4

Commuting Reduction and Semidirect Product Theory

In this chapter we develop two of the basic results on reduction by stages, namely the case of commuting reduction and semidirect product reduction. While one could view these as special cases of more general theorems to follow in the next chapter, it is worthwhile to see them on their own as more structured preludes to more general cases. In addition, these cases are important in applications as well as for the historical development of the subject.

4.1 Commuting Reduction

Theorems on reduction by stages have been given in various special instances by a number of authors, starting with time-honored observations in mechanics such as the following: When you want to reduce the dynamics of a rigid body moving in space, *first* you can pass to center of mass coordinates (that is, reduce by translations) and *second* you can pass to body coordinates (that is, reduce by the rotation group). For other problems, such as a rigid body in a fluid (see Leonard and Marsden [1997]) this process is not so simple; one does *not* simply pass to center of mass coordinates to get rid of translations. This shows that the general problem of reducing by the Euclidean group is a bit more subtle than one may think at first when one is considering say, particle mechanics. In any case, such procedures correspond to a reduction by stages result for *semidirect products*.

But we are getting ahead of ourselves; we need to step back and look first at an even simpler case in some detail, namely the case of a *direct product*. For instance, for a symmetric heavy top, the symmetry group is $S^1 \times S^1$, with the first S^1 being the symmetry of rotations about the vertical axis of gravity and the second S^1 being rotations about the symmetry axis of the body. These two group actions commute.

The version of commuting reduction given in Marsden and Weinstein [1974], p. 127 states that for two commuting group actions, one could reduce by them in succession and in either order and the result is the same as reducing by the *direct* product group. One version of this result, which we will go through rather carefully in a way that facilitates its generalization, is given in the following development.

The set up is as follows: Let P be a symplectic manifold, K be a Lie group (with Lie algebra $\mathfrak{k}$) acting symplectically on P and having an equivariant momentum map $\mathbf{J}_K : P \to \mathfrak{k}^*$. Let G be another group (with Lie algebra $\mathfrak{g}$) acting on P with an equivariant momentum map $\mathbf{J}_G : P \to \mathfrak{g}^*$. The first main assumption is

C1. The actions of G and K on P commute.

It follows that there is a well-defined action of $G \times K$ on P given by $(g, k) \cdot z = g \cdot (k \cdot z) = k \cdot (g \cdot z)$. Next, we claim that

$$\mathbf{J}_{G\times K} := \mathbf{J}_G \times \mathbf{J}_K : P \to (\mathfrak{g} \times \mathfrak{k})^* = \mathfrak{g}^* \times \mathfrak{k}^*$$

is a momentum map for the action of $G \times K$ on P. Indeed, for $\xi \in \mathfrak{g}$ and $\eta \in \mathfrak{k}$, we have

$$(\xi, \eta)_P(z) = \xi_P(z) + \eta_P(z),$$

as follows by noting that $\exp(t(\xi, \eta)) = (\exp(t\xi), \exp(t\eta))$. Note that

$$\mathbf{J}_{G\times K}^{(\xi,\eta)} = \mathbf{J}_G^{\xi} + \mathbf{J}_K^{\eta}.$$

Therefore,

$$\mathbf{i}_{(\xi,\eta)_P}\Omega = \mathbf{i}_{\xi_P}\Omega + \mathbf{i}_{\eta_P}\Omega = \mathbf{d}\mathbf{J}_G^{\xi} + \mathbf{d}\mathbf{J}_K^{\eta} = \mathbf{d}\mathbf{J}_{G\times K}^{(\xi,\eta)},$$

This proves the claim.

To ensure that $\mathbf{J}_{G\times K}$ is an equivariant momentum map, we make an additional hypothesis.

C2. $\mathbf{J}_G$ is K-invariant and $\mathbf{J}_K$ is G-invariant.

There are some remarks to be made about this condition. First of all, if $P = T^*Q$ and the actions are lifted from commuting actions on Q, then we assert that the condition **C2** automatically holds. This is because, in the cotangent case, we can use the explicit formula for the equivariant momentum maps $\mathbf{J}_G$ and $\mathbf{J}_K$. Let $k \in K$, $\alpha_q \in T_q^*Q$, and $\xi \in \mathfrak{g}$. Then

$$\begin{aligned}\langle \mathbf{J}_G(k \cdot \alpha_q), \xi \rangle &= \langle k \cdot \alpha_q, \xi_Q(k \cdot q) \rangle \\ &= \langle k \cdot \alpha_q, k \cdot \xi_Q(q) \rangle \\ &= \langle \alpha_q, \xi_Q(q) \rangle \\ &= \langle \mathbf{J}_G(\alpha_q), \xi \rangle .\end{aligned}$$

There is a similar argument for $\mathbf{J}_K$. This proves our assertion.

The second remark we wish to make is that in a sense, one needs to only assume that "half" of **C2** holds. Namely, we claim that if $\mathbf{J}_K$ is G-invariant and K is connected, then $\mathbf{J}_G$ is K-invariant. Indeed, $\mathbf{d} \langle \mathbf{J}_K, \eta \rangle \cdot \xi_P = 0$ for all $\xi \in \mathfrak{g}$ and $\eta \in \mathfrak{k}$ and hence

$$\begin{aligned}\mathbf{d} \langle \mathbf{J}_G, \xi \rangle \cdot \eta_P &= \mathbf{d} \langle \mathbf{J}_G, \xi \rangle \cdot X_{\langle \mathbf{J}_K, \eta \rangle} = \{ \langle \mathbf{J}_G, \xi \rangle , \langle \mathbf{J}_K, \eta \rangle \} \\ &= -\mathbf{d} \langle \mathbf{J}_K, \eta \rangle \cdot X_{\langle \mathbf{J}_G, \xi \rangle} = -\mathbf{d} \langle \mathbf{J}_K, \eta \rangle \cdot \xi_P = 0,\end{aligned}$$

from which we conclude K-invariance of $\mathbf{J}_G$ by connectedness of K, which proves the claim.

Now we have the ingredients needed to get an equivariant momentum map.

4.1.1 Proposition. *Under hypotheses* **C1** *and* **C2**, $\mathbf{J}_{G \times K}$ *is an equivariant momentum map for the action of* $G \times K$ *on* P.

Proof. For all $z \in P$ and $(g, k) \in G \times K$ we have

$$\begin{aligned}(\mathbf{J}_G \times \mathbf{J}_K) ((g, k) \cdot z) &= (\mathbf{J}_G(g \cdot k \cdot z), \, \mathbf{J}_K(g \cdot k \cdot z)) \\ &= (g \cdot \mathbf{J}_G(z), \, k \cdot \mathbf{J}_K(z)) \\ &= (g, k) \cdot (\mathbf{J}_G \times \mathbf{J}_K) (z),\end{aligned}$$

where we have used equivariance of each of $\mathbf{J}_G$ and $\mathbf{J}_K$, the fact that the actions commute (condition **C1**), and condition **C2**, the invariance of $\mathbf{J}_G$ and $\mathbf{J}_K$. ■

We need one more assumption.

C3. The action of $G \times K$ on P is free and proper.

Let $(\mu, \nu) \in \mathfrak{g}^* \times \mathfrak{k}^*$ be given. Since we have a simple product, the isotropy group is $(G \times K)_{(\mu,\nu)} = G_\mu \times K_\nu$. Our goal is to show that the "one-shot" reduced space

$$P_{(\mu,\nu)} = (\mathbf{J}_G \times \mathbf{J}_K)^{-1}(\mu, \nu)/(G_\mu \times K_\nu)$$

is symplectically diffeomorphic to the space obtained by first reducing by K at ν to form the first reduced space $P_\nu = \mathbf{J}_K^{-1}(\nu)/K_\nu$ and then reducing this space by the G action. Note that the actions of K and G on P are free and proper as a consequence of **C3**.

Warning. If each of the actions of G and K are free, this need not imply, conversely, that the action of $G \times K$ is free. For example, Let $G = K = S^1$ act on $\mathbb{R}^2$ minus the origin by standard rotations. The actions obviously commute, each one is free, but the product action is not free since $(e^{i\theta}, e^{-i\theta})z = z$ for any θ and any nonzero $z \in \mathbb{R}^2$.

Another example where this occurs is in the Lagrange top, that is, a rigid body with an axis of symmetry, rotating about a fixed point on that axis, and moving in a gravitational field. There are two commuting S^1 symmetry groups acting on the phase space $T^*\mathrm{SO}(3)$. These two actions are given by (the cotangent lift of) left translation corresponding to rotations about the axis of gravity and the other by right translation about the axis of symmetry; these two actions clearly commute. The corresponding integrals of motion lead to the complete integrability of the problem. One can reduce by the action of these groups either together or one following the other with the same final reduced space. In this problem, one should omit the "vertical" state of rotation of the body in order for the action of $S^1 \times S^1$ to be free, even though each action separately is free; see, for instance, Lewis, Ratiu, Simo, and Marsden [1992].

To carry out the second stage reduction, we need the following.

4.1.2 Lemma. *The group G induces a free and proper symplectic action on P_ν, and the map $\mathbf{J}_\nu : P_\nu \to \mathfrak{g}^*$ naturally induced by $\mathbf{J}_G$ is an equivariant momentum map for this action.*

Proof. Let the action of $g \in G$ on P be denoted by $\Psi_g : P \to P$. Since these maps commute with the action of K and leave the momentum map $\mathbf{J}_K$ invariant by hypothesis **C2**, there are well-defined induced maps $\Psi_g^\nu : \mathbf{J}_K^{-1}(\nu) \to \mathbf{J}_K^{-1}(\nu)$ and $\Psi_{g,\nu} : P_\nu \to P_\nu$, which then define smooth actions of G on $\mathbf{J}_K^{-1}(\nu)$ and on P_ν.

Let $\pi_\nu : \mathbf{J}_K^{-1}(\nu) \to P_\nu$ denote the natural projection and $i_\nu : \mathbf{J}_K^{-1}(\nu) \to P$ be the inclusion. We have by construction, $\Psi_{g,\nu} \circ \pi_\nu = \pi_\nu \circ \Psi_g^\nu$ and $\Psi_g \circ i_\nu = i_\nu \circ \Psi_g^\nu$.

Recall from Theorem 1.1.3 that the symplectic form Ω_ν on the reduced space P_ν is characterized by $i_\nu^*\Omega = \pi_\nu^*\Omega_\nu$. Therefore,

$$\pi_\nu^* \Psi_{g,\nu}^* \Omega_\nu = (\Psi_g^\nu)^* \pi_\nu^* \Omega_\nu = (\Psi_g^\nu)^* i_\nu^* \Omega = i_\nu^* \Psi_g^* \Omega = i_\nu^* \Omega = \pi_\nu^* \Omega_\nu .$$

Since π_ν is a surjective submersion, we conclude that

$$\Psi_{g,\nu}^* \Omega_\nu = \Omega_\nu .$$

Thus, we have a symplectic action of G on P_ν.

Since $\mathbf{J}_G$ is invariant under K and hence under K_ν, there is an induced map $\mathbf{J}_\nu : P_\nu \to \mathfrak{g}^*$ satisfying $\mathbf{J}_\nu \circ \pi_\nu = \mathbf{J}_G \circ i_\nu$. We now check that this is the momentum map for the action of G on P_ν. To do this, first note that for all $\xi \in \mathfrak{g}$, the vector fields ξ_P and ξ_{P_ν} are π_ν-related. We have

$$\pi_\nu^* \left(\mathbf{i}_{\xi_{P_\nu}} \Omega_\nu \right) = \mathbf{i}_{\xi_P} i_\nu^* \Omega = i_\nu^* \left(\mathbf{i}_{\xi_P} \Omega \right) = i_\nu^* \left(\mathbf{d} \left\langle \mathbf{J}_G, \xi \right\rangle \right) = \pi_\nu^* \left(\mathbf{d} \left\langle \mathbf{J}_\nu, \xi \right\rangle \right) .$$

Again, since π_ν is a surjective submersion, we conclude that

$$\mathbf{i}_{\xi_{P_\nu}} \Omega_\nu = \mathbf{d} \langle \mathbf{J}_\nu, \xi \rangle$$

and hence $\mathbf{J}_\nu$ is the momentum map for the G action on P_ν. Equivariance of $\mathbf{J}_\nu$ follows from that for $\mathbf{J}_G$, by a diagram chasing argument as above, using the relation $\mathbf{J}_\nu \circ \pi_\nu = \mathbf{J}_G \circ i_\nu$ and the relations between the actions of G on P, $\mathbf{J}_K^{-1}(\nu)$, and on P_ν.

We next prove that the action of G on P_ν is free and proper. First note that the action of G on $\mathbf{J}_K^{-1}(\nu)$ is free and proper. For $z \in \mathbf{J}_K^{-1}(\nu)$, let its class be denoted $[z]_\nu := \pi_\nu(z)$. The G action in this notation is simply $g[z]_\nu = [gz]_\nu$. To check freeness, assume that $[gz]_\nu = [z]_\nu$. Thus, there is a $k \in K_\nu$ such that $kgz = z$. But $kgz = (g,k)z$ and hence freeness of the action of $G \times K$ (condition **C3**) implies that $g = e, k = e$. Thus, the action of G on P_ν is free.

