2

Motivations

In this chapter, we provide several examples from biology, physics, and mathematics including topology and stochastic processes, which have motivated the development of the theory of evolution algebras.

2.1 Examples from Biology

2.1.1 Asexual propagation

Prokaryotes are nonsexual reproductive organisms. Prokaryotic cells, unlike eukaryotic cells, do not have nuclei. The genetic material (DNA) is concentrated in a region called the nucleoid, with no membrane to separate this region from the rest of the cell. In prokaryote inheritance, there is no mitosis and meiosis. Instead, prokaryotes reproduce by binary fission. That is, after the prokaryotic chromosome duplicates and the cell enlarges, the enlarged cell becomes two small cells divided by a cell wall. Basically, the genetic information passed from one generation to the next should be conserved because of the strictness of DNA self-replication. However, there are still many possible factors in the environment that can induce the change of genetic information from generation to generation. The inheritance of prokaryotes is then not Mendelian. The first factor is DNA mutation. The second factor is related to gene recombination between a prokaryotic gene and a viral gene, for example bacteriophage λ's gene. This process of recombination between a prokaryotic gene and a viral gene is called gene transduction. For the detailed process of transduction, please refer to Nell Campbell [15]. The third factor comes from conjugation induced by sex plasmids. That is a direct transfer of genetic material between two prokaryotic cells. The most extensively studied case is Escherichia coli. Figure 2.1 depicts the division of bacterial cell from the book [15].

Now, let's mathematically formulate the asexual reproduction process. Suppose that we have n genetically distinct prokaryotes, denoting them by

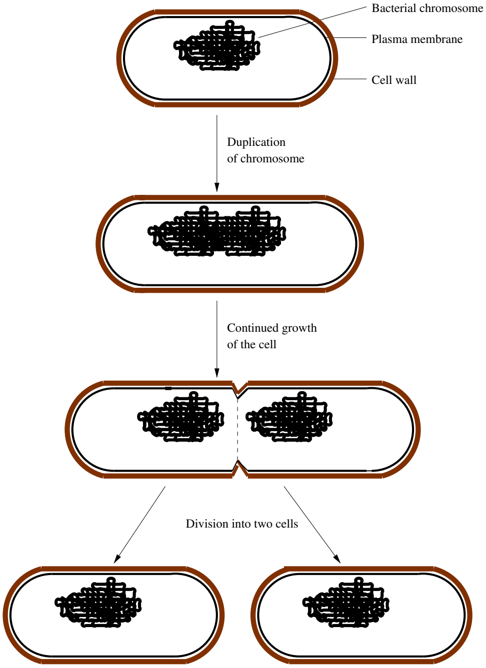

Fig. 2.1. Bacterial cell division

$p_1, p_2, \ldots, p_n$. We also suppose that the same environmental conditions are maintained from generation to generation. We look at changes in gene frequencies over two generation. We can view it either from the population standpoint or from the individual standpoint. To this end, we can set the following relations:

$$\begin{cases} p_i \cdot p_i = \sum_{k=1}^{n} c_{ik} p_k, \\ \quad p_i \cdot p_j = 0, i \neq j. \end{cases}$$

Here, we view the multiplication as asexual reproduction.

2.1.2 Gametic algebras in asexual inheritance

Let us recall some basic facts in general genetic algebras first [22]. Consider an infinitely large, randomly mating population of diploid individuals, with individuals differing genetically at one or several autosomal loci. Let $a_1, a_2, \ldots, a_n$ be the genetically distinct gametes produced by the population. By random union of gametes a_i and a_j, zygotes of type $a_i a_j$ are formed. Assume that a zygote $a_i a_j$ produces a number γ_{ijk} of gametes of type a_k, which survive in the next generation, $k, i, j = 1, 2, \ldots, n$. In the absence of selection, we assume all zygotes have the same fertility, and every zygote produces the same number of surviving gametes. Thus, one can have the probability that a zygote $a_i a_j$

produces a gamete a_k by number γ_{ijk}, still denoting γ_{ijk} as the probability that satisfies $\sum_{k=1}^{n} \gamma_{ijk} = 1$. The frequency of gamete a_k produced by the total population is $\sum_{i,j=1}^{n} v_i \gamma_{ijk} v_j$ if the gamete frequency vector of parental generation is $(v_1, v_2, \ldots, v_n)$. Now, the gamete algebra is defined on the linear space spanned by these gametes $a_1, a_2, \ldots, a_n$ over the real number field by the following multiplication table

