2

Existence of the Integrated Density of States

Intuitively, the integrated density of states (IDS) measures how many electron energy levels can be found below a given energy per unit volume of a solid. An alternative name for this quantity is spectral distribution function. It can be used to calculate the free energy and hence all basic thermodynamic quantities of the corresponding non-interacting many-particle system.

To define the IDS mathematically one uses an exhaustion procedure. More precisely, one takes an increasing sequence Λ_l of open subsets of $\mathbb{R}^d$ such that each Λ_l has finite volume and $\bigcup_l \Lambda_l = \mathbb{R}^d$. Then the operator H_ω^l, which is the restriction of H_ω to Λ_l with Dirichlet boundary conditions, is selfadjoint, bounded below and its spectrum consists of discrete eigenvalues $\lambda_1(H_\omega^l) \leq \lambda_2(H_\omega^l) \leq \cdots \leq \lambda_n(H_\omega^l) \to \infty$. Here $\lambda_n = \lambda_{n+1}$ means that the eigenvalue is degenerate and we take this into account in the enumeration.

The *normalised eigenvalue counting function* or *finite volume integrated density of states* N_ω^l is defined as

$$N_\omega^l(E) := \frac{\#\{n|\, \lambda_n(H_\omega^l) < E\}}{|\Lambda_l|} \tag{2.1}$$

The numerator can equally well be expressed using the trace of the spectral projection $P_\omega^l(I)$ associated to the operator H_ω^l and an energy interval I, namely

$$\#\{n|\, \lambda_n(H_\omega^l) < E\} = \mathrm{Tr}\Big[P_\omega^l(]-\infty, E[)\Big]$$

Note that $N_\omega^l \colon \mathbb{R} \to [0, \infty[$ is a *distribution function* of a point measure for all $l \in \mathbb{N}$, i.e. $N_\omega^l(E) = \nu_\omega^l(]-\infty, E[)$. Here ν_ω^l is the *finite volume density of states measure* defined by

$$\nu_\omega^l(I) := |\Lambda_l|^{-1}\, \#\{n|\, \lambda_n(H_\omega^l) \in I\}$$

By definition, a distribution function is non-negative, left-continuous and non-decreasing. In particular, it has at most countably many points of discontinuity.

Under specific additional conditions on the random operator and the exhaustion sequence $\Lambda_l, l \in \mathbb{N}$ one can prove that

(i) For almost all $\omega \in \Omega$ the sequence N_ω^l converges to a distribution function N_ω as l goes to infinity. This means that we have $N_\omega^l(E) \to N_\omega(E)$ for all continuity points E of the limit distribution N_ω.

(ii) For almost all $\omega \in \Omega$ the distribution functions N_ω coincide, i.e. there is an ω-independent distribution function N such that $N = N_\omega$ for almost all ω. This function N is called the *integrated density of states*. Note that its independence of ω is not due to an explicit integration over the probability space Ω, but only to the exhaustion procedure. This is the reason why the IDS is called *self-averaging*.

(iii) In most cases there is a formula for the IDS as an expectation value of a trace per unit volume of a spectral projection. For $\mathbb{Z}^d$-ergodic operators it reads

$$N(E) := \mathbb{E}\left\{\mathrm{Tr}\big[\chi_\Lambda P_\omega(]-\infty, E[)\big]\right\} \tag{2.2}$$

Here Λ denotes the unit box $]0,1[^d$, which is the periodicity cell of the lattice $\mathbb{Z}^d$. Actually, one could choose certain other functions instead of χ_Λ, yielding all the same result, cf. Formula (2.15). The equality (2.2) holds for $\mathbb{R}^d$-ergodic operators, too. It is sometimes called *Pastur-Shubin trace formula*.

In the following we prove the properties of the IDS just mentioned by two methods. In Sects. 2.2–2.6 a detailed proof is given using the Laplace transforms of the distribution functions N_ω^l, while Sect. 2.7 is devoted to an alternative method of proof. It uses Dirichlet-Neumann bracketing estimates for Schrödinger operators, which carry over to the corresponding eigenvalue counting functions. These are thus super- or subadditive stochastic processes to which an ergodic theorem can be applied.

Actually the proof using Laplace transforms will apply to more general situations than discussed so far, namely to more general geometries than the Euclidean one. To be precise, we will consider random Schrödinger operators on Riemannian covering manifolds, where both the potential and the metric may depend on the randomness. This includes random Laplace-Beltrami operators.

We follow the presentation and proofs in [392, 327]. The general strategy we use was developed by Pastur and Shubin in [384] and [431] for random and almost-periodic operators in Euclidean space. A particular idea of this approach is to prove the convergence of the Laplace transforms $\mathcal{L}_\omega^l$ of the normalised finite volume eigenvalue counting functions N_ω^l instead of proving the convergence of N_ω^l directly. This is actually the main difference to the second approach we outline in Sect. 2.7, which is taken from [254]. The Pastur-Shubin strategy seems to be better suited for geometries with an underlying group structure which is non-abelian.

Indeed, one of the differences between random operators on manifolds and those on $\mathbb{R}^d$ is that the operator is equivariant with respect to a group which does not need to be commutative. This means that one has to use a non-abelian ergodic theorem to derive the convergence of the distribution functions N^l_ω or, alternatively, of their Laplace transforms $\mathcal{L}^l_\omega$. This imposes some restriction on the strategy of the proof since ergodic theorems which apply to non-abelian groups need more restrictive assumptions than their counterparts for commutative groups, cf. also Remark 2.6.2. For processes which are not additive, but only super- or subadditive, there is a non-abelian maximal ergodic theorem at disposal (cf. 6.4.1 Theorem in [313]) but so far no pointwise theorem. This is also the reason why the Dirichlet-Neumann bracketing approach of Sect. 2.7 does not seem applicable to random operators living on a covering manifold with non-abelian deck-transformation group (covering transformation group).

2.1 Schrödinger Operators on Manifolds: Motivation

In this chapter we study the IDS of random Schrödinger operators on manifolds. Let us first explain the physical motivation for this task.

Consider a particle or a system of particles which are constrained to a sub-manifold of the ambient (configuration) space. The classical and quantum Hamiltonians for such systems have been studied e.g. in [367, 173] (see also the references therein). To arrive at an effective Hamiltonian describing the constrained motion on the sub-manifold, a limiting procedure is used: a (sequence of) confining high-barrier potential(s) is added to the Hamiltonian defined on the ambient space to restrict the particle (system) to the sub-manifold. In [367, 173] one can find a discussion of the similarities and differences between the obtained effective quantum Hamiltonian and its classical analogue.

A important feature of the effective quantum Hamiltonian is the appearance of a so-called *extra-potential* depending on the extrinsic curvature of the sub-manifold and the curvature of the ambient space. This means that even if we disregard external electric forces the relevant quantum mechanical Hamiltonian of the constrained system is not the pure Laplacian but contains (in general) a potential energy term. This fact explains the existence of curvature-induced bound states in quantum waveguides and layers, see [157, 138, 343, 139] and the references therein.

As is mentioned in [367], the study of effective Hamiltonians of constrained systems is motivated by specific physical applications. They include stiff molecular bonds in (clusters of) rigid molecules and molecular systems evolving along reaction paths. From the point of view of the present work quantum wires, wave guides and layers are particularly interesting physical examples. Indeed, for these models (in contrast to quantum dots) at least one dimension of the constraint sub-manifold is of macroscopic size. Moreover, it is natural to assume that the resulting Hamiltonian exhibits some form of translation invariance in the macroscopic direction. E.g. it may be periodic, quasi-periodic or — in the case of a random model — stationary.

For random quantum waveguides and layers the existence of dense point spectrum is expected, cf. the discussion of localisation in Sect. 1.3. Indeed, for a specific type of random waveguide embedded in the Euclidean plane this has been rigorously proven in [274, 275]. The question of spectral localisation due to random geometries has been raised already in [109]. There the behaviour of Laplace-Beltrami operators under non-smooth perturbations of the metric is studied.

While the motivations presented above stem from solid state physics a further stimulus to study the IDS comes from within mathematics itself: For various geometries with a group action it makes sense to define translation invariant or periodic operators. This applies to covering manifolds, Cayley graphs and more generally quasi-transitive graphs, as well as for CW-complexes. These carry naturally defined Laplace operators on functions and more generally on p-forms. Related objects are magnetic Laplacians and Schrödinger operators, for which it is also makes sense to formulate an equivariance condition.

Here, by the term periodic we mean the property that there is a subgroup of the automorphism group of the geometric space such that the operator is invariant under conjugation with unitary transformations which are associated to elements of the subgroup.

Due to the wealth of possibilities of the geometric structure, here even Laplacians without any random perturbation may exhibit intriguing spectral properties, part of which is captured by the IDS, respectively the spectral distribution functions of Laplacians on forms. Instances of such features are L^2-Betti numbers, Novikov-Shubin invariants and other geometric L^2-invariants, the jumps of the IDS, and the gap structure of the spectrum.

Geometric L^2-invariants describe the behaviour of the spectral distribution function at energies near the spectral bottom. For instance, the p^{th} L^2-Betti number is the size of the jump at zero energy of the distribution function of the Laplacian on p-forms, see for instance [30, 118, 126]. Novikov-Shubin invariants correspond to characteristic exponents of the asymptotic behaviour of the IDS near zero, cf. e.g. [381, 380, 144, 208, 344, 160, 433, 347, 382, 25, 24]. L^2-torsion is a generalisation of ordinary torsion and has an analytic as well as a combinatorial variant. These invariants have been introduced in [348, 344, 356, 77] and studied in [349, 75, 76, 103, 71, 321, 272, 62].

Another interesting feature of some periodic Laplace-Beltrami operators is the existence of L^2-eigenfunctions, a phenomenon which cannot happen in Euclidean space. Since the IDS is a spectral measure of the periodic operator, the set of discontinuities of this function is precisely the set of eigenvalues of the operator. These issues have been studied for instance in [126, 461, 292, 462, 325]. For more details see the discussion in Remark 3.1.3.

The analysis of the gap structure of the spectrum of periodic operators of Schrödinger type is a further topic which has attracted attention. More precisely, one is interested whether the spectrum in interrupted by *spectral gaps*, i.e. intervals on the real line which belong to the resolvent set. In case there are gaps: can one establish upper and lower bounds for the width and number of gaps and the spectral bands separating them? For different types of periodic or gauge-invariant elliptic differential operators on manifolds spectral gaps have been analysed in [463, 70, 69, 238, 393, 394]. See Example 2.2.5 for a particular case. Even for periodic Schrödinger operators in Euclidean space it is not trivial to characterise the gap structure. This is illustrated by works devoted to the Bethe-Sommerfeld conjecture, e.g. [451, 446, 447, 448, 211]. For almost periodic operators the situation is even more difficult and additional questions arise like the gap labelling problem, see [37, 477, 240, 39, 42, 237, 38] and the references therein. Although the gap structure of the spectrum is a mathematically intriguing question for its own sake, it is also important from the physical point of view. The features of gaps in the energy spectrum are relevant for the conductance properties of the physical system cf. e.g.[339].

The periodic operators on manifolds discussed so far are generalised by their random analogues studied in this chapter.

2.2 Random Schrödinger Operators on Manifolds: Definitions

Let us explain the geometric setting in which we are working precisely: let X be a complete d-dimensional Riemannian manifold with metric g_0. We denote the volume form of g_0 by vol_0. Let Γ be a discrete, finitely generated subgroup of the isometries of (X, g_0) which acts freely and properly discontinuously on X such that the quotient $M := X/\Gamma$ is a compact (d-dimensional) Riemannian manifold. Let $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$ be a probability space on which Γ acts by measure preserving transformations. Assume moreover that the action of Γ on Ω is ergodic. Now we are in the position to define what we mean by a random metric and consequently a random Laplace-Beltrami operator.

