14

Optimal Orbit Reduction

As we already pointed out the main difference between the point and orbit reduced spaces is the invariance properties of the submanifolds out of which they are constructed. More specifically, if we mimic in the optimal context the standard orbit reduction procedure, the optimal orbit reduced space that we should study is $G \cdot \mathcal{J}^{-1}(\rho)/G = \mathcal{J}^{-1}(\mathcal{O}_\rho)/G$, where $\mathcal{O}_\rho := G \cdot \rho \subset M/A'_G$. The following pages constitute an in–depth study of this quotient and its relation with new (pre)–symplectic manifolds that can be used to reproduce the classical orbit reduction program and expressions.

14.1 The Space for Optimal Orbit Reduction

The first question that we have to tackle is: is there a canonical smooth structure for $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ and $\mathcal{J}^{-1}(\mathcal{O}_\rho)/G$ that we can use to carry out the orbit reduction scheme in this framework?

We will first show that there is an affirmative answer for the smooth structure of $\mathcal{J}^{-1}(\mathcal{O}_\rho)$. The main idea that we will prove in the following paragraphs is that *$\mathcal{J}^{-1}(\mathcal{O}_\rho)$ can be naturally endowed with the unique smooth structure that makes it into an initial submanifold of M*. We start with the following proposition.

14.1.1 Proposition. *Let $(M, \{\cdot,\cdot\})$ be a smooth Poisson manifold and G be a Lie group acting canonically and properly on M. Let $\mathcal{J} : M \to M/A'_G$ be the optimal momentum map associated to this action. Then,* (i). *The generalized distribution D on M defined by $D(m) := \mathfrak{g} \cdot m + A'_G(m)$, for all $m \in M$, is integrable.*

(ii). *Let $m \in M$ be such that $\mathcal{J}(m) = \rho$, then $G^0 \cdot \mathcal{J}^{-1}(\rho)$ is the maximal integral submanifold of D going through the point m. The symbol G^0 denotes the connected component of G containing the identity.*

Proof. (i). The distribution D can be written as the span of globally defined vector fields on M, that is,

$$D = \operatorname{span}\{\xi_M,\, X_f \mid \xi \in \mathfrak{g} \text{ and } f \in C^\infty(M)^G\}. \tag{14.1.1}$$

By the Frobenius–Stefan–Sussman Theorem (see Stefan [1974a,b] and Sussman [1973]), the integrability of D can be proved by showing that this distribution is invariant by the flows of the vector fields in (14.1.1) that we used to generate it. Let $f, l \in C^\infty(M)^G$, $\xi, \eta \in \mathfrak{g}$, F_t be the flow of X_l, and H_t be the flow of η_M. Recall that η_M is a complete vector field such that $H_t(m) = \exp t\eta \cdot m$, for all $t \in \mathbb{R}$ and $m \in M$. Now, the integrability of A'_G guarantees that $T_mF_t \cdot X_f(m) \in A'_G(F_t(m)) \subset D(F_t(m))$. Also, the G–equivariance of F_t and the invariance of the function f imply that $T_mF_t \cdot \xi_M(m) = \xi_M(F_t(m))$ and $T_mH_t \cdot X_f(m) = X_f(H_t(m))$. Finally,

$$\begin{aligned} T_mH_t \cdot \xi_M(m) &= \left.\frac{d}{ds}\right|_{s=0} \exp t\eta \exp s\xi \cdot m \\ = \left.\frac{d}{ds}\right|_{s=0} &\exp t\eta \exp s\xi \exp -t\eta \exp t\eta \cdot m = (\mathrm{Ad}_{\exp t\eta}\, \xi)_M(\exp t\eta \cdot m), \end{aligned}$$

which proves that D is integrable.
(ii) As D is integrable and is generated by the vector fields (14.1.1), its maximal integral submanifolds coincide with the orbits of the action of the pseudogroup constructed by finite composition of flows of the vector fields in (14.1.1), that is, for any $m \in M$, the integral leaf $\mathcal{L}_m$ of D that goes through m is:

$$\mathcal{L}_m = \{F_{t_1} \circ \cdots \circ F_{t_n}(m) \mid \text{with } F_{t_i} \text{ the flow of a vector field in (14.1.1)}\}.$$

Given that $[X_f, \xi_M] = 0$ for all $f \in C^\infty(M)^G$ and $\xi \in \mathfrak{g}$, the previous expression can be rewritten as

$$\begin{aligned} \mathcal{L}_m = \{H_{t_1} \circ \cdots \circ H_{t_j} \circ G_{s_1} \circ \cdots \circ G_{s_k}(m) \\ \mid G_{s_i} \text{ flow of } f_i \in C^\infty(M)^G, \text{ and } H_{t_i} \text{ flow of } \xi^i_M,\, \xi^i \in \mathfrak{g}\}. \end{aligned}$$

Therefore, $\mathcal{L}_m = G^0 \cdot \mathcal{J}^{-1}(\rho)$, as required. ■

As we already said, a general fact about integrable generalized distributions Dazord [1985] states that the smooth structure on a subset of M that makes it into a maximal integral manifold of a given distribution coincides with the *unique* smooth structure that makes it into an initial submanifold of M. Therefore, the previous proposition shows that the sets $G^0 \cdot \mathcal{J}^{-1}(\rho)$ are initial submanifolds of M.

14.1.2 Proposition. *Suppose that we have the same setup as in Proposition 14.1.1. If either G_ρ is closed in G or, more generally, G_ρ acts properly on $\mathcal{J}^{-1}(\rho)$, then:*

(i) *The G_ρ action on the product $G \times \mathcal{J}^{-1}(\rho)$ defined by $h \cdot (g, z) := (gh, h^{-1} \cdot z)$ is free and proper and therefore, the corresponding orbit space $G \times \mathcal{J}^{-1}(\rho)/G_\rho =: G \times_{G_\rho} \mathcal{J}^{-1}(\rho)$ is a smooth regular quotient manifold. We will denote by $\pi_{G_\rho} : G \times \mathcal{J}^{-1}(\rho) \to G \times_{G_\rho} \mathcal{J}^{-1}(\rho)$ the canonical surjective submersion.*

(ii) *The mapping $i : G \times_{G_\rho} \mathcal{J}^{-1}(\rho) \to M$ defined by $i([g, z]) := g \cdot z$ is an injective immersion onto $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ such that, for any $[g, z] \in G \times_{G_\rho} \mathcal{J}^{-1}(\rho)$, $T_{[g,z]}i \cdot T_{[g,z]}(G \times_{G_\rho} \mathcal{J}^{-1}(\rho)) = D(g \cdot z)$. On other words $i(G \times_{G_\rho} \mathcal{J}^{-1}(\rho)) = \mathcal{J}^{-1}(\mathcal{O}_\rho)$ is an integral submanifold of D.*

Proof. (i). It is easy to check that G_ρ is closed in G if and only if the action of G_ρ on G by right translations is proper. Additionally, if G_ρ is closed in G then the G_ρ–action on $\mathcal{J}^{-1}(\rho)$ is proper. In any case, if the action of G_ρ on either G, or on $\mathcal{J}^{-1}(\rho)$, or on both, is proper, so is the action on the product $G \times \mathcal{J}^{-1}(\rho)$ in the statement of the proposition. As to the freeness, it is inherited from the freeness of the G_ρ–action on G.
(ii). First of all, the map i is clearly well-defined and smooth since it is the projection onto the orbit space $G \times_{G_\rho} \mathcal{J}^{-1}(\rho)$ of the G_ρ–invariant smooth map $G \times \mathcal{J}^{-1}(\rho) \to M$ given by $(g, z) \longmapsto g \cdot z$. It is also injective because if $[g, z], [g', z'] \in G \times_{G_\rho} \mathcal{J}^{-1}(\rho)$ are such that $i([g, z]) = i([g', z'])$, then $g \cdot z = g' \cdot z'$ or, analogously, $g^{-1}g' \cdot z' = z$, which implies that $g^{-1}g' \in G_\rho$. Consequently, $[g, z] = [gg^{-1}g', (g')^{-1}g \cdot z] = [g', z']$, as required.

Finally, we check that i is an immersion. Let $[g, z] \in G \times_{G_\rho} \mathcal{J}^{-1}(\rho)$ arbitrary and $\xi \in \mathfrak{g}$, $f \in C^\infty(M)^G$ be such that

$$T_{[g,z]}i \cdot T_{(g,z)}\pi_{G_\rho} \cdot (T_e L_g(\xi), X_f(z)) = 0.$$

If we denote by F_t the flow of X_f we can rewrite this equality as

$$\left.\frac{d}{dt}\right|_{t=0} g \exp t\xi \cdot F_t(z) = 0 \quad \text{or equivalently,} \quad T_z\Phi_g(X_f(z) + \xi_M(z)) = 0.$$

Hence $X_f(z) = -\xi_M(z)$ which by (13.1.7) implies that $\xi \in \mathfrak{g}_\rho$ and therefore $T_{(g,z)}\pi_{G_\rho} \cdot (T_e L_g(\xi), X_f(z)) = T_{(g,z)}\pi_{G_\rho} \cdot (T_e L_g(\xi), -\xi_M(z)) = 0$, as required.

Given that for any $\xi \in \mathfrak{g}$, $f \in C^\infty(M)^G$, and $[g, z] \in G \times_{G_\rho} \mathcal{J}^{-1}(\rho)$ we see that $T_{[g,z]}i \cdot T_{(g,z)}\pi_{G_\rho} \cdot (T_e L_g(\xi), X_f(z)) = (\mathrm{Ad}_g\, \xi)_M(g \cdot z) + X_f(g \cdot z)$, it is clear that $T_{[g,z]}i \cdot T_{[g,z]}(G \times_{G_\rho} \mathcal{J}^{-1}(\rho)) = D(g \cdot z)$ and thereby $i(G \times_{G_\rho} \mathcal{J}^{-1}(\rho)) = \mathcal{J}^{-1}(\mathcal{O}_\rho)$ is an integral submanifold of D. ■

By using the previous propositions we will now show that, in the presence of the standard hypotheses for reduction, $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ is an initial submanifold of M whose connected components are the also initial submanifolds $gG^0 \cdot \mathcal{J}^{-1}(\rho)$, $g \in G$. We start with the following definition:

14.1.3 Definition. *Let $(M, \{\cdot, \cdot\})$ be a smooth Poisson manifold and G be a Lie group acting canonically and properly on M. Let $\mathcal{J} : M \to M/A'_G$ be the optimal momentum map associated to this action and $\rho \in M/A'_G$. Suppose that G_ρ acts properly on $\mathcal{J}^{-1}(\rho)$. In these circumstances, by Proposition 14.1.2, the twist product $G \times_{G_\rho} \mathcal{J}^{-1}(\rho)$ has a canonical smooth structure. Consider in the set $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ the smooth structure that makes the bijection $G \times_{G_\rho} \mathcal{J}^{-1}(\rho) \to \mathcal{J}^{-1}(\mathcal{O}_\rho)$ given by $(g, z) \to g \cdot z$ into a diffeomorphism. We will refer to this structure as the **initial smooth structure** of $\mathcal{J}^{-1}(\mathcal{O}_\rho)$.*

The following theorem justifies the choice of terminology in the previous definition and why we will be able to refer to the smooth structure there introduced as *THE* initial smooth structure of $\mathcal{J}^{-1}(\mathcal{O}_\rho)$.

14.1.4 Theorem. *Suppose that we are in the same setup as in Definition 14.1.3. Then, the set $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ endowed with the initial smooth structure is an actual initial submanifold of M that can be decomposed as a disjoint union of connected components as*

$$\mathcal{J}^{-1}(\mathcal{O}_\rho) = \dot{\bigcup_{[g] \in G/(G^0 G_\rho)}} gG^0 \cdot \mathcal{J}^{-1}(\rho). \tag{14.1.2}$$

Each connected component of $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ is a maximal integral submanifold of the distribution D defined in Proposition 14.1.1. If, additionally, the subgroup G_ρ is closed in G, the topology on $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ induced by its initial smooth structure coincides with the initial topology induced by the map $\mathcal{J}_{\mathcal{J}^{-1}(\mathcal{O}_\rho)} : \mathcal{J}^{-1}(\mathcal{O}_\rho) \to \mathcal{O}_\rho$ given by $z \longmapsto \mathcal{J}(z)$, where the orbit $\mathcal{O}_\rho$ is endowed with the smooth structure coming from the homogeneous manifold G/G_ρ. Finally, notice that (14.1.2) implies that $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ has as many connected components as the cardinality of the homogeneous manifold $G/(G^0 G_\rho)$.