To prove properness, let $[z_n]_\nu \to [z]_\nu$ and $[g_n z_n]_\nu \to [z']_\nu$. Since the action of K_ν on $\mathbf{J}_K^{-1}(\nu)$ is free and proper, by the definition of the quotient topology, and the fact that proper actions have slices (see the discussions in, for example, [MTA] and Duistermaat and Kolk [1999]), there are sequences $k_n, k_n' \in K_\nu$ such that $k_n z_n \to z$ and $k_n g_n z_n = g_n k_n z_n \to z'$ (since the actions commute). By properness of the original action, this implies that a subsequence of g_n converges. ■

With the above ingredients, we can now form the second reduced space, namely $(P_\nu)_\mu = \mathbf{J}_\nu^{-1}(\mu)/G_\mu$.

4.1.3 Theorem (Commuting Reduction Theorem)**.** *Under the hypotheses* **C1, C2, C3***,* $P_{(\mu,\nu)}$ *and* $(P_\nu)_\mu$ *are symplectically diffeomorphic.*

Proof. Composing the inclusion map

$$j : (\mathbf{J}_G \times \mathbf{J}_K)^{-1}(\mu,\nu) \to \mathbf{J}_K^{-1}(\nu)$$

with π_ν gives the map

$$\pi_\nu \circ j : (\mathbf{J}_G \times \mathbf{J}_K)^{-1}(\mu,\nu) \to P_\nu.$$

This map takes values in $\mathbf{J}_\nu^{-1}(\mu)$ because of the relation $\mathbf{J}_\nu \circ \pi_\nu = \mathbf{J}_G \circ i_\nu$. Thus, we get a map

$$\kappa_\nu : (\mathbf{J}_G \times \mathbf{J}_K)^{-1}(\mu,\nu) \to \mathbf{J}_\nu^{-1}(\mu).$$

such that $(i_\nu)_\mu \circ \kappa_\nu = \pi_\nu \circ j$, where we use the notation $(i_\nu)_\mu$ for the inclusion $\mathbf{J}_\nu^{-1}(\mu) \hookrightarrow P_\nu$. The map κ_ν is equivariant with respect to the action of $G_\mu \times K_\nu$ on the domain and G_μ on the range. Thus, it induces a map

$$[\kappa_\nu] : P_{(\mu,\nu)} \to (P_\nu)_\mu.$$

To show that this map is symplectic, it is enough to show that

$$\pi^*_{(\mu,\nu)}\left([\kappa_\nu]^*(\Omega_\nu)_\mu\right) = \pi^*_{(\mu,\nu)}\Omega_{(\mu,\nu)}, \tag{4.1.1}$$

where we use self-explanatory notation; $\Omega_{(\mu,\nu)}$ is the symplectic form on $P_{(\mu,\nu)}$, $\pi_{(\mu,\nu)} : (\mathbf{J}_G \times \mathbf{J}_K)^{-1}(\mu,\nu) \to P_{(\mu,\nu)}$ is the projection, $(\pi_\nu)_\mu : \mathbf{J}_\nu^{-1}(\mu) \to (P_\nu)_\mu$ is the projection, and $(\Omega_\nu)_\mu$ is the reduced symplectic form on $(P_\nu)_\mu$. It is enough to establish equation (4.1.1) since $\pi_{(\mu,\nu)}$ is a surjective submersion. The right hand side of (4.1.1) is given by

$$i^*_{(\mu,\nu)}\Omega$$

by the unique characterization of the reduced symplectic form $\Omega_{(\mu,\nu)}$. The left side is

$$\pi^*_{(\mu,\nu)}\left([\kappa_\nu]^*(\Omega_\nu)_\mu\right) = \kappa_\nu^*(\pi_\nu)^*_\mu(\Omega_\nu)_\mu = \kappa_\nu^*(i_\nu)^*_\mu\Omega_\nu$$

because of the relation $[\kappa_\nu]\circ\pi_{(\mu,\nu)} = (\pi_\nu)_\mu\circ\kappa_\nu$ and the unique characterization of the reduced symplectic form $(\Omega_\nu)_\mu$. However, since $(i_\nu)_\mu\circ\kappa_\nu = \pi_\nu\circ j$, we get

$$\kappa_\nu^*(i_\nu)^*_\mu\Omega_\nu = j^*\pi_\nu^*\Omega_\nu = j^*i_\nu^*\Omega,$$

by the unique characterization of the reduced symplectic form Ω_ν. Since $i_\nu \circ j = i_{(\mu,\nu)}$ we get the desired equality. Thus, $[\kappa_\nu] : P_{(\mu,\nu)} \to (P_\nu)_\mu$ is a symplectic map.

We will show that this map is a diffeomorphism by constructing an inverse. We begin by defining a map

$$\phi : \mathbf{J}_\nu^{-1}(\mu) \to P_{(\mu,\nu)},$$

as follows. Choose an equivalence class $[z]_\nu \in \mathbf{J}_\nu^{-1}(\mu) \subset P_\nu$ for $z \in \mathbf{J}_K^{-1}(\nu)$. The equivalence relation is that associated with the map π_ν; that is, with the action of K_ν. For each such point, we have $z \in (\mathbf{J}_G \times \mathbf{J}_K)^{-1}(\mu,\nu)$ since by construction $z \in \mathbf{J}_K^{-1}(\nu)$ and also

$$\mathbf{J}_G(z) = (\mathbf{J}_G \circ i_\nu)(z) = \mathbf{J}_\nu([z]_\nu) = \mu.$$

Hence, it makes sense to consider the class $[z]_{(\mu,\nu)} \in P_{(\mu,\nu)}$. The result is independent of the representative, since any other representative of the same class has the form $k \cdot z$ where $k \in K_\nu$. This produces the same class in $P_{(\mu,\nu)}$ since for this latter space, the quotient is by $G_\mu \times K_\nu$. The map ϕ is therefore well-defined.

This map ϕ is G_μ–invariant, and so it defines a quotient map

$$[\phi] : (P_\nu)_\mu \to P_{(\mu,\nu)}.$$

Chasing the definitions shows that this map is the inverse of the map $[\kappa_\nu]$. Thus, both are bijections. Since $[\kappa_\nu]$ is smooth and symplectic, it is an immersion. A dimension count shows that $(P_\nu)_\mu$ and $P_{(\mu,\nu)}$ have the same dimension. Thus, $[\kappa_\nu]$ is a bijective local diffeomorphism, so it is a diffeomorphism. ■

The above theorem on commuting reduction may be viewed in the general context discussed in §3.1 by taking $M = G \times K$ with the normal subgroup N being chosen to be either G or K, so that the quotient group of M is the other group.

It is instructive to build up to the general reduction by stages theorem by giving direct proofs of some simpler special cases, such as the one at hand and the case of semidirect products treated in §4.2; these special cases not only point the way to the general case, but contain interesting constructions that are relevant to these more specific cases. The general case has some subtleties not shared by these simpler cases which will be spelled out as we proceed.

4.2 Semidirect Products

Background and Literature. In some applications one has two symmetry groups that do not commute and thus the commuting reduction by stages theorem does not apply. In this more general situation, it matters in what order one performs the reduction.

The main result covering the case of semidirect products is due to Marsden, Ratiu and Weinstein [1984a,b], with important previous versions (more or less in chronological order) due to Sudarshan and Mukunda [1974], Vinogradov and Kupershmidt [1977], Ratiu [1980b], Guillemin and Sternberg [1980], Ratiu [1981, 1982], Marsden [1982], Marsden, Weinstein, Ratiu, Schmid, and Spencer [1982], Holm and Kupershmidt [1983a] and Guillemin and Sternberg [1984].

The general theory of semidirect products was motivated by several examples of physical interest, such as the Poisson structure for compressible fluids and magnetohydrodynamics. These examples are discussed in the original papers cited and references therein. Another, and very useful, concrete application of this theory is to underwater vehicle dynamics; see Leonard and Marsden [1997].

Generalities on Semidirect Products. We begin by recalling some definitions and properties of semidirect products. Let V be a vector space and assume that the Lie group G (with Lie algebra $\mathfrak{g}$) acts (on the left) by linear maps on V, and hence G also acts (also on the left) on its dual space V^*, the action by an element g on V^* being the transpose of the action of g^{-1} on V. As sets, the semidirect product $S = G \circledS V$ is the Cartesian product $S = G \times V$ and group multiplication is given by

$$(g_1, v_1)(g_2, v_2) = (g_1 g_2, v_1 + g_1 v_2),$$

where the action of $g \in G$ on $v \in V$ is denoted simply as gv. The identity element is $(e, 0)$ and the inverse of (g, v) is given by $(g, v)^{-1} = (g^{-1}, -g^{-1}v)$.

The Lie algebra of S is the semidirect product Lie algebra $\mathfrak{s} = \mathfrak{g} Ⓢ V$. The bracket is given by

$$[(\xi_1, v_1), (\xi_2, v_2)] = ([\xi_1, \xi_2], \xi_1 v_2 - \xi_2 v_1)\,,$$

where we denote the induced action of $\mathfrak{g}$ on V by concatenation, as in $\xi_1 v_2$.

Perhaps the most basic example of a semidirect product is the Euclidean group SE(3) of $\mathbb{R}^3$, which is studied in, for example, [MandS] and which will be treated in some detail in §4.4.

We will need the formulas for the adjoint action and the coadjoint action for semidirect products. Denoting these and other actions by simple concatenation, they are given by

$$(g, v)(\xi, u) = (g\xi, gu - \rho_v(g\xi)) \tag{4.2.1}$$

and

$$(g, v)(\mu, a) = (g\mu + \rho_v^*(ga), ga), \tag{4.2.2}$$

where $(g, v) \in S = G \times V$, $(\xi, u) \in \mathfrak{s} = \mathfrak{g} \times V$, $(\mu, a) \in \mathfrak{s}^* = \mathfrak{g}^* \times V^*$, and where $\rho_v : \mathfrak{g} \to V$ is defined by $\rho_v(\xi) = \xi v$, the infinitesimal action of ξ on v. The map $\rho_v^* : V^* \to \mathfrak{g}^*$ is the dual of the map ρ_v. The symbol ga denotes the (left) dual action of G on V^*, that is, the inverse of the dual isomorphism induced by $g \in G$ on V. The corresponding (left) action on the dual space is denoted by ξa for $a \in V^*$, that is,

$$\langle \xi a, v \rangle := - \langle a, \xi v \rangle\ .$$

Lie-Poisson Brackets and Hamiltonian Vector Fields. Recall from [MandS] that the Lie-Poisson bracket on the dual of a Lie algebra $\mathfrak{g}^*$ comes with two signs and is given on two functions F, K of $\mu \in \mathfrak{g}^*$ by

$$\{F, K\}_\pm(\mu) = \pm \left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta K}{\delta \mu} \right] \right\rangle \tag{4.2.3}$$

Recall also that this bracket is obtained naturally from the canonical bracket on T^*G by taking quotients—this is the Lie-Poisson reduction theorem that is found in [MandS], Chapter 13. The minus sign corresponds to reduction by the left action and the plus sign to reduction by the right action.

Next, we give the formula for the $\pm$ Lie-Poisson bracket on a semidirect product; namely, for $F, K : s^* \to \mathbb{R}$, their ***semidirect product bracket*** is given by:

$$\{F, K\}_\pm(\mu, a) = \pm \left\langle \mu, \left[\frac{\delta F}{\delta \mu}, \frac{\delta K}{\delta \mu} \right] \right\rangle \pm \left\langle a, \frac{\delta F}{\delta \mu} \cdot \frac{\delta K}{\delta a} - \frac{\delta K}{\delta \mu} \cdot \frac{\delta F}{\delta a} \right\rangle, \tag{4.2.4}$$

where $\delta F / \delta \mu \in \mathfrak{g}$, $\delta F / \delta a \in V$ are the functional derivatives. Also, one verifies that the Hamiltonian vector field of a smooth function $H : s^* \to \mathbb{R}$ is given by

$$X_H(\mu, a) = \mp \left(\mathrm{ad}^*_{\delta H/\delta\mu}\, \mu - \rho^*_{\delta H/\delta a} a,\ \frac{\delta H}{\delta \mu} \cdot a \right). \tag{4.2.5}$$

Semidirect Product Reduction Theorem—Statement. We next state the "classical" semidirect product reduction theorem and following this, we give a more general theorem concerning actions by semidirect products on symplectic manifolds. The strategy will be to obtain the classical result as a special case of the more general result, which we shall prove in detail.

The semidirect product reduction theorem states, roughly speaking, that for a semidirect product $S = G \circledS V$, one can first reduce T^*S by V and then by G and one gets the same result as reducing by S. The precise statement of the classical semidirect product reduction theorem is as follows.