Definition 2.2.1. *Let $\{g_\omega\}_{\omega\in\Omega}$ be a family of Riemannian metrics on X. Denote the corresponding volume forms by vol_ω. We call the family $\{g_\omega\}_{\omega\in\Omega}$ a* random metric *on (X, g_0) if the following five properties are satisfied:*

(2.3) *The map $\Omega \times TX \to \mathbb{R}$, $(\omega, v) \mapsto g_\omega(v, v)$ is jointly measurable.*

(2.4) *There is a $C_g \in]0, \infty[$ such that*

$$C_g^{-1}\, g_0(v, v) \le g_\omega(v, v) \le C_g\, g_0(v, v) \quad \text{for all} \quad v \in TX.$$

(2.5) *There is a $C_\rho \in]0, \infty[$ such that*

$$|\nabla_0\, \rho_\omega(x)|_0 \le C_\rho \text{ for all } x \in X,$$

where ∇_0 denotes the gradient with respect to g_0, ρ_ω is the unique smooth density of vol_0 with respect to vol_ω, and $|v|_0^2 = g_0(v, v)$.

(2.6) *There is a uniform lower bound $(d-1)K \in \mathbb{R}$ for the Ricci curvatures of all Riemannian manifolds (X, g_ω). Explicitly, $\mathrm{Ric}(g_\omega) \ge (d-1)K g_\omega$ for all $\omega \in \Omega$ and on the whole of X.*

(2.7) *The metrics are compatible in the sense that the deck transformations*

$$\gamma\colon (X, g_\omega) \to (X, g_{\gamma\omega}), \quad \gamma\colon x \mapsto \gamma x$$

are isometries.

Property (2.7) implies in particular that the induced maps

$$U_{(\omega,\gamma)}\colon L^2(X, \mathrm{vol}_{\gamma^{-1}\omega}) \to L^2(X, \mathrm{vol}_\omega), \quad (U_{(\omega,\gamma)}f)(x) = f(\gamma^{-1}x)$$

are unitary operators. The density ρ_ω appearing in (2.5) satisfies by definition

$$\int_X f(x)\, d\mathrm{vol}_0(x) = \int_X f(x)\rho_\omega(x)\, d\mathrm{vol}_\omega(x).$$

It is a smooth function and can be written as

$$\rho_\omega(x) = \big(\det g_0(e^i_\omega, e^j_\omega)\big)^{1/2} = \Big(\det g_\omega(e^i_0, e^j_0)\Big)^{-1/2}$$

Here $e^1_0, \dots, e^d_0$ denotes any basis of T_xX which is orthonormal with respect to the scalar product $g_0(x)$, and $e^1_\omega, \dots, e^d_\omega \in T_xX$ is any basis orthonormal with respect to $g_\omega(x)$. It follows from (2.4) that

$$C_g^{-d/2} \le \rho_\omega(x) \le C_g^{d/2} \quad \text{for all } x \in X,\ \omega \in \Omega \tag{2.8}$$

which in turn, together with property (2.5) and the chain rule, implies

$$|\nabla_0\, \rho_\omega^{\pm 1/2}(x)|_0 \le C_g^{3d/4}\, |\nabla_0\, \rho_\omega(x)|_0 \quad \text{for all } x \in X,\ \omega \in \Omega \tag{2.9}$$

Moreover, for any measurable $\Lambda \subset X$ by (2.8) we have the volume estimate

$$C_g^{-d/2}\, \mathrm{vol}_0(\Lambda) \le \mathrm{vol}_\omega(\Lambda) \le C_g^{d/2}\, \mathrm{vol}_0(\Lambda) \tag{2.10}$$

We denote the Laplace-Beltrami operator with respect to the metric g_ω by Δ_ω.

Associated to the random metric just described we define a random family of operators.

Definition 2.2.2. *Let $\{g_\omega\}$ be a random metric on (X, g_0). Let $V\colon \Omega \times X \to \mathbb{R}$ be a jointly measurable mapping such that for all $\omega \in \Omega$ the* potential *$V_\omega := V(\omega, \cdot) \ge 0$ is in $L^1_{\mathrm{loc}}(X)$. For each $\omega \in \Omega$ let $H_\omega = -\Delta_\omega + V_\omega$ be a Schrödinger operator defined on a dense subspace $\mathcal{D}_\omega$ of the Hilbert space $L^2(X, \mathrm{vol}_\omega)$. The family $\{H_\omega\}_{\omega\in\Omega}$ is called a* random Schrödinger operator *if it satisfies for all $\gamma \in \Gamma$ and $\omega \in \Omega$ the following* equivariance *condition*

$$H_\omega = U_{(\omega,\gamma)} H_{\gamma^{-1}\omega} U^*_{(\omega,\gamma)} \tag{2.11}$$

Remark 2.2.3 (Restrictions, quadratic forms and selfadjointness). Some remarks are in order why the sum of the Laplace-Beltrami operator and the potential is selfadjoint. We consider the two cases of an operator on the whole manifold X and on a proper open subset of X simultaneously. The set of all smooth functions with compact support in an open set $\Lambda \subset X$ is denoted by $C_c^\infty(\Lambda)$. For each $\omega \in \Omega$ we define the quadratic form

$$\widetilde{Q}(H_\omega^\Lambda) \colon C_c^\infty(\Lambda) \times C_c^\infty(\Lambda) \to \mathbb{R}, \tag{2.12}$$

$$(f,h) \mapsto \int_\Lambda g_\omega(x)\Big(\nabla f(x), \nabla h(x)\Big)\, d\mathrm{vol}_\omega(x) + \int_\Lambda f(x) V_\omega(x) h(x)\, d\mathrm{vol}_\omega(x)$$

We infer from Theorem 1.8.1 in [108] that this quadratic from is closable and its closure $Q(H_\omega^\Lambda)$ gives rise to a densely defined, non-negative selfadjoint operator H_ω^Λ. Actually, $Q(H_\omega^\Lambda)$ is the form sum of the quadratic forms of the negative Laplacian and the potential. By the very definition, $C_c^\infty(\Lambda)$ is dense in the domain of $Q(H_\omega^\Lambda)$ for all ω. The result in [108] is stated for the Euclidean case $X = \mathbb{R}^d$ but the proof works equally well for general Riemannian manifolds.

The unique selfadjoint operator associated to the above quadratic form is called *Schrödinger operator with Dirichlet boundary conditions*. It is the Friedrichs extension of the restriction $H_\omega^\Lambda|_{C_c^\infty(\Lambda)}$. If the potential term is absent we call it negative *Dirichlet Laplacian*.

There are special subsets of the manifold which will play a prominent role later:

Definition 2.2.4. *For an $x \in X$ the set $O(x) := \{y \in X \mid \exists \gamma \in \Gamma : y = \gamma x\}$ is called the Γ-orbit of x. The relation $x \sim y \Longleftrightarrow O(x) \cap O(y) \neq \emptyset$ partitions X into equivalence classes. A subset $\mathcal{F} \subset X$ is called Γ-*fundamental domain* if it contains exactly one element of each equivalence class.*

In [2, Sect. 3] it is explained how to obtain a connected, polyhedral Γ-fundamental domain $\mathcal{F} \subset X$ by lifting simplices of a triangularisation of M in a suitable manner. $\mathcal{F}$ consists of finitely many smooth images of simplices which can overlap only at their boundaries. In particular, it has piecewise smooth boundary.

To illustrate the above definitions we will look at some examples. Firstly, we consider covering manifolds with abelian deck-transformation group.

Example 2.2.5 (Abelian covering manifolds). Consider a covering manifold (X, g_0) with a finitely generated, abelian subgroup Γ of the isometries of X. If the number of generators of the group Γ equals r, it is isomorphic to $\mathbb{Z}^{r_0} \times \mathbb{Z}_{p_1}^{r_0} \times \ldots \mathbb{Z}_{p_n}^{r_n}$. Here $\sum r_i = r$ and $\mathbb{Z}_p$ is the cyclic group of order p. Assume as above that the quotient X/Γ is compact. Periodic Laplace-Beltrami and Schrödinger operators on such spaces have been analysed e.g. in [462, 393, 394].

In the following we will discus some examples studied by Post in [393, 394]. The aim of these papers was to construct covering manifolds, such that the corresponding Laplace operator has open spectral gaps. More precisely, for any given natural number N, manifolds are constructed with at least N spectral gaps. For technical reasons the study is restricted to abelian coverings. In this case the Floquet decomposition of the periodic operator can be used effectively. Post studies two classes of examples with spectral gaps. In the first case a conformal perturbation of a given covering manifold is used to open up gaps in the energy spectrum of the Laplacian. The second type of examples in [394] is of more interest to us. There, one starts with infinitely many translated copies of a compact manifold and joins them by cylinders to form a periodic network of ‘pipes’. By shrinking the radius of the connecting cylinders, more and more gaps emerge in the spectrum. Such manifolds have a non-trivial fundamental group and are thus topologically not equivalent to $\mathbb{R}^d$. On the other hand their deck-transformation group is rather easy to understand, since it is abelian. In particular, it is amenable (cf. Definition 2.3.4), which is a crucial condition in the study conducted later in this chapter. Some of the examples in [393, 394] are manifolds which can be embedded in $\mathbb{R}^3$ as surfaces. They can be thought of as periodic quantum waveguides and networks. See [393] for some very illustrative figures.

Furthermore, in [394] perturbations techniques for Laplace operators on covering manifolds have been developed, respectively carried over from earlier versions suited for compact manifolds, cf. [85, 22, 177]. They include conformal perturbations and local geometric deformations. Floquet decomposition is used to reduce the problem to an operator on a fundamental domain with quasi-periodic boundary conditions and discrete spectrum. Thereafter the min-max principle is applied to geometric perturbations of the Laplacian.

Related random perturbations of Laplacians are studied in [326, 325] (cf. also Example 2.2.7). In particular a Wegner estimate for such operators is derived.

Now we give an instance of a covering manifold X with non-abelian deck-transformation group Γ.

Example 2.2.6 (Heisenberg group). The Heisenberg group H_3 is the manifold of 3×3-matrices given by

$$H_3 = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R} \right\} \tag{2.13}$$

equipped with a left-invariant metric. The Lie-group H_3 is diffeomorphic to $\mathbb{R}^3$. Its group structure is not abelian, but nilpotent.

The subset $\Gamma = H_3 \cap M(3, \mathbb{Z})$ forms a discrete subgroup. It acts from the left on H_3 by isometries and the quotient manifold H_3/Γ is compact.

Next we give examples of a random potential and a random metric which give rise to a random Schrödinger operator as in Definition 2.2.2. Both have an underlying structure which resembles alloy-type models (in Euclidean space).

Example 2.2.7. (a) Consider the case where the metric is fixed, i.e. $g_\omega = g_0$ for all $\omega \in \Omega$, and only the potential depends on the randomness in the following way:

$$V_\omega(x) := \sum_{\gamma \in \Gamma} q_\gamma(\omega)\, u(\gamma^{-1}x), \tag{2.14}$$

Here $u : X \to \mathbb{R}$ is a bounded, compactly supported measurable function and $q_\gamma \colon \Omega \to \mathbb{R}$ is a sequence of independent, identically distributed random variables. By considerations as in Remark 1.2.2 the random operator $H_\omega := -\Delta + V_\omega, \omega \in \Omega$ is seen to satisfy the equivariance condition.

(b) Consider the situation where the metric has an alloy like structure. Let (g_0, X) be a Riemannian covering manifold and let a family of metrics $\{g_\omega\}_\omega$ be given by

$$g_\omega(x) = \left(\sum_{\gamma \in \Gamma} r_\gamma(\omega)\, u(\gamma^{-1}x)\right) g_0(x)$$

where $u \in C_c^\infty(X)$ and the $r_\gamma \colon \Omega \to]0, \infty[, \gamma \in \Gamma$ are a collection of independent, identically distributed random variables. Similarly as in the previous example one sees that the operators Δ_ω are equivariant.

Operators of the above type are discussed in [325].

2.3 Non-Randomness of Spectra and Existence of the IDS

Here we state the main theorems on the non-randomness of the spectral components and the existence and the non-randomness of the IDS. They refer to random Schrödinger operators as defined in 2.2.2.

Theorem 2.3.1. *There exists a subset Ω' of full measure in $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$ and subsets of the real line Σ and $\Sigma_\bullet$, where $\bullet \in \{disc, ess, ac, sc, pp\}$ such that for all $\omega \in \Omega'$*

$$\sigma(H_\omega) = \Sigma \quad \text{and} \quad \sigma_\bullet(H_\omega) = \Sigma_\bullet$$

for any $\bullet = disc, ess, ac, sc, pp$. If Γ is infinite, $\Sigma_{disc} = \emptyset$.

