2

Wavelet and Gabor Frames

In this section we give a brief survey of the main notations, definitions, and results from frame theory, wavelet analysis, and time-frequency analysis which will be used throughout the book. We conclude this chapter with a section on amalgam spaces in the setting of locally compact groups, since in the sequel amalgam spaces will be employed in different group settings.

2.1 Frame Theory

In this section we briefly recall the definition and basic properties of frames and Schauder bases in Hilbert spaces. For more information on frame theory we refer to the various books and papers authored by Casazza [14], Christensen [20, 21], Daubechies [41], Heil and Walnut [76], and Young [128], and concerning Schauder bases theory we refer to Heil [70], Lindenstrauss and Tzafriri [99], Marti [101], Singer [114], and Young [128].

Let $\mathcal{H}$ be a separable Hilbert space, and let I be an indexing set. A sequence $\{f_i\}_{i\in I} \subseteq \mathcal{H}$ is a *frame* for $\mathcal{H}$ if there exist constants $0 < A \leq B < \infty$ such that

$$A\,\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\,\|f\|^2 \quad \text{for all } f \in \mathcal{H}. \tag{2.1}$$

The constants A and B are called *lower* and *upper frame bounds*, respectively. If A and B can be chosen such that $A = B$, then $\{f_i\}_{i\in I}$ is a *tight frame*. If we can take $A = B = 1$, it is called a *Parseval frame*.

Let $\{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$. Then the *frame operator*

$$Sf = \sum_{i\in I} \langle f, f_i\rangle\, f_i$$

is a bounded, positive, and invertible mapping of $\mathcal{H}$ onto itself, which satisfies

$$A\,\mathrm{Id} \leq S \leq B\,\mathrm{Id},$$

where Id denotes the identity operator. The *canonical dual frame* is $\{\tilde{f}_i\}_{i\in I}$, where $\tilde{f}_i = S^{-1}f_i$. For each $f \in \mathcal{H}$ we have the *frame expansions*

$$f = \sum_{i\in I} \langle f, f_i\rangle\, \tilde{f}_i = \sum_{i\in I} \langle f, \tilde{f}_i\rangle f_i.$$

In the special case that $\{f_i\}_{i\in I}$ forms a Parseval frame, the frame operator S is the identity, the dual frame coincides with the frame itself, and the frame expansions reduce to $f = \sum_{i\in I} \langle f, f_i\rangle\, f_i$.

A sequence which satisfies the upper frame bound estimate in (2.1), but not necessarily the lower estimate, is called a *Bessel sequence* and B is a *Bessel bound*. In this case,

$$\Big\|\sum_{i\in I} c_i f_i\Big\|^2 \le B \sum_{i\in I} |c_i|^2 \qquad \text{for any } (c_i)_{i\in I} \in \ell^2(I). \tag{2.2}$$

In particular, $\|f_i\|^2 \le B$ for every $i \in I$.

A sequence $\{f_i\}_{i\in\mathbb{N}}$ is a *Schauder basis* for $\mathcal{H}$ if for each $f \in \mathcal{H}$ there exist unique scalars $c_i(f)$, $i \in \mathbb{N}$, such that

$$f = \sum_{i=1}^{\infty} c_i(f) f_i. \tag{2.3}$$

Then there exists a unique biorthogonal system $\{\tilde{f}_i\}_{i\in\mathbb{N}}$ in $\mathcal{H}$, which is also a Schauder basis, called the *dual basis*, and which satisfies

$$f = \sum_{i=1}^{\infty} \langle f, f_i\rangle\ \tilde{f}_i = \sum_{i=1}^{\infty} \langle f, \tilde{f}_i\rangle\, f_i \quad \text{for all } f \in \mathcal{H}.$$

The associated *partial sum operators* are $S_N(f) = \sum_{i=1}^{N} \langle f, \tilde{f}_i\rangle\, f_i$ for $f \in \mathcal{H}$. The *basis constant* is the finite number $C = \sup_N \|S_N\|$. If for each $f \in \mathcal{H}$ the series $f = \sum_i c_i(f)\, f_i$ converges with respect to any ordering of the indices, then $\{f_i\}_{i\in\mathbb{N}}$ is called an *unconditional basis*. Consequently, for a Schauder basis the ordering in (2.3) can be crucial. If $0 < \inf_i \|f_i\| \le \sup_i \|f_i\| < \infty$ then $\{f_i\}_{i\in\mathbb{N}}$ is a *bounded basis*. A sequence $\{f_i\}_{i\in\mathbb{N}}$ which is a Schauder basis for its closed linear span within $\mathcal{H}$, denoted by $\overline{\text{span}}_{i\in\mathbb{N}}\{f_i\}$, is called a *Schauder basic sequence*.

