3

The Krasnoselskij Iteration

It is well known that if T is assumed to be only a nonexpansive map, then the Picard iterations $\{T^n x_0\}_{n\,\geq 0}$ need no longer converge (to a fixed point of T). In fact, in general, T need not have a fixed point, as shown by Exercises 1.15, 1.16 and 1.19.

It is the purpose of this chapter to survey some old and new results on the approximation of fixed points for nonexpansive and pseudocontractive type operators by means of Krasnoselskij iteration.

The key idea in introducing Krasnoselskij iteration is the fact that, if T_λ is the averaged mapping associated to T, then if T is nonexpansive, so is T_λ, and both have the same fixed point set, see Exercise 3.3. Furthermore, T_λ has much more asymptotic behavior than the original mapping T.

Krasnoselskij was the first to notice the regularizing effect of T_λ in the case of a uniformly convex Banach space, see also the Bibliographical Comments at the end of this chapter.

3.1 Nonexpansive Operators in Hilbert Spaces

We begin this section by proving the Browder-Gohde-Kirk fixed point theorem (Theorem 1.2), which is a basic fixed point existence result for nonexpansive operators. The proof will be given in a Hilbert space setting, suitable to many convergence theorems for the Krasnoselskij iteration.

Theorem 3.1. *Let C be a closed bounded convex subset of the Hilbert space H and $T : C \to C$ be a nonexpansive operator. Then T has at least one fixed point.*

Proof. For a fixed element v_0 in C and a number s with $0 < s < 1$, we denote

$$U_s(x) = (1 - s)v_0 + s\,Tx\ , \quad x \in C.$$

Since C is convex and closed, we deduce that $U_s : C \to C$ is a $s-$contraction and, in virtue of Theorem 1.1, it has a unique fixed point, say u_s. On the other hand, since C is closed, convex and bounded in the Hilbert space H, it is weakly compact. Hence we may find a sequence $\{s_j\}$ in (0,1) such that $s_j \to 1$ (as $j \to \infty$) and $u_j = u_{s_j}$ converges weakly to an element p of H.

Since C is weakly closed, p lies in C. We shall prove that p is a fixed point of T. If u is any arbitrary point in H, we have

$$\|u_j - u\|^2 = \|(u_j - p) + (p - u)\|^2 = \|u_j - p\|^2 + \|p - u\|^2 + 2\langle u_j - p, p - u\rangle,$$

where

$$2\langle u_j - p, p - u\rangle \to 0 \quad (\text{as } j \to \infty),$$

since $u_j - p$ converges weakly to zero in H. Setting $u = Tp$ above, we obtain

$$\lim_{j\to\infty}\left(\|u_j - Tp\|^2 - \|u_j - p\|^2\right) = \|p - Tp\|^2 .$$

Moreover, since $s_j \to 1$ and $U_{s_j} u_j = u_j$, we have

$$Tu_j - u_j = [s_j Tu_j + (1 - s_j)\, v_0] - u_j + (1 - s_j)[Tu_j - v_0] =$$

$$= (U_{s_j} u_j - u_j) + (1 - s_j)(Tu_j - v_0) = 0 + (1 - s_j)(Tu_j - v_0) \to 0,$$

as $j \to \infty$, and therefore $\lim_{j\to\infty} \|Tu_j - u_j\| = 0$.

On the other hand, since T is nonexpansive, we have

$$\|Tu_j - Tp\| \le \|u_j - p\|$$

and hence

$$\|u_j - Tp\| \le \|u_j - Tu_j\| + \|Tu_j - Tp\| \le \|u_j - Tu_j\| + \|u_j - p\| .$$

Thus

$$\limsup\,(\|u_j - Tp\| - \|u_j - p\|) \le \lim_{j\to\infty} \|u_j - Tu_j\| = 0$$

and, due to the boundedness of C, we have also

$$\limsup\left(\|u_j - Tp\|^2 - \|u_j - p\|^2\right) =$$

$$= \limsup\,(\|u_j - Tp\| - \|u_j - p\|)\,(\|u_j - Tp\| + \|u_j - p\|) \le 0,$$

which yields

$$\lim_{j\to\infty}\left(\|u_j - Tp\|^2 - \|u_j - p\|^2\right) = 0$$

and hence

$$\|p - Tp\|^2 = 0,$$

that is, p is a fixed point of T. □

Remark. Even if the proof of Theorem 3.1 is more constructive than the corresponding version of this result in uniformly convex Banach spaces (Theorem 1.2), it does not provide a method for computation of fixed points.