The theorem is proven in [328], see Theorem 5.1. The arguments go to a large part along the lines of [388, 319, 255]. Compare also the literature on almost-periodic Schrödinger operators, for instance [431, 32].

For the proof of the theorem one has to find random variables which encode the spectrum of $\{H_\omega\}_\omega$ and which are invariant under the action of Γ. By ergodicity they will be constant almost surely. The natural random variables to use are spectral projections, more precisely, their traces. However, since $\mathbb{R}$ is uncountable and one has to deal also with the different spectral components, some care is needed.

Random operators introduced in Definition 2.2.2 are naturally affiliated to a von Neumann algebra of operators which we specify in

Definition 2.3.2. *A family* $\{B_\omega\}_{\omega\in\Omega}$ *of bounded operators* $B_\omega\colon L^2(X, \mathrm{vol}_\omega) \to L^2(X, \mathrm{vol}_\omega)$ *is called a* bounded random operator *if it satisfies:*

(i) $\omega \mapsto \langle g_\omega, B_\omega f_\omega\rangle$ *is measurable for arbitrary* $f, g \in L^2(\Omega \times X, \mathbb{P} \circ \mathrm{vol})$.
(ii) *There exists a* ω*-uniform bound on the norms* $\|B_\omega\|$ *for almost all* $\omega \in \Omega$.
(iii) *For all* $\omega \in \Omega, \gamma \in \Gamma$ *the equivariance condition*

$$B_\omega = U_{(\omega,\gamma)} B_{\gamma^{-1}\omega} U^*_{(\omega,\gamma)}$$

holds.

By the results of Sect. 2.4, $\{F(H_\omega)\}_\omega$ is a bounded random operator for any measurable, bounded function F.

It turns out that (equivalence classes of) bounded random operators form a von Neumann algebra. More precisely, consider two bounded random operators $\{A_\omega\}_\omega$ and $\{B_\omega\}_\omega$ as *equivalent* if they differ only on a subset of Ω of measure zero. Each equivalence class gives rise to a bounded operator on $L^2(\Omega \times X, \mathbb{P} \circ \mathrm{vol})$ by $(Bf)(\omega, x) := B_\omega f_\omega(x)$, see Appendix A in [328]. This set of operators is a von Neumann algebra $\mathcal{N}$ by Theorem 3.1 in [328]. On $\mathcal{N}$ a trace τ of type II_∞ is given by

$$\tau(B) := \mathbb{E}\,[\mathrm{Tr}(\chi_{\mathcal{F}}\, B_\bullet)]$$

Here $\mathrm{Tr} := \mathrm{Tr}_\omega$ denotes the trace on the Hilbert space $L^2(X, \mathrm{vol}_\omega)$. Actually, for any choice of $u\colon \Omega \times X \to \mathbb{R}^+$ with $\sum_{\gamma\in\Gamma} u_{\gamma^{-1}\omega}(\gamma^{-1}x) \equiv 1$ for all $(\omega, x) \in \Omega \times X$ we have

$$\tau(B) = \mathbb{E}\,[\mathrm{Tr}(u_\bullet\, B_\bullet)] \tag{2.15}$$

In analogy with the case of operators which are Γ-invariant [30] we call τ the *Γ-trace.* The spectral projections $\{P_\omega(]-\infty, \lambda[)\}_\omega$ of $\{H_\omega\}_\omega$ onto the interval $]-\infty, \lambda[$ form a bounded random operator. Thus, it corresponds to an element of $\mathcal{N}$ which we denote by $P(]-\infty, \lambda[)$. Consider the normalised Γ-trace of P

$$N_H(\lambda) := \frac{\tau(P(]-\infty, \lambda[)}{\mathbb{E}\,[\mathrm{vol}_\bullet(\mathcal{F})]} \tag{2.16}$$

The following is Theorem 3 in [327], see also [328].

Theorem 2.3.3. $P(]-\infty, \lambda[)$ *is the spectral projection of the direct integral operator*

$$H := \int_\Omega^\oplus H_\omega\, d\mathbb{P}(\omega)$$

and N_H is the distribution function of its spectral measure. In particular, the almost sure spectrum Σ of $\{H_\omega\}_\omega$ coincides with the points of increase

$$\{\lambda \in \mathbb{R} | \, N_H(\lambda + \varepsilon) > N_H(\lambda - \varepsilon) \text{ for all } \varepsilon > 0\}$$

of N_H.

That the IDS can be expressed in terms of a trace on a von Neumann Algebra was known long ago. In [430] and [431] Shubin establishes this relation for almost-periodic elliptic differential operators in Euclidean space.

We want to describe the self-averaging IDS by an exhaustion of the whole manifold X along a sequence $\Lambda_l \nearrow X$, $l \in \mathbb{N}$ of subsets of X. To ensure the existence of a sequence of subsets which is appropriate for the exhaustion procedure, we have to impose additional conditions on the group Γ, which will be discussed next.

Definition 2.3.4. *A group Γ is called* amenable *if it has a left invariant mean m_L.*

Amenability enters as a key notion in Definition 2.3.6 and Theorem 2.3.8. For readers acquainted only with Euclidean geometry, its role is motivated in Remark 2.3.10.

Under some conditions on the group, amenability can be expressed in other ways. A locally compact group Γ is amenable if for any $\varepsilon > 0$ and compact $K \subset \Gamma$ there is a compact $G \subset \Gamma$ such that

$$m_L(G \Delta KG) < \varepsilon \, m_L(G)$$

where m_L denotes the left invariant Haar measure, cf. Theorem 4.13 in [390]. This is a geometric description of amenability of Γ. If Γ is a discrete, finitely generated group we chose m_L to be the counting measure and write instead $|\cdot|$. In this case Γ is amenable if and only if a *Følner sequence* exists:

Definition 2.3.5. *Let Γ be a discrete, finitely generated group.*

(i) *A sequence $\{I_l\}_l$ of finite, non-empty subsets of Γ is called a* Følner sequence *if for any finite $K \subset \Gamma$ and $\varepsilon > 0$*

$$|I_l \triangle K I_l| \leq \varepsilon \, |I_l|$$

for all l large enough.

(ii) *We say that a sequence $I_l \subset \Gamma, l \in \mathbb{N}$ of finite sets has the* Tempelman *or* doubling property *if it obeys*

$$\sup_{l \in \mathbb{N}} \frac{|I_l I_l^{-1}|}{|I_l|} < \infty$$

(iii) *We say that a sequence $I_l \subset \Gamma, l \in \mathbb{N}$ of finite sets has the* Shulman property *if it obeys*

$$\sup_{l \in \mathbb{N}} \frac{|I_l I_{l-1}^{-1}|}{|I_l|} < \infty$$

(iv) *A Følner sequence* $\{I_l\}_l$ *is called a* tempered Følner sequence *if it has the Shulman property.*

In our setting Γ is discrete and finitely generated. (Actually, $K := \{\gamma \in \Gamma \mid \overline{\gamma\mathcal{F}} \cap \overline{\mathcal{F}} \neq \emptyset\}$ is a finite generator set for Γ. This follows from the fact that the quotient manifold X/Γ is compact, cf. Sect. 3 in [2].) Under this circumstances a Følner sequence exists if and only if there is a sequence of finite, non-empty sets $J_l \subset \Gamma, l \in \mathbb{N}$ such that $\lim_{l\to\infty} \frac{|J_l \triangle \gamma J_l|}{|J_l|} = 0$ for all $\gamma \in \Gamma$. Moreover, for discrete, finitely generated, amenable groups there exists a Følner sequence which is increasing and exhausts Γ, cf. Theorem 4 in [1].

Both properties (ii) and (iii) control the growth of the group Γ. Lindenstrauss observed in [342] that each Følner sequence has a tempered subsequence. Note that this implies that every amenable group contains a tempered Følner sequence. One of the deep results of Lindenstrauss' paper is, that this condition is actually sufficient for a pointwise ergodic theorem, cf. Theorem 2.6.1. Earlier it was known that such theorems can be established under the more restrictive Tempelman property [470, 313, 471]. Shulman [434] first realised the usefulness of the relaxed condition (iii).

In the class of countably generated, discrete groups there are several properties which ensure amenability. Abelian groups are amenable. More generally, all solvable groups and groups of subexponential growth, in particular nilpotent groups, are amenable. This includes the (discrete) Heisenberg group considered in Example 2.2.6. Subgroups and quotient groups of amenable groups are amenable. On the other hand, the free group with two generators is not amenable.

For the discussion of combinatorial properties of Følner sequences in discrete amenable groups see [1].

Any finite subset $I \subset \Gamma$ defines a corresponding set

$$\phi(I) := \operatorname{int}\Big(\bigcup_{\gamma\in I} \gamma\overline{\mathcal{F}}\Big) \subset X$$

where $\operatorname{int}(\cdot)$ stands for the open interior of a set.

In the following we will need some notation for the thickened boundary. Denote by d_0 the distance function on X associated to the Riemannian metric g_0. For $h > 0$, let $\partial_h\Lambda := \{x \in X \mid d_0(x, \partial\Lambda) \le h\}$ be the *boundary tube of width* h and Λ_h be the interior of the set $\Lambda \setminus \partial_h\Lambda$.

Definition 2.3.6. *(a) A sequence* $\{\Lambda_l\}_l$ *of subsets of* X *is called* admissible exhaustion *if there exists an increasing, tempered Følner sequence* $\{I_l\}_l$ *with* $\bigcup_l I_l = \Gamma$ *such that* $\Lambda_l = \phi(I_l^{-1})$, $l \in \mathbb{N}$.

(b) A sequence $\Lambda_l, l \in \mathbb{N}$ *of subsets of* (X, g_0) *is said to satisfy the* van Hove property *[478] if*

$$\lim_{l\to\infty} \frac{\mathrm{vol}_0(\partial_h\Lambda_l)}{\mathrm{vol}_0(\Lambda_l)} = 0 \textit{ for all } h > 0 \tag{2.17}$$

In our setting amenability of Γ ensures that an admissible exhaustion always exists. It is easy to see (cf. e.g. Lemma 2.4 in [392]) that every admissible exhaustion satisfies the van Hove property. Inequality (2.10) implies that for a sequence with the van Hove property

$$\lim_{l\to\infty}\frac{\mathrm{vol}_\omega(\partial_h\Lambda_l)}{\mathrm{vol}_\omega(\Lambda_l)}=0 \text{ for all } h>0$$

holds for all $\omega\in\Omega$. Let us remark that one could require for the sets Λ_l in the exhaustion sequence to have smooth boundary, cf. Definition 2.1 in [392]. Such sequences exist for any X with amenable deck-transformation group Γ, as well. This may be of interest, if one wants to study Laplacians with Neumann boundary conditions.

For groups of polynomial growth it is possible to construct analoga of admissible exhaustions by taking metric open balls $B_{r_l}(o)$ around a fixed point $o\in X$ with increasing radii $r_1,\dots,r_n,\dots\to\infty$, cf. Theorem 1.5 in [392].

Remark 2.3.7. In our setting it is always possible to chose the sequences $\{I_l\}_l$ and $\{\Lambda_l\}_l$ in such a way that they exhaust the group, respectively the manifold. However, this is not necessary for our results.

A simple instance where $\cup_l\Lambda_l\neq X$ can be given in one space dimension. Let $X=\mathbb{R}$, $\Gamma=\mathbb{Z}$, $I_l=\{1-l,\dots,0\}$, $\mathcal{F}=[0,1[$ and consequently $\Lambda_l=]0,l[$. One can use this sequence of sets to define the IDS of random Schrödinger operators although $\cup_l\Lambda_l=[0,\infty[$. A non-trivial example where the sets Λ_l do not exhaust X can be found in [464, 466]. There Sznitman considers random Schrödinger operators in hyperbolic spaces. In that setting the approach presented here does not work due to lack of amenability. Sznitman constructs the IDS by choosing a sequence of balls Λ_l which converges to a horoball of the hyperbolic space. The resulting IDS corresponds to the restriction of the random operator to the horoball and not to the one on the whole space.

We denote by H^l_ω the Dirichlet restriction of H_ω to Λ_l, cf. Remark 2.2.3, and define the finite volume IDS by the formula

$$N^l_\omega(\lambda):=\mathrm{vol}_\omega(\Lambda_l)^{-1}\#\{n\mid\lambda_n(H^l_\omega)<\lambda\}$$

Now we are able to state the result of [327] on the existence of a self-averaging IDS.