We conclude this section with the following well-known result concerning the relationship between Schauder bases, Riesz bases, and frames (compare Casazza [14] or Christensen [21]).

Proposition 2.1. *The following three statements are equivalent:*

(i) $\{f_i\}_{i\in\mathbb{N}}$ *is a Schauder basis and a frame for* $\mathcal{H}$*,*

(ii) $\{f_i\}_{i\in\mathbb{N}}$ *is a Riesz basis for* $\mathcal{H}$*,*

(iii) $\{f_i\}_{i\in\mathbb{N}}$ *is a bounded unconditional basis for* $\mathcal{H}$*.*

2.2 Wavelet Analysis

In this section we will focus on the basic definitions, notations, and results in wavelet analysis which will be used in the sequel. For more information on wavelet theory we refer the reader to the books by Chui [24], Daubechies [41], and Hernández and Weiss [79], and the papers authored by Heil and Walnut [76] and Weiss and Wilson [124]. Most of the following definitions can be generalized to higher dimensions, but since in this book we focus on the one-dimensional situation, we just state the one-dimensional definitions. Let $\mathbb{A} = \mathbb{R}^+ \times \mathbb{R}$ denote the *affine group*, endowed with the multiplication

$$(a, b) \cdot (x, y) = \big(ax, \tfrac{b}{x} + y\big).$$

The identity element of $\mathbb{A}$ is $e = (1, 0)$, and inverses are given by

$$(a, b)^{-1} = \big(\tfrac{1}{a}, -ab\big).$$

The left-invariant Haar measure on $\mathbb{A}$ is $\mu_{\mathbb{A}} = \frac{dx}{x}\,dy$. We denote the norm and inner product on $L^2(\mathbb{A})$ with respect to this Haar measure by $\|\cdot\|_{L^2(\mathbb{A})}$ and $\langle \cdot, \cdot \rangle_{L^2(\mathbb{A})}$, respectively, whereas the norm and inner product on $L^2(\mathbb{R})$ will be denoted by $\|\cdot\|$ or $\|\cdot\|_2$ and $\langle \cdot, \cdot \rangle$.

Let σ be the unitary representation of $\mathbb{A}$ on $L^2(\mathbb{R})$ defined by

$$(\sigma(a, b)\psi)(x) = \tfrac{1}{\sqrt{a}}\psi(\tfrac{x}{a} - b) = D_a T_b \psi(x),$$

where D_a denotes the *dilation operator* $D_a f(x) = \frac{1}{\sqrt{a}} f(\frac{x}{a})$ and T_b denotes the *translation operator* $T_b f(x) = f(x - b)$.

For $f \in L^1(\mathbb{R}^d)$, we will use the following convention for the Fourier transform:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \langle x, \xi \rangle}\, dx.$$

Its extension to a unitary mapping from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$ will also be denoted by $\hat{f}$. The inverse Fourier transform shall be denoted by $f^\vee$.

Given $\psi \in L^2(\mathbb{R})$, called an *analyzing wavelet*, the *continuous wavelet transform (CWT)* $W_\psi f$ of $f \in L^2(\mathbb{R})$ with respect to ψ is

$$W_\psi f(a, b) = \big\langle f, \sigma(a, b)\psi \big\rangle = \big\langle f, D_a T_b \psi \big\rangle = \tfrac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(x)\, \overline{\psi(\tfrac{x}{a} - b)}\, dx.$$

We have

$$|W_\psi f(a, b)| \le \|f\|_2 \, \|\psi\|_2 \quad \text{for all } (a, b) \in \mathbb{A}$$

and $W_\psi f \in C(\mathbb{A})$. However, W_ψ does not map $L^2(\mathbb{R})$ into $L^2(\mathbb{A})$ for each $\psi \in L^2(\mathbb{R})$. We say that $\psi \in L^2(\mathbb{R})$ is *admissible* if the *admissibility constant* C_ψ defined by