4.2.1 Theorem (Semidirect Product Reduction Theorem)**.** *As above, let $S = G \circledS V$, choose $\sigma = (\mu, a) \in \mathfrak{g}^* \times V^*$ and reduce T^*S by the action of S at σ, which, by Corollary* 1.1.4 *gives the coadjoint orbit $\mathcal{O}_\sigma$ through $\sigma \in \mathfrak{s}^*$. Then, there is a symplectic diffeomorphism between $\mathcal{O}_\sigma$ and the reduced space obtained by reducing T^*G by the subgroup G_a (the isotropy of G for its action on V^* at the point $a \in V^*$) at the point $\mu|\mathfrak{g}_a$, where $\mathfrak{g}_a$ is the Lie algebra of G_a.*

Remark. Note that in the semidirect product reduction theorem, only $a \in V$ and $\mu|\mathfrak{g}_a \in \mathfrak{g}_a^*$ are used in the equivalent description of the coadjoint orbit. Thus, one gets, as a corollary, the interesting fact that the semidirect product coadjoint orbits through $\sigma_1 = (\mu_1, a_1)$ and $\sigma_2 = (\mu_2, a_2)$ are symplectically diffeomorphic whenever $a_1 = a_2 = a$ and $\mu_1|\mathfrak{g}_a = \mu_2|\mathfrak{g}_a$. We shall see a similar phenomenon in more general situations of group extensions later.

The preceding result will next be shown to be a special case of a theorem we shall prove on reduction by stages for semidirect products acting on a symplectic manifold

Semidirect Product Actions. We now set the stage for the statement of the more general reduction by stages result for semidirect product actions. Start with a free and proper symplectic action of a semidirect product $S = G \circledS V$ on a symplectic manifold P and assume that this action has an equivariant momentum map $\mathbf{J}_S : P \to \mathfrak{s}^*$. Since V is a (closed, normal) subgroup of S, it also acts on P and has a momentum map $\mathbf{J}_V : P \to V^*$ given by

$$\mathbf{J}_V = i_V^* \circ \mathbf{J}_S \,,$$

where $i_V : V \to \mathfrak{s}$, given by $v \mapsto (0, v)$, is the inclusion where and $i_V^* : \mathfrak{s}^* \to V^*$ is its dual.

We carry out the reduction of P by S at a value $\sigma = (\mu, a)$ of the momentum map $\mathbf{J}_S$ for S (it is a regular value because of the freeness assumption) in two stages using the following procedure.

- First, reduce P by V at the value a (it follows from the freeness assumption that this too is a regular value) to get the first reduced space $P_a = \mathbf{J}_V^{-1}(a)/V$.

- Form the isotropy subgroup G_a consisting of elements of G that leave the point a fixed, using the action of G on V^*.

We shall show shortly that G_a acts freely and properly on P_a and has an induced equivariant momentum map $\mathbf{J}_a : P_a \to \mathfrak{g}_a^*$, where $\mathfrak{g}_a$ is the Lie algebra of G_a.

- Second, reduce P_a at the point $\mu_a := \mu|\mathfrak{g}_a$ to get the second reduced space $(P_a)_{\mu_a} = \mathbf{J}_a^{-1}(\mu_a)/(G_a)_{\mu_a}$.

4.2.2 Theorem (Reduction by Stages for Semidirect Product Actions)**.** *The two-stage reduced space $(P_a)_{\mu_a}$ is symplectically diffeomorphic to the "all-at-once" reduced space P_σ obtained by reducing P by the whole group S at the point $\sigma = (\mu, a)$.*

We have made the free and proper assumption on the action of S in this case that is the analog of the hypothesis **SRFree** in the symplectic reduction Theorem 1.1.3. One can also make hypotheses analogous to **SRRegular**, but these assumptions would need to be imposed at each of the stages. We have used the free and proper assumption since, as we shall see, it is automatically inherited in each of the two stages.

Special Cases. We recover the classical semidirect product reduction Theorem 4.2.1 by choosing $P = T^*S$ and using the fact that the first reduced space, namely reduction by V, is just T^*G with its canonical symplectic structure. We shall go through this in detail in §4.3.

The commuting reduction theorem for the case in which K is a vector space results from semidirect product reduction when we take the action of G on K to be trivial. The fact that the full commuting reduction theorem is not literally as special case suggests that there is a generalization of the semidirect product reduction theorem to the case in which V is replaced by a general Lie group. We give, in fact, more general results in this direction later. Note that in the commuting reduction theorem, what we called ν is called a in the semidirect product reduction theorem.

The original papers of Marsden, Ratiu and Weinstein [1984a,b] give a direct proof of Theorem 4.2.1 along lines somewhat different than we shall present here. The proofs we give in this book have the advantage that they work for more general reduction by stages theorems.

Classifying Orbits. Combined with the cotangent bundle reduction theorem (as mentioned in the introductory chapter, the reader may consult either [FofM] or [LonM] for an exposition), the semidirect product reduction theorem is a very useful tool. For example, using these techniques, one sees readily that the generic coadjoint orbits for the Euclidean group are cotangent bundles of spheres with the associated coadjoint orbit symplectic structure given by the canonical structure plus a magnetic term. We shall discuss this problem in detail starting with the Euclidean group in §4.4.

Reducing Dynamics. There is a method for reducing dynamics that is associated with the geometry of the semidirect product reduction theorem. One can start with a Hamiltonian on either of the phase spaces $(P_a)_{\mu_a}$ or P_σ and induce one (and hence its associated dynamics) on the other space in a natural way.

Another view of reducing dynamics that is useful in many applications is as follows: one starts with a Hamiltonian H_a on T^*G that depends parametrically on a variable $a \in V^*$; this parametric dependence identifies the space V^*, and hence V. The Hamiltonian, regarded as a map $H : T^*G \times V^* \to \mathbb{R}$ should be invariant on T^*G under the action of G on $T^*G \times V^*$. This condition is equivalent to the invariance of the corresponding function on $T^*S = T^*G \times V \times V^*$ extended to be constant in the variable V, under the action of the semidirect product. This observation allows one to identify the reduced dynamics of H_a on T^*Q reduced by G_a with a Hamiltonian system on $\mathfrak{s}^*$ or, if one prefers, on the coadjoint orbits of $\mathfrak{s}^*$. For example, this observation is extremely useful in underwater vehicle dynamics (again, see Leonard and Marsden [1997]).

The Momentum Map for the V-action. We now work towards a proof of reduction by stages for semidirect product actions, Theorem 4.2.2. We first elaborate on the constructions in the statement of the theorem.

Thus, we start by considering a given symplectic action of S on a symplectic manifold P and assume that this action has an equivariant momentum map $\mathbf{J}_S : P \to \mathfrak{s}^*$. Since V is a (normal) subgroup of S, it also acts on P and has a momentum map $\mathbf{J}_V : P \to V^*$ given by

$$\mathbf{J}_V = i_V^* \circ \mathbf{J}_S,$$

where $i_V : V \to \mathfrak{s}$ is the inclusion $v \mapsto (0, v)$ and $i_V^* : \mathfrak{s}^* \to V^*$ is its dual. We think of this merely as saying that $\mathbf{J}_V$ is the second component of $\mathbf{J}_S$.

We can also regard G as a subgroup of S by $g \mapsto (g, 0)$. Thus, G also has an equivariant momentum map $\mathbf{J}_G : P \to \mathfrak{g}^*$ that is the first component of $\mathbf{J}_S$ but this will play a secondary role in what follows.

Equivariance of $\mathbf{J}_S$ under G implies the following relation for $\mathbf{J}_V$:

$$\mathbf{J}_V(gz) = g\mathbf{J}_V(z), \tag{4.2.6}$$

where $z \in P$ and we denote the appropriate action of $g \in G$ on an element by concatenation, as before. To prove equation (4.2.6), one uses the fact that for the coadjoint action of S on $\mathfrak{s}^*$ the second component of that action is just the dual of the given action of G on V, which is evident from equation (4.2.2).

The Reduction by Stages Construction. We now elaborate on the reduction by stages construction given in Theorem 4.2.2. An important step will be to show that the construction is, in fact, well-defined.

The "one-shot" reduction step is, in principle, straightforward: one carries out reduction of P by S at a regular value $\sigma = (\mu, a)$ of the momentum map $\mathbf{J}_S$ for S.

On the other hand, in reduction by stages, one carries out the reduction in the following two stages (see Figure 4.2.1).

- First, reduce P by V at the value $a \in V^*$. Since the action of S was assumed to be free and proper, so is the action by V and hence a is a regular value. Thus, we get the reduced manifold $P_a = \mathbf{J}_V^{-1}(a)/V$. Since the reduction is by an Abelian group, the quotient is taken using the whole of V. We will denote the projection to the reduced space by
$$\pi_a : \mathbf{J}_V^{-1}(a) \to P_a.$$
- Second, form the group G_a consisting of elements of G that leave the point a fixed using the induced action of G on V^*. We will need to show that the group G_a acts on P_a and has an induced equivariant momentum map $\mathbf{J}_a : P_a \to \mathfrak{g}_a^*$, where $\mathfrak{g}_a$ is the Lie algebra of G_a.
- Third, using this action of G_a, reduce P_a at the point $\mu_a := \mu|\mathfrak{g}_a$ to get the reduced manifold $(P_a)_{\mu_a} = \mathbf{J}_a^{-1}(\mu_a)/(G_a)_{\mu_a}$.

To prove the result, we will systematically check these claims and after doing this, we will set up the symplectic isomorphism.

Inducing an Action. We first check that we get a free and proper symplectic action of G_a on the V–reduced space P_a. We do this in the following lemmas.

4.2.3 Lemma. *The group G_a leaves the set $\mathbf{J}_V^{-1}(a)$ invariant.*

Proof. Suppose that $\mathbf{J}_V(z) = a$ and that $g \in G$ leaves a invariant. Then by the equivariance relation (4.2.6) noted above, we have
$$\mathbf{J}_V(gz) = g\mathbf{J}_V(z) = ga = a.$$
Thus, G_a acts on the set $\mathbf{J}_V^{-1}(a)$. ▼

4.2.4 Lemma. *The action of G_a on $\mathbf{J}_V^{-1}(a)$ constructed in the preceding lemma, induces a free and proper action Ψ^a on the quotient space $P_a = \mathbf{J}_V^{-1}(a)/V$.*

Proof. If we let elements of the quotient space be denoted by $[z]_a$, regarded as equivalence classes (relative to the action of G_a), then we claim that $g[z]_a = [gz]_a$ defines the action. We first show that it is well-defined.

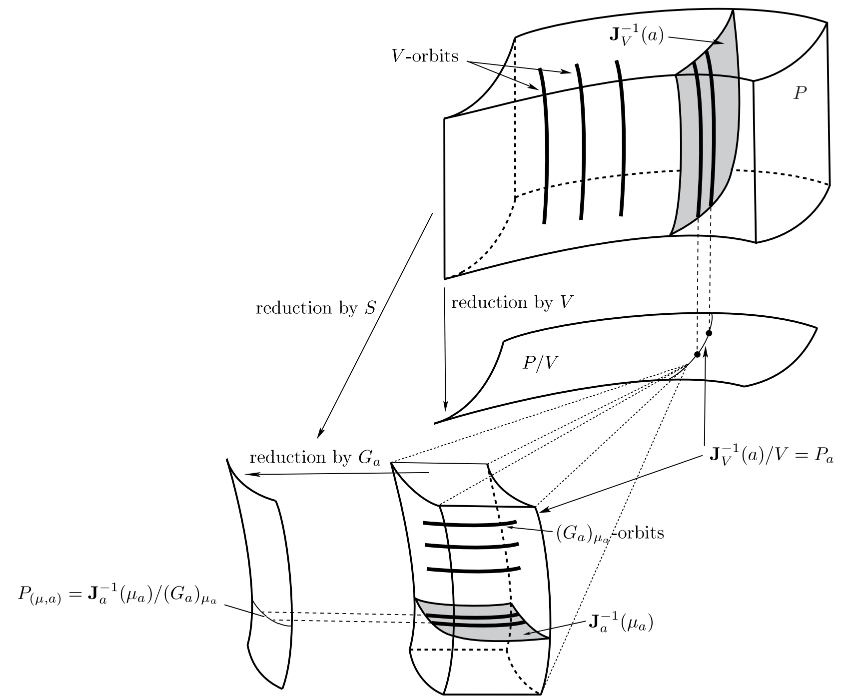

FIGURE 4.2.1. A schematic of reduction by stages for semidirect products.

Indeed, for any $v \in V$ we have $[z]_a = [vz]_a$, so that identifying $v = (e, v)$ and $g = (g, 0)$ in the semidirect product, it follows that

$$\begin{aligned}[gvz]_a &= [(g,0)(e,v)z]_a = [(g,gv)z]_a \\ &= [(e,gv)(g,0)z]_a = [(gv)(gz)]_a \\ &= [gz]_a.\end{aligned}$$

Thus, the action

$$\Psi^a : (g, [z]_a) \in G_a \times P_a \mapsto [gz]_a \in P_a$$

of G_a on the V–reduced space P_a is well-defined.