Proof. First of all notice that the sets $gG^0 \cdot \mathcal{J}^{-1}(\rho)$ are clearly maximal integral submanifolds of D by part (ii) in Proposition 14.1.1. As a corollary of this, they are the connected components of $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ endowed with the smooth structure in Definition 14.1.4. Indeed, let S be the connected component of $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ that contains $gG^0 \cdot \mathcal{J}^{-1}(\rho)$, that is, $gG^0 \cdot \mathcal{J}^{-1}(\rho) \subset S \subset \mathcal{J}^{-1}(\mathcal{O}_\rho)$. As $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ is a manifold, it is locally connected, and therefore its connected components are open and closed. In particular, since S is an open connected subset of $\mathcal{J}^{-1}(\mathcal{O}_\rho)$, part (ii) in Proposition 14.1.2 shows that S is a connected integral submanifold of D. By the maximality of $gG^0 \cdot \mathcal{J}^{-1}(\rho)$ as an integral submanifold of D,

$gG^0 \cdot \mathcal{J}^{-1}(\rho) = S$, necessarily. The set $gG^0 \cdot \mathcal{J}^{-1}(\rho)$ is therefore a connected component of $\mathcal{J}^{-1}(\mathcal{O}_\rho)$. As it is a leaf of a smooth integrable distribution on M, it is also an initial submanifold of M Dazord [1985] of dimension $d = \dim \mathcal{J}^{-1}(\mathcal{O}_\rho) = \dim G + \dim \mathcal{J}^{-1}(\rho) - \dim G_\rho$.

We now show that $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ with the smooth structure in Definition 14.1.4 is an initial submanifold of M. First of all part (ii) in Proposition 14.1.2 shows that $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ is an injectively immersed submanifold of M. The initial character can be obtained as a consequence of the fact that its connected components are initial together with the following elementary lemma:

14.1.5 Lemma. *Let N be an injectively immersed submanifold of the smooth manifold M. Suppose that N can be written as the disjoint union of a family $\{S_\alpha\}_{\alpha \in I}$ of open subsets of N such that each S_α is an initial submanifold of M. Then, N is initial.*

Proof. Let $i_N : N \hookrightarrow M$ and $i_\alpha : S_\alpha \hookrightarrow N$ be the injections. Let Z be an arbitrary smooth manifold and $f : Z \to M$ be a smooth map such that $f(Z) \subset N$. As the sets S_α are open and partition N, the manifold Z can be written as a disjoint union of open sets $Z_\alpha := f^{-1}(S_\alpha)$, that is

$$Z = \dot{\bigcup_{\alpha \in I}} f^{-1}(S_\alpha).$$

Given that for each index α the map $f_\alpha : Z_\alpha \to M$ obtained by restriction of f to Z_α is smooth, the corresponding map $\bar{f}_\alpha : Z_\alpha \to S_\alpha$ defined by the identity $i_\alpha \circ \bar{f}_\alpha = f_\alpha$ is also smooth by the initial character of S_α. Let $\bar{f} : Z \to N$ be the map obtained by union of the mappings $\bar{f}_\alpha$. This map is smooth and satisfies that $i_N \circ \bar{f} = f$ which proves that N is initial. $\blacktriangledown$

We now prove Expression (14.1.2). First of all notice that as G^0 is normal in G, the set $G^0 G_\rho$ is a (possibly non-closed) subgroup of G. We obviously have that

$$\mathcal{J}^{-1}(\mathcal{O}_\rho) = \bigcup_{g \in G} gG^0 \mathcal{J}^{-1}(\rho). \tag{14.1.3}$$

Moreover, if g and $g' \in G$ are such that $[g] = [g'] \in G/(G^0 G_\rho)$ then we can write that $g' = ghk$ with $h \in G^0$ and $k \in G_\rho$. Consequently, $g' G^0 \mathcal{J}^{-1}(\rho) = ghkG^0 \mathcal{J}^{-1}(\rho) = gh(G^0 k)\mathcal{J}^{-1}(\rho) = g(hG^0)(k\mathcal{J}^{-1}(\rho)) = gG^0 \mathcal{J}^{-1}(\rho)$, which implies that (14.1.3) can be refined to

$$\mathcal{J}^{-1}(\mathcal{O}_\rho) = \bigcup_{[g] \in G/(G^0 G_\rho)} gG^0 \mathcal{J}^{-1}(\rho). \tag{14.1.4}$$

It only remains to be shown that this union is disjoint: let $gh \cdot z = lh' \cdot z'$ with $h, h' \in G^0$ and $z, z' \in \mathcal{J}^{-1}(\rho)$. If we apply $\mathcal{J}$ to both sides of this equality we obtain that $gh \cdot \rho = lh' \cdot \rho$. Hence, $(h')^{-1} l^{-1} gh \in G_\rho$ and

$l^{-1}g \in h'G_\rho h^{-1} \subset G^0 G_\rho$. This implies that $[l] = [g] \in G/(G^0 G_\rho)$ and $gG^0 \mathcal{J}^{-1}(\rho) = lG^0 \mathcal{J}^{-1}(\rho)$, as required.

We finally show that when G_ρ is closed in G, the topology on $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ induced by its initial smooth structure coincides with the initial topology induced by the map $\mathcal{J}_{\mathcal{J}^{-1}(\mathcal{O}_\rho)} : \mathcal{J}^{-1}(\mathcal{O}_\rho) \to \mathcal{O}_\rho$ on $\mathcal{J}^{-1}(\mathcal{O}_\rho)$. Recall first that this topology is characterized by the fact that for any topological space Z and any map $\phi : Z \to \mathcal{J}^{-1}(\mathcal{O}_\rho)$ the map $\phi : Z \to \mathcal{J}^{-1}(\mathcal{O}_\rho)$ is continuous if and only if $\mathcal{J}_{\mathcal{J}^{-1}(\mathcal{O}_\rho)} \circ \phi$ is continuous. Moreover, as the family

$$\{\mathcal{J}^{-1}_{\mathcal{J}^{-1}(\mathcal{O}_\rho)}(U) \mid U \text{ open subset of } \mathcal{O}_\rho\}$$

is a subbase of this topology, the initial topology on $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ induced by the map $\mathcal{J}_{\mathcal{J}^{-1}(\mathcal{O}_\rho)}$ is first countable. We prove that this topology coincides with the topology induced by the initial smooth structure on $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ by showing that the map

$$f : G \times_{G_\rho} \mathcal{J}^{-1}(\rho) \to \mathcal{J}^{-1}(\mathcal{O}_\rho), \quad \text{where} \quad f([g, z]) := g \cdot z$$

is a homeomorphism when we consider $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ as a topological space with the initial topology induced by $\mathcal{J}_{\mathcal{J}^{-1}(\mathcal{O}_\rho)}$. Indeed, f is continuous if and only if the map $G \times_{G_\rho} \mathcal{J}^{-1}(\rho) \to \mathcal{O}_\rho$ given by $[g, z] \mapsto g \cdot \rho$ is continuous, which in turn is equivalent to the continuity of the map $G \times \mathcal{J}^{-1}(\rho) \to G/G_\rho$ defined by $(g, z) \longmapsto gG_\rho$, which is true. We now show that the inverse

$$f^{-1} : \mathcal{J}^{-1}(\mathcal{O}_\rho) \to G \times_{G_\rho} \mathcal{J}^{-1}(\rho)$$

of f given by $g \cdot z \mapsto [g, z]$ is continuous. Since the initial topology on $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ induced by $\mathcal{J}_{\mathcal{J}^{-1}(\mathcal{O}_\rho)}$ is first countable it suffices to show that for any convergent sequence $\{z_n\} \subset \mathcal{J}^{-1}(\mathcal{O}_\rho) \to z \in \mathcal{J}^{-1}(\mathcal{O}_\rho)$, we have

$$\lim_{n\to\infty} f^{-1}(z_n) = f^{-1}(\lim_{n\to\infty} z_n) = f^{-1}(z).$$

Indeed, as $\mathcal{J}_{\mathcal{J}^{-1}(\mathcal{O}_\rho)}$ is continuous, the sequence $\{\mathcal{J}(z_n) = g_n \cdot \rho\} \subset \mathcal{O}_\rho$ converges in $\mathcal{O}_\rho$ to $\mathcal{J}(z) = g \cdot \rho$, for some $g \in G$. Let $j : \mathcal{O}_\rho \to G/G_\rho$ be the standard diffeomorphism and $\sigma : U_{gG_\rho} \subset G/G_\rho \to G$ be a local smooth section of the submersion $G \to G/G_\rho$ in a neighborhood U_{gG_ρ} of $gG_\rho \in G/G_\rho$. Let $V := \mathcal{J}^{-1}_{\mathcal{J}^{-1}(\mathcal{O}_\rho)}(j^{-1}(U_{gG_\rho}))$. V is an open neighborhood of z in $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ because

$$j \circ \mathcal{J}_{\mathcal{J}^{-1}(\mathcal{O}_\rho)}(z) = j(g \cdot \rho) = gG_\rho \in U_{gG_\rho}.$$

We now notice that for any $m \in V$ we can write

$$f^{-1}(m) = [\sigma \circ j \circ \mathcal{J}_{\mathcal{J}^{-1}(\mathcal{O}_\rho)}(m), (\sigma \circ j \circ \mathcal{J}_{\mathcal{J}^{-1}(\mathcal{O}_\rho)}(m))^{-1} \cdot m].$$

Consequently, since

$$\begin{aligned}\lim_{n\to\infty} f^{-1}(z_n) &= \lim_{n\to\infty} [\sigma \circ j \circ \mathcal{J}_{\mathcal{J}^{-1}(\mathcal{O}_\rho)}(z_n), (\sigma \circ j \circ \mathcal{J}_{\mathcal{J}^{-1}(\mathcal{O}_\rho)}(z_n))^{-1} \cdot z_n] \\ &= [\sigma \circ j \circ \mathcal{J}_{\mathcal{J}^{-1}(\mathcal{O}_\rho)}(z), (\sigma \circ j \circ \mathcal{J}_{\mathcal{J}^{-1}(\mathcal{O}_\rho)}(z))^{-1} \cdot z] = f^{-1}(z),\end{aligned}$$

the continuity of f^{-1} is guaranteed. ■

14.2 The Symplectic Orbit Reduction Quotient

We will know show that the quotient $\mathcal{J}^{-1}(\mathcal{O}_\rho)/G$ can be endowed with a smooth structure that makes it into a regular quotient manifold, that is, the projection $\pi_{\mathcal{O}_\rho} : \mathcal{J}^{-1}(\mathcal{O}_\rho) \to \mathcal{J}^{-1}(\mathcal{O}_\rho)/G$ is a smooth submersion. We will carry this out under the same hypotheses present in Definition 14.1.3, that is, G_ρ acts properly on $\mathcal{J}^{-1}(\rho)$.