Theorem 2.3.8. *Let $\{H_\omega\}_\omega$ be a random Schrödinger operator and Γ an amenable group. For any admissible exhaustion $\{\Lambda_l\}_l$ there exists a set $\Omega'\subset\Omega$ of full measure such that*

$$\lim_{l\to\infty}N^l_\omega(\lambda)=N_H(\lambda), \tag{2.18}$$

for every $\omega\in\Omega'$ and every continuity point $\lambda\in\mathbb{R}$ of N_H.

Definition 2.3.9. *The distribution function defined by the limit in* (2.18) *is called* integrated density of states*.*

Thus all properties (i)–(iii) on page 14 can be established for the model under study. In particular, formula (2.18) is a variant of the Pastur-Shubin trace formula in the context of manifolds. Theorem 2.3.8 is proven in Sects. 2.4–2.6. It recovers in particular the result of Adachi and Sunada [2] on the existence of the IDS of periodic Schrödinger operators on manifolds.

Remark 2.3.10. Let us motivate, for readers acquainted only with Euclidean space, why it is natural that the amenability requirement enters the theorem. In the theory of random operators and in statistical mechanics one often considers a sequence of sets $\Lambda_l, l \in \mathbb{N}$ which tends to the whole space. Even in Euclidean geometry it is known that the exhaustion sequence $\Lambda_l, l \in \mathbb{N}$ needs to tend to $\mathbb{R}^d$ in an appropriate way, e.g. in the sense of van Hove or Fisher [417]. Convergence in the sense of van Hove [478] means that

$$\lim_{l\to\infty} \frac{|\partial_\varepsilon \Lambda_l|}{|\Lambda_l|} = 0 \tag{2.19}$$

for all positive ε.

If one chooses the sequence $\Lambda_l, l \in \mathbb{N}$ badly, one cannot expect the convergence of the finite volume IDS' N_ω^l to a limit as $l \to \infty$. In a non-amenable geometry, any exhaustion sequence is bad, since (2.19) cannot be satisfied, cf. Proposition 1.1 in [2].

Remark 2.3.11. We have assumed the potentials V_ω to be non-negative and some of our proofs will rely on this fact.

However, the *statements* of Theorem 2.3.1 on the non-randomness of the spectrum and Theorem 2.3.8 on the existence of the IDS carry over to V_ω which are uniformly bounded below by a constant C not depending on $\omega \in \Omega$. Indeed, in this case our results directly apply to the shifted operator family $\{H_\omega - C\}_{\omega\in\Omega}$. This implies immediately the same statements for the original operators, since the spectral properties we are considering transform trivially if a constant is added to the operator.

Remark 2.3.12 (Uniform convergence of the IDS). For many types of random Hamiltonians on discrete geometric structures the convergence (2.18) of the IDS is actually uniform in the spectral parameter almost surely, cf. [323, 330, 147, 148, 324, 329]. Note that this statement is non-trivial, since the IDS may have discontinuities, as discussed in Remark 3.1.3.

Uniform convergence of the IDS has also been established for certain types of random Schrödinger operators on metric graphs, cf. [209].

2.4 Measurability

Since we want to study the operators H_ω^Λ as random variables we need a notion of measurability. To this aim, we extend the definition introduced by

Kirsch and Martinelli [255] for random operators on a fixed Hilbert space to families of operators where the spaces and domains of definition vary with $\omega \in \Omega$.

To distinguish between the scalar products of the different L^2-spaces we denote by $\langle \cdot, \cdot \rangle_0$ the scalar product on $L^2(\Lambda, \mathrm{vol}_0)$ and by $\| \cdot \|_0$ the corresponding norm. Similarly, $\langle \cdot, \cdot \rangle_\omega$ and $\| \cdot \|_\omega$ are the scaler product and the norm, respectively, of $L^2(\Lambda, \mathrm{vol}_\omega)$.

Definition 2.4.1. *Consider a family of selfadjoint operators* $\{H_\omega\}_\omega$*, where the domain of* H_ω *is a dense subspace* $\mathcal{D}_\omega$ *of* $L^2(\Lambda, \mathrm{vol}_\omega)$*. The family* $\{H_\omega\}_\omega$ *is called a* measurable family of operators *if*

$$\omega \mapsto \langle f_\omega, F(H_\omega) f_\omega \rangle_\omega \tag{2.20}$$

is measurable for all measurable and bounded $F\colon \mathbb{R} \to \mathbb{C}$ *and all measurable functions* $f\colon \Omega \times \Lambda \to \mathbb{R}$ *with* $f(\omega, \cdot) = f_\omega \in L^2(\Lambda, \mathrm{vol}_\omega)$ *for every* $\omega \in \Omega$*.*

Theorem 2.4.2. *A random Schrödinger operator* $\{H_\omega\}_{\omega \in \Omega}$ *as in Definition 2.2.2 is a measurable family of operators. The same applies to the Dirichlet restrictions* $\{H_\omega^\Lambda\}_{\omega \in \Omega}$ *to any open subset* Λ *of* X*.*

For the proof of this theorem we need some preliminary considerations.

As the next lemma will show, assumption (2.4) in our setting implies that it is sufficient to establish the weak measurability (2.20) for functions f which are constant in ω. To formulate the precise statement, we first note that the Hilbert spaces $L^2(\Lambda, \mathrm{vol}_0)$ and $L^2(\Lambda, \mathrm{vol}_\omega)$ coincide as sets for all $\omega \in \Omega$, though not in their scalar products. Thus it makes sense to speak about a function $f_\omega \equiv f$ as an element of $L^2(\Lambda, \mathrm{vol}_\omega)$ "=" $L^2(\Lambda, \mathrm{vol}_0)$.

Lemma 2.4.3. *A random Schrödinger operator* $\{H_\omega\}_\omega$ *is measurable if and only if*

$$\omega \mapsto \langle f, F(H_\omega) f \rangle_\omega \text{ is measurable} \tag{2.21}$$

for all measurable and bounded $F\colon \mathbb{R} \to \mathbb{C}$ *and all* $f \in L^2(\Lambda, \mathrm{vol}_0)$*.*

Proof. To see this, note that (2.21) implies the same statement if we replace $f(x)$ by $h(\omega, x) = g(\omega) f(x)$ where $g \in L^2(\Omega)$ and $f \in L^2(\Lambda, \mathrm{vol}_0)$. Such functions form a total set in $L^2(\Omega \times \Lambda, \mathbb{P} \circ \mathrm{vol})$.

Now, consider a measurable $h\colon \Omega \times \Lambda \to \mathbb{R}$ such that $h(\omega, \cdot) \in L^2(\Lambda, \mathrm{vol}_\omega)$ for every $\omega \in \Omega$. Then $h_n(\omega, x) := \chi_{h,n}(\omega)\, h(\omega, x)$ is in $L^2(\Omega \times \Lambda, \mathbb{P} \circ \mathrm{vol})$ where $\chi_{h,n}$ denotes the characteristic function of the set $\{\omega \in \Omega | \, \|h(\omega)\|_{L^2(\Lambda, \mathrm{vol}_\omega)} \le n\}$. Since $\chi_{h,n} \to 1$ pointwise on Ω for $n \to \infty$ we obtain

$$\langle h_n(\omega), F(H_\omega) h_n(\omega) \rangle_\omega \to \langle h(\omega), F(H_\omega) h(\omega) \rangle_\omega$$

which shows that $\{H_\omega\}_\omega$ is a measurable family of operators. □

To prove Theorem 2.4.2 we will pull all operators H_ω^Λ onto the same Hilbert space using the unitary transformation S_ω induced by the density ρ_ω

$$S_\omega \colon L^2(\Lambda, \mathrm{vol}_0) \to L^2(\Lambda, \mathrm{vol}_\omega), \quad (S_\omega f)(x) = \rho_\omega^{1/2}(x) f(x)$$

The transformed operators are

$$A_\omega := -S_\omega^{-1}\, \Delta_\omega^\Lambda\, S_\omega \tag{2.22}$$
$$A_\omega \colon S_\omega^{-1}\, \mathcal{D}(\Delta_\omega^\Lambda) \subset L^2(\Lambda, \mathrm{vol}_0) \longrightarrow L^2(\Lambda, \mathrm{vol}_0)$$

The domain of definition $S_\omega^{-1}\, \mathcal{D}(\Delta_\omega^\Lambda)$ is dense in $L^2(\Lambda, \mathrm{vol}_0)$ and contains all smooth functions of compact support in Λ.

The first fact we infer for the operators $A_\omega, \omega \in \Omega$ is that they are uniformly bounded with respect to each other, at least in the sense of quadratic forms. This is the content of Proposition 3.4 in [327] which we quote without proof.

Denote the quadratic forms associated to the operators $-\Delta_0^\Lambda$, respectively A_ω, by Q_0 and Q_ω, and the corresponding *quadratic form domains* by $\mathcal{D}(Q_0)$ and $\mathcal{D}(Q_\omega)$.

Proposition 2.4.4. *Let $\mathcal{D} \subset L^2(\Lambda, \mathrm{vol}_0)$ be the closure of $C_c^\infty(\Lambda)$ with respect to the norm $\big(Q_0(f,f) + \|f\|_0^2\big)^{1/2}$. Then*

$$\mathcal{D} = \mathcal{D}(Q_0) = \mathcal{D}(Q_\omega)$$

and there exists a constant C_A such that

$$C_A^{-1}\left(Q_0(f,f) + \|f\|_0^2\right) \le Q_\omega(f,f) + \|f\|_0^2 \le C_A\left(Q_0(f,f) + \|f\|_0^2\right) \tag{2.23}$$

for all $f \in \mathcal{D}$ and $\omega \in \Omega$.

In the proof of this proposition the bound (2.5) — more precisely (2.9) — on the gradient of the density ρ_ω is needed. It seems to be a technical assumption and in fact dispensable by using a trick from [109], at least if Λ is precompact or of finite volume.

Since we are now dealing with a family of operators on a fixed Hilbert space, we are in the position to apply the theory developed in [255]. The following result is an extension of Proposition 3 there. It suits our purposes and shows that our notion of measurability is compatible with the one in [255].

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{D} \subset \mathcal{H}$ a (fixed) dense subset and $B_\omega \colon \mathcal{D} \to \mathcal{H}$, $\omega \in \Omega$ non-negative operators. Denote by $\tilde{\Sigma} = \overline{\bigcup_\omega \sigma(B_\omega)}$ the closure of the union of all spectra, and by $\tilde{\Sigma}^c$ its complement. To establish the measurability of the family $\{B_\omega\}_\omega$ one can use one of the following classes of test functions:

- $\mathcal{F}_1 = \{\chi_{]-\infty,\lambda[} | \, \lambda \ge 0\}$,
- $\mathcal{F}_2 = \{x \mapsto \mathrm{e}^{\mathrm{i}tx} | \, t \in \mathbb{R}\}$,
- $\mathcal{F}_3 = \{x \mapsto \mathrm{e}^{-tx} | \, t \ge 0\}$,

- $\mathcal{F}_4 = \{x \mapsto (z-x)^{-1} | \, z \in \mathbb{C} \setminus \tilde{\Sigma}\}$,
- $\mathcal{F}_5 = \mathcal{F}_4(z_0) = \{x \mapsto (z_0 - x)^{-1}\}$ for a fixed $z_0 \in \mathbb{C} \setminus \tilde{\Sigma}$,
- $\mathcal{F}_6 = C_b = \{f : \mathbb{R} \to \mathbb{C} | \, f$ bounded, continuous$\}$,
- $\mathcal{F}_7 = \{f : \mathbb{R} \to \mathbb{C} | \, f$ bounded, measurable$\}$.

The following proposition says, that it does not matter which of the above sets of functions one chooses for testing the measurability of $\{B_\omega\}_\omega$.

Proposition 2.4.5. *For $i = 1, \ldots, 7$ the following statements are equivalent:*

$(\mathbf{F}_i)$ $\qquad \omega \mapsto \langle f, F(B_\omega)h \rangle_{\mathcal{H}}$ *is measurable for all $f, h \in \mathcal{H}$ and $F \in \mathcal{F}_i$*

Proof. It is obvious that $(\mathbf{F}_4) \Rightarrow (\mathbf{F}_5)$, $(\mathbf{F}_7) \Rightarrow (\mathbf{F}_6)$, and $(\mathbf{F}_6) \Rightarrow (\mathbf{F}_3)$. The equivalence of $(\mathbf{F}_1)$, $(\mathbf{F}_2)$ and $(\mathbf{F}_4)$ can be found in [255].