This action is free because if $[gz]_a = [z]_a$, then there is a $v \in V$ such that $vgz = z$. Since $vg = (g, v)$, freeness of the S-action implies that $g = e$ and $v = 0$.

To show properness, assume $[z_n]_a \to [z]_a$ and that $[g_n z_n]_a \to [z']_a$. We must find a convergent subsequence $g_{n_p} \in G_a$. There are sequences $v_n \in V$ and $v'_n \in V$ such that $v_n z_n = (e, v_n)z_n \to z$ and $v'_n g_n z_n = (g_n, v'_n)z_n \to z'$. Write

$$\begin{aligned}(g_n, v'_n)z_n &= (g_n, v'_n)(e, v_n)^{-1}(e, v_n)z_n \\ &= (g_n, v'_n - v_n)(e, v_n)z_n\end{aligned}$$

Thus, $(g_n, v'_n - v_n)$ has a convergent subsequence, by properness of the S action on P and hence the first components also form a convergent subsequence. Since G_a is closed and $g_n \in G_a$, we get a convergent subsequence in G_a. ▼

The Induced Action is Symplectic. Our next task is to show that the induced action just obtained is symplectic.

4.2.5 Lemma. *The action Ψ^a of G_a on the quotient space $P_a = \mathbf{J}_V^{-1}(a)/V$ constructed in the preceding lemma, is symplectic.*

Proof. Let $\pi_a : \mathbf{J}_V^{-1}(a) \to P_a$ denote the natural projection and let the inclusion be denoted $i_a : \mathbf{J}_V^{-1}(a) \to P$. Denote by $\Psi_g : P \to P$ the action of $g \in G$ on P. The preceding lemma 4.2.4 shows that

$$(i_a \circ \Psi_g)|\mathbf{J}_V^{-1}(a) = \Psi_g \circ i_a$$

for any $g \in G_a$. By construction, $\Psi_g^a \circ \pi_a = (\pi_a \circ \Psi_g)|\mathbf{J}_V^{-1}(a)$. The unique characterization $i_a^*\Omega = \pi_a^*\Omega_a$ of the reduced symplectic form Ω_a on P_a yields

$$\pi_a^*(\Psi_g^a)^*\Omega_a = \Psi_g^*\pi_a^*\Omega_a = \Psi_g^* i_a^*\Omega = i_a^*\Psi_g^*\Omega = i_a^*\Omega = \pi_a^*\Omega_a.$$

Since π_a is a surjective submersion, we conclude that

$$(\Psi_g^a)^*\Omega_a = \Omega_a.$$

Thus, the action of G_a on P_a is symplectic. ▼

An Induced Momentum Map. We next check that the symplectic action obtained in the preceding lemma has an equivariant momentum map that we shall call the ***induced momentum map***. As we shall see later, in more general cases, this turns out to be a critical step; in particular, one needs to be cautious because for central extensions, for instance, the momentum map induced at this step *need not be equivariant*—the fact that one gets an equivariant momentum map in this case is a special feature of semidirect products, about which we shall have more to say later.

4.2.6 Lemma. *The symplectic action Ψ^a on the quotient space $P_a = \mathbf{J}_V^{-1}(a)/V$ has an equivariant momentum map.*

Proof. We first show that the composition of the restriction $\mathbf{J}_S|\mathbf{J}_V^{-1}(a)$ with the projection to $\mathfrak{g}_a^*$ induces a well-defined map $\mathbf{J}_a : P_a \to \mathfrak{g}_a^*$. To check this, note that for $z \in \mathbf{J}_V^{-1}(a)$, and $\xi \in \mathfrak{g}_a$, equivariance gives

$$\langle \mathbf{J}_S(vz), \xi \rangle = \langle v\mathbf{J}_S(z), \xi \rangle = \langle (e,v)\mathbf{J}_S(z), \xi \rangle = \left\langle \mathbf{J}_S(z), (e,v)^{-1}(\xi, 0) \right\rangle.$$

In this equation, the symbol $(e,v)^{-1}(\xi,0)$ means the adjoint action of the group element $(e,v)^{-1} = (e,-v)$ on the Lie algebra element $(\xi,0)$. Thus,

$(e, v)^{-1}(\xi, 0) = (\xi, \xi v)$, and so, continuing the above calculation, and using the fact that $\mathbf{J}_V(z) = a$, we get

$$\begin{aligned}\langle \mathbf{J}_S(vz), \xi \rangle &= \langle \mathbf{J}_S(z), (\xi, \xi v) \rangle = \langle \mathbf{J}_G(z), \xi \rangle + \langle \mathbf{J}_V(z), \xi v \rangle \\ &= \langle \mathbf{J}_G(z), \xi \rangle - \langle \xi a, v \rangle = \langle \mathbf{J}_G(z), \xi \rangle .\end{aligned}$$

In this calculation, the term $\langle \xi a, v \rangle$ is zero since $\xi \in \mathfrak{g}_a$. Thus, we have shown that the expression

$$\langle \mathbf{J}_a([z]_a), \xi \rangle = \langle \mathbf{J}_G(z), \xi \rangle$$

for $\xi \in \mathfrak{g}_a$ is well-defined. Here, $[z]_a \in P_a$ denotes the V–orbit of $z \in \mathbf{J}_V^{-1}(a)$. This expression may be written as

$$\mathbf{J}_a \circ \pi_a = \iota_a^* \circ \mathbf{J}_G \circ i_a,$$

where $\iota_a : \mathfrak{g}_a \to \mathfrak{g}$ is the inclusion map and $\iota_a^* : \mathfrak{g}^* \to \mathfrak{g}_a^*$ is its dual.

Next, we show that the map $\mathbf{J}_a$ is the momentum map of the G_a–action on P_a. Since the vector fields $\xi_P|(\mathbf{J}_V^{-1}(a))$ and ξ_{P_a} are π_a–related for all $\xi \in \mathfrak{g}_a$, we have

$$\pi_a^* \left(\mathbf{i}_{\xi_{P_a}} \Omega_a \right) = \mathbf{i}_{\xi_P} i_a^* \Omega = i_a^* \left(\mathbf{i}_{\xi_P} \Omega \right) = i_a^* \left(\mathbf{d} \left\langle \mathbf{J}_G, \xi \right\rangle \right) = \pi_a^* \left(\mathbf{d} \left\langle \mathbf{J}_a, \xi \right\rangle \right).$$

Again, since π_a is a surjective submersion, it follows that

$$\mathbf{i}_{\xi_{P_a}} \Omega_a = \mathbf{d} \left\langle \mathbf{J}_a, \xi \right\rangle$$

and hence $\mathbf{J}_a$ is the momentum map for the G_a action on P_a.

Equivariance of $\mathbf{J}_a$ follows from that for $\mathbf{J}_G$, by a diagram chasing argument as above, using the identity $\mathbf{J}_a \circ \pi_a = \iota_a^* \circ \mathbf{J}_G \circ i_a$ and the relations between the actions of G on $\mathbf{J}_V^{-1}(a)$ and of G_a on P_a. ▼

Proof of Theorem4.2.2. Having established the preliminary facts in the preceding lemmas, we are ready to prove the main reduction by stages theorem for semidirect products.

Let $\sigma = (\mu, a)$. Start with the inclusion map

$$j : \mathbf{J}_S^{-1}(\sigma) \to \mathbf{J}_V^{-1}(a)$$

which makes sense since the second component of σ is a. Composing this map with π_a, we get the smooth map

$$\pi_a \circ j : \mathbf{J}_S^{-1}(\sigma) \to P_a.$$

This map takes values in $\mathbf{J}_a^{-1}(\mu_a)$ because of the relation $\mathbf{J}_a \circ \pi_a = \iota_a^* \circ \mathbf{J}_G \circ i_a$ and $\mu_a = \iota_a^*(\mu)$. Thus, we can regard it as a map

$$\pi_a \circ j : \mathbf{J}_S^{-1}(\sigma) \to \mathbf{J}_a^{-1}(\mu_a).$$

We assert that projection onto the first factor defines a smooth Lie group homomorphism $\psi : S_\sigma \to (G_a)_{\mu_a}$. In fact, the first component g of $(g, v) \in S_\sigma$ lies in $(G_a)_{\mu_a}$ because

$$(\mu, a) = (g, v)(\mu, a) = (g\mu + \rho_v^*(ga), ga)$$

implies, from the second component, that $g \in G_a$ and from the first component, the identity $\iota_a^* \rho_v^* a = 0$, and the G_a–equivariance of the map ι_a, that g also leaves μ_a invariant. This proves the assertion.

The map $\pi_a \circ j$ is equivariant with respect to the action of S_σ on the domain and $(G_a)_{\mu_a}$ on the range via the homomorphism ψ. Thus, $\pi_a \circ j$ induces a smooth map

$$[\pi_a \circ j] : P_\sigma \to (P_a)_{\mu_a}.$$

Diagram chasing, as above, shows that this map is symplectic.

We will show that this map $[\pi_a \circ j]$ is a diffeomorphism by constructing an inverse. We will begin by showing how to define a map

$$\phi : \mathbf{J}_a^{-1}(\mu_a) \to P_\sigma.$$

Refer to Figure 4.2.2 for the spaces involved.

$$\begin{tikzcd}
\mathbf{J}_s^{-1}(\sigma) \arrow[rr, "j"] \arrow[dd, "\pi_\sigma"'] \arrow[ddr, "\pi_a \circ j"] & & \mathbf{J}_V^{-1}(a) \arrow[r, "i_a"] \arrow[dd, "\pi_a"] & P \arrow[r, "\mathbf{J}_s"] \arrow[d, "\mathbf{J}_V"'] \arrow[dr, "\mathbf{J}_G"] & \mathfrak{s}^* \ni \sigma \arrow[d] \\
& & & a \in V^* & \mathfrak{g}^* \ni \mu \arrow[d, "\iota_a^*"'] \\
P_\sigma \arrow[dr, "{[\pi_a \circ j]}"'] & \mathbf{J}_a^{-1}(\mu_a) \arrow[l, "\phi"'] \arrow[r, hook, "l_{\mu_a}"'] \arrow[d, "\pi_{\mu_a}"] & P_a \arrow[rr, "\mathbf{J}_a"'] & & \mathfrak{g}_a^* \ni \mu_a \\
& (P_a)_{\mu_a} & & &
\end{tikzcd}$$

FIGURE 4.2.2. Maps that are used in the proof of the semidirect product reduction theorem.

To do this, take an equivalence class $[z]_a \in \mathbf{J}_a^{-1}(\mu_a) \subset P_a$ for $z \in \mathbf{J}_V^{-1}(a)$, that is, the V–orbit of z. For each such point, we will try to choose some $v \in V$ such that $vz \in \mathbf{J}_S^{-1}(\sigma)$. For this to hold, we must have

$$(\mu, a) = \mathbf{J}_S(vz).$$

By equivariance, the right hand side equals

$$\begin{aligned} v\mathbf{J}_S(z) &= (e,v)(\mathbf{J}_G(z),\mathbf{J}_V(z)) \\ &= (e,v)(\mathbf{J}_G(z),a) \\ &= (\mathbf{J}_G(z)+\rho_v^*(a),a). \end{aligned}$$

Thus, we require that

$$\mu = \mathbf{J}_G(z)+\rho_v^*(a).$$

That this is possible follows from the next lemma.

4.2.7 Lemma (Annihilator Lemma). *If* $\mathfrak{g}_a^\circ = \{\nu \in \mathfrak{g}^* \mid \nu|\mathfrak{g}_a = 0\}$ *denotes the annihilator of* $\mathfrak{g}_a$ *in* $\mathfrak{g}^*$*, then*

$$\mathfrak{g}_a^\circ = \{\rho_v^* a \mid v \in V\}.$$

Proof. The identity we showed above, namely $\iota_a^*\rho_v^* a = 0$, shows that

$$\mathfrak{g}_a^\circ \supset \{\rho_v^* a \mid v \in V\}.$$

Now we use the following elementary fact from linear algebra. Let E and F be vector spaces, and $F_0 \subset F$ a subspace. Let $T : E \to F^*$ be a linear map whose range lies in the annihilator F_0° of F_0 and such that every element $f \in F$ that annihilates the range of T is in F_0. Then T maps *onto* F_0°. [1]

In our case, we choose $E = V$, $F = \mathfrak{g}$, $F_0 = \mathfrak{g}_a$, and we let $T : V \to \mathfrak{g}^*$ be defined by $T(v) = \rho_v^*(a)$. To verify the hypothesis, note that we have already shown that the range of T lies in the annihilator of $\mathfrak{g}_a$. Let $\xi \in \mathfrak{g}$ annihilate the range of T. Thus, for all $v \in V$,

$$0 = \langle \xi, \rho_v^* a\rangle = \langle \rho_v \xi, a\rangle = \langle \xi v, a\rangle = -\langle v, \xi a\rangle,$$

and so $\xi \in \mathfrak{g}_a$ as required. Thus, the lemma is proved. ▼

We apply the lemma to the element $\nu = \mu - \mathbf{J}_G(z)$, which is an element in the annihilator of $\mathfrak{g}_a$ because $[z]_a \in \mathbf{J}_a^{-1}(\mu_a)$ and hence $\iota_a^*(\mathbf{J}_G(p)) = \mu_a$. Thus, there is a $v \in V$ such that $\mu - \mathbf{J}_G(z) = \rho_v^* a$.