First of all notice that as $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ is an initial G–invariant submanifold of M, the G–action on $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ is smooth. We will prove that $\mathcal{J}^{-1}(\mathcal{O}_\rho)/G$ is a regular quotient manifold by showing that this action is actually proper and satisfies that all the isotropy subgroups are conjugate to a given one. Indeed, recall that the initial manifold structure on $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ is the one that makes it G–equivariantly diffeomorphic to the twist product $G \times_{G_\rho} \mathcal{J}^{-1}(\rho)$ when we take in this space the G–action given by the expression $g \cdot [h, z] := [gh, z]$, $g \in G$, $[h, z] \in G \times_{G_\rho} \mathcal{J}^{-1}(\rho)$. Therefore, it suffices to show that this G–action has the desired properties. First of all this action is proper since a general property about twist products (see [HRed]) says that the G–action on $G \times_{G_\rho} \mathcal{J}^{-1}(\rho)$ is proper if and only if the G_ρ–action on $\mathcal{J}^{-1}(\rho)$ is proper, which we supposed as a hypothesis. We now look at the isotropies of this action: in Proposition 13.2.1 we saw that all the elements in $\mathcal{J}^{-1}(\rho)$ have the same G–isotropy, call it H. As $H \subset G_\rho$, this is also their G_ρ–isotropy. Now, using a standard property of the isotropies of twist products (see [HRed]), we have

$$G_{[g,z]} = g(G_\rho)_z g^{-1} = gHg^{-1},$$

for any $[g, z] \in G \times_{G_\rho} \mathcal{J}^{-1}(\rho)$, as required.

The quotient manifold $\mathcal{J}^{-1}(\mathcal{O}_\rho)/G$ is naturally diffeomorphic to the symplectic point reduced space. Indeed,

$$\mathcal{J}^{-1}(\mathcal{O}_\rho)/G \simeq G \times_{G_\rho} \mathcal{J}^{-1}(\rho)/G \simeq \mathcal{J}^{-1}(\rho)/G_\rho.$$

This diffeomorphism can be explicitly implemented as follows. Let $l_\rho : \mathcal{J}^{-1}(\rho) \to \mathcal{J}^{-1}(\mathcal{O}_\rho)$ be the inclusion. As the inclusion $\mathcal{J}^{-1}(\rho) \hookrightarrow M$ is smooth and $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ is initial l_ρ is smooth. Also, since l_ρ is (G_ρ, G) equivariant it drops to a unique smooth map $L_\rho : \mathcal{J}^{-1}(\rho)/G_\rho \to \mathcal{J}^{-1}(\mathcal{O}_\rho)/G$ that makes the following diagram

$$\begin{array}{ccc}
\mathcal{J}^{-1}(\rho) & \xrightarrow{l_\rho} & \mathcal{J}^{-1}(\mathcal{O}_\rho) \\
{\scriptstyle \pi_\rho}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{\mathcal{O}_\rho}} \\
\mathcal{J}^{-1}(\rho)/G_\rho & \xrightarrow{L_\rho} & \mathcal{J}^{-1}(\mathcal{O}_\rho)/G.
\end{array}$$

commutative. L_ρ is a smooth bijection. In order to show that its inverse is also smooth we will think of $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ as $G \times_{G_\rho} \mathcal{J}^{-1}(\rho)$. First of all notice that the projection $G \times \mathcal{J}^{-1}(\rho) \to \mathcal{J}^{-1}(\rho)$ is G_ρ–(anti)equivariant and therefore induces a smooth map $G \times_{G_\rho} \mathcal{J}^{-1}(\rho) \to \mathcal{J}^{-1}(\rho)/G_\rho$ given by $[g, z] \mapsto [z]$, $[g, z] \in G \times_{G_\rho} \mathcal{J}^{-1}(\rho)$. This map is G–invariant and therefore drops to another smooth mapping $G \times_{G_\rho} \mathcal{J}^{-1}(\rho)/G \to \mathcal{J}^{-1}(\rho)/G_\rho$ that coincides with L_ρ^{-1}, the inverse of L_ρ, which is consequently a diffeomorphism.

The orbit reduced space $\mathcal{J}^{-1}(\mathcal{O}_\rho)/G$ can be therefore trivially endowed with a symplectic structure $\omega_{\mathcal{O}_\rho}$ by defining $\omega_{\mathcal{O}_\rho} := (L_\rho^{-1})^* \omega_\rho$. We put together all the facts that we just proved in the following theorem–definition:

14.2.1 Theorem (Optimal Orbit Reduction by Poisson Actions)**.** *Suppose that $(M, \{\cdot, \cdot\})$ is a smooth Poisson manifold and G is a Lie group acting canonically and properly on M. Let $\mathcal{J} : M \to M/A'_G$ be the optimal momentum map associated to this action and $\rho \in M/A'_G$. Suppose that G_ρ acts properly on $\mathcal{J}^{-1}(\rho)$. If we denote $\mathcal{O}_\rho := G \cdot \rho$, then:*

(i) *There is a unique smooth structure on $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ that makes it into an initial submanifold of M.*

(ii) *The G–action on $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ by restriction of the G–action on M is smooth and proper and all its isotropy subgroups are conjugate to a given compact isotropy subgroup of the G–action on M.*

(iii) *The quotient $M_{\mathcal{O}_\rho} := \mathcal{J}^{-1}(\mathcal{O}_\rho)/G$ admits a unique smooth structure that makes the projection $\pi_{\mathcal{O}_\rho} : \mathcal{J}^{-1}(\mathcal{O}_\rho) \to \mathcal{J}^{-1}(\mathcal{O}_\rho)/G$ a surjective submersion.*

(iv) *The quotient $M_{\mathcal{O}_\rho} := \mathcal{J}^{-1}(\mathcal{O}_\rho)/G$ admits a unique symplectic structure $\omega_{\mathcal{O}_\rho}$ that makes it symplectomorphic to the point reduced space M_ρ. We will refer to the pair $(M_{\mathcal{O}_\rho}, \omega_{\mathcal{O}_\rho})$ as the **(optimal) orbit reduced space** of $(M, \{\cdot, \cdot\})$ at $\mathcal{O}_\rho$.*

In this setup we can easily formulate an analog of Theorem 13.5.3.

14.2.2 Theorem (Optimal orbit reduction of G–equivariant Poisson dynamics)**.** *Let $(M, \{\cdot, \cdot\})$ be a smooth Poisson manifold and G be a Lie group acting canonically and properly on M. Let $\mathcal{J} : M \to M/A'_G$ be the optimal momentum map associated and $\rho \in M/A'_G$ be such that G_ρ acts properly on $\mathcal{J}^{-1}(\rho)$. Let $h \in C^\infty(M)^G$ be a G–invariant function on M and X_h be the associated G–equivariant Hamiltonian vector field on M. Then,*

(i) *The flow F_t of X_h leaves $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ invariant, commutes with the G–action, and therefore induces a flow $F_t^{\mathcal{O}_\rho}$ on $M_{\mathcal{O}_\rho}$ uniquely determined by the relation*

$$\pi_{\mathcal{O}_\rho} \circ F_t \circ i_{\mathcal{O}_\rho} = F_t^{\mathcal{O}_\rho} \circ \pi_{\mathcal{O}_\rho},$$

where $i_{\mathcal{O}_\rho} : \mathcal{J}^{-1}(\mathcal{O}_\rho) \hookrightarrow M$ is the inclusion.

(ii) *The flow* $F_t^{\mathcal{O}_\rho}$ *in* $(M_{\mathcal{O}_\rho}, \omega_{\mathcal{O}_\rho})$ *is Hamiltonian with the Hamiltonian function* $h_{\mathcal{O}_\rho} \in C^\infty(M_{\mathcal{O}_\rho})$ *given by the equality* $h_{\mathcal{O}_\rho} \circ \pi_{\mathcal{O}_\rho} = h \circ i_{\mathcal{O}_\rho}$.

(iii) *Let* $k \in C^\infty(M)^G$ *be another* G*–invariant function on* M *and* $\{\cdot,\cdot\}_{\mathcal{O}_\rho}$ *be the Poisson bracket associated to the symplectic form* $\omega_{\mathcal{O}_\rho}$ *on* $M_{\mathcal{O}_\rho}$. *Then,* $\{h,k\}_{\mathcal{O}_\rho} = \{h_{\mathcal{O}_\rho}, k_{\mathcal{O}_\rho}\}_{\mathcal{O}_\rho}$.

We conclude this section with a brief description of the orbit version of the regularized reduced spaces introduced in Definition 13.6.2 for the symplectic case. If we follow the prescription introduced in Section 14.1 using the L^ρ–action on M_H^ρ we are first supposed to study the set $\mathcal{J}_{L^\rho}^{-1}(L^\rho \cdot \sigma)$. The initial smooth structure on this set induced by the twist product $L^\rho \times_{L_\sigma^\rho} \mathcal{J}_{L^\rho}^{-1}(\sigma)$ makes it into an initial submanifold of M_H^ρ. Moreover, if we use the statements in Proposition 13.6.2 it is easy to see that

$$\mathcal{J}_{L^\rho}^{-1}(L^\rho \cdot \sigma) = L^\rho \cdot \mathcal{J}_{L^\rho}^{-1}(\sigma) = N(H)^\rho \cdot \mathcal{J}^{-1}(\rho) = \mathcal{J}^{-1}(\mathcal{N}_\rho),$$

with $\mathcal{N}_\rho := N(H)^\rho \cdot \rho \subset M/A'_G$.

The set $\mathcal{J}_{L^\rho}^{-1}(L^\rho \cdot \sigma) = \mathcal{J}^{-1}(\mathcal{N}_\rho)$ is an embedded submanifold of $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ (since $\mathcal{J}^{-1}(\mathcal{N}_\rho) \simeq N(H)^\rho \times_{G_\rho} \mathcal{J}^{-1}(\rho)$ is embedded in $G \times_{G_\rho} \mathcal{J}^{-1}(\rho) \simeq \mathcal{J}^{-1}(\mathcal{O}_\rho)$). Moreover, a simple diagram chasing shows that the symplectic quotient $(\mathcal{J}_{L^\rho}^{-1}(L^\rho \cdot \sigma)/L^\rho, (\omega|_{M_H^\rho})_{L^\rho\cdot\sigma})$ is naturally symplectomorphic to the orbit reduced space $(\mathcal{J}^{-1}(\mathcal{O}_\rho)/G, \omega_{\mathcal{O}_\rho})$. We will say that $(\mathcal{J}_{L^\rho}^{-1}(L^\rho \cdot \sigma)/L^\rho, (\omega|_{M_H^\rho})_{L^\rho\cdot\sigma})$ is ***an orbit regularization*** of $(\mathcal{J}^{-1}(\mathcal{O}_\rho)/G, \omega_{\mathcal{O}_\rho})$.

We finally show that

$$\mathcal{J}^{-1}(\mathcal{O}_\rho) = \dot{\bigcup}_{[g]\in G/N(H)^\rho} \mathcal{J}^{-1}(\mathcal{N}_{g\cdot\rho}). \tag{14.2.1}$$

The equality is a straightforward consequence of the fact that for any $g \in G$,

$$\begin{aligned} M_{gHg^{-1}}^{g\rho} &= \Phi_g(M_H^\rho), \\ N(gHg^{-1})^{g\rho} &= gN(H)^\rho g^{-1}, \\ \mathcal{J}^{-1}(\mathcal{N}_{g\cdot\rho}) &= gN(H)^\rho \mathcal{J}^{-1}(\rho). \end{aligned}$$

The last relation implies that if $g, g' \in G$ are such that $[g] = [g'] \in G/N(H)^\rho$, then $\mathcal{J}^{-1}(\mathcal{N}_{g\cdot\rho}) = \mathcal{J}^{-1}(\mathcal{N}_{g'\cdot\rho})$. We now show that the union in (14.2.1) is indeed disjoint: let $gn\cdot z \in \mathcal{J}^{-1}(\mathcal{N}_{g\cdot\rho})$ and $g'n'\cdot z' \in \mathcal{J}^{-1}(\mathcal{N}_{g'\cdot\rho})$ be such that $gn \cdot z = g'n' \cdot z'$, with $g, g' \in G$, $n, n' \in N(H)^\rho$, and $z, z' \in \mathcal{J}^{-1}(\rho)$. Since $gn \cdot z = g'n' \cdot z'$, we necessarily have that $G_{gn\cdot z} = G_{g'n'\cdot z'}$ which implies that $gHg^{-1} = g'H(g')^{-1}$, and hence $g^{-1}g' \in N(H)$. We now recall that M_H^ρ is the accessible set going through z or z' of the integrable generalized distribution B'_G defined by

$$B'_G := \operatorname{span}\{X \in \mathfrak{X}(U)^G \mid U \text{ open } G\text{–invariant set in } M\},$$

where the symbol $\mathfrak{X}(U)^G$ denotes the set of G–equivariant vector fields defined on U. Let $\mathcal{B}'_G$ be the pseudogroup of transformations of M consisting of the G–equivariant flows of the vector fields that span B'_G. Now, as the points $n \cdot z, n' \cdot z' \in M_H^\rho$, there exists $\mathcal{F}_T \in \mathcal{B}'_G$ such that $n' \cdot z' = \mathcal{F}_T(n \cdot z)$, hence $(g')^{-1} gn \cdot z = \mathcal{F}_T(n \cdot z)$. Moreover, as any element in M_H^ρ can be written as $\mathcal{G}_T(n \cdot z)$ with $\mathcal{G}_T \in \mathcal{B}'_G$, we have

$$(g')^{-1} g \cdot \mathcal{G}_T(n \cdot z) = \mathcal{G}_T((g')^{-1} gn \cdot z) = \mathcal{G}_T(\mathcal{F}_T(n \cdot z)) \in M_H^\rho,$$

which implies that $(g')^{-1} g \in N(H)^\rho$ and therefore $[g] = [g'] \in G/N(H)^\rho$, as required.