$$a_i a_j = \sum_{k=1}^{n} \gamma_{ijk} a_k, \qquad i, j = 1, 2, \ldots, n,$$

and then linear extension onto the whole space. However, when we consider the asexual inheritance, the interpretation $a_i a_j$ as a zygote does not make sense biologically if $a_i \neq a_j$. But, $a_i a_i = a_i^2$ can still be interpreted as self-replication. Therefore, in asexual inheritance, we can use the following relations to define an algebra

$$\begin{cases} a_i \cdot a_i = \sum_{k=1}^{n} \gamma_{ik} a_k, \\ a_i \cdot a_j = 0, \quad i \neq j. \end{cases}$$

In the asexual inheritance, $a_i a_j$ is no longer a zygote; actually, it does not exist. Mathematically, we set $a_i a_j = 0$. Of course, this case is not of Mendelian inheritance.

2.1.3 The Wright-Fisher model

In population genetics, one often considers evolutionary behavior of a diploid population with a fixed size N. Suppose that the individuals in this population are monoecious and that no selective differences exist between two alleles A_1 and A_2 possible at a certain locus A. There are, $g_1, g_2, \ldots, g_n$, $n = 2N$ genes in the population in any generation. If we do not pay attention to genealogical relations, it is sufficient to know the number X of A_1 gene in each generation for understanding population evolutionary behavior. Clearly in any generation, X takes one of the values $0, 1, \ldots, 2N$, and we denote the value assumed by X in generation t by $X(t)$. We must assume some specific model that describes the way in which the genes in generation $t+1$ are derived from the genes in generation t. The Wright-Fisher model [2] [16] assumes that the genes in generation $t+1$ are derived by sampling with replacement from the genes of generation t. This means that the number $X(t+1)$ is a binomial random variable with index n and parameter $\frac{X(t)}{n}$. More explicitly, given $X(t) = k$, the probability p_{kl} of $X(t+1) = l$ is given by

$$p_{kl} = \binom{n}{l} \left(\frac{k}{n}\right)^{l} \left(1 - \frac{k}{n}\right)^{n-l}.$$

It is clear that $X(t)$ has markovian properties. Now, if we just overlook the details of the reproduction process and consider these probabilities as numbers, we may say that a certain gene, name it g_i in generation t, can reproduce

p_{ij} genes g_j in generation $t+1$. So, we focus on each individual gene to study its reproduction from the population level. Of course, the crossing of genes does not make any sense genetically, although the "replication" of a gene has certain biological meanings. Therefore, this viewpoint suggests the following symbolical formulae

$$\begin{cases} g_i \cdot g_i = \sum_{j=1}^{n} m_{ij} g_j \\ g_i \cdot g_j = 0, \quad i \neq j \end{cases},$$

where m_{ij} is the number of "offspring" of g_i. We will study a simple case that includes selection as a parameter in Example 7.

2.2 Examples from Physics

2.2.1 Particles moving in a discrete space

Consider a particle moving in a discrete space, for example, in a graph G. Suppose it starts at vertex v_i, then, which vertex will be its second position depends on which neighbor of v_i this particle prefers to. We may attach a preference coefficient to each edge from v_i to its neighbor v_j. For instance, we use w_{ij} as the preference coefficient, which is not necessarily a probability. Thus, the second position will be the vertex that this particle most prefers to. This particle will move on the graph continuously. If the particle stop at some vertex, its trace would be a path with the maximum of the total preference coefficient. Now, a question we need to ask is that how one can describe the motion of the particle algebraically and how one can find a path with the maximum of the total preference coefficients once the starting vertex and the end vertex are given. To discuss these problems, we can set up an algebraic model by giving the generator set and the defining relations as follows.

Let the vertex set $V = \{v_1, v_2, \ldots, v_r\}$ be the generator set, the defining relations are given:

$$\begin{cases} v_i \cdot v_i = \sum_j w_{ij} v_j \\ v_i \cdot v_j = 0, \quad i \neq j \end{cases},$$

where preference coefficients w_{ij} and w_{ji} may be different, and $i, j = 1, 2, \ldots, r$. In this content a path with the maximum of the total preference coefficient is just a principal power of an element in the algebra; we will see this point later on.