The above argument shows how to construct v so that $vz \in \mathbf{J}_S^{-1}(\sigma)$. We then claim that we can define the map

$$\phi : [z]_a \in \mathbf{J}_a^{-1}(\mu_a) \mapsto [vz]_\sigma \in P_\sigma,$$

where $v \in V$ has been chosen as above and $[vz]_\sigma$ is the S_σ–equivalence class in P_σ of vz.

[1] We are phrasing things this way so that the basic framework will also apply in the infinite dimensional case, with the understanding that at this point one would invoke arguments used in the Fredholm alternative theorem. In the finite dimensional case, the result may be proved by a dimension count.

To show that the map ϕ so constructed is well-defined, we replace z by another representative uz of the same class $[z]_a$; here u is an arbitrary element of V. Following the above procedure, choose v_1 so that $\mathbf{J}_S(v_1uz) = \sigma$. Now we must show that $[vz]_\sigma = [v_1uz]_\sigma$. In other words, we must show that there is a group element $(g, w) \in S_\sigma$ such that

$$(g, w)(e, v)z = (e, v_1)(e, u)z.$$

This will hold if we can show that $(g, w) := (e, v_1)(e, u)(e, v)^{-1} \in S_\sigma$. However, by construction, $\mathbf{J}_S(vz) = \sigma = \mathbf{J}_S(v_1uz)$; in other words, we have

$$\sigma = (\mu, a) = (e, v)\mathbf{J}_S(z) = (e, v_1)(e, u)\mathbf{J}_S(z).$$

Thus, by isolating $\mathbf{J}_S(z)$, we get $(e, v)^{-1}\sigma = (e, u)^{-1}(e, v_1)^{-1}\sigma$ and so the element $(g, w) = (e, v_1)(e, u)(e, v)^{-1}$ belongs to S_σ. Thus, the map ϕ is well-defined.

The strategy for proving smoothness of ϕ is to choose a local trivialization of the V bundle $\mathbf{J}_V^{-1}(a) \to \mathbf{J}_a^{-1}(\mu_a)$ and define a local section which takes values in the image of $\mathbf{J}_S^{-1}(\sigma)$ under the embedding j. Smoothness of the local section follows by using a complement to the kernel of the linear map $v \mapsto \rho_v^*(a)$ that defines the solution v of the equation $\rho_v^*(a) = \mu - \mathbf{J}_G(z)$. Using such a complement depending smoothly on the data creates a uniquely defined smooth selection of a solution.

Next, we show that the map ϕ is $(G_a)_{\mu_a}$–invariant. To see this, let $[z]_a \in \mathbf{J}_a^{-1}(\mu_a)$ and $g_0 \in (G_a)_{\mu_a}$. Choose $v \in V$ so that $vz \in \mathbf{J}_S^{-1}(\sigma)$ and let $u \in V$ be chosen so that $ug_0z \in \mathbf{J}_S^{-1}(\sigma)$. We must show that $[vz]_\sigma = [ug_0z]_\sigma$. Thus, we must find an element $(g, w) \in S_\sigma$ such that

$$(g, w)(e, v)z = (e, u)(g_0, 0)z.$$

This will hold if we can show that $(g, w) := (e, u)(g_0, 0)(e, v)^{-1} \in S_\sigma$. Since $\sigma = \mathbf{J}_S(vz) = \mathbf{J}_S(ug_0z)$, by equivariance of $\mathbf{J}_S$ we get,

$$\sigma = (e, v)\mathbf{J}_S(z) = (e, u)(g_0, 0)\mathbf{J}_S(z).$$

Isolating $\mathbf{J}_S(z)$, this implies that

$$(e, v)^{-1}\sigma = (g_0, 0)^{-1}(e, u)^{-1}\sigma,$$

which means that indeed $(g, w) = (e, u)(g_0, 0)(e, v)^{-1} \in S_\sigma$. Hence ϕ is $(G_a)_{\mu_a}$–invariant, and so induces a well-defined map

$$[\phi] : (P_a)_{\mu_a} \to P_\sigma.$$

Chasing the definitions shows that $[\phi]$ is the inverse of the map $[\pi_a \circ j]$.

Smoothness of $[\phi]$ follows from smoothness of ϕ since the quotient by the group action, π_a is a smooth surjective submersion. Thus, both $[\pi_a \circ j]$ and ϕ are symplectic diffeomorphisms. ■

In this framework, one can also, of course, reduce the dynamics of a given invariant Hamiltonian as was done for the case of reduction by T^*S by stages.

Remarks.

1. Choose $P = T^*S$ in the preceding theorem, with the cotangent action of S on T^*S induced by left translations of S on itself. Reducing T^*S by the action of V gives a space naturally diffeomorphic to T^*G—this may be checked directly, but we will detail the *real* reason this is so in the next section. Thus, the reduction by stages theorem gives as a corollary, the semidirect product reduction Theorem 4.2.1.

2. The original proof of Theorem 4.2.1 in Marsden, Ratiu and Weinstein [1984a,b] essentially used the map $[\phi]$ constructed above to obtain the required symplectic diffeomorphism. However, the generalization presented here to obtain reduction by stages for semidirect product actions, required an essential modification of the original method.

3. In the following section we shall give some details concerning reduction by stages for SE(3), the special Euclidean group of $\mathbb{R}^3$. This illustrates some important aspects and applications of the classical semidirect product reduction Theorem 4.2.1.

4. We briefly describe two examples that require the more general result of Theorem 4.2.2.

   (a) First, consider a pseudo-rigid body in a fluid; that is, a body which can undergo linear deformations and moving in potential flow, as was the case for rigid bodies in potential flow in Leonard and Marsden [1997]. Here the phase space is $P = T^* \mathrm{GE}(3)$ (where GE(3) is the semidirect product $\mathrm{GL}(3) \circledS \mathbb{R}^3$) and the symmetry group we want to reduce by is SE(3); it acts on GE(3) on the left by composition and hence on $T^* \mathrm{GE}(3)$ by cotangent lift. According to the general theory, we can reduce by the action of $\mathbb{R}^3$ first and then by SO(3). This example has the interesting feature that the center of mass need not move uniformly along a straight line, so the first reduction by translations is not trivial. The same thing happens for a rigid body moving in a fluid.

   (b) A second, more sophisticated example is a fully elastic body, in which case, P is the cotangent bundle of the space of all embeddings of a reference configuration into $\mathbb{R}^3$ (as in Marsden and Hughes [1983]) and we take the group again to be SE(3) acting by composition on the left. Again, one can perform reduction in two stages.

   As we have mentioned before, the reduction by stages philosophy is quite helpful in understanding the dynamics and stability of underwater vehicle dynamics, as in Leonard and Marsden [1997].

4.3 Cotangent Bundle Reduction and Semidirect Products

The purpose of this section is to couple the semidirect product reduction theorem with cotangent bundle reduction to obtain a more detailed structure of the reduced spaces for the right cotangent lifted action of $G \circledS V$ on $T^* (G \circledS V)$. Of course, by Theorem 1.2.3 on reduction to coadjoint orbits, these reduced spaces are the coadjoint orbits of the group $G \circledS V$.

To carry out this program, we first construct a mechanical connection on the bundle $G \circledS V \to G$ and prove that this connection is flat. This will allow us to identify (equivariantly) the first (V-reduced) space with $(T^*G, \Omega_{\mathrm{can}})$. We will then be in a position to apply cotangent bundle reduction again to complete the orbit classification.

Notation. As in the preceding section, let $S = G \circledS V$ be the semidirect product of a Lie group G and a vector space V on which G acts, with multiplication

$$(g, v)(h, w) = (gh, v + gw), \tag{4.3.1}$$

where $g, h \in G$ and $v, w \in V$. The identity element is $(e, 0)$ and inversion is given by $(g, v)^{-1} = (g^{-1}, -g^{-1}v)$. Recall that the Lie algebra of S is the semidirect product $\mathfrak{s} = \mathfrak{g} \circledS V$ with the commutator

$$[(\xi, v), (\eta, w)] = ([\xi, \eta], \, \xi w - \eta v) \, , \tag{4.3.2}$$

where $\xi, \eta \in \mathfrak{g}$ and $v, w \in V$.

In what follows it is convenient to explicitly introduce the homomorphism $\phi : G \to \mathrm{Aut}(V)$ defining the given G-representation on V and to recall that we identify V with $\{e\} \times V$, a closed normal Lie subgroup of $G \circledS V$.

The adjoint representation of S on $\mathfrak{s}$ given in equation (4.2.1) restricts to the S-representation on V given by $\mathrm{Ad}_{(g,v)} u = gu$ for any $g \in G$ and $u, v \in V$. Its derivative with respect to the group variable (g, v) in the direction $(\xi, w) \in \mathfrak{s}$ is $\mathrm{ad}_{(\xi,w)} u = \xi u$.

The Mechanical Connection. Let $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mathfrak{g}}$ and $\langle\!\langle \cdot, \cdot \rangle\!\rangle_V$ be two positive definite inner products on the Lie algebra $\mathfrak{g}$ and on the vector space V, respectively. Then

$$\langle\!\langle (\xi, v), (\eta, w) \rangle\!\rangle_{\mathfrak{s}} = \langle\!\langle \xi, \eta \rangle\!\rangle_{\mathfrak{g}} + \langle\!\langle v, w \rangle\!\rangle_V \, , \tag{4.3.3}$$

for any $(\xi, v), (\eta, w) \in \mathfrak{s}$, defines a positive definite inner product on $\mathfrak{s}$. Since the spaces $\mathfrak{g} \times \{0\}$ and $\{0\} \times V$ are orthogonal, the orthogonal $\langle\!\langle \, , \rangle\!\rangle_{\mathfrak{s}}$-projection $\mathcal{P}_V : \mathfrak{s} = \mathfrak{g} \circledS V \to V$ is simply the projection on the second factor.

Extend the inner product (4.3.3) on $\mathfrak{s}$ to a *right*-invariant Riemannian metric on S by setting

$$\begin{aligned}&\langle\!\langle (X_g, u), (Y_g, w) \rangle\!\rangle_{(g,v)} \\ &\quad = \left\langle\!\left\langle T_{(g,v)} R_{(g,v)^{-1}}(X_g, u), T_{(g,v)} R_{(g,v)^{-1}}(Y_g, w) \right\rangle\!\right\rangle_{\mathfrak{s}}, \end{aligned} \tag{4.3.4}$$

where $(g, v) \in S$, $(X_g, u), (Y_g, w) \in T_{(g,v)}S$, and $R_{(g,v)}$ is right translation[2] on S. The derivative of $R_{(h,w)}$ is readily computed from (4.3.1) to be

$$T_{(g,v)} R_{(h,w)}(X_g, u) = (X_g \cdot h,\ u + T_g \phi^w(X_g)), \tag{4.3.5}$$

where $(X_g, w) \in T_{(g,v)}S$, $X_g \cdot h := T_g R_h(X_g)$, R_h is the right translation on G, and $\phi^w : G \to V$ is given by $\phi^w(g) := gw$. In particular

$$T_{(g,v)} R_{(g,v)^{-1}}(X_g, u) = \left(X_g \cdot g^{-1},\ u - (X_g \cdot g^{-1})v\right), \tag{4.3.6}$$

a formula that is useful in the subsequent computations.

The hypotheses of Theorem 2.1.15 hold for the bundle $G \circledS V \to V$ and hence the mechanical connection $\mathcal{A}^V \in \Omega(G \circledS V; V)$ associated to the Riemannian metric $\langle\!\langle\, , \rangle\!\rangle_{\mathfrak{s}}$ is given by formula (2.1.15) which in this case becomes

$$\begin{aligned}\mathcal{A}^V(g, v)\,(X_g, u) &= \mathrm{Ad}_{(g,v)^{-1}}\left(\mathcal{P}_V T_{(g,v)} R_{(g,v)^{-1}}(X_g, u)\right) \\ &= \mathrm{Ad}_{(g^{-1}, -g^{-1}v)}\left(\mathcal{P}_V\left(X_g \cdot g^{-1},\ u - (X \cdot g^{-1})v\right)\right) \\ &= g^{-1}\left(u - (X_g \cdot g^{-1})v\right), \end{aligned} \tag{4.3.7}$$

where $(g, v) \in S$ and $(X_g, u) \in T_{(g,v)}S$.

Notice that the connection $\mathcal{A}^V$ is *not* S-invariant. In contrast, the same construction for *central* extensions yields an invariant but nonflat mechanical connection. As we shall see later, invariance in this case will follow from the equivariance equation (2.1.16).