14.3 The Polar Reduced Spaces

As we already pointed out, the standard theory of orbit reduction provides a characterization of the symplectic form of the orbit reduced spaces in terms of the symplectic structures of the corresponding coadjoint orbits that, from the dual pairs point of view, play the role of the symplectic leaves of the Poisson manifold in duality, namely $\mathbf{J}(M) \subset \mathfrak{g}^*$.

We will now show that when the group of Poisson transformations A_G is von Neumann (actually we just need weakly von Neumann), that is, when the diagram

$$(M/G, \{\cdot,\cdot\}_{M/A_G}) \overset{\pi_{A_G}}{\longleftarrow} (M, \{\cdot,\cdot\}) \overset{\mathcal{J}}{\longrightarrow} (M/A'_G, \{\cdot,\cdot\}_{M/A'_G})$$

is a dual pair in the sense of Definition 13.3.3, the classical picture can be reproduced in this context. More specifically, in this section we will show that:

- The symplectic leaves of $(M/A'_G, \{\cdot,\cdot\}_{M/A'_G})$ admit a smooth presymplectic structure that generalizes the Kostant–Kirillov–Souriau symplectic structure in the coadjoint orbits of the dual of a Lie algebra in the sense that they are homogeneous presymplectic manifolds. We will refer to these “generalized coadjoint orbits” as ***polar reduced spaces***.
- The presymplectic structure of the polar reduced spaces is related to the symplectic form of the orbit reduced spaces introduced in the previous section via an equality that holds strong resemblance with the classical expression (13.5.1). Also, it is possible to provide a very explicit characterization of the situations in which the polar reduced spaces are actually symplectic.
- When the manifold M is symplectic, the polar reduced space decomposes as a union of embedded symplectic submanifolds that correspond to the polar reduced spaces of the regularizations of the orbit reduced space. Each of these symplectic manifolds is a homogeneous manifold and we will refer to them as the ***regularized polar reduced subspaces***.

We start with a proposition that spells out the smooth structure of the polar reduced spaces. ***In this section*** we use a stronger hypothesis on G_ρ with respect to the one we used in the previous section, namely, ***we will assume that*** G_ρ ***is closed in*** G which, as we point out in the proof of Proposition 14.1.2, ***implies that the*** G_ρ ***action on*** $\mathcal{J}^{-1}(\rho)$ ***is proper***.

14.3.1 Proposition. *Let* $(M, \{\cdot, \cdot\})$ *be a smooth Poisson manifold and* G *be a Lie group acting canonically and properly on* M*. Let* $\mathcal{J} : M \to M/A'_G$ *be the optimal momentum map associated to this action and* $\rho \in M/A'_G$*. Suppose that* G_ρ *is closed in* G*. Then, the polar distribution* A'_G *restricts to a smooth integrable regular distribution on* $\mathcal{J}^{-1}(\mathcal{O}_\rho)$*, that we will also denote by* A'_G*. The leaf space* $M'_{\mathcal{O}_\rho} := \mathcal{J}^{-1}(\mathcal{O}_\rho)/A'_G$ *admits a unique smooth structure that makes it into a regular quotient manifold and diffeomorphic to the homogeneous manifold* G/G_ρ*. With this smooth structure the projection*

$$\mathcal{J}_{\mathcal{O}_\rho} : \mathcal{J}^{-1}(\mathcal{O}_\rho) \to \mathcal{J}^{-1}(\mathcal{O}_\rho)/A'_G$$

is a smooth surjective submersion. We will refer to $M'_{\mathcal{O}_\rho}$ *as the* ***polar reduced space***.

Proof. Let $m \in \mathcal{J}^{-1}(\mathcal{O}_\rho)$. By Proposition 14.1.2 we have $T_m\mathcal{J}^{-1}(\mathcal{O}_\rho) = D(m) = \mathfrak{g}\cdot m + A'_G(m)$, which implies that the restriction of A'_G to $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ is tangent to it. Consequently, as $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ is an immersed submanifold of M, there exists for each Hamiltonian vector field $X_f \in \mathfrak{X}(M)$, $f \in C^\infty(M)^G$, a vector field $X'_f \in \mathfrak{X}(\mathcal{J}^{-1}(\mathcal{O}_\rho))$ such that

$$Ti_{\mathcal{O}_\rho} \circ X'_f = X_f \circ i_{\mathcal{O}_\rho},$$

with $i_{\mathcal{O}_\rho} : \mathcal{J}^{-1}(\mathcal{O}_\rho) \hookrightarrow M$ the injection. The restriction $A'_G|_{\mathcal{J}^{-1}(\mathcal{O}_\rho)}$ of A'_G to $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ is generated by the vector fields of the form X'_f and it is therefore smooth. It is also integrable since for any point $m = g \cdot z \in \mathcal{J}^{-1}(\mathcal{O}_\rho)$, $z \in \mathcal{J}^{-1}(\rho)$, the embedded submanifold $\mathcal{J}^{-1}(g \cdot \rho)$ of $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ is the maximal integral submanifold of $A'_G|_{\mathcal{J}^{-1}(\mathcal{O}_\rho)}$. This is so because the flows F_t and F'_t of X_f and X'_f, respectively, satisfy that $i_{\mathcal{O}_\rho} \circ F'_t = F_t \circ i_{\mathcal{O}_\rho}$. It is then clear that $A'_G|_{\mathcal{J}^{-1}(\mathcal{O}_\rho)}$ has constant rank since $\dim A'_G|_{\mathcal{J}^{-1}(\mathcal{O}_\rho)} = \dim \mathcal{J}^{-1}(\rho)$. This all shows that the leaf space $\mathcal{J}^{-1}(\mathcal{O}_\rho)/A'_G$ is well-defined.

In order to show that the leaf space $\mathcal{J}^{-1}(\mathcal{O}_\rho)/A'_G$ is a regular quotient manifold we first notice that

$$\mathcal{J}^{-1}(\mathcal{O}_\rho)/A'_G \simeq (G \times_{G_\rho} \mathcal{J}^{-1}(\rho))/A'_G$$

is in bijection with the quotient G/G_ρ that, by the hypothesis on the closedness of G_ρ is a smooth homogeneous manifold. Take in $M'_{\mathcal{O}_\rho} := \mathcal{J}^{-1}(\mathcal{O}_\rho)/A'_G$ the smooth structure that makes the bijection with G/G_ρ a diffeomorphism. It turns out that that smooth structure is the unique one that makes $M'_{\mathcal{O}_\rho}$ into a regular quotient manifold since it can be readily verified that the map

$$\begin{array}{rccc} \mathcal{J}_{\mathcal{O}_\rho}: & \mathcal{J}^{-1}(\mathcal{O}_\rho) \simeq G \times_{G_\rho} \mathcal{J}^{-1}(\rho) & \longrightarrow & \mathcal{J}^{-1}(\mathcal{O}_\rho)/A'_G \simeq G/G_\rho \\ & [g, z] & \longmapsto & gG_\rho \end{array}$$

is a surjective submersion. ■

We now introduce the regularized polar reduced subspaces of $M'_{\mathcal{O}_\rho}$, available when M is symplectic. We retake the ideas and notations introduced just above (14.2.1). Let $(\mathcal{J}_{L^\rho}^{-1}(L^\rho \cdot \sigma)/L^\rho, (\omega|_{M_H^\rho})_{L^\rho \cdot \sigma})$ be an orbit regularization of $(\mathcal{J}^{-1}(\mathcal{O}_\rho)/G, \omega_{\mathcal{O}_\rho})$. A straightforward application of Proposition 13.6.1 implies that the reduced space polar to

$$(\mathcal{J}_{L^\rho}^{-1}(L^\rho \cdot \sigma)/L^\rho, (\omega|_{M_H^\rho})_{L^\rho \cdot \sigma})$$

equals

$$\mathcal{J}_{L^\rho}^{-1}(L^\rho \cdot \sigma)/A'_{L^\rho} = \mathcal{J}^{-1}(\mathcal{N}_\rho)/A'_G$$

which is naturally diffeomorphic to $N(H)^\rho/G_\rho$. We will say that

$$\mathcal{J}^{-1}(\mathcal{N}_\rho)/A'_G$$

is ***a regularized polar reduced subspace*** of $M'_{\mathcal{O}_\rho}$. We will write

$$M'_{\mathcal{N}_\rho} := \mathcal{J}^{-1}(\mathcal{N}_\rho)/A'_G$$

and denote by $\mathcal{J}_{\mathcal{N}_\rho} : \mathcal{J}^{-1}(\mathcal{N}_\rho) \to \mathcal{J}^{-1}(\mathcal{N}_\rho)/A'_G$ the canonical projection. Notice that the spaces $M'_{\mathcal{N}_\rho}$ are embedded submanifolds of $M'_{\mathcal{O}_\rho}$. Finally, the decomposition (14.2.1) implies that the polar reduced space can be written as the following disjoint union of regularized polar reduced subspaces:

$$\begin{aligned} M'_{\mathcal{O}_\rho} &= \mathcal{J}^{-1}(\mathcal{O}_\rho)/A'_G \\ &= \dot{\bigcup}_{[g] \in G/N(H)^\rho} \mathcal{J}^{-1}(\mathcal{N}_{g \cdot \rho})/A'_G \\ &= \dot{\bigcup}_{[g] \in G/N(H)^\rho} M'_{\mathcal{N}_{g \cdot \rho}}. \end{aligned} \tag{14.3.1}$$

Equivalently, we have

$$G/G_\rho = \dot{\bigcup}_{[g] \in G/N(H)^\rho} gN(H)^\rho/G_\rho, \tag{14.3.2}$$

where the quotient $gN(H)^\rho/G_\rho$ denotes the orbit space of the free and proper action of G_ρ on $gN(H)^\rho$ by $h \cdot gn := gnh$, $h \in G_\rho$, $n \in N(H)^\rho$.