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\xi)|^2}{|\xi|}\, d\xi$$

is finite. This condition is also sometimes called the *admissibility condition.* In particular, this is equivalent to the condition that both integrals

$$C_\psi^- = \int_{-\infty}^{0} \frac{|\hat{\psi}(\xi)|^2}{|\xi|}\, d\xi \quad \text{and} \quad C_\psi^+ = \int_{0}^{\infty} \frac{|\hat{\psi}(\xi)|^2}{|\xi|}\, d\xi$$

are finite. We further set

$$L_A^2(\mathbb{R}) = \{\psi \in L^2(\mathbb{R}) : \psi \text{ is admissible}\}.$$

Note that if $\psi \in L^1(\mathbb{R}) \cap L_A^2(\mathbb{R})$, then we must have $\hat{\psi}(0) = 0$, since $\hat{\psi}$ is continuous. If ψ is admissible, then W_ψ maps $L^2(\mathbb{R})$ into $L^2(\mathbb{A})$, cf. Heil and Walnut [76, Cor. 3.3.6]. Precisely, we have that if $\psi \in L_A^2(\mathbb{R})$ and $f \in L^2(\mathbb{R})$, then

$$\|W_\psi f\|_{L^2(\mathbb{A})}^2 = C_\psi^+ \int_0^\infty |\hat{f}(\xi)|^2\, d\xi + C_\psi^- \int_{-\infty}^0 |\hat{f}(\xi)|^2\, d\xi \le C_\psi\, \|f\|_2^2 .$$

Furthermore, the roles of f and ψ can be interchanged by using the relation $W_f\psi(a,b) = \overline{W_\psi f((a,b)^{-1})}$. We remark that this lack of symmetry is due to the fact that the Haar measure on $\mathbb{A}$ is not unimodular.

The next lemma lists several useful equivalent forms of the CWT:

Lemma 2.2. *If $f, \psi \in L^2(\mathbb{R})$, then*

$$\begin{aligned} W_\psi f(a,b) &= \tfrac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(x)\, \overline{\psi(\tfrac{x-ab}{a})}\, dx \\ &= \sqrt{a} \int_{-\infty}^{\infty} \hat{f}(\xi)\, \overline{\hat{\psi}(a\xi)}\, e^{2\pi i ab\xi}\, d\xi \\ &= (\hat{f} \cdot D_{a^{-1}} \overline{\hat{\psi}})^\vee(ab). \end{aligned}$$

The Besov spaces $B_{p,q}^\alpha(\mathbb{R})$, where $\alpha > 0$ and $1 \le p, q \le \infty$, are the natural function spaces associated with the CWT, namely, their norms quantify time-scale concentration of functions or distributions. They consist of functions in $L^p(\mathbb{R})$ with "smoothness α," with the parameter q allowing for a finer graduation of the quantification of smoothness. There are many equivalent definitions of the Besov spaces, and we refer to Triebel [122] for more information. An important fact is that equivalent norms for the Besov spaces can be formulated in terms of the discrete wavelet transform (see Meyer [102]) or the continuous wavelet transform (see Perrier and Basdevant [106]).

For practical purposes, however, discrete wavelet systems are needed, i.e., wavelet systems $\{\sigma(a,b)\psi\}_{(a,b)\in\Lambda}$, where Λ does not equal $\mathbb{A}$, but instead is just a sequence in $\mathbb{A}$. We remark that although Λ will always denote a countable sequence of points in $\mathbb{A}$ and not merely a subset, for simplicity we will write $\Lambda \subseteq \mathbb{A}$. In particular, this means that we allow repetitions of points. Further recall that the *disjoint union* $S = \bigcup_{i=1}^n S_i$ of a finite collection of sequences $S_1, \ldots, S_n$ contained in some set is the sequence $S = \{s_{11}, \ldots, s_{1n}, s_{21}, \ldots, s_{2n}, \ldots\}$, where each S_i is indexed as $S_i = \{s_{ki}\}_{k \in \mathbb{N}}$, i.e., S is the sequence obtained by amalgamating $S_1, \ldots, S_n$.