2.2.2 Flows in a discrete space (networks)

Let us recall some basic definitions in a type of network flow theory. Let $G = (V, E)$ be a multigraph, $s, t \in V$ be two fixed vertices, and $c : \overrightarrow{E} \to N$ be a map, where N is the set of the natural numbers with zero. We call c a

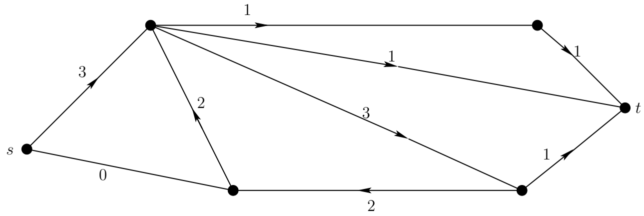

Fig. 2.2. Example of networks

capacity function on G and the tuple (G, s, t, c) a network, where $\overrightarrow{E}$ is the set of directed edges of G. Let us see an example of networks, Fig. 2.2.

Note that c is defined independently for the two directions of an edge. A function $f : \overrightarrow{E} \to R$ is a flow in the network (G, s, t, c) if it satisfies the following three conditions

(F1) $f(e, x, y) = -f(e, y, x)$, for all $(e, x, y) \in \overrightarrow{E}$ with $x \neq y$;
(F2) $f(v, V) = 0$, for all $v \in V - \{s, t\}$;
(F3) $f(\overrightarrow{e}) \leq c(\overrightarrow{e})$, for all $\overrightarrow{e} \in \overrightarrow{E}$.

Now, let us denote the capacity from vertex x to vertex y by c_{xy}, which is given by the capacity function $c(e, x, y) = c_{xy}$. We define an algebra $A(G, s, t, c)$ by generators and defining relations. The generator set is V and the defining relations are given by

$$\begin{cases} x \cdot x = \sum_y c_{xy} y \\ x \cdot y = 0, \quad x \neq y \end{cases},$$

where x and y are vertices. In the algebra $A\left(G, s, t, c\right)$, a flow is just an antisymmetric linear map. The interesting thing is that the requirement for Kirchhoff's law for a flow is automatically satisfied in the algebra.

2.2.3 Feynman graphs

Here let us recall some basic concepts in elementary particle physics. A Feynman graph [17] is a graph, each edge of which topologically represents a propagation of a free elementary particle and each vertex of which represents an interaction of elementary particles. Here, we regard a Feynman graph as an abstract object. A Feynman graph may have some extraordinary edges, called external edges, in addition to the ordinary edges, which are called internal edges. Every external edge has only one end point. A vertex is called an external vertex if at least one external edge is incident with it. Vertices other than external vertices are called internal vertices. According to the total number n of external edges, connected Feynman graphs have various names. For $n = 0$, they are called vacuum polarization graphs; $n = 1$, tadpole graphs; $n = 2$, self-energy graphs; $n = 3$, vertex graphs; $n = 4$, two-particle scattering graphs; and $n = 5$, one-particle production graphs. There are many issues in the theory of the Feynman integral that can be addressed. But here as an example to show that there exists an algebraic structure, we only mention one problem. To find some supporting properties of the Feynman integral, we need to discuss the so-called transport problem in a Feynman graph. That is, to transport given loads placed at some of vertices to the remainders as requested in such a way that when carrying a load along a edge l it does not exceed the capacity assigned to l. Similar to the previous example about the flows in a discrete space (networks), once we define an algebraic model as we did in the previous example, we will have a simple version of the original problem. So, our algebraic model can provide some insight into the theory of the Feynman integral. Below, is an example of a Feynman graph, Fig. 2.3, which yields a peculiar solution to the Landau equations and its corresponding algebra.

Denote their vertices as v_1, v_2, v_3, v_4, and two "infinite" vertices ε_1 and ε_2. The algebra corresponding to this self-energy Feynman graph is a quotient algebra whose generator set is $\{v_1, v_2, v_3, v_4, \varepsilon_1, \varepsilon_2\}$ and whose defining relations are given by

$$\begin{aligned}
v_1^2 &= a_{12}v_2,\ v_2^2 = p\varepsilon_1,\\
v_3^2 &= a_{31}v_1 + a_{32}v_2,\\
v_4^2 &= a_{41}v_1 + a_{43}v_3 - p\varepsilon_2,\\
\varepsilon_1^2 &= \varepsilon_1,\ \varepsilon_2^2 = \varepsilon_2,\\
0 &= v_i \cdot v_j,\ i \neq j,\\
0 &= \varepsilon_1 \cdot \varepsilon_2.
\end{aligned}$$

Here, coefficients a_{ij} and p are numbers that have physical significance.