Before we state our next result we need some terminology. We will denote by $C^\infty\left(\mathcal{J}^{-1}(\mathcal{O}_\rho)/A'_G\right)$ the set of smooth real valued functions on $M'_{\mathcal{O}_\rho}$ with the smooth structure introduced in Proposition 14.3.1. Recall now that, as we pointed out in (13.1.3), there is a notion of smooth function on M/A'_G, namely

$$C^\infty(M/A'_G) := \{f \in C^0(M/A'_G) \mid f \circ \mathcal{J} \in C^\infty(M)^{A'_G}\}.$$

Analogously, for each open A'_G–invariant subset U of M we can define

$$C^\infty(U/A'_G) := \{f \in C^0(U/A'_G) \mid f \circ \mathcal{J}|_U \in C^\infty(U)^{A'_G}\}.$$

We define the set of ***Whitney smooth functions*** $W^\infty\left(\mathcal{J}^{-1}(\mathcal{O}_\rho)/A'_G\right)$ on $\mathcal{J}^{-1}(\mathcal{O}_\rho)/A'_G$ as

$$\begin{aligned} &W^\infty\left(\mathcal{J}^{-1}(\mathcal{O}_\rho)/A'_G\right) \\ &\qquad := \{f : M'_{\mathcal{O}_\rho} \to \mathbb{R} \mid f = F|_{M'_{\mathcal{O}_\rho}}, \text{ with } F \in C^\infty(M/A'_G)\}. \end{aligned}$$

The definitions and the fact that $\mathcal{J}_{\mathcal{O}_\rho}$ is a submersion imply that

$$W^\infty\left(\mathcal{J}^{-1}(\mathcal{O}_\rho)/A'_G\right) \subset C^\infty\left(\mathcal{J}^{-1}(\mathcal{O}_\rho)/A'_G\right).$$

Indeed, let $f \in W^\infty\left(\mathcal{J}^{-1}(\mathcal{O}_\rho)/A'_G\right)$ arbitrary. By definition, there exist $F \in C^\infty(M/A'_G)$ such that $f = F|_{M'_{\mathcal{O}_\rho}}$. As $F \in C^\infty(M/A'_G)$ we have $F \circ \mathcal{J} \in C^\infty(M)$. Also, as $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ is an immersed initial submanifold of M, the injection $i_{\mathcal{O}_\rho} : \mathcal{J}^{-1}(\mathcal{O}_\rho) \hookrightarrow M$ is smooth, and therefore so is $F \circ \mathcal{J} \circ i_{\mathcal{O}_\rho} = F \circ \mathcal{J}_{\mathcal{O}_\rho}$. Consequently, $f \circ \mathcal{J}_{\mathcal{O}_\rho} = F \circ \mathcal{J}_{\mathcal{O}_\rho}$ is smooth. As $\mathcal{J}_{\mathcal{O}_\rho}$ is a submersion f is necessarily smooth, that is, $f \in C^\infty\left(\mathcal{J}^{-1}(\mathcal{O}_\rho)/A'_G\right)$, as required.

14.3.2 Definition. *We say that $M'_{\mathcal{O}_\rho}$ is **Whitney spanned** when the differentials of its Whitney smooth functions span its cotangent bundle, that is,*

$$\operatorname{span}\{\mathbf{d}f(\sigma) \mid f \in W^\infty(M'_{\mathcal{O}_\rho})\} = T^*_\sigma M'_{\mathcal{O}_\rho}, \quad \textit{for all} \quad \sigma \in M'_{\mathcal{O}_\rho}.$$

A sufficient (but not necessary!) condition for $M'_{\mathcal{O}_\rho}$ to be Whitney spanned is that $W^\infty(M'_{\mathcal{O}_\rho}) = C^\infty(M'_{\mathcal{O}_\rho})$.

We are now in the position to state the main results of this section.

14.3.3 Theorem (Polar reduction of a Poisson manifold)**.** *Let $(M, \{\cdot,\cdot\})$ be a smooth Poisson manifold and G be a Lie group acting canonically and properly on M. Let $\mathcal{J} : M \to M/A'_G$ be the optimal momentum map associated to this action and $\rho \in M/A'_G$ be such that G_ρ is closed in G. If A_G is weakly von Neumann then, for each point $z \in \mathcal{J}^{-1}(\mathcal{O}_\rho)$ and vectors $v, w \in T_z\mathcal{J}^{-1}(\mathcal{O}_\rho)$, there exists an open A'_G–invariant neighborhood U of z and two smooth functions $f, g \in C^\infty(U)$ such that $v = X_f(z)$ and $w =$*

$X_g(z)$. Moreover, there is a unique presymplectic form $\omega'_{\mathcal{O}_\rho}$ on the polar reduced space $M'_{\mathcal{O}_\rho}$ that satisfies

$$\{f,g\}|_U(z) = \pi^*_{\mathcal{O}_\rho}\omega_{\mathcal{O}_\rho}(z)(v,w) + \mathcal{J}^*_{\mathcal{O}_\rho}\omega'_{\mathcal{O}_\rho}(z)(v,w) \tag{14.3.3}$$

If $M'_{\mathcal{O}_\rho}$ is Whitney spanned then the form $\omega'_{\mathcal{O}_\rho}$ is symplectic.

Remark. It can be proved that when A_G is von Neumann and A'_G satisfies the extension property the symplecticity of $\omega'_{\mathcal{O}_\rho}$ is equivalent to $M'_{\mathcal{O}_\rho}$ being Whitney spanned.

When the Poisson manifold $(M,\{\cdot,\cdot\})$ is actually a symplectic manifold with symplectic form ω the von Neumann condition in the previous result is no longer needed. Moreover, the conditions under which the form $\omega'_{\mathcal{O}_\rho}$ is symplectic can be completely characterized and the regularized polar subspaces appear as symplectic submanifolds of the polar space that contains them.

14.3.4 Theorem (Polar reduction of a symplectic manifold)**.** *Let (M,ω) be a smooth symplectic manifold and G be a Lie group acting canonically and properly on M. Let $\mathcal{J} : M \to M/A'_G$ be the optimal momentum map associated to this action and $\rho \in M/A'_G$ be such that G_ρ is closed in G.*

(i) *There is a unique presymplectic form $\omega'_{\mathcal{O}_\rho}$ on the polar reduced space $M'_{\mathcal{O}_\rho} \simeq G/G_\rho$ that satisfies*

$$i^*_{\mathcal{O}_\rho}\omega = \pi^*_{\mathcal{O}_\rho}\omega_{\mathcal{O}_\rho} + \mathcal{J}^*_{\mathcal{O}_\rho}\omega'_{\mathcal{O}_\rho}. \tag{14.3.4}$$

The form $\omega'_{\mathcal{O}_\rho}$ is symplectic if and only if for one point $z \in \mathcal{J}^{-1}(\mathcal{O}_\rho)$ (and hence for all) we have

$$\mathfrak{g}\cdot z \cap (\mathfrak{g}\cdot z)^\omega \subset T_z M_{G_z} \tag{14.3.5}$$

(ii) *Let $M'_{\mathcal{N}_\rho} = \mathcal{J}^{-1}(\mathcal{N}_\rho)/A'_G \simeq N(H)^\rho/G_\rho$ be a regularized polar reduced subspace of $M'_{\mathcal{O}_\rho}$. Let*

$$j_{\mathcal{N}_\rho} : \mathcal{J}^{-1}(\mathcal{N}_\rho)/A'_G \hookrightarrow \mathcal{J}^{-1}(\mathcal{O}_\rho)/A'_G$$

be the inclusion and $\omega'_{\mathcal{O}_\rho}$ the presymplectic form defined in (i)*. Then, the form*

$$\omega'_{\mathcal{N}_\rho} := j^*_{\mathcal{N}_\rho}\omega'_{\mathcal{O}_\rho} \tag{14.3.6}$$

is symplectic, that is, the regularized polar subspaces are symplectic submanifolds of the polar space that contains them.

Remark. The characterization (14.3.5) of the symplecticity of $\omega'_{\mathcal{O}_\rho}$ admits a particularly convenient reformulation when the G–action on the symplectic manifold (M,ω) admits an equivariant momentum map $\mathbf{J} : M \to \mathfrak{g}^*$.

Indeed, let $z \in M$ be such that $\mathbf{J}(z) = \mu \in \mathfrak{g}^*$ and $G_z = H$. Then, if the symbol G_μ denotes the coadjoint isotropy of μ, (14.3.5) is equivalent to

$$\mathfrak{g} \cdot z \cap (\mathfrak{g} \cdot z)^\omega = \mathfrak{g}_\mu \cdot z \subset T_z M_H,$$

which in turn amounts to $\mathfrak{g}_\mu \cdot z \subset \mathfrak{g}_\mu \cdot z \cap T_z M_H = \mathrm{Lie}(N(H) \cap G_\mu) \cdot z$. Let $N_{G_\mu}(H) := N(H) \cap G_\mu$. With this notation, the condition can be rewritten as $\mathfrak{g}_\mu + \mathfrak{h} \subset \mathrm{Lie}(N_{G_\mu}(H)) + \mathfrak{h} \subset \mathfrak{g}_\mu$ or, equivalently, as

$$\mathfrak{g}_\mu = \mathrm{Lie}(N_{G_\mu}(H)). \tag{14.3.7}$$

Proof of Theorem 14.3.3.. Since A_G is weakly von Neumann, we see that for any $z \in M$ $\mathfrak{g} \cdot z \subset A''_G(z)$ or, equivalently, that for any $z \in M$ and any $\xi \in \mathfrak{g}$, there is a A'_G–invariant neighborhood U of z and a function $F \in C^\infty(U/A'_G)$ such that $\xi_M(z) = X_{F \circ \mathcal{J}}(z)$. Consequently, for any vector $v \in T_z \mathcal{J}^{-1}(\mathcal{O}_\rho)$ there exists $f \in C^\infty(M)^G$ and $F \in C^\infty(U/A'_G)$ (shrink U if necessary) such that

$$v = X_f(z) + X_{F \circ \mathcal{J}}(z) = X_{f|_U + F \circ \mathcal{J}}(z).$$

Let $w \in T_z \mathcal{J}^{-1}(\mathcal{O}_\rho)$, $l \in C^\infty(M)^G$, and $L \in C^\infty(U/A'_G)$ be such that

$$w = X_l(z) + X_{L \circ \mathcal{J}}(z) = X_{l|_U + L \circ \mathcal{J}}(z).$$

The expression (14.3.3) can then be rewritten as

$$\begin{aligned}
\mathcal{J}^*_{\mathcal{O}_\rho} \omega'_{\mathcal{O}_\rho}(z)(v, w) &= \mathcal{J}^*_{\mathcal{O}_\rho} \omega'_{\mathcal{O}_\rho}(z)(X_{f|_U + F \circ \mathcal{J}}(z), X_{l|_U + L \circ \mathcal{J}}(z)) \\
&= \{f + F \circ \mathcal{J}, l + L \circ \mathcal{J}\}|_U(z) \\
&\quad - \pi^*_{\mathcal{O}_\rho} \omega_{\mathcal{O}_\rho}(z)(X_{f|_U + F \circ \mathcal{J}}(z), X_{l|_U + L \circ \mathcal{J}}(z)) \\
&= \{F \circ \mathcal{J}, L \circ \mathcal{J}\}|_U(z)
\end{aligned} \tag{14.3.8}$$

We now show that $\omega'_{\mathcal{O}_\rho}$ is well-defined. Indeed, let $z' \in \mathcal{J}^{-1}(\mathcal{O}_\rho)$ and $v', w' \in T_{z'} \mathcal{J}^{-1}(\mathcal{O}_\rho)$ be such that $T_z \mathcal{J}_{\mathcal{O}_\rho} \cdot v = T_{z'} \mathcal{J}_{\mathcal{O}_\rho} \cdot v'$ and $T_z \mathcal{J}_{\mathcal{O}_\rho} \cdot w = T_{z'} \mathcal{J}_{\mathcal{O}_\rho} \cdot w'$. First of all these equalities imply the existence of an element $\mathcal{F}_T$ in the polar pseudogroup of A_G such that $z' = \mathcal{F}_T(z)$. As $\mathcal{F}_T$ is a local diffeomorphism such that $\mathcal{J}_{\mathcal{O}_\rho} \circ \mathcal{F}_T = \mathcal{J}_{\mathcal{O}_\rho}$, we get $T_z \mathcal{J}_{\mathcal{O}_\rho} = T_{z'} \mathcal{J}_{\mathcal{O}_\rho} \cdot T_z \mathcal{F}_T$. Now, we can rewrite the conditions $T_z \mathcal{J}_{\mathcal{O}_\rho} \cdot v = T_{z'} \mathcal{J}_{\mathcal{O}_\rho} \cdot v'$ and $T_z \mathcal{J}_{\mathcal{O}_\rho} \cdot w = T_{z'} \mathcal{J}_{\mathcal{O}_\rho} \cdot w'$ as $T_{z'} \mathcal{J}_{\mathcal{O}_\rho} \cdot T_z \mathcal{F}_T \cdot v = T_{z'} \mathcal{J}_{\mathcal{O}_\rho} \cdot v'$ and $T_{z'} \mathcal{J}_{\mathcal{O}_\rho} \cdot T_z \mathcal{F}_T \cdot w = T_{z'} \mathcal{J}_{\mathcal{O}_\rho} \cdot w'$, respectively, which implies the existence of two functions $f', l' \in C^\infty(M)^G$ such that

$$\begin{aligned}
v' &= T_z \mathcal{F}_T(X_f(z) + X_{F \circ \mathcal{J}}(z)) + X_{f'}(\mathcal{F}_T(z)) \\
w' &= T_z \mathcal{F}_T(X_l(z) + X_{L \circ \mathcal{J}}(z)) + X_{l'}(\mathcal{F}_T(z))
\end{aligned}$$

or, equivalently:

$$\begin{aligned}
v' &= X_{f \circ \mathcal{F}_{-T}}(\mathcal{F}_T(z)) + X_{F \circ \mathcal{J}}(\mathcal{F}_T(z)) + X_{f'}(\mathcal{F}_T(z)) \\
w' &= X_{l \circ \mathcal{F}_{-T}}(\mathcal{F}_T(z)) + X_{L \circ \mathcal{J}}(\mathcal{F}_T(z)) + X_{l'}(\mathcal{F}_T(z)).
\end{aligned}$$