Definition 2.3. *(a) Given an analyzing wavelet $\psi \in L^2(\mathbb{R})$, a sequence of* time-scale indices $\Lambda \subseteq \mathbb{A}$*, and a weight function $w : \Lambda \to \mathbb{R}^+$, the* weighted (irregular) wavelet system *generated by ψ, Λ, and w is defined by*

$$\begin{aligned}\mathcal{W}(\psi, \Lambda, w) &= \{w(a,b)^{\frac{1}{2}}\, \sigma(a,b)\psi\}_{(a,b)\in\Lambda} \\ &= \{w(a,b)^{\frac{1}{2}}\, D_a T_b \psi\}_{(a,b)\in\Lambda} \\ &= \{w(a,b)^{\frac{1}{2}}\, \tfrac{1}{\sqrt{a}}\psi(\tfrac{x}{a} - b)\}_{(a,b)\in\Lambda}.\end{aligned}$$

If $w = 1$ we omit writing it.

(b) Let $\psi_1, \ldots, \psi_L \in L^2(\mathbb{R})$, and let $\Lambda_1, \ldots, \Lambda_L \subseteq \mathbb{A}$ with associated weight functions $w_\ell : \Lambda_\ell \to \mathbb{R}^+$ for $\ell = 1, \ldots, L$ be given. Then the weighted (irregular) wavelet system *generated by $\{(\psi_\ell, \Lambda_\ell, w_\ell)\}_{\ell=1}^L$ is the disjoint union*

$$\bigcup_{\ell=1}^{L} \mathcal{W}(\psi_\ell, \Lambda_\ell, w_\ell).$$

This definition of weighted wavelet systems includes as special cases the classical affine systems, the quasi-affine systems, and the co-affine systems (defined below). In particular, it is important to allow the case of nonconstant weights in order to obtain the quasi-affine systems.

The most often employed and studied wavelet systems are the *classical affine systems*

$$\mathcal{W}(\psi, \{(a^j, bk)\}_{j,k\in\mathbb{Z}}),$$

where $\psi \in L^2(\mathbb{R})$ and $a > 1$, $b > 0$. Since these systems lack the property of being shift-invariant, i.e., of being invariant under integer translations, the so-called *quasi-affine systems*

$$\mathcal{W}(\psi, \{(a^j, bk)\}_{j<0,k\in\mathbb{Z}} \cup \{(a^j, a^{-j}bk)\}_{j\geq 0,k\in\mathbb{Z}}, w),$$

where

$$\begin{aligned} w(a^j, bk) &= 1, & j < 0,\ k \in \mathbb{Z}, \\ w(a^j, a^{-j}bk) &= a^{-j}, & j \geq 0,\ k \in \mathbb{Z}, \end{aligned}$$

were developed for $a \in \mathbb{Z}$, $b > 0$ by Ron and Shen [109] and for $a \in \mathbb{Q}$ by Bownik [11]. Further contributions werde made by Chui, Shi, and Stöckler [35], Gressman, Labate, Weiss, and Wilson [61], and Johnson [88]. In [61] Gressman, Labate, Weiss, and Wilson also studied classical affine systems for

$b = 1$ with interchanged ordering of dilation and translation, i.e., wavelet systems of the form

$$\{T_k D_{a^j} \psi\}_{j,k\in\mathbb{Z}} = \{D_{a^j} T_{a^{-j}k} \psi\}_{j,k\in\mathbb{Z}},$$

where $a > 1$. This amounts, in the terminology of this book and letting $b > 0$ be arbitrary, to taking

$$\mathcal{W}(\psi, \{(a^j, a^{-j}bk)\}_{j,k\in\mathbb{Z}}).$$

These are the so-called *co-affine systems*. We also refer to Johnson [90].

Recently a general notion of oversampled affine systems was introduced by Hernández, Labate, Weiss, and Wilson [78] and extended by Johnson [89], which includes not only the classical affine, but also the quasi-affine and co-affine systems as special cases.

Definition 2.4. *Given $\psi \in L^2(\mathbb{R})$, $a > 1$, $b > 0$, and $\{r_j\}_{j\in\mathbb{Z}} \subseteq \mathbb{R}^+$, an* oversampled affine system *is a weighted wavelet system of the form $\mathcal{W}(\psi, \Lambda, w)$ with*

$$\Lambda = \left\{(a^j, \tfrac{bk}{r_j})\right\}_{j,k\in\mathbb{Z}} \qquad \text{and} \qquad w(a^j, \tfrac{bk}{r_j}) = \tfrac{1}{r_j}.$$

Example 2.5. The following are special cases of oversampled affine systems.

(i) The classical affine systems are obtained by setting $r_j \equiv 1$.