The Flatness Calculation. The "reason" why the first reduced space is so simple is that the mechanical connection $\mathcal{A}^V$ is flat—that is, its curvature is zero. This is a direct consequence of Theorem 2.1.16 as will be shown below. Let $(X_g, \bar{u}), (Y_g, \bar{w}) \in T_{(g,v)}S$ and let

$$\begin{aligned}(\xi, u) &= T_{(g,v)} R_{(g,v)^{-1}}(X_g, \bar{u}) \\ (\eta, w) &= T_{(g,v)} R_{(g,v)^{-1}}(Y_g, \bar{v}),\end{aligned}$$

each of which is an element of $\mathfrak{s}$. We compute the curvature of the mechanical connection $\mathcal{A}^V$ with the assistance of the equation $\mathrm{ad}_{(\xi,w)}\, u = \xi u$,

[2]Our choice of *right translations* is motivated by infinite dimensional applications to diffeomorphism groups. Of course, there is a left invariant analogue of the constructions given here.

using the formula (2.1.17), which in this case becomes

$$
\begin{aligned}
&\operatorname{curv}_{\mathcal{A}^V}\left((X_g, \bar{u}), (Y_g, \bar{w})\right) \\
&= \operatorname{Ad}_{(g,v)^{-1}}\left(-\operatorname{ad}_{(\xi,u)}\mathcal{P}_V(\eta, w) + \operatorname{ad}_{(\eta,w)}\mathcal{P}_V(\xi, u) + \mathcal{P}_V\left[(\xi, u), (\eta, w)\right] + 0\right) \\
&= \operatorname{Ad}_{(g,v)^{-1}}\left(-\xi w + \eta u + \xi w - \eta u\right) = 0.
\end{aligned}
$$

We summarize this discussion in the following theorem.

4.3.1 Theorem. *The mechanical connection $\mathcal{A}^V$ defined on the right principal V-bundle $S \to G$ by formula (4.3.7) is flat.*

Remarks. If one's goal is simply to pick a connection on the the principal V-bundle $S \to G$ in order to realize the first reduced space as T^*G with the canonical structure, then one may use the *trivial* connection associated with the product structure $S = G \times V$, so that the connection 1-form is simply projection to V. This connection has the needed equivariance properties to realize the reduced space as T^*G and identifies the resulting action of G_a as the right action on T^*G. On the other hand, in more general situations in which the bundles may not be trivial, it is the mechanical connection which is used in the construction and so it is of interest to use it here as well. In particular, in the second stage reduction, one needs a connection on the (generally) nontrivial bundle $G \to G/G_a$ and such a connection is naturally induced by the mechanical connection.

Cotangent Bundle Structure of the Orbits. We are now ready to establish the extent to which coadjoint orbits of $G \circledS V$ are cotangent bundles (possibly with magnetic terms). We will illustrate the methods with SE(3) in §4.4. As we have mentioned, the strategy is to combine the reduction by stages theorem with the cotangent bundle reduction theorem. In the course of doing this, we recover a result of Ratiu [1980a, 1981, 1982] regarding the embedding of the semidirect product coadjoint orbits into cotangent bundles with magnetic terms, but will provide a different proof here based on connections. We consider here the cotangent lift of right translation of S on T^*S (see Theorem 4.2.1) and all connections are the mechanical connections associated to the right invariant metrics induced on S and G by the inner products $\langle\!\langle\, , \rangle\!\rangle_{\mathfrak{s}}$ and $\langle\!\langle\, , \rangle\!\rangle_{\mathfrak{g}}$, respectively..

4.3.2 Theorem. *Let $S = G \circledS V$ and $\mathbf{J}_V : T^*S \to V^*$, $\mathbf{J}_V(\alpha_g, a) = g^{-1}a$, be the momentum map of the cotangent lift of right translation of V on S, where $\langle g^{-1}a, u\rangle := \langle a, gu\rangle$ for any $u \in V$, $a \in V^*$, and $g \in G$. Let $a \in \mathbf{J}_V(T^*S) \subset V^*$ and reduce T^*S at a. There is a right G_a-equivariant symplectic diffeomorphism*

$$
(T^*S)_a := \mathbf{J}_V^{-1}(a)/V \simeq (T^*G, \Omega_a)\,, \tag{4.3.8}
$$

where $\Omega_a = \Omega_{\mathrm{can}}$ is the canonical symplectic form. Furthermore, let $\sigma = (\mu, a) \in \mathfrak{s}^* \times V^*$ and reduce T^*S by the cotangent lift of right translation of

S on itself at σ obtaining the coadjoint orbit $\mathcal{O}_\sigma$ through σ endowed with the plus orbit symplectic form. Let $\mathbf{J}_a : T^*G \to \mathfrak{g}_a^*$ be the momentum map of the cotangent lift of right translation of the isotropy subgroup $G_a = \{g \in G \mid ga = g\}$ on G, $\mathfrak{g}_a = \{\xi \in \mathfrak{g} \mid \xi a = 0\}$ the Lie algebra of G_a, and $\mu_a = \mu|_{\mathfrak{g}_a}$. Then there is a symplectic diffeomorphism

$$\mathcal{O}_\sigma \simeq (T^*G)_{\mu_a} := \mathbf{J}_a^{-1}(\mu_a)/\left(G_a\right)_{\mu_a} \tag{4.3.9}$$

and a symplectic embedding

$$(T^*G)_{\mu_a} \hookrightarrow \left(T^*\left(G/\left(G_a\right)_{\mu_a}\right), \Omega_{\mu_a}\right),$$

where $\Omega_{\mu_a} = \Omega_{\text{can}} - \pi^*\mathcal{B}_{\mu_a}$ with $\mathcal{B}_{\mu_a}$ a closed two-form on $G/\left(G_a\right)_{\mu_a}$ calculated in Theorem 4.3.3. The image of this embedding is a vector subbundle of $T^*\left(G/\left(G_a\right)_{\mu_a}\right)$. If G_a is Abelian, in which case $(G_a)_{\mu_a} = G_a$, this embedding is a diffeomorphism onto $T^*(G/G_a)$.

Proof. The fact that the spaces in (4.3.8) are symplectomorphic is a consequence of the standard cotangent bundle reduction theorem for Abelian symmetry groups in §2.2 combined with Theorem 4.3.1. As we have seen in §2.2, the symplectomorphism is induced by the shift map (which, recall, is also the projection to the horizontal part):

$$\text{shift}_a : \mathbf{J}_V^{-1}(a) \to \mathbf{J}_V^{-1}(0), \qquad \text{shift}_a(p_{(g,v)}) = p_{(g,v)} - \left\langle a, \mathcal{A}^V(g,v)\right\rangle.$$

To show the equivariance it only suffices to check that

$$\text{shift}_a\left(p_{(g,v)} \cdot (h,0)\right) = \left(\text{shift}_a(p_{(g,v)})\right) \cdot (h,0), \tag{4.3.10}$$

for any $p_{(g,v)} \in T^*_{(g,v)}S$ and $h \in G_a$. However, if $(X,u) \in T_{(gh,v)}S$, formulas (4.3.7), (4.3.5), and $ha = a$ imply

$$\begin{aligned}
\left\langle a, \mathcal{A}^V((g,v)(h,0))(X,u)\right\rangle &= \left\langle a, (gh)^{-1}\big(u - (X \cdot (gh)^{-1})v\big)\right\rangle \\
&= \left\langle ha, g^{-1}\big(u - (X \cdot (gh)^{-1})v\big)\right\rangle \\
&= \left\langle a, g^{-1}\big(u - ((X \cdot h^{-1}) \cdot g^{-1})v\big)\right\rangle \\
&= \left\langle a, \mathcal{A}^V(g,v)(X \cdot h^{-1}, u)\right\rangle \\
&= \left\langle a, \mathcal{A}^V(g,v)\big((X,u) \cdot (h,0)^{-1}\big)\right\rangle,
\end{aligned}$$

which proves (4.3.10).

The fact that the map in (4.3.9) is a symplectomorphism follows from Theorem 4.2.1 and the G_a-equivariance in (4.3.8).

The rest of the statement is a direct consequence of the Cotangent Bundle Reduction Theorem 2.2.1: the magnetic term of the cotangent bundle $T^*\left(G/\left(G_a\right)_{\mu_a}\right)$ is the μ_a-component $\mathcal{B}_{\mu_a} := \langle \mu_a, \mathcal{B}\rangle$ of the curvature $\mathcal{B}$ of the mechanical connection $\mathcal{A}^{G_a}$ on the right principal bundle $G \to G/G_a$ associated to the inner product $\langle\!\langle\, , \rangle\!\rangle_{\mathfrak{g}}$ (see Proposition 2.2.5). ∎

Calculation of $\mathcal{A}^{G_a}$ and $\mathbf{d}\mathcal{A}^{G_a}$. As promised in the preceding theorem, we now derive formulas for the mechanical connection and its curvature on the right principal G_a-bundle $G \to G/G_a$.

4.3.3 Theorem. *The mechanical connection on the right principal bundle $G \to G/G_a$ associated to the inner product $\langle\!\langle\, , \rangle\!\rangle_{\mathfrak{g}}$ is given by*

$$\begin{aligned}&\mathcal{A}^{G_a}(g)(X_g) = \mathbb{P}_a(T_gL_{g^{-1}}X_g)\\&+\left(\left(\mathbb{P}_a \circ \mathrm{Ad}_g^T \circ \mathrm{Ad}_g\right)\Big|_{\mathfrak{g}_a}\right)^{-1}\left(\mathbb{P}_a \circ \mathrm{Ad}_g^T \circ \mathrm{Ad}_g\right)\left(\mathbb{P}_a^{\perp}(T_gL_{g^{-1}}X_g)\right), \quad (4.3.11)\end{aligned}$$

where $\mathbb{P}_a : \mathfrak{g} \to \mathfrak{g}_a$ and $\mathbb{P}_a^{\perp} : \mathfrak{g} \to \mathfrak{g}_a^{\perp}$ are the orthogonal projections relative to the inner product $\langle\!\langle\, , \rangle\!\rangle_{\mathfrak{g}}$. Let $\mathcal{A}_{\mu_a}^{G_a} := \left\langle \mu_a, \mathcal{A}^{G_a}\right\rangle \in \Omega^1(G)$ be the μ_a-component of $\mathcal{A}^{G_a}$. The two-form $\mathcal{B}_{\mu_a} \in \Omega^2(G)$ is obtained by dropping $\mathbf{d}\mathcal{A}_{\mu_a}^{G_a}$ to the quotient $G/\left(G_a\right)_{\mu_a}$.

If $\mathrm{Ad}_g^T \circ \mathrm{Ad}_g$ leaves $\mathfrak{g}_a$ invariant, where $\mathrm{Ad}_g^T : \mathfrak{g} \to \mathfrak{g}$ is the transpose (adjoint) of Ad_g relative to $\langle\!\langle\, , \rangle\!\rangle_{\mathfrak{g}}$ (this holds, in particular, when $\langle\!\langle\, , \rangle\!\rangle_{\mathfrak{g}}$ is Ad*-invariant, which can always be achieved if G is compact), the formulas for the connection and its differential simplify to*

$$\mathcal{A}^{G_a} = \mathbb{P}_a \circ \theta^L \qquad (4.3.12)$$

and

$$\mathbf{d}\mathcal{A}^{G_a}(g)(X_g, Y_g) = -\mathbb{P}_a\left([T_gL_{g^{-1}}X_g, T_gL_{g^{-1}}Y_g]\right), \qquad (4.3.13)$$

where θ^L, defined by $\theta^L(X_g) = T_gL_{g^{-1}}X_g$, is the left-invariant Maurer-Cartan form on G (see Theorem 2.1.14).

Proof. We first compute the locked inertia tensor for the right action of G_a on G. Let $\langle\!\langle\, , \rangle\!\rangle_g$ denote the right invariant extension of the inner product $\langle\!\langle\, , \rangle\!\rangle_{\mathfrak{g}}$ to an inner product on T_gG, so that $\langle\!\langle\, , \rangle\!\rangle_e = \langle\!\langle\, , \rangle\!\rangle_{\mathfrak{g}}$ and let $\xi, \eta \in \mathfrak{g}_a$. By definition, the locked inertia tensor is given by

$$\begin{aligned}\langle \mathbb{I}(g)(\xi), \eta\rangle &= \langle\!\langle \xi_G(g), \eta_G(g)\rangle\!\rangle_g = \langle\!\langle T_eL_g\xi, T_eL_g\eta\rangle\!\rangle_g = \langle\!\langle \mathrm{Ad}_g\,\xi, \mathrm{Ad}_g\,\eta\rangle\!\rangle_e\\&= \left\langle\!\!\left\langle \mathrm{Ad}_g^T \circ \mathrm{Ad}_g\,\xi, \eta\right\rangle\!\!\right\rangle_e = \left\langle\!\!\left\langle (\mathbb{P}_a \circ \mathrm{Ad}_g^T \circ \mathrm{Ad}_g)(\xi), \eta\right\rangle\!\!\right\rangle_e.\end{aligned}$$

Thus,

$$\mathbb{I}(g)(\xi) = \left\langle\!\!\left\langle (\mathbb{P}_a \circ \mathrm{Ad}_g^T \circ \mathrm{Ad}_g)(\xi), \cdot\right\rangle\!\!\right\rangle_e \in \mathfrak{g}_a^*. \qquad (4.3.14)$$

Since the action is free, $\mathbb{I}(g)$ is invertible for every $g \in G$ and hence we conclude that $\left(\mathbb{P}_a \circ \mathrm{Ad}_g^T \circ \mathrm{Ad}_g\right)\Big|_{\mathfrak{g}_a} : \mathfrak{g}_a \to \mathfrak{g}_a$ is an isomorphism.