Therefore, using (14.3.3), we arrive at

$$\begin{aligned}
&\mathcal{J}_{\mathcal{O}_\rho}^{*}\omega'_{\mathcal{O}_\rho}(z')(v',w')\\
&\quad=\{f\circ\mathcal{F}_{-T}+F\circ\mathcal{J}+f',l\circ\mathcal{F}_{-T}+L\circ\mathcal{J}+l'\}|_V(\mathcal{F}_T(z))\\
&\qquad-\pi_{\mathcal{O}_\rho}^{*}\omega_{\mathcal{O}_\rho}(\mathcal{F}_T(z))(X_{f\circ\mathcal{F}_{-T}|_V+F\circ\mathcal{J}+f'}(z),X_{l\circ\mathcal{F}_{-T}|_V+L\circ\mathcal{J}+l'}(z))\\
&\quad=\{F\circ\mathcal{J},L\circ\mathcal{J}\}|_V(\mathcal{F}_T(z))=\{F\circ\mathcal{J},L\circ\mathcal{J}\}|_U(z)\\
&\quad=\mathcal{J}_{\mathcal{O}_\rho}^{*}\omega'_{\mathcal{O}_\rho}(z)(v,w),
\end{aligned}$$

where $V=U\cap\mathcal{F}_T(\mathrm{Dom}(\mathcal{F}_T))=\mathcal{F}_T(U\cap\mathrm{Dom}(\mathcal{F}_T))$. Hence, the form $\omega'_{\mathcal{O}_\rho}$ is well-defined. The closedness and skew symmetric character of $\omega'_{\mathcal{O}_\rho}$ is obtained as a consequence of $\mathcal{J}_{\mathcal{O}_\rho}$ being a surjective submersion, $\omega_{\mathcal{O}_\rho}$ being closed and skew symmetric, and the $\{\cdot,\cdot\}$ being a Poisson bracket. An equivalent fashion to realize this is by writing $\omega'_{\mathcal{O}_\rho}$ in terms of the symplectic structure of the leaves of M. Indeed, as A_G is weakly von Neumann, each connected component of $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ lies in a single symplectic leaf of $(M,\{\cdot,\cdot\})$. In order to simplify the exposition suppose that $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ is connected and let $\mathcal{L}_{\mathcal{O}_\rho}$ be the unique symplectic leaf of M that contains it (otherwise one has just to proceed connected component by connected component). Let $i_{\mathcal{L}_{\mathcal{O}_\rho}}:\mathcal{J}^{-1}(\mathcal{O}_\rho)\to\mathcal{L}_{\mathcal{O}_\rho}$ be the natural injection. Given that $i_{\mathcal{O}_\rho}:\mathcal{J}^{-1}(\mathcal{O}_\rho)\to M$ is smooth and $\mathcal{L}_{\mathcal{O}_\rho}$ is an initial submanifold of M, the map $i_{\mathcal{L}_{\mathcal{O}_\rho}}$ is therefore smooth. If we denote by $\omega_{\mathcal{L}_{\mathcal{O}_\rho}}$ the symplectic form of the leaf $\mathcal{L}_{\mathcal{O}_\rho}$, expression (14.3.4) can be rewritten as:

$$i_{\mathcal{L}_{\mathcal{O}_\rho}}^{*}\omega_{\mathcal{L}_{\mathcal{O}_\rho}}=\pi_{\mathcal{O}_\rho}^{*}\omega_{\mathcal{O}_\rho}+\mathcal{J}_{\mathcal{O}_\rho}^{*}\omega'_{\mathcal{O}_\rho}. \tag{14.3.9}$$

The antisymmetry and closedness of $\omega'_{\mathcal{O}_\rho}$ appears then as a consequence of the antisymmetry and closedness of $\omega_{\mathcal{O}_\rho}$ and $\omega_{\mathcal{L}_{\mathcal{O}_\rho}}$.

It just remains to be shown that if $M'_{\mathcal{O}_\rho}$ is Whitney spanned then the form $\omega'_{\mathcal{O}_\rho}$ is non degenerate. Let $z\in\mathcal{J}^{-1}(\mathcal{O}_\rho)$ and $v\in T_z\mathcal{J}^{-1}(\mathcal{O}_\rho)$ be such that

$$\omega'_{\mathcal{O}_\rho}(\mathcal{J}_{\mathcal{O}_\rho}(z))(T_z\mathcal{J}_{\mathcal{O}_\rho}\cdot v,T_z\mathcal{J}_{\mathcal{O}_\rho}\cdot w)=0,\quad\text{for all}\quad w\in T_z\mathcal{J}^{-1}(\mathcal{O}_\rho). \tag{14.3.10}$$

Take now $f\in C^\infty(M)^G$ and $F\in C^\infty(U/A'_G)$ such that $v=X_f(z)+X_{F\circ\mathcal{J}}(z)$. Condition (14.3.10) is equivalent to requiring that

$$\omega'_{\mathcal{O}_\rho}(\mathcal{J}_{\mathcal{O}_\rho}(z))(T_z\mathcal{J}_{\mathcal{O}_\rho}\cdot X_{F\circ\mathcal{J}}(z),T_z\mathcal{J}_{\mathcal{O}_\rho}\cdot X_{L\circ\mathcal{J}}(z))=0, \tag{14.3.11}$$

for all $L\in C^\infty(V/A'_G)$ and all open A'_G–invariant neighborhoods V of z. By (14.3.8) we can rewrite (14.3.11) as

$$\{F\circ\mathcal{J},L\circ\mathcal{J}\}|_{U\cap V}(z)=0. \tag{14.3.12}$$

Now, notice that for any $h \in W^\infty(M'_{\mathcal{O}_\rho})$ there exists a function $H \in C^\infty(M/A'_G)$ such that $H|_{M'_{\mathcal{O}_\rho}} = h$. Moreover, by (14.3.12) we obtain:

$$\begin{aligned}\mathbf{d}h(\mathcal{J}_{\mathcal{O}_\rho}(z)) \cdot (T_z\mathcal{J}_{\mathcal{O}_\rho} \cdot X_{F\circ\mathcal{J}}(z)) \\ = \mathbf{d}(h \circ \mathcal{J}_{\mathcal{O}_\rho})(z) \cdot X_{F\circ\mathcal{J}}(z) = \mathbf{d}(H \circ \mathcal{J})(z) \cdot X_{F\circ\mathcal{J}}(z) = 0.\end{aligned}$$

Given that the previous equality holds for any $h \in W^\infty(M'_{\mathcal{O}_\rho})$ and $M'_{\mathcal{O}_\rho}$ is Whitney spanned, we obtain

$$T_z\mathcal{J}_{\mathcal{O}_\rho} \cdot X_{F\circ\mathcal{J}}(z) = T_z\mathcal{J}_{\mathcal{O}_\rho} \cdot v = 0,$$

as required.

Proof of Theorem 14.3.4. (i) The well–definedness and presymplectic character of $\omega'_{\mathcal{O}_\rho}$ in this case can be obtained as a consequence of Theorem 14.3.3. This is particularly evident when we think of $\omega'_{\mathcal{O}_\rho}$ as the form characterized by equality (14.3.9) and we recall that in the symplectic case $\omega_{\mathcal{L}_{\mathcal{O}_\rho}} = \omega$.

It just remains to be shown that the form $\omega'_{\mathcal{O}_\rho}$ is non degenerate if and only if condition (14.3.5) holds. We proceed by showing first that if condition (14.3.5) holds for the point $z \in \mathcal{J}^{-1}(\mathcal{O}_\rho)$ then it holds for all the points in $\mathcal{J}^{-1}(\mathcal{O}_\rho)$. We will then prove that (14.3.5) at the point z is equivalent to the non degeneracy of $\omega'_{\mathcal{O}_\rho}$ at $\mathcal{J}_{\mathcal{O}_\rho}(z)$.

Suppose first that the point $z \in \mathcal{J}^{-1}(\mathcal{O}_\rho)$ is such that $\mathfrak{g} \cdot z \cap (\mathfrak{g} \cdot z)^\omega \subset T_zM_{G_z}$. Notice now that any element in $\mathcal{J}^{-1}(\mathcal{O}_\rho)$ can be written as $\Phi_g(\mathcal{F}_T(z))$ with $g \in G$ and $\mathcal{F}_T$ in the polar pseudogroup of A_G. It is easy to show that the relation

$$\mathfrak{g} \cdot (\Phi_g(\mathcal{F}_T(z))) \cap (\mathfrak{g} \cdot (\Phi_g(\mathcal{F}_T(z))))^\omega \subset T_{\Phi_g(\mathcal{F}_T(z))}M_{G_{\Phi_g(\mathcal{F}_T(z))}}$$

is equivalent to $T_z(\Phi \circ \mathcal{F}_T)(\mathfrak{g} \cdot z \cap (\mathfrak{g} \cdot z)^\omega) \subset T_z(\Phi \circ \mathcal{F}_T)M_{G_z}$ and therefore to $\mathfrak{g} \cdot z \cap (\mathfrak{g} \cdot z)^\omega \subset T_zM_{G_z}$.

Let now $v \in T_z\mathcal{J}^{-1}(\mathcal{O}_\rho)$ be such that

$$\omega'_{\mathcal{O}_\rho}(\mathcal{J}_{\mathcal{O}_\rho}(z))(T_z\mathcal{J}_{\mathcal{O}_\rho} \cdot v, T_z\mathcal{J}_{\mathcal{O}_\rho} \cdot w) = 0, \quad \text{for all} \quad w \in T_z\mathcal{J}^{-1}(\mathcal{O}_\rho). \tag{14.3.13}$$

Take now $f \in C^\infty(M)^G$ and $\xi \in \mathfrak{g}$ such that $v = X_f(z) + \xi_M(z)$. Condition (14.3.13) is equivalent to having that

$$\omega'_{\mathcal{O}_\rho}(\mathcal{J}_{\mathcal{O}_\rho}(z))(T_z\mathcal{J}_{\mathcal{O}_\rho} \cdot \xi_M(z), T_z\mathcal{J}_{\mathcal{O}_\rho} \cdot \eta_M(z)) = 0, \qquad \text{for all } \eta \in \mathfrak{g}$$

which by (14.3.4) can be rewritten as

$$\omega(z)(\xi_M(z), \eta_M(z)) = 0, \qquad \text{for all } \eta \in \mathfrak{g},$$

and thereby amounts to having that $\xi_M(z) \in \mathfrak{g} \cdot z \cap (\mathfrak{g} \cdot z)^\omega$. Hence, $\omega'_{\mathcal{O}_\rho}(\mathcal{J}_{\mathcal{O}_\rho}(z))$ is non degenerate if and only if $\xi_M(z) \in \ker T_z\mathcal{J}_{\mathcal{O}_\rho} = A'_G(z)$.