(ii) The quasi-affine systems of Ron and Shen [109] are obtained when a is an integer, $b = 1$, and

$$r_j = \begin{cases} 1, & j < 0, \\ a^j, & j \geq 0. \end{cases}$$

(iii) The quasi-affine systems of Bownik [11] are obtained when $a = \frac{p}{q}$ is rational, $b = 1$, and

$$r_j = \begin{cases} q^{-j}, & j < 0, \\ p^j, & j \geq 0. \end{cases}$$

(iv) The co-affine systems of Gressman, Labate, Weiss, and Wilson [61] are obtained by setting $r_j = a^j$ and $b = 1$.

2.3 Time-Frequency Analysis

As in the section before, we will state the basic definitions, notations, and results from time-frequency analysis as far as we will need them later. We mention the books by Daubechies [41], Feichtinger and Strohmer [56, 57], and Gröchenig [63] as references for further details.

In time-frequency analysis, time-frequency shifts play the role that time-scale shifts play in the wavelet setting. The time-frequency plane is actually the Heisenberg group $\mathbb{H} = \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{T}$ endowed with the multiplication

$$(a, b, c)(a', b', c') = (a + a', b + b', cc' e^{-2\pi i \langle a, b' \rangle}).$$

The toral component will later be ignored. This group is equipped with the so-called Schrödinger representation, which is the irreducible unitary representation of $\mathbb{H}$ on $L^2(\mathbb{R}^d)$ defined by

$$(\rho(a, b, c)g)(x) = c\, e^{2\pi i \langle b, x \rangle} g(x - a) = cM_b T_a g(x),$$

where M_b denotes the *modulation operator* $M_b f(x) = e^{2\pi i \langle b, x \rangle} f(x)$.

The analogue of the continuous wavelet transform is the *Short-Time Fourier transform (STFT)* of $f \in L^2(\mathbb{R}^d)$ with respect to $g \in L^2(\mathbb{R}^d)$ given by

$$V_g f(a, b) = \langle f, \rho(a, b, 1)g \rangle = \langle f, M_b T_a g \rangle = \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i \langle b, x \rangle} \overline{g(x - a)}\, dx$$

for $(a, b) \in \mathbb{R}^d \times \mathbb{R}^d = \mathbb{R}^{2d}$. We have

$$|V_g f(a, b)| \leq \|f\|_2 \, \|g\|_2 \quad \text{for all } (a, b) \in \mathbb{R}^{2d}$$

and $V_g f \in C(\mathbb{R}^{2d})$. Unlike the wavelet case, all vectors in $L^2(\mathbb{R})$ are admissible, because the Haar measure on the Heisenberg group is unimodular. The STFT V_g maps $L^2(\mathbb{R}^d)$ into $L^2(\mathbb{R}^{2d})$ for all $g \in L^2(\mathbb{R}^d)$. Given $g_1, g_2 \in L^2(\mathbb{R}^d)$ and $f_1, f_2 \in L^2(\mathbb{R}^d)$, the *orthogonality relations* for the STFT are

$$\left\langle V_{g_1} f_1,\, V_{g_2} f_2 \right\rangle = C_{g_1, g_2} \left\langle f_1, f_2 \right\rangle, \tag{2.4}$$

where $C_{g_1, g_2} = \int_{-\infty}^{\infty} \overline{g_1(x)}\, g_2(x)\, dx$. There is a great deal of symmetry between g and f in the STFT; more precisely, $V_f g(a, b) = e^{-2\pi i \langle a, b \rangle} \overline{V_g f(-a, -b)}$, hence, in particular, $\|V_g f\|_2 = \|V_f g\|_2$.

The modulation spaces are the natural function spaces associated with the STFT, namely, their norms quantify time-frequency concentration of functions or distributions in the same way that the Besov space norms quantify time-scale concentration. In particular, the *modulation space* $M^1(\mathbb{R}^d)$ consists of all functions $f \in L^1(\mathbb{R}^d)$ for which the following norm is finite:

$$\|f\|_{M^1(\mathbb{R}^d)} = \|V_g f\|_{L^1(\mathbb{R}^{2d})} = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |V_g f(a, b)|\, db\, da,$$

where g is any nonzero Schwartz-class function (each choice of g yields the same space under an equivalent norm). This modulation space was first defined by Feichtinger in [47] and is therefore also called the *Feichtinger algebra* (and is sometimes denoted by S_0); it is an algebra under both pointwise multiplication and convolution, and has many other remarkable properties. Notice that $M^1(\mathbb{R}^d) \subseteq L^2(\mathbb{R}^d)$. Moreover, we have the following well-known result on the relation between integrable STFTs and generators being in $M^1(\mathbb{R}^d)$.