Next, we compute the value $\mathbf{J}\left(\langle\!\langle X_g, \cdot \rangle\!\rangle_g\right) \in \mathfrak{g}_a^*$ of the G_a-momentum map $\mathbf{J} : T^*G \to \mathfrak{g}_a^*$. For $\xi \in \mathfrak{g}_a$ we have

$$\begin{aligned}\left\langle \mathbf{J}\left(\langle\!\langle X_g, \cdot \rangle\!\rangle_g\right), \xi \right\rangle &= \langle\!\langle X_g, \xi_G(g) \rangle\!\rangle = \langle\!\langle X_g, T_e L_g \xi \rangle\!\rangle_g \\ &= \left\langle\!\left\langle T_g R_{g^{-1}} X_g, \operatorname{Ad}_g \xi \right\rangle\!\right\rangle_e \\ &= \left\langle\!\!\left\langle \operatorname{Ad}_g^T (T_g R_{g^{-1}} X_g), \xi \right\rangle\!\!\right\rangle_e \\ &= \left\langle\!\!\left\langle \langle(\mathbb{P}_a \circ \operatorname{Ad}_g^T \circ \operatorname{Ad}_g)(T_g L_{g^{-1}} X_g), \xi \right\rangle\!\!\right\rangle_e .\end{aligned}$$

We conclude that

$$\begin{aligned}\mathbf{J}\left(\langle\!\langle X_g, \cdot \rangle\!\rangle_g\right) &= \left\langle\!\!\left\langle (\mathbb{P}_a \circ \operatorname{Ad}_g^T \circ \operatorname{Ad}_g)(T_g L_{g^{-1}} X_g), \cdot \right\rangle\!\!\right\rangle_e \\ &= \left\langle\!\!\left\langle (\mathbb{P}_a \circ \operatorname{Ad}_g^T \circ \operatorname{Ad}_g)(\mathbb{P}_a T_g L_{g^{-1}} X_g), \cdot \right\rangle\!\!\right\rangle_e \\ &\quad + \left\langle\!\!\left\langle (\mathbb{P}_a \circ \operatorname{Ad}_g^T \circ \operatorname{Ad}_g)(\mathbb{P}_a^\perp T_g L_{g^{-1}} X_g), \cdot \right\rangle\!\!\right\rangle_e . \qquad (4.3.15)\end{aligned}$$

Using (4.3.14) and (4.3.15) in the definition (2.1.4) of he mechanical connection yields (4.3.11).

Now assume that $\operatorname{Ad}_g^T \circ \operatorname{Ad}_g$ leaves $\mathfrak{g}_a$ invariant; since this linear operator is symmetric, it also leaves its orthogonal complement invariant. This implies that the second summand in (4.3.15) vanishes and hence

$$\mathbf{J}\left(\langle\!\langle X_g, \cdot \rangle\!\rangle_g\right) = \left\langle\!\!\left\langle \left(\mathbb{P}_a \circ \operatorname{Ad}_g^T \circ \operatorname{Ad}_g\right)\left(\mathbb{P}_a(g^{-1} \cdot X_g)\right), \cdot \right\rangle\!\!\right\rangle_e . \qquad (4.3.16)$$

Combining (4.3.14) and (4.3.16), we get

$$\mathcal{A}^{G_a}(g)(X_g) = \left(\mathbb{I}(g)^{-1} \circ \mathbf{J}\right)\left(\langle\!\langle X_g, \cdot \rangle\!\rangle_g\right) = (\mathbb{P}_a \circ \theta^L)(X_g).$$

To compute $\mathbf{d}\mathcal{A}^{G_a}(g)(X_g, Y_g)$, extend X_g, Y_g to left invariant vector fields $\bar{X}, \bar{Y}$. Then,

$$\begin{aligned}\mathbf{d}\mathcal{A}^{G_a}(g)(X_g, Y_g) &= \bar{X}[\mathcal{A}^{G_a}(\bar{Y})](g) - \bar{Y}[\mathcal{A}^{G_a}(\bar{X})](g) - \mathcal{A}^{G_a}(g)([\bar{X}, \bar{Y}](g)) \\ &= -\mathbb{P}_a([T_g L_g^{-1} \cdot X_g, T_g L_g^{-1} \cdot Y_g]), \qquad (4.3.17)\end{aligned}$$

where we have used the fact that $\mathcal{A}^{G_a}(\bar{Y})$, for example, is constant from the preceding expression for $\mathcal{A}^{G_a}$ and left invariance, and so the first two terms vanish. ■

4.4 Example: The Euclidean Group

This section uses the results of the preceding section to classify the coadjoint orbits of the Euclidean group SE(3). We will also make use of mechanical connections and their curvatures to compute the the coadjoint orbit symplectic forms.

A Right Invariant Metric on SE(3)**.** Identify

$$\mathfrak{se}(3) \simeq \mathfrak{so}(3) \oplus \mathbb{R}^3$$

and define the natural inner product at the identity (see (4.3.3))

$$\langle\!\langle (X,a),(Y,b)\rangle\!\rangle_{(I,0)} = -\frac{1}{2}\operatorname{tr}(XY) + \langle\!\langle a,b\rangle\!\rangle\,,$$

where, on the right hand side, $\langle\!\langle \cdot,\cdot\rangle\!\rangle$ denotes the Euclidean inner product. Requiring right invariance of the metric and use of the equation (4.3.6) gives

$$\begin{aligned}
&\langle\!\langle (X_A,a_A),(Y_A,b_A)\rangle\!\rangle_{(A,\alpha)}\\
&= \left\langle\!\!\left\langle (X_A\cdot A^{-1}, a_A - (X_A\cdot A^{-1})\alpha),\, (Y_A\cdot A^{-1}, b_A - (Y_A\cdot A^{-1})\alpha)\right\rangle\!\!\right\rangle_{(I,0)}\\
&= -\frac{1}{2}\operatorname{tr}(X_A\cdot A^{-1}\cdot Y_A\cdot A^{-1}) + \left\langle\!\!\left\langle (X_A\cdot A^{-1})\alpha, (Y_A\cdot A^{-1})\alpha\right\rangle\!\!\right\rangle\\
&\qquad - \left\langle\!\!\left\langle (X_A\cdot A^{-1})\alpha, b_A\right\rangle\!\!\right\rangle - \left\langle\!\!\left\langle (Y_A\cdot A^{-1})\alpha, a_A\right\rangle\!\!\right\rangle + \langle\!\langle a_A, b_A\rangle\!\rangle\,. \qquad (4.4.1)
\end{aligned}$$

The mechanical connection for the principal $\mathbb{R}^3$-bundle $\mathrm{SE}(3)\to\mathrm{SO}(3)$, is given by (4.3.7):

$$\mathcal{A}^{\mathbb{R}^3}(A,\alpha)(X_A,a_A) = A^{-1}\left(a_A - (X_A\cdot A^{-1})\alpha\right)$$

and from Theorem 4.3.1, we see that $\operatorname{curv}\mathcal{A}^{\mathbb{R}^3} = \mathbf{d}\mathcal{A}^{\mathbb{R}^3} = 0$.

First Reduction. We first reduce by the $\mathbb{R}^3$-cotangent lifted action. Let $a \in \mathbb{R}^{3^*} = \mathbb{R}^3$. By Theorem 4.3.2, the cotangent bundle reduction theorem for semidirect products, we know that the first reduced space $(T^*\,\mathrm{SE}(3))_a = \mathbf{J}^{-1}_{\mathbb{R}^3}(a)/\mathbb{R}^3$ is symplectically diffeomorphic to the cotangent bundle $(T^*\,\mathrm{SO}(3),\Omega_{\mathrm{can}})$.

Second Reduction. We first take the easy case in which $a=0$. Then $G_a = \mathrm{SO}(3)$. Reduction by the SO(3)-action therefore gives coadjoint orbits of SO(3). Thus $\mathcal{O}_{(a=0,\mu)} = S^2_\mu$, the two-sphere passing through $\mu\in\mathbb{R}^3$.

Next, assume $a\neq 0$. Then the group $\mathrm{SE}(3)_a/\mathbb{R}^3_a \simeq \mathrm{SO}(3)_a \simeq S^1$ acts (on the *right*) on the first reduced space, $(T^*\,\mathrm{SO}(3),\Omega_{\mathrm{can}})$. Note that the map $[A]\in\mathrm{SO}(3)/\,\mathrm{SO}(3)_a \mapsto Aa \in S^2_a$, the two-sphere passing through $a\in\mathbb{R}^3$, is a diffeomorphism. Depending on whether $\mu=0$ or $\mu\neq 0$, we have to consider two further subcases.

Suppose that $\mu=0$. Reducing by the $\mathrm{SO}(3)_a = S^1$-action at $\mu_a = 0$ gives, by another application of the cotangent bundle reduction theorem for Abelian groups, the symplectic manifold $(T^*S^2_a,\Omega_{\mathrm{can}})$ (see also Theorem 4.3.2).

Finally, consider the subcase $\mu\neq 0$. The group $G_a = \mathrm{SO}(3)_a = S^1$ acts by cotangent lift of right translation on $T^*\,\mathrm{SO}(3)$. The S^1-principal bundle $\mathrm{SO}(3) \to \mathrm{SO}(3)/\mathrm{SO}(3)_a \simeq S^2_a$ naturally inherits a metric from the principal $\mathbb{R}^3$-bundle $\mathrm{SE}(3) \to \mathrm{SO}(3)$, which is $\mathrm{SO}(3)$-invariant.

Let us compute the curvature of the mechanical connection on the bundle $\mathrm{SO}(3) \to S^2_a$. It is convenient to use the Lie algebra isomorphism $x \in \mathbb{R}^3 \mapsto \widehat{x} \in \mathfrak{so}(3)$ defined by the cross product, namely, $\widehat{x}u = x \times u$. The inner product on $\mathfrak{so}(3)$

$$\langle\!\langle \widehat{x}, \widehat{y} \rangle\!\rangle_I := \langle\!\langle x, y \rangle\!\rangle = -\frac{1}{2}\operatorname{tr}(\widehat{x}\,\widehat{y}),$$

where $x, y \in \mathbb{R}^3$, induces the right invariant Riemannian metric $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ on $\mathrm{SO}(3)$ given on the tangent space at A by

$$\langle\!\langle X_A, Y_A \rangle\!\rangle_A = -\frac{1}{2}\operatorname{tr}(X_A \cdot A^{-1} \cdot Y_A \cdot A^{-1}),$$

where $X_A, Y_A \in T_A\,\mathrm{SO}(3)$. The Lie algebra of $\mathrm{SO}(3)_a \simeq S^1$ is $\operatorname{span}\{a\} \cong \mathbb{R}$, so the infinitesimal generator of $u \in \mathbb{R}$ is

$$u_{\mathrm{SO}(3)}(A) = \left.\frac{d}{dt}\right|_{t=0} A\exp(tu\widehat{a}) = uA\widehat{a}. \tag{4.4.2}$$

By right invariance of the metric and the identity $A\widehat{a}A^{-1} = \widehat{Aa}$, we get for any $u, v \in \mathbb{R}$,

$$\begin{aligned} \langle\!\langle \mathbb{I}(A)u\widehat{a}, v\widehat{a} \rangle\!\rangle &= uv\,\langle\!\langle A\widehat{a}, A\widehat{a} \rangle\!\rangle_A = uv\,\left\langle\!\!\left\langle A\widehat{a}A^{-1}, A\widehat{a}A^{-1} \right\rangle\!\!\right\rangle_I \\ &= uv\left\langle\!\!\!\left\langle \widehat{Aa}, \widehat{Aa} \right\rangle\!\!\!\right\rangle_I = uv\,\langle\!\langle Aa, Aa \rangle\!\rangle = \langle\!\langle ua, va \rangle\!\rangle\,. \end{aligned} \tag{4.4.3}$$

To identify from this formula the locked inertia tensor $\mathbb{I}(A) : \operatorname{span}\{\widehat{a}\} \to \operatorname{span}\{a\}^*$ as a linear map from $\operatorname{span}\{\widehat{a}\}$ to $\mathbb{R}^3$ and to determine its one-dimensional range, we will make use of the isomorphism $\bar{\mu} \in \mathfrak{so}(3)^* \mapsto \mu \in \mathbb{R}^3$ given by $\langle \bar{\mu}, \widehat{x} \rangle = \langle\!\langle \mu, x \rangle\!\rangle$ for any $x \in \mathbb{R}^3$.