Suppose now that condition (14.3.5) holds; then, as $\xi_M(z) \in \mathfrak{g}\cdot z \cap (\mathfrak{g}\cdot z)^\omega$ we have $\xi_M(z) \in T_z M_{G_z}$. Using (13.1.1) we can conclude that $\xi_M(z) \in A'_G(z)$, as required. Conversely, suppose that $\omega'_{\mathcal{O}_\rho}$ is symplectic. The previous equalities immediately imply that $\mathfrak{g}\cdot z \cap (\mathfrak{g}\cdot z)^\omega \subset A'_G(z) \subset T_z M_{G_z}$, as required.

(ii) The form $\omega'_{\mathcal{N}_\rho}$ is clearly closed and antisymmetric. We now show that it is non degenerate. Recall firs that the tangent space to $T_z\mathcal{J}^{-1}(\mathcal{N}_\rho)$ at a given point $z \in \mathcal{J}^{-1}(\mathcal{N}_\rho)$ is given by the vectors of the form

$$v = X_f(z) + \xi_M(z),$$

with $f \in C^\infty(M)^G$ and $\xi \in \mathrm{Lie}(N(H)^\rho)$. Let $v = X_f(z) + \xi_M(z) \in T_z\mathcal{J}^{-1}(\mathcal{N}_\rho)$ be such that

$$\mathcal{J}^*_{\mathcal{N}_\rho}(j^*_{\mathcal{N}_\rho}\omega'_{\mathcal{O}_\rho})(z)(X_f(z) + \xi_M(z), X_g(z) + \eta_M(z)) = 0,$$

for all $\eta \in \mathrm{Lie}(N(H)^\rho)$ and $g \in C^\infty(M)^G$.

If we plug into the previous expression the definition of the form $\omega'_{\mathcal{O}_\rho}$ we obtain

$$\omega(z)(\xi_M(z), \eta_M(z)) = 0,$$

for all $\eta \in \mathrm{Lie}(N(H)^\rho)$, that is,

$$\begin{aligned}\xi_M(z) &\in (\mathrm{Lie}(N(H)^\rho)\cdot z) \cap (\mathrm{Lie}(N(H)^\rho)\cdot z)^\omega \\ &= (\mathrm{Lie}(N(H)^\rho)\cdot z) \cap (\mathrm{Lie}(N(H)^\rho)\cdot z)^{\omega|_{M_H^\rho}} \\ &= (\mathrm{Lie}(N(H)^\rho/H)\cdot z) \cap A'_{N(H)^\rho/H}(z),\end{aligned}$$

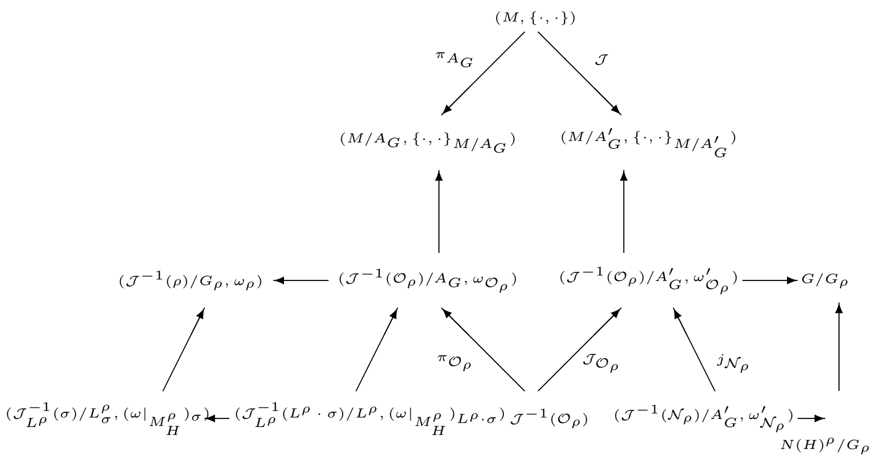
where the last equality follows from (13.1.1) and the freeness of the natural $N(H)^\rho/H$–action on M_H^ρ. We now recall (see Lemma 4.4 in Ortega and Ratiu [2002]) that any $N(H)^\rho/H$–invariant function on M_H^ρ admits a local extension to a G–invariant function on M, hence $\xi_M(z) \in (\mathrm{Lie}(N(H)^\rho/H)\cdot z) \cap A'_G(z)$, and consequently $T_z\mathcal{J}_{\mathcal{O}_\rho}\cdot \xi_M(z) = T_z\mathcal{J}_{\mathcal{O}_\rho}\cdot v = T_z\mathcal{J}_{\mathcal{N}_\rho}\cdot v = 0$, as required. ■

14.4 Symplectic Leaves and the Reduction Diagram

Suppose that A'_G is completable so that the symplectic leaves of M/A'_G are well-defined. We recall that this is automatically the case when (M,ω) is symplectic and the G–group action is proper (see Ortega [2003a]). Assume also that A_G is von Neumann so that the diagram $(M/G, \{\cdot,\cdot\}_{M/A_G}) \overset{\pi_{A_G}}{\leftarrow} (M, \{\cdot,\cdot\}) \overset{\mathcal{J}}{\rightarrow} (M/A'_G, \{\cdot,\cdot\}_{M/A'_G})$ constitutes a dual pair.

Notice that by Definition 13.4.1, the symplectic leaves of M/A_G and M/A'_G coincide with the connected components of the orbit reduced spaces $M_{\mathcal{O}_\rho}$ and polar reduced spaces $M'_{\mathcal{O}_\rho}$, that we studied in sections 14.2 and 14.3, respectively. We saw that whenever G_ρ is closed in G and the Whitney spanning condition is satisfied these spaces are actual symplectic manifolds. When M is symplectic, the symplecticity of the leaves of M/A'_G is characterized by condition (14.3.5) or even by (14.3.7), provided that the G–action has an associated standard equivariant momentum map $\mathbf{J} : M \to \mathfrak{g}^*$. Moreover, when $M_{\mathcal{O}_\rho}$ and $M'_{\mathcal{O}_\rho}$ are corresponding leaves, their symplectic structures are connected to each other by an identity that naturally generalizes the classical relation that we recalled in (13.5.2).

The following diagram represents all the spaces that we worked with and their relations. The part of the diagram dealing with the regularized spaces refers only to the situation in which M is symplectic.



14.5 Orbit Reduction: Beyond Compact Groups

The approach to optimal orbit reduction developed in the last few sections sheds some light on how to carry out orbit reduction with a standard momentum map when the symmetry group is not compact. This absence of compactness poses some technical problems that have been tackled by various people over the years using different approaches. Since these problems already arise in the free actions case we will restrict ourselves to this situation. More specifically we will assume that we have a Lie group G (not necessarily compact) acting freely and canonically on the symplectic manifold (M, ω). We will suppose that this action has an associated coadjoint equivariant momentum map $\mathbf{J} : M \to \mathfrak{g}^*$. For the sake of simplicity in the exposition and in order to have a better identification with the material presented in the previous sections we will assume that $\mathbf{J}$ has connected fibers. This assumption is not fundamental. The reader interested in the general case with no connectedness hypothesis in the fibers and nonfree actions may want to check with [HRed].

In the presence of the hypotheses that we just stated, the momentum map $\mathbf{J}$ is a submersion that maps M onto an open coadjoint equivariant subset $\mathfrak{g}^*_{\mathbf{J}}$ of $\mathfrak{g}^*$. Moreover, any value $\mu \in \mathfrak{g}^*_{\mathbf{J}}$ of $\mathbf{J}$ is regular and has an associated smooth Marsden–Weinstein symplectic reduced space $\mathbf{J}^{-1}(\mu)/G_\mu$. What about the orbit reduced space $\mathbf{J}^{-1}(\mathcal{O}_\mu)/G$? When the Lie group G is compact there is no problem to canonically endow $\mathbf{J}^{-1}(\mathcal{O}_\mu)/G$ with a smooth structure. Indeed, in this case the coadjoint orbit $\mathcal{O}_\mu$ is an embedded submanifold of $\mathfrak{g}^*$ transverse to the momentum mapping. The Transversal Mapping Theorem ensures that $\mathbf{J}^{-1}(\mathcal{O}_\mu)$ is a G–invariant embedded submanifold of M and hence the quotient $\mathbf{J}^{-1}(\mathcal{O}_\mu)/G$ is smooth and symplectic with the form spelled out in (13.5.1). In the non compact case this argument breaks down due to the non embedded character of $\mathcal{O}_\mu$ in $\mathfrak{g}^*$. In trying to fix this problem this has lead to the assumption of locally closedness on the coadjoint orbits that one can see in a number of papers (see for instance Bates and Lerman [1997]). Nevertheless, this hypothesis is not needed to carry out point reduction, and therefore makes the two approaches non equivalent. The first work where this hypothesis has been eliminated is Cushman and Śniatycki [2001]. In this paper the authors use a combination of distribution theory with Sikorski differential spaces to show that the orbit reduced space is a symplectic manifold. Nevertheless, the first reference where the standard formula (13.5.1) appears at this level of generality is Blaom [2001]. In that paper the author only deals with the free case. Nevertheless the use of a standard technique of reduction to the isotropy type manifolds that the reader can find in Sjamaar and Lerman [1991]; Ortega [1998]; Cushman and Śniatycki [2001]; Ortega and Ratiu [2004a] generalizes the results of Blaom [2001] to singular situations.

In the next few paragraphs we will illustrate Theorem 14.3.3 by showing that the results in Cushman and Śniatycki [2001]; Blaom [2001] can be obtained as a corollary of it.

We start by identifying in this setup all the elements in that result. First of all, we note that the polar distribution satisfies $A'_G = \ker T\mathbf{J}$ (see Ortega and Ratiu [2002]) and the connectedness hypothesis on the fibers of $\mathbf{J}$ implies that the optimal momentum map $\mathcal{J} : M \to M/A'_G$ in this case can be identified with $\mathbf{J} : M \to \mathfrak{g}^*_{\mathbf{J}}$. This immediately implies that for any $\mu \in \mathfrak{g}^*_{\mathbf{J}} \simeq M/A'_G$, the isotropy G_μ is closed in G and, by Theorem 14.1.4 there is a unique smooth structure on $\mathbf{J}^{-1}(\mathcal{O}_\mu)$ that makes it into an initial submanifold of M and, at the same time, an integral manifold of the distribution $D = A'_G + \mathfrak{g}\cdot m = \ker T\mathbf{J} + \mathfrak{g}\cdot m$. This structure coincides with the one given in Blaom [2001]. Also, by Theorem 14.2.1, the quotient $\mathbf{J}^{-1}(\mathcal{O}_\mu)/G$ admits a unique symplectic structure $\omega_{\mathcal{O}_\mu}$ that makes it symplectomorphic to the Marsden–Weinstein point reduced space $(\mathbf{J}^{-1}(\mu)/G_\mu, \omega_\mu)$. It remains to be shown that we can use (14.3.4) in this case and that the resulting formula coincides with the standard one (13.5.1) provided by Blaom [2001]. An analysis of the polar reduced space in this setup will provide an affirmative answer to this question.