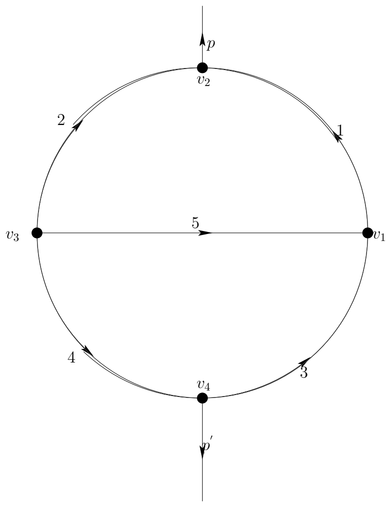

Fig. 2.3. Example of Feynman graph

2.3 Examples from Topology

2.3.1 Motions of particles in a 3-manifold

Consider a particle moving in the space (a 3-manifold M, compact or non-compact), and fix a time period t_1 to record the positions of the particle, the recorded trace of the particle is an embedded graph. There is a triangulation of the 3-manifold whose skeleton is the graph. To describe the motion, we may define

$$\begin{cases} v_i \cdot v_i = \sum_j a_{ij} v_j \\ v_i \cdot v_j = 0, \quad i \neq j, \end{cases}$$

where v_i is a vertex of the triangulation. The coefficient a_{ij} may be related to properties of the 3-manifold. For example, when the manifold carries a geometrical structure, a_{ij} may be related to the Gaussian curvature (could be negative) along the curved edge. We use these relations to define an algebra $A(M, t_1)$. This algebra will give information about the motion of the particle. When the time period of the recording is changed to t_2, we will obtain another algebra $A(M, t_2)$. Let's take an infinite sequence of time interval for recording, we will have a sequence of algebras $A(M, t_k)$. When the time interval goes to zero, we could ask what is the limit of the sequence $A(M, t_k)$. It is obvious that the sequence of these algebras reflects the properties of the manifold M. In Chapter 6, we give a different sequence of evolution algebras and an interesting conjecture related to 3-manifolds.

2.3.2 Random walks on braids with negative probabilities

In the low-dimensional topology, there is an extensive literature on the Burau representation. Jones, in his paper "Hecke algebra representation of braid groups and link polynomials" [27], offered a probabilistic interpretation of the Burau representation. We quote from this paper (with a small correction):

"For positive braids there is also a mechanical interpretation of the Burau matrix: lay the braid out flat and make it into a bowling alley with n lanes, the lanes going over each other according to the braid. If a ball travelling along a lane has probability $1 - t$ of falling off the top lane (and continuing in the lane below) at every crossing, then the (i, j) entry of the (nonreduced) Burau matrix is the probability that a ball bowled in the ith lane will end up in the jth."

Lin, Tian, and Wang, in their paper "Burau representation and random walks on string links" [28], generalized this idea to string links. Let's quote from their paper about the assignment of probability (weight) at each crossing for random walks:

(1) If we come to a positive crossing on the upper segment, the weight is $1-t$ if we choose to jump down and t otherwise; and
(2) If we come to a negative crossing on the upper segment, the weight is $1-\bar{t}$ if we choose to jump down and $\bar{t}$ otherwise, where $\bar{t} = t^{-1}$".

Now, we can see there are negative probabilities involved in this kind of random walks on braids. We will not go through their model here.

2.4 Examples from Probability Theory

2.4.1 Stochastic processes

Consider a stochastic process that moves through a countable set S of states. At stage n, the process decides where to go next by a random mechanism that depends only on the current state, and not on the previous history or even by the time n. These processes are called Markov chains on countable state spaces. Precisely, let X_n be a discrete-time Markov chain with state space $S = \{s_i \mid i \in \Lambda\}$, the transition probability be given by $p_{ij} = \Pr\{X_{n+1} = s_j \mid X_n = s_i\}$. Here we first consider stationary Markov chains. Then, we can reformulate such a Markov chain by an algebra. Taking the generator set as S, and the defining relations as follows

$$\begin{cases} s_i \cdot s_i = \sum_j p_{ij} s_j \\ s_i \cdot s_j = 0, \quad i \neq j \end{cases},$$

then we obtain a quotient algebra. As examples, we will study these algebras in detail in Chapter 4 of the book.