The projection $\mathbb{R}^3 \to \operatorname{span}\{a\}$ is given by $x \mapsto \frac{\langle\!\langle x,a \rangle\!\rangle}{\|a\|^2}a$ and composing it with the isomorphism $\widehat{x} \in \mathfrak{so}(3) \mapsto x \in \mathbb{R}^3$ gives the projection $\widehat{x} \in \mathfrak{so}(3) \mapsto \frac{\langle\!\langle x,a \rangle\!\rangle}{\|a\|^2}a \in \operatorname{span}\{a\}$. The dual $\operatorname{span}\{a\}^* \to \mathfrak{so}(3)^*$ of this map composed with the isomorphism $\bar{\mu} \in \mathfrak{so}(3)^* \mapsto \mu \in \mathbb{R}^3$ gives the embedding $\kappa : \varphi \in \operatorname{span}\{a\}^* \mapsto \frac{\langle \varphi, \widehat{a} \rangle}{\|a\|^2}a \in \operatorname{span}\{a\} \subset \mathbb{R}^3$. This isomorphism κ which identifies $\operatorname{span}\{a\}^*$ with $\operatorname{span}\{a\}$ is thus characterized by

$$\langle\!\langle \kappa(\varphi), a \rangle\!\rangle = \langle \varphi, \widehat{a} \rangle\,. \tag{4.4.4}$$

Thus, by (4.4.3), we get

$$\langle\!\langle \kappa\left(\mathbb{I}(A)u\widehat{a}\right), a \rangle\!\rangle = \langle\!\langle \mathbb{I}(A)u\widehat{a}, \widehat{a} \rangle\!\rangle = \langle\!\langle ua, a \rangle\!\rangle\,.$$

Therefore, identifying via κ the spaces $\operatorname{span}\{a\}^*$ and $\operatorname{span}\{a\}$, formula (4.4.3) shows that $\mathbb{I}(A) : \operatorname{span}\{\widehat{a}\} \to \operatorname{span}\{a\}$ is given by

$$\mathbb{I}(A)\widehat{a} = a. \tag{4.4.5}$$

Taking $u \in \mathbb{R}$, the $\mathrm{SO}(3)_a$-momentum map $\mathbf{J} : T^* \mathrm{SO}(3) \to \mathrm{span}\{a\}^*$ is given by

$$\begin{aligned}\left\langle \mathbf{J}\left(\langle\!\langle X_A, \cdot \rangle\!\rangle_A\right), u\widehat{a} \right\rangle &= \left\langle\!\left\langle X_A, u_{\mathrm{SO}(3)}(A) \right\rangle\!\right\rangle_A = u \left\langle\!\left\langle X_A, A\widehat{a} \right\rangle\!\right\rangle_A \\ &= u \left\langle\!\left\langle X_A \cdot A^{-1}, A\widehat{a}A^{-1} \right\rangle\!\right\rangle_I \\ &= u \left\langle\!\left\langle \mathrm{Ad}_{A^{-1}}(X_A \cdot A^{-1}), \widehat{a} \right\rangle\!\right\rangle_I \\ &= u \left\langle\!\left\langle A^{-1} \cdot X_A, \widehat{a} \right\rangle\!\right\rangle_I \\ &= \left(\frac{1}{\|a\|^2} \left\langle\!\left\langle A^{-1} \cdot X_A, \widehat{a} \right\rangle\!\right\rangle_I a, ua \right),\end{aligned}$$

so that, identifying $\mathrm{span}\{a\}^*$ with $\mathrm{span}\{a\}$ via the map κ, equation (4.4.4) gives

$$\mathbf{J}\left(\langle\!\langle X_A, \cdot \rangle\!\rangle_A\right) = \frac{1}{\|a\|^2} \left\langle\!\left\langle A^{-1} \cdot X_A, \widehat{a} \right\rangle\!\right\rangle_I a \in \mathrm{span}\{a\}. \tag{4.4.6}$$

Therefore, by (4.4.5) and (4.4.6), the mechanical $\mathrm{span}\{\widehat{a}\}$-valued connection one-form has the expression

$$\begin{aligned}\mathcal{A}^{\mathrm{SO}(3)_a}(A)(X_A) &:= \left(\mathbb{I}(A)^{-1} \circ \mathbf{J}\right)\left(\langle\!\langle X_A, \cdot \rangle\!\rangle_A\right) \\ &= \frac{1}{\|a\|^2} \left\langle\!\left\langle A^{-1} \cdot X_A, \widehat{a} \right\rangle\!\right\rangle_I \widehat{a}.\end{aligned} \tag{4.4.7}$$

If $\mu \in \mathbb{R}^3$, then $\bar{\mu}|_{\mathrm{span}\{a\}} \in \mathrm{span}\{a\}^*$ and hence $\kappa\left(\bar{\mu}|_{\mathrm{span}\{a\}}\right) = \frac{\langle\!\langle \mu, a \rangle\!\rangle}{\|a\|^2} a \in \mathrm{span}\{a\}$, which says that if we identify $\mathrm{span}\{a\}^*$ with $\mathrm{span}\{a\}$ via κ then $\mu_a = \frac{\langle\!\langle \mu, a \rangle\!\rangle}{\|a\|^2} a \in \mathrm{span}\{a\}$.

From (4.4.7) we see that if $\mu \in \mathbb{R}^3$, the μ_a-component $\mathcal{A}^{\mathrm{SO}(3)_a}_{\mu_a}$ of $\mathcal{A}^{\mathrm{SO}(3)_a}$ is given by

$$\begin{aligned}\mathcal{A}^{\mathrm{SO}(3)_a}_{\mu_a}(A)(X_A) &= \left\langle \bar{\mu}|_{\mathrm{span}\{\widehat{a}\}}, \mathcal{A}^{\mathrm{SO}(3)_a}(A)(X_A) \right\rangle \\ &= \left\langle \bar{\mu}, \frac{1}{\|a\|^2} \left\langle\!\left\langle A^{-1} \cdot X_A, \widehat{a} \right\rangle\!\right\rangle_I \widehat{a} \right\rangle \\ &= \frac{\langle\!\langle \mu, a \rangle\!\rangle}{\|a\|^2} \left\langle\!\left\langle A^{-1} \cdot X_A, \widehat{a} \right\rangle\!\right\rangle_I.\end{aligned} \tag{4.4.8}$$

To find the magnetic term we need to compute $\mathbf{d}\mathcal{A}^{\mathrm{SO}(3)_a}_{\mu_a}(A)(X_A, Y_A)$ for $A \in \mathrm{SO}(3)$ and $X_A, Y_A \in T_A \mathrm{SO}(3)$. Let $X_A = \widehat{x} \cdot A, Y_A = \widehat{y} \cdot A \in T_A \mathrm{SO}(3)$. Denote by $\bar{X}, \bar{Y}$ the right invariant vector fields whose values at I are $\widehat{x}$ and $\widehat{y}$ respectively. Then (4.4.7) and the identities $A^{-1}\widehat{x}A = \widehat{A^{-1}x}$, $\langle\!\langle \widehat{x}, \widehat{y} \rangle\!\rangle_I =$

$\langle\langle x, y \rangle\rangle$, imply

$$\begin{aligned} Y_A \left[\mathcal{A}^{\mathrm{SO}(3)_a}_{\mu_a}(\bar{X}) \right] &= \left. \frac{d}{dt} \right|_{t=0} \mathcal{A}^{\mathrm{SO}(3)_a}_{\mu_a}(\bar{X})((\exp t\widehat{y})A) \\ &= \left. \frac{d}{dt} \right|_{t=0} \mathcal{A}^{\mathrm{SO}(3)_a}_{\mu_a}((\exp t\widehat{y})A)(\widehat{x} \cdot (\exp t\widehat{y})A) \\ &= \left. \frac{d}{dt} \right|_{t=0} \frac{\langle\langle \mu, a \rangle\rangle}{\|a\|^2} \left\langle\left\langle A^{-1} \exp(-t\widehat{y})\widehat{x}(\exp t\widehat{y})A,\, \widehat{a} \right\rangle\right\rangle_I \\ &= \left. \frac{d}{dt} \right|_{t=0} \frac{\langle\langle \mu, a \rangle\rangle}{\|a\|^2} \left\langle\left\langle A^{-1} \exp(-t\widehat{y})x, a \right\rangle\right\rangle \\ &= \frac{\langle\langle \mu, a \rangle\rangle}{\|a\|^2} \left\langle\left\langle A^{-1}(x \times y), a \right\rangle\right\rangle . \end{aligned} \tag{4.4.9}$$

Similarly

$$X_A \left[\mathcal{A}^{\mathrm{SO}(3)_a}_{\mu_a}(\bar{Y}) \right] = -\frac{\langle\langle \mu, a \rangle\rangle}{\|a\|^2} \left\langle\left\langle A^{-1}(x \times y), a \right\rangle\right\rangle . \tag{4.4.10}$$

Finally, since $[\bar{X}, \bar{Y}])(A) = -[\widehat{x}, \widehat{y}] \cdot A$ (because $\bar{X}, \bar{Y}$ are *right* invariant vector fields), formula (4.4.7) yields

$$\begin{aligned} \mathcal{A}^{\mathrm{SO}(3)_a}_{\mu_a}\left(\left[\bar{X}, \bar{Y} \right] \right)(A) &= \mathcal{A}^{\mathrm{SO}(3)_a}_{\mu_a}(A)(-[\widehat{x}, \widehat{y}] \cdot A) \\ &= -\frac{\langle\langle \mu, a \rangle\rangle}{\|a\|^2} \left\langle\left\langle A^{-1}[\widehat{x}, \widehat{y}]A, \widehat{a} \right\rangle\right\rangle_I \\ &= -\frac{\langle\langle \mu, a \rangle\rangle}{\|a\|^2} \left\langle\left\langle A^{-1}(x \times y), a \right\rangle\right\rangle . \end{aligned} \tag{4.4.11}$$

Formulas (4.4.9), (4.4.10), and (4.4.11) therefore give

$$\begin{aligned} &\mathbf{d}\mathcal{A}^{\mathrm{SO}(3)_a}_{\mu_a}(A)(X_A, Y_A) \\ &\quad = X_A \left[\mathcal{A}^{\mathrm{SO}(3)_a}_{\mu_a}(\bar{Y}) \right] - Y_A \left[\mathcal{A}^{\mathrm{SO}(3)_a}_{\mu_a}(\bar{X}) \right] - \mathcal{A}^{\mathrm{SO}(3)_a}_{\mu_a}\left(\left[\bar{X}, \bar{Y} \right] \right)(A) \\ &\quad = -\frac{\langle\langle \mu, a \rangle\rangle}{\|a\|^2} \left\langle\left\langle A^{-1}(x \times y), a \right\rangle\right\rangle , \end{aligned} \tag{4.4.12}$$

where $X_A = \widehat{x} \cdot A$, $Y_A = \widehat{y} \cdot A \in T_A\,\mathrm{SO}(3)$. Note that this equation agrees with the result of Theorem 4.3.3.

This two-form on SO(3) clearly induces a two-form $\mathcal{B}_{\mu_a}$, the magnetic term, on the sphere S^2_a by

$$\mathcal{B}_{\mu_a}(Aa)(x \times Aa, y \times Aa) = -\frac{\langle\langle \mu, a \rangle\rangle}{\|a\|^2} \langle\langle x \times y, Aa \rangle\rangle . \tag{4.4.13}$$

Invoking the cotangent bundle reduction theorem we classify the orbits of SE(3) as follows.

4.4.1 Theorem. *The coadjoint orbits of* $\mathrm{SE}(3)$ *are of the following types.*

- $\mathcal{O}_{(a=0,\mu)} \simeq (S^2_\mu, \omega_\mu)$
- $\mathcal{O}_{(a\neq 0,\mu=0)} \simeq (T^*S^2_a, \Omega_{\mathrm{can}})$
- $\mathcal{O}_{(a\neq 0,\mu\neq 0)} \simeq (T^*S^2_a, \Omega_{\mathrm{can}} - \pi^*\mathcal{B}_{\mu_a})$

where ω_μ *is the orbit symplectic form on the sphere* S^2_μ *of radius* $\|\mu\|$, $\mu_a = \frac{\langle\langle \mu, a\rangle\rangle}{\|a\|^2} a \in \mathrm{span}\{a\}$ *is the orthogonal projection of* μ *to* $\mathrm{span}\{a\}$, $\pi : T^*S^2_a \to S^2_a$ *is the cotangent bundle projection,* Ω_{can} *is the canonical symplectic structure on* $T^*S^2_a$*, and the two-form* $\mathcal{B}_{\mu_a}$ *on the sphere* S^2_a *of radius* $\|a\|$ *is given by formula* (4.4.13).