To show $(\mathbf{F}_5) \Rightarrow (\mathbf{F}_4)$, consider the set

$$Z := \{z \in \tilde{\Sigma}^c | \, \omega \mapsto (z - H_\omega)^{-1} \text{ is weakly measurable}\}$$

in the topological space $\tilde{\Sigma}^c$. It is closed, since $z_n \to z$ implies the convergence of the resolvents, see e.g. [410, Theorem VI.5]. A similar argument using the resolvent equation and a Neumann series expansion shows that $z \in Z$ implies $B_\delta(z) \subset Z$ where $\delta := d(z, \tilde{\Sigma})$. Since $\tilde{\Sigma}^c$ is connected, $Z = \tilde{\Sigma}^c$ follows.

$(\mathbf{F}_3) \Rightarrow (\mathbf{F}_1)$: By the Stone-Weierstrass Theorem, see e.g. [410, Thm. IV.9], applied to $C([0,\infty])$ it follows that $\mathcal{F}_3$ is dense in the set of functions $\{f \in C([0,\infty]) \, | \, f(\infty) = 0\} = C_\infty([0,\infty[)$. We may approximate any $\chi_{]-\infty,\lambda[}$ pointwise by a monotone increasing sequence $0 \le f_n, n \in \mathbb{N}$ in $C_\infty(\mathbb{R})$. Polarisation, the spectral theorem, and the monotone convergence theorem for integrals imply that $\chi_{]-\infty,\lambda[}(H_\omega)$ is weakly measurable. An analogous argument shows $(\mathbf{F}_1) \Rightarrow (\mathbf{F}_7)$, since any non-negative $f \in \mathcal{F}_7$ can be approximated monotonously pointwise by non-negative step functions $f_n, n \in \mathbb{N}$. □

We use the following proposition taken from [458] (Prop. 1.2.6.) to show that $\{A_\omega\}_\omega$ is a measurable family of operators.

Proposition 2.4.6. *Let $B_\omega, \omega \in \Omega$ and B_0 be non-negative operators on a Hilbert space $\mathcal{H}$. Let Q_ω, $\omega \in \Omega$ and Q_0 be the associated closed quadratic forms with the following properties:*

(2.25) *Q_ω, $\omega \in \Omega$ and Q_0 are defined on the same dense subset $\mathcal{D} \subset \mathcal{H}$.*
(2.26) *There is a constant $C > 0$ such that*

$$C^{-1} \left(Q_0(f,f) + \|f\|_0^2\right) \le Q_\omega(f,f) + \|f\|_0^2 \le C \left(Q_0(f,f) + \|f\|_0^2\right)$$

for all $\omega \in \Omega$ and $f \in \mathcal{D}$.
(2.27) *For every $f \in \mathcal{D}$ the map $\omega \mapsto Q_\omega(f,f)$ is measurable.*

Then the family $\{B_\omega\}_\omega$ of operators satisfies the equivalent properties of Proposition 2.4.5.

By property ($\mathbf{F}_7$) and Lemma 2.4.3, $\{B_\omega\}_\omega$ as in Proposition 2.4.6 is a measurable family of operators.

We apply the proposition to $B_\omega = A_\omega$, where $\{A_\omega\}_\omega$ is defined in (2.22). To do so, we check that the properties (2.25)–(2.27) are satisfied: Properties (2.25) and (2.26) follow from Proposition 2.4.4. Property (2.27) is obvious for $f \in C_c^\infty(\Lambda)$ and follows by approximation for all $f \in \mathcal{D}$, since $C_c^\infty(\Lambda)$ is dense, again by Proposition 2.4.4.

Proof (of Theorem 2.4.2). We already know that the transformed 'kinetic' part A_ω, $\omega \in \Omega$ of the Hamiltonian is measurable. To deal with the singular potential we introduce the cut off

$$V_{\omega,n}(x) := \min\{n, V_\omega(x)\} \text{ for } n \in \mathbb{N} \text{ and } \omega \in \Omega$$

The auxiliary potential $V_{\omega,n}$ is bounded and in particular its domain of definition is the whole Hilbert space $L^2(\Lambda, \mathrm{vol}_0)$. Thus the operator sum

$$A_{\omega,n} := A_\omega + V_{\omega,n}, \quad \omega \in \Omega$$

is well defined and [255, Prop. 4] implies that it forms a measurable family of operators. To recover the unbounded potential V_ω, we consider the semigroups $\omega \mapsto \exp(-tA_{\omega,n}), t > 0$ which are weakly measurable.

The quadratic forms of $A_{\omega,n}$ converge monotonously to the form of $A_\omega^\infty := A_\omega + V_\omega$. Now Theorems VIII.3.13a and IX.2.16 in [239] imply that the semigroups of $A_{\omega,n}$ converge weakly towards the one of A_ω^∞ for $n \to \infty$. Thus $\exp(-tA_\omega^\infty)$ is weakly measurable, which implies the measurability of the family A_ω^∞.

Finally, since S_ω is multiplication with the measurable function $(x, \omega) \mapsto \rho_\omega(x)$, this implies the measurability of the family $H_\omega = S_\omega A_\omega^\infty S_\omega^{-1}$, $\omega \in \Omega$.
□

For later use let us note that the trace of measurable operators is measurable. More precisely we will need the fact that for Λ of finite volume the mappings

$$\omega \mapsto \mathrm{Tr}(\chi_\Lambda \mathrm{e}^{-tH_\omega}) \quad \text{and} \quad \omega \mapsto \mathrm{Tr}(\mathrm{e}^{-tH_\omega^\Lambda}) \tag{2.28}$$

are measurable. Note that one can chose an orthonormal basis for $L^2(\Lambda, \mathrm{vol}_\omega)$ which depends in a measurable way on ω, cf. for instance Lemma II.2.1 in [117]. Thus (2.28) follows immediately from the Definition 2.4.1 of measurable operators.

2.5 Bounds on the Heat Kernels Uniform in ω

This paragraph is devoted to heat kernel estimates of the Schrödinger operators H_ω. It consists of four parts. Firstly we discuss existence of L^2-kernels of $\mathrm{e}^{-tH_\omega}, t > 0$ and derive rough upper bounds relying on results in [108].

Secondly, we infer Gaussian off-diagonal decay estimates of the kernels using estimates derived in [332]. We then present an idea of H. Weyl to derive the *principle of not feeling the boundary*, and finally we state a proposition which summarises the information on the heat kernel needed in the next section.

We have to control the dependence on the metric and potential of all these estimates since both the metric and the potential vary with the random parameter $\omega \in \Omega$.

As H_ω is non-negative, the semigroup $\mathrm{e}^{-tH_\omega}, t > 0$ consists of contractions. Moreover, the semigroup satisfies some nice properties formulated in the following definition which enable us to derive estimates on the corresponding heat kernel.

Definition 2.5.1. *Let $\Lambda \subset X$ be open and μ a σ-finite Borel measure on Λ. Let A be a real, non-negative, selfadjoint operator on the Hilbert space $L^2(\Lambda, \mu)$. The semigroup $\mathrm{e}^{-tA}, t > 0$ is called* positivity preserving *if $\mathrm{e}^{-tA} f \geq 0$ for any $0 \leq f \in L^2(\Lambda, \mu)$ and $t > 0$. Furthermore, $\mathrm{e}^{-tA}, t > 0$ is called a* Markov semigroup*, if it is well defined on $L^\infty(\Lambda, \mu)$ and the two following properties hold*

$$\mathrm{e}^{-tA} \colon L^2(\Lambda, \mu) \longrightarrow L^2(\Lambda, \mu) \qquad \textit{is positivity preserving for every } t > 0 \tag{2.29}$$

$$\mathrm{e}^{-tA} \colon L^\infty(\Lambda, \mu) \to L^\infty(\Lambda, \mu) \qquad \textit{is a contraction for every } t > 0 \tag{2.30}$$

In this case A is called a Dirichlet form.

A Markov semigroup e^{-tA} is called ultracontractive *if*

$$\mathrm{e}^{-tA} \colon L^2(\Lambda, \mu) \to L^\infty(\Lambda, \mu) \ \textit{is bounded for all } t > 0 \tag{2.31}$$

The above (2.29) and (2.30) are called *Beurling-Deny conditions* [44, 45].

We infer from [108] the following facts: A Markov semigroup is a contraction on $L^p(\Lambda, \mu)$ for all $1 \leq p \leq \infty$ (and all $t > 0$). For all $\omega \in \Omega$ the Schrödinger operator H_ω^Λ on $L^2(\Lambda, \mathrm{vol}_\omega)$ is a Dirichlet form, [108, Thm. 1.3.5]. There the proof is given for $X = \mathbb{R}^d$, but it applies to manifolds, too. By Sobolev embedding estimates and the spectral theorem $\mathrm{e}^{t\Delta_\omega^\Lambda}$ is ultracontractive. Thus by Lemma 2.1.2 in [108] each $\mathrm{e}^{t\Delta_\omega^\Lambda}$ has a kernel, which we denote by $k_\omega^\Lambda(t, \cdot, \cdot)$, such that for almost all $x, y \in \Lambda$

$$0 \leq k_\omega^\Lambda(t, x, y) \leq \|\mathrm{e}^{t\Delta_\omega^\Lambda}\|_{1,\infty} =: C_\omega^\Lambda(t) \tag{2.32}$$

Here $\|B\|_{1,\infty}$ denotes the norm of $B \colon L^1 \to L^\infty$. For $\Lambda = X$ we use the abbreviation $k_\omega = k_\omega^X$.

To derive an analogous estimate to (2.32) for the full Schrödinger operator with potential we make use of the *Feynman-Kac formula*. Using the symbol $\mathbf{E}_x$ for the expectation with respect to the *Brownian motion* b_t starting in $x \in X$ the formula reads

$$(\mathrm{e}^{-tH_\omega} f)(x) = \mathbf{E}_x \left(\mathrm{e}^{-\int_0^t V_\omega(b_s)\, ds} f(b_t) \right)$$

For a stochastically complete manifold X and bounded, continuous V_ω the formula is proven, for instance, in Theorem IX.7A in [150]. It extends to general non-negative potentials which are in L^1_{loc} using semigroup and integral convergence theorems similarly as in the proof of Theorem X.68 in [407]. Since we consider (geodesically) complete manifolds whose Ricci curvature is bounded below, they are all stochastically complete, cf. for instance [206] or Theorem 4.2.4 in [217].

Since the potential is non-negative, the Feynman-Kac formula implies for non-negative $f \in L^1(\Lambda, \text{vol}_\omega)$

$$0 \le \big(\mathrm{e}^{-tH_\omega^\Lambda} f\big)(x) \le \big(\mathrm{e}^{t\Delta_\omega^\Lambda} f\big)(x) \le C_\omega^\Lambda(t)\, \|f\|_{L^1}$$

for almost every $x \in \Lambda$. Thus $\mathrm{e}^{-tH_\omega^\Lambda} : L^1(\Lambda, \text{vol}_\omega) \to L^\infty(\Lambda, \text{vol}_\omega)$ has the same bound $C_\omega^\Lambda(t)$ as the semigroup where the potential is absent. This yields the pointwise estimate on the kernel $k_{H_\omega}^\Lambda(t, \cdot, \cdot)$ of $\mathrm{e}^{-tH_\omega^\Lambda}$:

$$0 \le k_{H_\omega}^\Lambda(t, x, y) \le C_\omega^\Lambda(t) \quad \text{for almost every } x, y \in X. \tag{2.33}$$

In the following we derive sharper upper bounds on the kernels which imply their decay in the distance between the two space arguments x and y. Such estimates have been proven by Li and Yau [332] for *fundamental solutions* of the heat equation. One would naturally expect that the fundamental solution and the L^2-heat kernel of the semigroup coincide under some regularity assumptions. This is actually the case as has been proven for instance in [119] for vanishing, and in [327] for smooth, non-negative potentials. The proof in the last cited source uses that H_ω is a Dirichlet form.