Proposition 2.6. *The following conditions are equivalent.*

(i) $f \in M^1(\mathbb{R}^d)$.
(ii) $V_f f \in L^1(\mathbb{R}^{2d})$.

For more details on modulation spaces we refer the reader to Gröchenig [63].

As in wavelet theory discrete versions of the continuous Gabor systems are of considerable interest. Notice that also in this case, although Λ will always denote a sequence of points in $\mathbb{R}^d$ and not merely a subset, for simplicity we will write $\Lambda \subseteq \mathbb{R}^d$. Recall the definition of a disjoint union of sequences from Section 2.2.

Definition 2.7. *(a) Given a generator* $g \in L^2(\mathbb{R})$*, a sequence of* time-frequency indices $\Lambda \subseteq \mathbb{R}^{2d}$*, and a weight function* $w : \Lambda \to \mathbb{R}^+$*, the* weighted (irregular) Gabor system *generated by* g*,* Λ*, and* w *is defined by*

$$\begin{aligned}\mathcal{G}(g, \Lambda, w) &= \{w(a,b)^{\frac{1}{2}}\, \rho(a,b,1)g\}_{(a,b)\in\Lambda} \\ &= \{w(a,b)^{\frac{1}{2}}\, M_b T_a g\}_{(a,b)\in\Lambda} \\ &= \{w(a,b)^{\frac{1}{2}}\, e^{2\pi i \langle b,x\rangle} g(x-a)\}_{(a,b)\in\Lambda}.\end{aligned}$$

If $w = 1$ *we omit writing it.*

(b) Let $g_1, \dots, g_L \in L^2(\mathbb{R}^d)$*, and let* $\Lambda_1, \dots, \Lambda_L \subseteq \mathbb{R}^{2d}$ *with associated weight functions* $w_\ell : \Lambda_\ell \to \mathbb{R}^+$ *for* $\ell = 1, \dots, L$ *be given. Then the* weighted (irregular) Gabor system *generated by* $\{(g_\ell, \Lambda_\ell, w_\ell)\}_{\ell=1}^L$ *is the disjoint union*

$$\bigcup_{\ell=1}^{L} \mathcal{G}(g_\ell, \Lambda_\ell, w_\ell).$$

This definition of weighted Gabor systems includes as special cases the so-called *regular Gabor systems*, which are Gabor systems of the form $\mathcal{G}(g, a\mathbb{Z}^d \times b\mathbb{Z}^d)$, where $a, b > 0$.

One tool for studying irregular nonweighted Gabor systems is the notion of Beurling density. For $h > 0$ and $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, we let $Q_h(x)$ denote the cube centered at x with side length h, i.e., $Q_h(x) = \prod_{j=1}^d \left[x_j - \frac{h}{2}, x_j + \frac{h}{2}\right)$. Then the *upper Beurling density* of a sequence Λ in $\mathbb{R}^d$ is defined by

$$\mathcal{D}^+(\Lambda) = \limsup_{h\to\infty} \sup_{x\in\mathbb{R}^d} \frac{\#(\Lambda \cap Q_h(x))}{h^d},$$

and its *lower Beurling density* is

$$\mathcal{D}^-(\Lambda) = \liminf_{h\to\infty} \inf_{x\in\mathbb{R}^d} \frac{\#(\Lambda \cap Q_h(x))}{h^d}.$$

If $\mathcal{D}^-(\Lambda) = \mathcal{D}^+(\Lambda)$, then we say that Λ has *uniform Beurling density* and denote this density by $\mathcal{D}(\Lambda)$. For example, the lattice $\Lambda = a\mathbb{Z} \times b\mathbb{Z} \subseteq \mathbb{R}^2$, $a, b > 0$ has the uniform Beurling density $\mathcal{D}(\Lambda) = \frac{1}{ab}$.