By Proposition 14.3.1 the polar reduced space $\mathbf{J}(\mathcal{O}_\mu)/A'_G$ is endowed with the only smooth structure that makes it diffeomorphic to the homogeneous space $G/G_\mu \simeq \mathcal{O}_\mu$. Hence, in this case $\mathbf{J}_{\mathcal{O}_\mu} : \mathbf{J}^{-1}(\mathcal{O}_\mu) \to \mathcal{O}_\mu$ is the map given by $\mathbf{J}_{\mathcal{O}_\mu}(z) := \mathbf{J}(z)$ which is smooth because the coadjoint orbits are always initial submanifolds of $\mathfrak{g}^*$. Therefore we can already compute the polar symplectic form $\omega'_{\mathcal{O}_\mu}$. By (14.3.4) we see for any $\xi, \eta \in \mathfrak{g}$ and any $z \in \mathbf{J}^{-1}(\mathcal{O}_\mu)$ (for simplicity in the exposition we take $\mathbf{J}(z) = \mu$):

$$\begin{aligned}\mathbf{J}^*_{\mathcal{O}_\mu}\omega'_{\mathcal{O}_\mu}(z)(\xi_M(z), \eta_M(z))& \\ = i^*_{\mathcal{O}_\mu}\omega(z)(\xi_M(z), \eta_M(z)) &- \pi^*_{\mathcal{O}_\mu}\omega_{\mathcal{O}_\mu}(z)(\xi_M(z), \eta_M(z)),\end{aligned}$$

or, equivalently:

$$\omega'_{\mathcal{O}_\mu}(\mu)(\mathrm{ad}^*_\xi \mu, \mathrm{ad}^*_\eta \mu) = \omega(z)(\xi_M(z), \eta_M(z)) = \langle \mathbf{J}(z), [\xi, \eta] \rangle = \langle \mu, [\xi, \eta] \rangle.$$

In conclusion, in this case the polar reduced form $\omega'_{\mathcal{O}_\mu}$ coincides with the "+"–Kostant–Kirillov–Souriau symplectic form on the coadjoint orbit $\mathcal{O}_\mu$. Therefore, the general optimal orbit reduction formula (14.3.4) coincides with the standard one (13.5.1).

14.6 Examples: Polar Reduction of the Coadjoint Action

We now provide two examples on how we can use the coadjoint action along with Theorems 14.3.3 and 14.3.4 to easily produce symplectic manifolds and symplectically decomposed presymplectic manifolds.

Coadjoint Orbits as Polar Reduced Spaces. Let G be a Lie group, $\mathfrak{g}$ be its Lie algebra, and $\mathfrak{g}^*$ be its dual considered as a Lie–Poisson space. In this elementary example we show how the coadjoint orbits appear as the polar reduced spaces of the coadjoint G–action on $\mathfrak{g}^*$.

A straightforward computation shows that the coadjoint action of G on the Lie–Poisson space $\mathfrak{g}^*$ is canonical. Moreover, the polar distribution $A'_G(\mu) = 0$ for all $\mu \in \mathfrak{g}^*$ and therefore the optimal momentum map $\mathcal{J} : \mathfrak{g}^* \to \mathfrak{g}^*$ is the identity map on $\mathfrak{g}^*$. This immediately implies that any open set $U \subset \mathfrak{g}^*$ is A'_G–invariant, that $C^\infty(U)^{A'_G} = C^\infty(U)$, and that therefore $\mathfrak{g} \cdot \mu \subset A''_G(\mu)$, for any $\mu \in \mathfrak{g}^*$. The coadjoint action on $\mathfrak{g}^*$ is therefore weakly von Neumann (actually, if G is connected A_G is von Neumann).

We now look at the corresponding reduced spaces. On one hand the orbit reduced spaces $\mathcal{J}^{-1}(\mathcal{O}_\rho)/G$ are the quotients $G\cdot\mu/G$ and therefore amount to points. At the same time, we have $\mathcal{J}^{-1}(\mathcal{O}_\rho)/A'_G = \mathcal{O}_\mu/A'_G = \mathcal{O}_\mu$, that is, the polar reduced spaces are the coadjoint orbits which, by Theorem 14.3.3, are symplectic. Indeed, the Whitney spanning condition necessary for the application of this result is satisfied since in this case

$$\operatorname{span}\{\mathbf{d}f(\mu) \mid f \in W^\infty(M'_{\mathcal{O}_\rho})\} = \operatorname{span}\{\mathbf{d}h|_{\mathcal{O}_\mu}(\mu) \mid h \in C^\infty(\mathfrak{g}^*)\} = T^*_\mu\mathcal{O}_\mu.$$

Note that the last equality is a consequence of the immersed character of the coadjoint orbits $\mathcal{O}_\mu$ as submanifolds of $\mathfrak{g}^*$ (the equality is easily proved using immersion charts around the point μ).

Symplectic Decomposition of Presymplectic Homogeneous Manifolds. Let G be a Lie group, $\mathfrak{g}$ be its Lie algebra, and $\mathfrak{g}^*$ be its dual. Let $\mathcal{O}_{\mu_1}$ and $\mathcal{O}_{\mu_2}$ be two coadjoint orbits of $\mathfrak{g}^*$ that we will consider as symplectic manifolds endowed with the KKS–symplectic forms $\omega_{\mathcal{O}_{\mu_1}}$ and $\omega_{\mathcal{O}_{\mu_2}}$, respectively. The cartesian product $\mathcal{O}_{\mu_1}\times\mathcal{O}_{\mu_2}$ is also a symplectic manifold with the sum symplectic form $\omega_{\mathcal{O}_{\mu_1}}+\omega_{\mathcal{O}_{\mu_2}}$. The diagonal action of G on $\mathcal{O}_{\mu_1}\times\mathcal{O}_{\mu_2}$ is canonical with respect to this symplectic structure and, moreover, it has an associated standard equivariant momentum map $\mathbf{J} : \mathcal{O}_{\mu_1}\times\mathcal{O}_{\mu_2} \to \mathfrak{g}^*$ given by $\mathbf{J}(\nu,\eta) = \nu+\eta$. We now suppose that this action is proper and we will study, in this particular case, the orbit and polar reduced spaces introduced in the previous sections.

We start by looking at the level sets of the optimal momentum map

$$\mathcal{J} : \mathcal{O}_{\mu_1}\times\mathcal{O}_{\mu_2} \to \mathcal{O}_{\mu_1}\times\mathcal{O}_{\mu_2}/A'_G.$$

A general result (see Theorem 3.6 in Ortega and Ratiu [2002]) states that in the presence of a standard momentum map the fibers of the optimal momentum map coincide with the connected components of the intersections of the level sets of the momentum map with the isotropy type submanifolds. Hence, in our case, if $\rho = \mathcal{J}(\mu_1,\mu_2)$, we have

$$\mathcal{J}^{-1}(\rho) = (\mathbf{J}^{-1}(\mu_1+\mu_2)\cap(\mathcal{O}_{\mu_1}\times\mathcal{O}_{\mu_2})_{G_{(\mu_1,\mu_2)}})_c, \tag{14.6.1}$$

where the subscript c in the previous expression stands for the connected component of $\mathbf{J}^{-1}(\mu_1+\mu_2)\cap(\mathcal{O}_{\mu_1}\times\mathcal{O}_{\mu_2})_{G_{(\mu_1,\mu_2)}}$ that contains $\mathcal{J}^{-1}(\rho)$. Given that the isotropy $G_{(\mu_1,\mu_2)} = G_{\mu_1}\cap G_{\mu_2}$, with G_{μ_1} and G_{μ_2} the coadjoint isotropies of μ_1 and μ_2, respectively, the expression (14.6.1) can be rewritten as

$$\begin{aligned}&\mathcal{J}^{-1}(\rho) = (\{(\operatorname{Ad}^*_{g^{-1}}\mu_1, \operatorname{Ad}^*_{h^{-1}}\mu_2) \mid g,h\in G, \text{ such that}\\ &\operatorname{Ad}^*_{g^{-1}}\mu_1+\operatorname{Ad}^*_{h^{-1}}\mu_2 = \mu_1+\mu_2,\ gG_{\mu_1}g^{-1}\cap hG_{\mu_2}h^{-1} = G_{\mu_1}\cap G_{\mu_2}\})_c.\end{aligned}$$

It is easy to show that in this case

$$G_\rho = N_{G_{\mu_1+\mu_2}}(G_{\mu_1}\cap G_{\mu_2})^c, \tag{14.6.2}$$

where the superscript c denotes the closed subgroup of

$$N_{G_{\mu_1+\mu_2}}(G_{\mu_1} \cap G_{\mu_2}) := N(G_{\mu_1} \cap G_{\mu_2}) \cap G_{\mu_1+\mu_2}$$

that leaves $\mathcal{J}^{-1}(\rho)$ invariant. Theorems 13.5.1 and 14.2.1 guarantee that the quotients $\mathcal{J}^{-1}(\rho)/G_\rho \simeq \mathcal{J}^{-1}(\mathcal{O}_\rho)/G$ are symplectic. Nevertheless, we will focus our attention in the corresponding polar reduced spaces.

According to Theorem 14.3.4 and to (14.6.2), the polar reduced space corresponding to $\mathcal{J}^{-1}(\mathcal{O}_\rho)/G$ is the homogeneous presymplectic manifold

$$G/N_{G_{\mu_1+\mu_2}}(G_{\mu_1} \cap G_{\mu_2})^c. \tag{14.6.3}$$

Expression (14.3.7) states that $G/N_{G_{\mu_1+\mu_2}}(G_{\mu_1} \cap G_{\mu_2})^c$ is symplectic if and only if

$$\mathfrak{g}_{\mu_1+\mu_2} = \operatorname{Lie}(N_{G_{\mu_1+\mu_2}}(G_{\mu_1} \cap G_{\mu_2})),$$

which is obviously true when, for instance, $G_{\mu_1} \cap G_{\mu_2}$ is a normal subgroup of $G_{\mu_1+\mu_2}$. In any case, using (14.3.2) we can write the polar reduced space (14.6.3) as a disjoint union of its regularized symplectic reduced subspaces that, that in this case are of the form

$$gN(G_{\mu_1} \cap G_{\mu_2})^\rho / N_{G_{\mu_1+\mu_2}}(G_{\mu_1} \cap G_{\mu_2})^c$$

with $g \in G$ and where the superscript ρ denotes the closed subgroup of $N(G_{\mu_1} \cap G_{\mu_2})$ that leaves invariant the connected component of $(\mathcal{O}_{\mu_1} \times \mathcal{O}_{\mu_2})_{G_{\mu_1} \cap G_{\mu_2}}$ that contains $\mathcal{J}^{-1}(\rho)$. More explicitly, we can write the following symplectic decomposition of the polar reduced space:

$$\begin{aligned} &G/N_{G_{\mu_1+\mu_2}}(G_{\mu_1} \cap G_{\mu_2})^c \\ &\qquad = \dot{\bigcup}_{[g] \in G/N(G_{\mu_1} \cap G_{\mu_2})^c} gN(G_{\mu_1} \cap G_{\mu_2})^c / N_{G_{\mu_1+\mu_2}}(G_{\mu_1} \cap G_{\mu_2})^c. \end{aligned}$$

What we just did in the previous paragraphs for two coadjoint orbits can be inductively generalized to n orbits. We collect the results of that construction under the form of a proposition.

14.6.1 Proposition. *Let G be a Lie group, $\mathfrak{g}$ be its Lie algebra, and $\mathfrak{g}^*$ be its dual. Let $\mu_1, \ldots, \mu_n \in \mathfrak{g}^*$. Then, the homogeneous manifold*

$$G/N_{G_{\mu_1+\cdots+\mu_n}}(G_{\mu_1} \cap \ldots \cap G_{\mu_n})^c \tag{14.6.4}$$

has a natural presymplectic structure that is nondegenerate if and only if

$$\mathfrak{g}_{\mu_1+\cdots+\mu_n} = \operatorname{Lie}(N_{G_{\mu_1+\cdots+\mu_n}}(G_{\mu_1} \cap \ldots \cap G_{\mu_n})).$$

Moreover, (14.6.4) can be written as a the following disjoint union of symplectic submanifolds

$$\begin{aligned} &G/N_{G_{\mu_1+\cdots+\mu_n}}(G_{\mu_1} \cap \ldots \cap G_{\mu_n})^c \\ &= \dot{\bigcup}_{[g] \in G/N(G_{\mu_1} \cap \ldots \cap G_{\mu_n})^\rho} gN(G_{\mu_1} \cap \ldots \cap G_{\mu_n})^\rho / N_{G_{\mu_1+\cdots+\mu_n}}(G_{\mu_1} \cap \ldots \cap G_{\mu_n})^c. \end{aligned}$$