To formulate the results of Li and Yau [332] which we will be using, we denote by $d_\omega : X \times X \to [0, \infty[$ the Riemannian distance function on X with respect to g_ω. Note that the following proposition concerns the heat kernel of the pure Laplacian.

Proposition 2.5.2. *For every $t > 0$ there exist constants $C(t) > 0$, $\alpha_t > 0$ such that*

$$k_\omega(t, x, y) \le C(t) \exp\big(- \alpha_t\, d_0^2(x, y)\big) \tag{2.34}$$

for all $\omega \in \Omega$ and $x, y \in X$.

Proof. For a fixed Schrödinger operator the estimate (with d_0 replaced by d_ω) is contained in Corollary 3.1 in [332]. There the upper bound is given explicitly in terms of the geometric bounds on the manifold. This enables one to show that properties (2.4), (2.8) and

$$C_g^{-1} d_0(x, y) \le d_\omega(x, y) \le C_g d_0(x, y)$$

ensure that the constants $C(t)$ and α_t in (2.34) may be chosen uniformly in ω. Moreover, for measuring the distance between the points x and y we may always replace d_ω by d_0 by increasing α_t. □

Let us collect various consequences of Proposition 2.5.2 which will be useful later on.

(i) The pointwise kernel bound on the left hand side of (2.33) can be chosen uniformly in $\omega \in \Omega$.

(ii) We stated Proposition 2.5.2 for the pure Laplacian, although Li and Yau treat the case of a Schrödinger operator with potential. The reason for this is that we want to avoid the regularity assumptions on the potential imposed in [332].
To recover from (2.34) the case where a (non-negative) potential is present we use again the Feynman-Kac formula. We need now a local version of the argument leading to (2.33). More precisely, we consider e^{-tH_ω} as an operator from $L^1(B_\varepsilon(y))$ to $L^\infty(B_\varepsilon(x))$ for small $\varepsilon > 0$. Thus, we obtain

$$0 \le k_{H_\omega}(t,x,y) \le C(t) \exp\big(-\alpha_t\, d_0^2(x,y)\big)$$

(iii) The estimates derived so far immediately carry over to the case where the entire manifold is replaced by an open subset $\Lambda \subset X$.

$$0 \le k^\Lambda_{H_\omega}(t,x,y) \le k_{H_\omega}(t,x,y)$$

This is due to *domain monotonicity*, see for example [108, Thm. 2.1.6] where this fact is proven using functional analytic tools. Another way to see that this estimate is true, is to use the probabilistic representation of the heat semigroup, cf. [35, 435].

(iv) The *Bishop volume comparison* theorem controls the growth of the volume of balls with radius r, see for instance [51], [84, Thm. III.6] or [72]. It tells us that the lower bound (2.6) on the Ricci curvature is sufficient to bound the growth of the volume of balls as r increases. The volume of the ball can be estimated by the volume of a ball with the same radius in a space with constant curvature K. The latter volume grows at most exponentially in the radius. For our purposes it is necessary to have an ω-uniform version of the volume growth estimate. Using Properties (2.4), (2.6) and (2.8) we obtain the uniform bound

$$\mathrm{vol}_\omega\big(\{y|\, d_\omega(x,y) < r\}\big) \le C_1\, e^{C_2 r} \quad \text{for all } x \in X$$

where C_1, C_2 do not depend on x and ω. This implies that for all exponents $p > 0$, there exists a $M_p(t) < \infty$ such that the moment estimate

$$\int_\Lambda [k^\Lambda_{H_\omega}(t,x,y)]^p \, d\mathrm{vol}_\omega(y) \le M_p(t)$$

holds uniformly in $\Lambda \subset X$ open, in $x \in \Lambda$ and $\omega \in \Omega$. We set $M(t) := M_1(t)$.

(v) The heat kernel estimates imply a uniform bound on the traces of the semigroup localised in space. Let $\Lambda \subset X$ be a (fixed) open set of finite volume. There exists a constant $C_{\mathrm{Tr}} = C_{\mathrm{Tr}}(\Lambda, t) > 0$ such that for all $\omega \in \Omega$

$$\mathrm{Tr}\big(\chi_\Lambda \, \mathrm{e}^{-tH_\omega}\big) \le C_{\mathrm{Tr}}$$

Intuitively this is the same as saying that $\int_\Lambda k_{H_\omega}(t,x,x)\, d\mathrm{vol}_\omega(x)$ is uniformly bounded. However, since the diagonal $\{(x,x) | \, x \in \Lambda\}$ is a set of measure zero, the integral does not make sense as long as we consider k_{H_ω} as an L^2-function. We do not want here to address the question of continuity of the kernel. Instead we use the semigroup property $\mathrm{e}^{-2tH_\omega} = \mathrm{e}^{-tH_\omega}\mathrm{e}^{-tH_\omega}, t > 0$ and selfadjointness to express the trace as

$$\mathrm{Tr}\big(\chi_\Lambda \, \mathrm{e}^{-tH_\omega}\big) = \int_\Lambda \int_\Lambda [k_{H_\omega}(t/2, x, y)]^2 \, d\mathrm{vol}_\omega(x) \, d\mathrm{vol}_\omega(y) \le M_2(t/2) \, \mathrm{vol}_\omega(\Lambda) \tag{2.35}$$

By (2.10) this is bounded uniformly in $\omega \in \Omega$. Applying domain monotonicity once more, we obtain

$$\mathrm{Tr}\big(\mathrm{e}^{-tH_\omega^\Lambda}\big) \le M_2(t/2) \, \mathrm{vol}_\omega(\Lambda) \le M_2(t/2) \, C_g^{d/2} \, \mathrm{vol}_0(\Lambda) \tag{2.36}$$

The following lemma is a maximum principle for Schrödinger operators with non-negative potentials. Combined with the off-diagonal decay estimates in Proposition 2.5.2 it will give us a proof of the principle of not feeling the boundary.

Lemma 2.5.3 (Maximum principle for heat equation with non-negative potential). *Let $\Lambda \subset X$ be open with compact closure, V be a non-negative function, and $u \in C([0,T[\times\overline{\Lambda}) \cap C^2(]0,T[\times\Lambda)$ be a solution of the heat equation $\frac{\partial}{\partial t}u + (-\Delta + V)u = 0$ on $]0,T[\times\Lambda$ with non-negative supremum $s = \sup\{u(t,x) \mid (t,x) \in [0,T[\times\overline{\Lambda}\}$. Then,*

$$s = \max\left\{ \max_{x\in\overline{\Lambda}} u(0,x), \sup_{[0,T[\times\partial\Lambda} u(t,x) \right\}$$

Note that regularity of V is not assumed explicitly, but implicitly by the requirements on u. They are e.g. satisfied if V is smooth. Indeed, in that case the heat kernel is smooth, as can be seen following the proof of [108, Thm. 5.2.1].

Now we are in the position to state the second, refined estimate on the heat kernels, the *principle of not feeling the boundary.* It is a formulation of the fact that the heat kernel of the Dirichlet-Laplacian on a (large) open set Λ does not differ much from the heat kernel associated to the Laplacian on the whole manifold, as long as one stays well inside Λ. As before, we derive this estimate first for the pure Laplacian and then show that it carries over to Schrödinger operators with non-negative potential.

Proposition 2.5.4. *For any fixed* $t, \varepsilon > 0$*, there exists an* $h = h(t, \varepsilon) > 0$ *such that for every open set* $\Lambda \subset X$ *and all* $\omega \in \Omega$

$$0 \le k_\omega(t,x,y) - k_\omega^\Lambda(t,x,y) \le \varepsilon$$

for all $x \in \Lambda, y \in \Lambda_h$.

Proof. The first inequality is a consequence of domain monotonicity. So we just have to prove the second one.

Fix $\omega \in \Omega$ and $t, \varepsilon > 0$. Choose $h > 0$ such that

$$C(t) \exp\Big(- \alpha_t \big(h/2\big)^2\Big) \le \varepsilon$$

Note that the choice is independent of ω. For any $y \in \Lambda_h$ and $0 < \delta < h/2$ denote by $B_\delta(y)$ the open d_0-ball around y with radius δ. Let $f_\delta \in C_0^\infty(B_\delta(y))$ be a non-negative approximation of the δ-distribution at y.

We consider now the time evolution of the initial value f under the two semigroups generated by Δ_ω and Δ_ω^Λ, respectively.

$$\begin{aligned} u_1(t,x) &:= \int_X k_\omega(t,x,z) f_\delta(z) d\mathrm{vol}_\omega(z) = \int_\Lambda k_\omega(t,x,z) f_\delta(z) d\mathrm{vol}_\omega(z). \\ u_2(t,x) &:= \int_\Lambda k_\omega^\Lambda(t,x,z) f_\delta(z) d\mathrm{vol}_\omega(z). \end{aligned}$$

The difference $u_1(t,x) - u_2(t,x)$ solves the heat equation $\frac{\partial}{\partial t} u = \Delta_\omega u$ and satisfies the initial condition $u_1(0,x) - u_2(0,x) = f_\delta(x) - f_\delta(x) = 0$ for all $x \in \Lambda$. Now, by domain monotonicity we know $k_\omega(t,x,z) - k_\omega^\Lambda(t,x,z) \ge 0$, thus

$$u_1(t,x) - u_2(t,x) = \int_\Lambda \big[k_\omega(t,x,z) - k_\omega^\Lambda(t,x,z)\big] f_\delta(z)\, d\mathrm{vol}_\omega(z) \ge 0$$

for all $t > 0$ and $x \in \Lambda$. The application of the maximum principle yields

$$u_1(t,x) - u_2(t,x) \le \max_{]0,t] \times \partial\Lambda} \{u_1(s,w) - u_2(s,w)\}. \tag{2.37}$$

The right hand side can be further estimated by:

$$\begin{aligned} u_1(s,w) - u_2(s,w) &\le \int_\Lambda k_\omega(s,w,z) f_\delta(z)\, d\mathrm{vol}_\omega(z) \\ &= \int_{\Lambda_{h/2}} k_\omega(s,w,z) f_\delta(z)\, d\mathrm{vol}_\omega(z). \end{aligned}$$

Since $w \in \partial\Lambda$ and $z \in \Lambda_{h/2}$, we conclude using Proposition 2.5.2:

$$\int_{\Lambda_{h/2}} k_\omega(s,w,z) f_\delta(z) d\mathrm{vol}_\omega(z) \le C(t) \exp\big(- \alpha_t (h/2)^2\big) \le \varepsilon$$

Since the bound is independent of δ we may take the limit $\delta \to 0$ which concludes the proof. $\square$

One can prove the principle of not feeling the boundary by other means too, see for instance [349, 122, 392]. This alternative approach uses information on the behaviour of solutions of the wave equation. Unlike the solutions of the heat equation, they do not have the unphysical property that their support spreads instantaneously to infinity. Actually, the solutions of the wave equation have finite propagation speed [468]. Fourier transforms and the spectral theorem turn this information into estimates on the difference of the solutions of the free and restricted heat equation. Sobolev estimates lead then to the principle of not feeling the boundary. See also Sect. 7 in [406].

Remark 2.5.5. Similarly as in Lemma 2.5.3, one can prove the proposition, if a potential is present. More precisely, Proposition 2.5.4 is valid for Schrödinger operators with potentials V such that for continuous initial and boundary values the solution of the heat equation $\frac{\partial}{\partial t}u = -(-\Delta_\omega + V)u$ is in $C([0,T[\times\overline{\Lambda}) \cap C^2(]0,T[\times\Lambda)$. However, Proposition 2.5.4 implies an analogous estimate for the case where a non-negative potential is present, similarly as in (ii) on page 33. This will be explained next.