It was shown by H. Landau [97, Lem. 4], that these densities do not depend on the particular choice of sets $Q_h(x)$, $h > 0$, $x \in \mathbb{R}^d$, in the sense that we can substitute these sets by sets $x + hU$, where $U \subseteq \mathbb{R}^d$ is a compact set with $|U| = 1$, i.e., of Lebesgue measure 1, whose boundary has measure zero, yet still obtain the same notion.

For more details on Beurling density and its connections to Gabor frames we refer the reader to the papers authored by Balan, Casazza, Heil, and Z. Landau [7] and by Christensen, Deng, and Heil [22]. New characterizations and an extension of the notion of Beurling density to weighted sequences can be found in Kutyniok [93]. This paper also contains a fundamental relationship between this density, the frame bounds, and the norm of the generator for weighted Gabor frames.

2.4 Amalgam Spaces

An amalgam space combines a local criterion for membership with a global criterion. The first amalgam spaces were introduced by Wiener in his study of generalized harmonic analysis [125, 126]. A comprehensive general theory of amalgam spaces on locally compact groups was introduced and extensively studied by Feichtinger and Gröchenig, e.g., [48, 52, 53, 50]. For an expository introduction to Wiener amalgams on $\mathbb{R}$ with extensive references to the original literature, we refer to Heil [71]. In the following we will give a brief survey of amalgam spaces of the type $W_G(L^\infty, L^p)$ and $W_G(C, L^p)$, where $1 \le p < \infty$ and G is a locally compact group. For the theory of locally compact groups we refer the reader to Folland [58] and Hewitt and Ross [80, 81].

Let G be a locally compact group and let μ_G denote a left-invariant Haar measure on G. Then the Wiener amalgam spaces $W_G(L^\infty, L^p)$ and $W_G(C, L^p)$ are defined as follows.

Definition 2.8. *Given* $1 \le p < \infty$, *the* amalgam space $W_G(L^\infty, L^p)$ *on the locally compact group* G *consists of all functions* $f \colon G \to \mathbb{C}$ *such that*

$$\|f\|_{W_G(L^\infty, L^p)} = \left(\int_G \operatorname{ess\,sup}_{a \in G} |f(a)\, \phi(x^{-1}a))|^p \, d\mu_G(x) \right)^{1/p} < \infty,$$

where ϕ *is a fixed continuous function with compact support satisfying* $0 \le \phi(x) \le 1$ *for all* $x \in G$, *and* $\phi(x) = 1$ *on some compact neighborhood of the identity. The amalgam space* $W_G(C, L^p)$ *is the closed subspace of* $W_G(L^\infty, L^p)$ *consisting of the continuous functions in* $W_G(L^\infty, L^p)$.

$W_G(L^\infty, L^p)$ is a Banach space, and its definition is independent of the choice of ϕ, in the sense that each choice of ϕ yields the same space under an equivalent norm. For proofs and more details, see Feichtinger and Gröchenig [52, 53].

The space $W_G(L^\infty, L^p)$ can be equipped with an equivalent discrete-type norm. For this, we first require some notation. Given some neighborhood U of the identity in G, a sequence $\{x_i\}_{i\in I}$ in G is called *U-dense*, if $\bigcup_{i\in I} x_i U = G$. It is called *$V$-separated*, if for some relatively compact neighborhood V of the identity the sets $\{x_i V\}_{i\in I}$ are pairwise disjoint. The sequence is called *relatively separated*, if it is the finite union of V-separated sequences.

Definition 2.9. *A sequence of continuous functions $\{\phi_i\}_{i\in I}$ on G is called a* bounded partition of unity*, or* BUPU*, if*

(i) $0 \le \phi_i(x) \le 1$ *for all* $i \in I$ *and* $x \in G$.
(ii) $\sum_{i\in I} \phi_i \equiv 1$.
(iii) *There exists a compact neighborhood U of the identity in G with nonempty interior and a U-dense, relatively separated sequence $\{x_i\}_{i\in I}$ such that* $\operatorname{supp}(\phi_i) \subseteq x_i U$ *for all $i \in I$.*

Then we have the following result from Feichtinger [49] (compare also Feichtinger and Gröchenig [52]).

Theorem 2.10. *If $\{\phi_i\}_{i\in I}$ is a BUPU, then*

$$\|f\|_{W_G(L^\infty,L^p)} \asymp \left(\sum_{i\in I} \|f\phi_i\|_\infty^p \right)^{\frac{1}{p}},$$

where $\asymp$ denotes the equivalence of norms.