-1}(\alpha_{l-2}) \in \Phi^-$, namely the reflection $w_{l-2} : \mathbf{e}_{l-2} - \mathbf{e}_{l-1} \mapsto \mathbf{e}_{l-1} - \mathbf{e}_{l-2}$. We need to show that this element is not in $W(l-1)$. Now $(\sigma_{w_{l-2}^{-1}})_{l-1} = 1$. So we just need to demonstrate that $w_{l-2}(l-1)w_{l-2}^{-1}w_{\Phi_2}(\alpha_{l-2}) \in \Phi^+$. The element $w_{l-2}(l-1)$ is again the element swapping $\mathbf{e}_{l-2}$ and $\mathbf{e}_l$. Then $w_{l-2}(l-1)w_{l-2}^{-1}w_{\Phi_2}(\alpha_{l-2}) = \mathbf{e}_{l-1} + \mathbf{e}_l \in \Phi^+$. Hence $w_{l-2} \notin W(l-1)$ and any element in $W(l-1)$ with $\Pi' = \{\alpha_{l-2}\}$ must have length greater than one. Recall that $\lambda(w)$ is the number of positive roots sent to negative roots by w. Hence it suffices to show at least one positive root outside of Π' which gets sent to a negative root. In the case that $\Pi' = \{\alpha_{l-2}, \alpha_{l-1}\}$, $\{\alpha_{l-2}, \alpha_l\}$ or $\{\alpha_{l-2}, \alpha_{l-1}, \alpha_l\}$ then since $\mathbf{e}_{l-2} - \mathbf{e}_{l-1}$ and $\mathbf{e}_{l-1} + \varepsilon\mathbf{e}_l$ are sent to negative roots (where $\varepsilon = \pm 1$ according to the choice of Π') then $\mathbf{e}_{l-2} + \varepsilon\mathbf{e}_l = (\mathbf{e}_{l-2} - \mathbf{e}_{l-1}) + (\mathbf{e}_{l-1} + \varepsilon\mathbf{e}_l)$ is also sent to a negative root. Hence $\lambda(w) > |\Pi'|$. □

This completes the analysis and confirms that $B_1(U) = a_{\beta-1,1}U$ where $a_{\beta-1,1} \ge 1$ (in fact it is possible to show that $a_{\beta-1,1} = 2$). Hence

$$-\frac{B_1(-1)}{(-1)A'(-1)} = -a_{\beta-1,1}$$

and so $\Re\left(-\frac{B_\gamma(\omega)}{\omega A'(\omega)}\right) < 0$, confirming Hypothesis 2. Therefore we can apply Theorem 5.13 to deduce that $P_G(s)$ has a natural boundary at $\beta_P = b_{l-2}/2 = l(l-1)/2 - 1$.

Corollary 6.13. *If $G = \mathrm{GO}_{2l}^+$ of type D_l then $Z_G(s)$ has abscissa of convergence at $\alpha_G = b_l + 1$ and a natural boundary at $\beta_P = b_{l-2}/2 = l(l-1)/2 - 1$.*