Consider $\mathrm{e}^{-tH_\omega} - \mathrm{e}^{-tH_\omega^\Lambda}$ as an operator from $L^1(\Lambda_h)$ to $L^\infty(\Lambda)$, and denote by τ_x^Λ the *first exit time* from Λ for a Brownian motion starting in x. By the Feynman-Kac formula, we have for $0 \le f \in L^1(\Lambda_h)$

$$[(\mathrm{e}^{-tH_\omega} - \mathrm{e}^{-tH_\omega^\Lambda})f](x) = \mathbf{E}_x\Big(\mathrm{e}^{-\int_0^t ds V(b_s)} f(b_t)\chi_{\{b \mid \tau_x^\Lambda \le t\}}\Big)$$
$$\le \mathbf{E}_x\Big(f(b_s)\chi_{\{b\mid \tau_x^\Lambda \le t\}}\Big) = \int [k_\omega(t,x,y) - k_\omega^\Lambda(t,x,y)] f(y)\, d\mathrm{vol}_\omega$$
$$\le \varepsilon \int f(y)\, d\mathrm{vol}_\omega$$

if we chose h as in Proposition 2.5.4. Thus for almost all $x \in \Lambda, y \in \Lambda_h$

$$k_{H_\omega}(t,x,y) - k_{H_\omega}^\Lambda(t,x,y) \le \|\mathrm{e}^{-tH_\omega} - \mathrm{e}^{-tH_\omega}\|_{L^1(\Lambda_h)\to L^\infty(\Lambda)} \le \varepsilon \tag{2.38}$$

The upper bounds on the heat kernel and the principle of not feeling the boundary enable us to prove a result on the traces of localised heat-semigroups: In the macroscopic limit, as Λ tends (in a nice way) to the whole of X, the two quantities

$$\mathrm{Tr}(\chi_\Lambda \mathrm{e}^{-tH_\omega}) \quad \text{and} \quad \mathrm{Tr}(\mathrm{e}^{-tH_\omega^\Lambda})$$

are approximately the same. The precise statement is contained in the following

Proposition 2.5.6. *Let $\{\Lambda_l\}_{l\in\mathbb{N}}$, be a sequence of subsets of X which satisfies the van Hove property 2.17 and let $\{H_\omega\}_\omega$ be a random Schrödinger operator. Then*

$$\lim_{l\to\infty} \sup_{\omega\in\Omega} \frac{1}{\mathrm{vol}_\omega(\Lambda_l)} \Big|\mathrm{Tr}(\chi_{\Lambda_l}\mathrm{e}^{-tH_\omega}) - \mathrm{Tr}(\mathrm{e}^{-tH_\omega^l})\Big| = 0$$

Proof. We consider first a fixed $l \in \mathbb{N}$ and abbreviate $\Lambda = \Lambda_l$. For the operator $\mathrm{e}^{-tH_\omega^\Lambda}$ we may write the trace in the same way as in (2.35) to obtain

$$\mathrm{Tr}(\mathrm{e}^{-tH_\omega^\Lambda}) = \int_\Lambda \int_\Lambda [k_{H_\omega}^\Lambda(t/2, x, y)]^2 d\mathrm{vol}_\omega(x) d\mathrm{vol}_\omega(y) \tag{2.39}$$

We express the difference of (2.35) and (2.39) using

$$(k_{H_\omega})^2 - (k_{H_\omega}^\Lambda)^2 = (k_{H_\omega} - k_{H_\omega}^\Lambda)(k_{H_\omega} + k_{H_\omega}^\Lambda)$$

Next we chose $h = h(t/2, \varepsilon) > 0$ as in Proposition 2.5.4 and decompose the integration domain according to

$$\Lambda \times \Lambda = (\Lambda \times \Lambda_h) \cup (\Lambda \times \partial_h \Lambda)$$

The difference of the traces can be now estimated as

$$\begin{aligned} 0 &\le \mathrm{Tr}(\chi_\Lambda \, \mathrm{e}^{-tH_\omega}) - \mathrm{Tr}(\mathrm{e}^{-tH_\omega^\Lambda}) \\ &= \int_\Lambda \int_{\Lambda_h} \left[k_{H_\omega}\left(\tfrac{t}{2}, x, y\right) - k_{H_\omega}^\Lambda\left(\tfrac{t}{2}, x, y\right)\right] \left[k_{H_\omega}\left(\tfrac{t}{2}, x, y\right) + k_{H_\omega}^\Lambda\left(\tfrac{t}{2}, x, y\right)\right] d\mathrm{vol}_\omega(x, y) \\ &+ \int_\Lambda \int_{\partial_h \Lambda} \left[k_{H_\omega}\left(\tfrac{t}{2}, x, y\right) - k_{H_\omega}^\Lambda\left(\tfrac{t}{2}, x, y\right)\right] \left[k_{H_\omega}\left(\tfrac{t}{2}, x, y\right) + k_{H_\omega}^\Lambda\left(\tfrac{t}{2}, x, y\right)\right] d\mathrm{vol}_\omega(x, y) \end{aligned} \tag{2.40}$$

The first term is bounded by $2M(t/2)\,\varepsilon\,\mathrm{vol}_\omega(\Lambda)$ and the second by

$$2M(t/2)\,C(t/2)\,\mathrm{vol}_\omega(\partial_h \Lambda)$$

It follows that

$$\frac{1}{\mathrm{vol}_\omega(\Lambda)} \Big(\mathrm{Tr}(\chi_\Lambda \, \mathrm{e}^{-tH_\omega}) - \mathrm{Tr}(\mathrm{e}^{-tH_\omega^\Lambda}) \Big) \le 2M(t/2)\varepsilon + 2M(t/2)C(t/2) \frac{\mathrm{vol}_\omega(\partial_h \Lambda)}{\mathrm{vol}_\omega(\Lambda)}$$

Now, we let l go to infinity. Since the sequence Λ_l satisfies the van Hove property (2.17) and since our bounds are uniform in ω, the proposition is proven. □

2.6 Laplace Transform and Ergodic Theorem

This section completes the proof of Theorem 2.3.8. It relies, apart from the results established in Sects. 2.4–2.5, on a general ergodic theorem and a criterion for the convergence of distribution functions.

Lindenstrauss proved in [342, 341] an ergodic theorem which applies to locally compact, second countable amenable groups. It includes as a special case the following statement for discrete groups.

Theorem 2.6.1. *Let Γ be an amenable discrete group and $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$ be a probability space. Assume that Γ acts ergodically on Ω by measure preserving transformations. Let $\{I_l\}_l$ be a tempered Følner sequence in Γ. Then for every $f \in L^1(\Omega)$*

$$\lim_{j\to\infty} \frac{1}{|I_l|} \sum_{\gamma \in I_l} f(\gamma\omega) = \mathbb{E}(f) \tag{2.41}$$

for almost all $\omega \in \Omega$.

In the application we have in mind $f \in L^\infty$, so the convergence holds in the L^1-topology, too.

Remark 2.6.2. Some background on previous results can be found for instance in Sect. 6.6 of Krengel' s book [313], in Tempelman's works [469, 470, 471] or some other sources [152, 204, 26, 151, 383]. The book [471] gives in Sect. 5.6 a survey of Shulman's results [434]. Mean ergodic theorems hold in more general circumstances, see for instance [313, Sect. 6.4] or [471, Ch. 6].

We will apply the ergodic theorem above not to the normalised eigenvalue counting functions N_ω^l, but to their Laplace transforms $\mathcal{L}_\omega^l$. The reason is, that the $\mathcal{L}_\omega^l$ are bounded, while the original N_ω^l are not. The following criterion of Pastur and Shubin [384, 431] says that it is actually sufficient to test the convergence of the Laplace transforms.

Lemma 2.6.3 (Pastur-Shubin convergence criterion). *Let N_n be a sequence of distribution functions such that*

(i) *there exists a $\lambda_0 \in \mathbb{R}$ such that $N_l(\lambda) = 0$ for all $\lambda \le \lambda_0$ and $l \in \mathbb{N}$,*
(ii) *there exists a function $C : \mathbb{R}^+ \to \mathbb{R}$ such that $\mathcal{L}^l(t) := \int e^{-\lambda t} dN_l(\lambda) \le C(t)$ for all $l \in \mathbb{N}$ and $t > 0$,*
(iii) $\lim_{l\to\infty} \mathcal{L}^l(t) =: \mathcal{L}(t)$ *exists for all $t > 0$.*

Then $\mathcal{L}$ is the Laplace transform of a distribution function N and for all continuity points λ of N we have

$$N(\lambda) := \lim_{l\to\infty} N_l(\lambda)$$

Finally, we present the proof of Theorem 2.3.8 on the existence of a self-averaging IDS:

Proof (of Theorem 2.3.8). We have to check the conditions in the previous lemma for the normalised eigenvalue counting functions N_ω^l. The first one is clearly satisfied for $\lambda_0 = 0$, since all operators we are dealing with are non-negative. To see (ii), express the Laplace transform by the trace of the heat semigroup

$$\mathcal{L}_\omega^l(t) = \frac{1}{\mathrm{vol}_\omega(\Lambda)} \sum_{n,\, \lambda_n \in \sigma} e^{-t\lambda_n} = \frac{1}{\mathrm{vol}_\omega(\Lambda)} \mathrm{Tr}(e^{-tH_\omega^l})$$

The sum extends over all eigenvalues λ_n of H_ω^l, counting multiplicities. Now, (2.36) implies condition (ii) of the Pastur-Shubin criterion.

To prove (iii) we will show for all $t > 0$ the convergence

$$\lim_{j\to\infty} \mathcal{L}_\omega^l(t) = \int_{\mathbb{R}} \mathrm{e}^{-t\lambda} dN_H(\lambda)$$

in (L^1 and) $\mathbb{P}$-almost sure-sense. For technical reasons we will deal separately with the convergence of the enumerator and denominator in

$$\mathcal{L}_\omega^l(t) = \mathrm{vol}_\omega(\Lambda_l)^{-1} \,\mathrm{Tr}(\mathrm{e}^{-tH_\omega^l})$$

However, we need *some* normalisation, to avoid divergences. Consider first the enumerator with an auxiliary normalisation

$$|I_l|^{-1} \,\mathrm{Tr}(\mathrm{e}^{-tH_\omega^l}) \tag{2.42}$$

Introduce for two sequences of random variables $a_l(\omega), b_l(\omega), l \in \mathbb{N}$ the equivalence relation $a_l \overset{j\to\infty}{\sim} b_l$ if they satisfy $a_l - b_l \to 0$ almost surely for $l \to \infty$. By Proposition 2.5.6, the equivariance, and Lindenstrauss' ergodic theorem 2.6.1

$$|I_l|^{-1} \,\mathrm{Tr}(\mathrm{e}^{-tH_\omega^l}) \overset{j\to\infty}{\sim} |I_l|^{-1} \,\mathrm{Tr}(\chi_{\Lambda_l} \,\mathrm{e}^{-tH_\omega}) = |I_l|^{-1} \sum_{\gamma\in I_l^{-1}} \mathrm{Tr}(\chi_{\gamma\mathcal{F}} \,\mathrm{e}^{-tH_\omega})$$

$$= \quad |I_l|^{-1} \sum_{\gamma\in I_l} \mathrm{Tr}(\chi_{\mathcal{F}} \,\mathrm{e}^{-tH_{\gamma\omega}}) \overset{j\to\infty}{\sim} \mathbb{E}\left\{\mathrm{Tr}(\chi_{\mathcal{F}} \,\mathrm{e}^{-tH_\bullet})\right\}$$

Similarly we infer for the normalised denominator

$$|I_l|^{-1}\mathrm{vol}_\omega(\Lambda_l) = |I_l|^{-1} \sum_{\gamma\in I_l^{-1}} \mathrm{vol}_\omega(\gamma\mathcal{F}) = |I_l|^{-1} \sum_{\gamma\in I_l} \mathrm{vol}_{\gamma\omega}(\mathcal{F}) \overset{j\to\infty}{\sim} \mathbb{E}\left\{\mathrm{vol}_\bullet(\mathcal{F})\right\}$$

Note that by (2.10) all terms in the above line are bounded from above and below uniformly in ω. By taking quotients we obtain

$$\mathcal{L}_\omega^l(t) = \frac{|I_l|^{-1} \,\mathrm{Tr}(\mathrm{e}^{-tH_\omega^l})}{|I_l|^{-1} \,\mathrm{vol}_\omega(\Lambda_l)} \overset{j\to\infty}{\sim} \frac{\mathbb{E}\left\{\mathrm{Tr}(\chi_{\mathcal{F}}\mathrm{e}^{-tH_\bullet})\right\}}{\mathbb{E}\left\{\mathrm{vol}_\bullet(\mathcal{F})\right\}}$$

Uniform boundedness implies that the convergence holds also in L^1-sense. The right hand side is the Laplace transform of N_H, see the proof of Theorem 6.1 of [328] for a detailed calculation. □

2.7 Approach Using Dirichlet-Neumann Bracketing

We outline an alternative proof of the existence of the IDS due to Kirsch and Martinelli [254]. It applies to random Schrödinger operators on $\mathbb{R}^d$. It relies on an ergodic theorem for superadditive processes by Akcoglu and

Krengel [17] and estimates on the number of bound states essentially implied by the Weyl asymptotics.

Let us explain the notion of a superadditive process in our context. Denote by Z the set of all multi-dimensional intervals or boxes Λ in $\mathbb{R}^d$ such that $\Lambda = \{x | \, a_j < x_j < b_j, \text{ for } j = 1, \dots, d\}$ for some $a, b \in \mathbb{Z}^d$ with $a_j < b_j$ for all $j = 1, \dots, d$. The restriction of H_ω to a $\Lambda \in Z$ with Dirichlet boundary conditions is denoted by H_ω^Λ and with Neumann boundary conditions by $H_\omega^{\Lambda,N}$. Consider a group $\{T_k\}_{k \in \mathbb{Z}^d}$ (or semigroup $\{T_k\}_{k \in \mathbb{N}_0^d}$) of measure preserving transformations on the probability space $(\Omega, \mathcal{B}_\Omega, \mathbb{P})$.

Definition 2.7.1. *A set function $F \colon Z \to L^1(\Omega)$ is called a (discrete)* super-additive process *(with respect to $\{T_k\}_k$) if the following conditions are satisfied*

$$F_\Lambda \circ T_k = F_{\Lambda + k} \text{ for all } k \in \mathbb{Z}^d \text{ (or } \mathbb{N}_0^d), \Lambda \in Z \tag{2.43}$$

$$\text{if } \Lambda_1, \dots, \Lambda_n \in Z \text{ such that } \Lambda := \operatorname{int}\Big(\bigcup_{k=1}^n \overline{\Lambda}_k\Big) \in Z \text{ then, } F_\Lambda \geq \sum_{k=1}^n F_{\Lambda_k} \tag{2.44}$$

$$\gamma := \gamma(F) := \sup_{\Lambda \in Z} |\Lambda|^{-1} \, \mathbb{E}\{F_\Lambda\} < \infty \tag{2.45}$$

F is called subadditive *if $-F$ is superadditive.*

Similarly one can define superadditive processes with respect to an action of $\mathbb{R}^d$ on Ω.

We formulate the main result of [17] in the way it suits our needs (see Theorem 2.4 and the Remark on page 59 in [17] and Sect. 6.2 in [313]).

Theorem 2.7.2. *Let F be a discrete superadditive process. For $l \in \mathbb{N}$ set $\Lambda_l :=]-l/2, l/2[^d$. Then the limit*

$$\lim_{l \to \infty} l^{-d} F_{\Lambda_l} \qquad \text{exists for almost all } \omega \in \Omega$$

If $\{T_k\}_k$ acts ergodically on $(\Omega, \mathbb{P})$ we have $\lim_{l \to \infty} l^{-d} F_{\Lambda_l}(\omega) = \gamma(F)$ almost surely.

More generally, one can replace the cubes $\Lambda_l, l \in \mathbb{N}$ by a so-called *regular sequence*, cf. [470, 17, 254] or Sect. 6.2 in [313].

To apply the superadditive ergodic theorem we consider for arbitrary, fixed $\lambda \in \mathbb{R}$ the processes given by the eigenvalue counting functions of the Dirichlet and Neumann Laplacian

$$F_\Lambda^D := F_\Lambda^D(\lambda, \omega) := \#\{n | \, \lambda_n(H_\omega^\Lambda) < \lambda\}, \quad \Lambda \in Z$$
$$F_\Lambda^N := F_\Lambda^N(\lambda, \omega) := \#\{n | \, \lambda_n(H_\omega^{\Lambda,N}) < \lambda\}, \quad \Lambda \in Z$$

where H_ω is a random operator as in Definition 1.2.3. Obviously for $\Lambda = \Lambda_l =]-l/2, l/2[^d$ we have $F_\Lambda^D(\lambda) = l^d N_\omega^l(\lambda)$. We will show that $F_\Lambda^D, \Lambda \in Z$ is a superadditive process, which is also true for $-F_\Lambda^N, \Lambda \in Z$. Property (2.43) follows from the equivariance of $\{H_\omega\}_\omega$, while (2.44) and (2.45) are implied by the following

Lemma 2.7.3. *Let H_ω be a random operator as in Definition 1.2.3 and λ a fixed energy value.*

(i) *For two cubes $\Lambda_{(1)} \subset \Lambda_{(2)}$ we have $F^D_{\Lambda_{(2)}} \geq F^D_{\Lambda_{(1)}}$ and $F^N_{\Lambda_{(1)}} \geq F^D_{\Lambda_{(1)}}$.*
(ii) *If $\Lambda_{(1)}, \Lambda_{(2)} \in Z$ are disjoint such that $\Lambda = \Lambda_{(1)} \cup \Lambda_{(2)} \cup M \in Z$ where $M \subset \mathbb{R}^d$ is a set of measure zero, then*

$$F^D_\Lambda \geq F^D_{\Lambda_{(1)}} + F^D_{\Lambda_{(2)}}$$
$$F^N_\Lambda \leq F^N_{\Lambda_{(1)}} + F^N_{\Lambda_{(2)}}$$

(iii) *There exists a $C_\lambda \in \mathbb{R}$ such that for all $\Lambda \in Z$ and $\omega \in \Omega$ we have $F^D_\Lambda(\omega) \leq C_\lambda |\Lambda|$.*

Proof. The first two statements are known as Dirichlet-Neumann bracketing and are stated e.g. in Proposition XIII.15.4 in [408]. See also Sect. I.5 in [84] for analogous results on manifolds. Lemma A.3.1 in the Appendix and its proof imply property (iii) with $C_\lambda = \left(\frac{\mathrm{e}}{2\pi d}\lambda\right)^{d/2}$. □

More background on bracketing techniques can be found in Sects. XIII.3, 15 and 16 in [408]. The Weyl type bounds are related to the Lieb-Thirring and Cwikel-Lieb-Rozenblum estimates for bound states [416, 334, 333, 101].

Now we can state the main result of [254].

Theorem 2.7.4. *There exists a set $\Omega' \subset \Omega$ of full measure such that*

$$N(\lambda) := \lim_{l\to\infty} N^l_\omega(\lambda) \tag{2.46}$$

exists for every $\omega \in \Omega'$ and every continuity point $\lambda \in \mathbb{R}$ of N.

Proof. For a fixed $\lambda \in \mathbb{R}$ one applies Theorem 2.7.2 to $F^D_\Lambda(\lambda,\omega)$, $\Lambda \in Z$, and denotes the corresponding $\gamma(F)$ by $\gamma(\lambda)$. By definition $F^D_\Lambda(\lambda,\omega) \leq F^D_\Lambda(\tilde\lambda,\omega)$ for all $\lambda \leq \tilde\lambda$ and all $\omega \in \Omega$, $\Lambda \in Z$. Thus $\lambda \mapsto \gamma(\lambda)$ is a non-decreasing function. It has at most a countable set of discontinuity points. We denote its complement by $\mathcal{C}$ and choose a dense countable set $S \subset \mathcal{C}$. Hence γ is continuous at each $\lambda \in S$.

Since our transformation group is ergodic, for each λ there is a set Ω_λ of measure one on which the convergence $\lim_{l\to\infty} l^{-d} F^D_{\Lambda_l}(\omega) = \gamma(\lambda)$ holds. Since S is countable, $\Omega' = \cap_{\lambda\in S}\Omega_\lambda$ has full measure, and the convergence statement of Theorem 2.7.2 holds for all $\lambda \in S$ and $\omega \in \Omega'$. Define the distribution function $N(\lambda) := \lim_{S\ni\tilde\lambda\nearrow\lambda} \gamma(\tilde\lambda)$. Thus, γ and N coincide on $\mathcal{C}$.

The monotonicity of $\lambda \mapsto F^D_{\Lambda_l}(\lambda,\omega)$ and the continuity of N on $\mathcal{C}$ imply the statement of the theorem. To see this, choose a sequence $\lambda_n \in S$, $\lambda_n \geq \lambda \in \mathcal{C}$, $\lim_{n\to\infty} \lambda_n = \lambda$. Then we have

$$l^{-d} F^D_{\Lambda_l}(\lambda,\omega) - N(\lambda) \leq l^{-d} F^D_{\Lambda_l}(\lambda_n,\omega) - N(\lambda_n) + N(\lambda_n) - N(\lambda).$$

For $\omega \in \Omega'$ and $\varepsilon > 0$ we choose first n sufficiently large s.t. $N(\lambda_n) - N(\lambda) \leq \varepsilon/2$ and then l sufficiently large s.t. $l^{-d} F^D_{\Lambda_l}(\lambda_n, \omega) - N(\lambda_n) \leq \varepsilon/2$. Thus one sees that

$$\limsup_{l\to\infty} l^{-d} F^D_{\Lambda_l}(\lambda, \omega) \leq N(\lambda).$$

Similarly one can choose a sequence $\lambda_n \in S$, $\lambda_n \leq \lambda \in \mathcal{C}$, $\lim_{n\to\infty} \lambda_n = \lambda$ and then show that $\liminf_{l\to\infty} l^{-d} F^D_{\Lambda_l}(\lambda, \omega) \geq N(\lambda)$. □

For models which satisfy both the conditions of the previous theorem and of 2.3.8 the two definitions of the IDS coincide.

Under certain regularity assumptions the theorem remains true if Neumann boundary conditions are used to define the IDS, cf. Theorem 3.3.(b) of [254]. In this case one works with the subadditive process $F^N_\Lambda(\lambda, \omega)$, $\Lambda \in Z$. There are versions of the above theorem for $\mathbb{R}^d$-ergodic potentials, cf. for instance [254, 224].

2.8 Independence of the Choice of Boundary Conditions

Consider again the more general setting of Schrödinger operators on a Riemannian covering manifold X. If the open subset $\Lambda \subset X$ of finite volume is sufficiently regular, the Neumann Laplacian $H^{\Lambda,N}_\omega$ on Λ has discrete spectrum. One condition which ensures this is the *extension property* of the domain Λ, see e.g. [108], which is in turn satisfied if the boundary $\partial\Lambda$ is piecewise smooth. Minimal conditions which ensure the extension property are discussed in Sect. VI.3 of [454]. Thus it is possible to define the normalised eigenvalue counting function

$$N^{\Lambda,N}_\omega(\lambda) := \frac{1}{|\Lambda|} \#\{n \in \mathbb{N} |\, \lambda_n(H^{\Lambda,N}_\omega) < \lambda\}$$

Let Λ_l be an admissible exhaustion $\Lambda_l \subset X, l \in \mathbb{N}$ of sets which all have the extension property. Consider the sequence of distribution functions $N^{l,N}_\omega := N^{\Lambda_l,N}_\omega$. It is natural to ask whether it converges almost surely, and, moreover, whether its limit coincides with N as defined in Theorem 2.3.8. If this is true, the IDS is independent of the choice of Dirichlet or Neumann boundary conditions used for its construction. This indicates that boundary effects are negligible in the macroscopic limit.

However, this turns out not to be true for all geometric situations. Sznitman studied in [464, 466] the IDS of a random Schrödinger operator on a horoball in hyperbolic space with potential generated by a Poissonian field. He showed that the IDS does depend on the choice of boundary condition used for its construction. Actually, he computes the Lifshitz asymptotics of the IDS at energies near the bottom of the spectrum and shows that it is different for Dirichlet and Neumann boundary conditions.

In contrast, in the case of Euclidean geometry $X = \mathbb{R}^d$, the question of boundary condition independence has been answered positively already some decades ago [43, 254, 432, 137] for a large class of $\mathbb{Z}^d$ or $\mathbb{R}^d$-ergodic random potentials. More recently, there has been interest in the same question if a magnetic field is included in the Hamiltonian, see also Sect. 5.7. In this case the coincidence of the IDS defined by the use of Dirichlet and Neumann boundary conditions was established for bounded (electric) potentials in [375], for non-negative potentials in [125], and for certain potentials assuming both arbitrarily large positive and negative values in [224] and [222]. The last mentioned approach seems to be extensible to non-Euclidean geometries.