Definition 3.1. Let H be a Hilbert space and C a subset of H. A mapping $T : C \to H$ is called *demicompact* if it has the property that whenever $\{u_n\}$ is a bounded sequence in H and $\{Tu_n - u_n\}$ is strongly convergent, then there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ which is strongly convergent.

We can give now a result on approximating fixed points of nonexpansive mappings by means of the Krasnoselskij iteration. To this end, we start by proving the next Lemma.

Lemma 3.1. *Let C be a bounded closed convex subset of a Hilbert space H and $T : C \to C$ be a nonexpansive and demicompact operator. Then the set F_T of fixed points of T is a nonempty convex set.*

Proof. Since T is nonexpansive, by Theorem 3.1, T has fixed points in C, that is, $F_T \neq \emptyset$. Furthermore, F_T is convex, i.e., when $x, y \in F_T$ and $\lambda \in [0, 1]$ we have

$$u_\lambda = (1 - \lambda)x + \lambda y \in F_T.$$

Indeed,

$$\| Tu_\lambda - x\| = \| Tu_\lambda - Tx\| \leq \| u_\lambda - x\| \quad \text{and} \quad \| Tu_\lambda - y\| \leq \| u_\lambda - y\|,$$

which imply that

$$\| x - y\| \leq \| x - Tu_\lambda\| + \| Tu_\lambda - y\| \leq \| x - y\|.$$

This shows that for some a, b with $0 \leq a,\ b \leq 1$, we have

$$x - Tu_\lambda = a(x - u_\lambda) \quad \text{and} \quad y - Tu_\lambda = b(y - u_\lambda)$$

from which it follows that $Tu_\lambda = u_\lambda \in F_T$. □

Theorem 3.2. *Let C be a bounded closed convex subset of a Hilbert space H and $T : C \to C$ be a nonexpansive and demicompact operator. Then the set F_T of fixed points of T is a nonempty convex set and for any given x_0 in C and any fixed number λ with $0 < \lambda < 1$, the Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty}$ given by*

$$x_{n+1} = (1 - \lambda)x_n + \lambda T x_n, \quad n = 0, 1, 2, \ldots \tag{1}$$

converges (strongly) to a fixed point of T.

Proof. The first part follows by Lemma 3.1.

For any $x_0 \in C$, the sequence $\{x_n\}_{n=0}^{\infty}$ given by (1) lies in C and is bounded. Let p be a fixed point of T, and, so of the averaged map U_λ, given by

$$U_\lambda = (1 - \lambda)I + \lambda T \quad (I \ = \text{the identity map}). \tag{2}$$

We first prove that the sequence $\{x_n - Tx_n\}_{n \in \mathbb{N}}$ converges strongly to zero. Indeed

$$x_{n+1} - p = (1-\lambda)\, x_n + \lambda\, T x_n - p = (1-\lambda)(x_n - p) + \lambda(T x_n - p).$$

On the other hand, for any constant a,

$$a(x_n - T x_n) = a(x_n - p) - a(T x_n - p).$$

Then

$$\| x_{n+1} - p\|^2 = (1-\lambda)^2 \, \| x_n - p\|^2 + \lambda^2 \, \| T x_n - p\|^2 + \\ +2\lambda(1-\lambda) \, \langle T x_n - p, x_n - p\rangle$$

and

$$a^2 \, \|x_n - T x_n\|^2 = a^2 \, \| x_n - p\|^2 + a^2 \, \| T x_n - p\|^2 - 2a^2 \, \langle T x_n - p, x_n - p\rangle \, .$$

Hence, summing up the corresponding sides of the preceding two inequalities and using the fact that T is nonexpansive and $Tp = p$, we get

$$\| x_{n+1} - p\|^2 + a^2 \, \| x_n - T x_n\|^2 \leq [2a^2 + \lambda^2 + (1-\lambda)^2] \cdot \| \, x_n - p\|^2 + \\ +2[\lambda(1-\lambda) - a^2] \cdot \langle T x_n - p, x_n - p\rangle \, .$$

If we choose now an a such that $a^2 \leq \lambda(1-\lambda)$, then from the last inequality we obtain

$$\| x_{n+1} - p\|^2 + a^2 \, \| x_n - T x_n\|^2 \leq \\ \leq \left(2a^2 + \lambda^2 + (1-\lambda)^2 + 2\lambda(1-\lambda) - 2a^2\right) \|x_n - p\|^2 = \|x_n - p\|^2$$

(we used the Cauchy-Schwarz inequality,

$$\langle T x_n - p, x_n - p\rangle \ \leq \ \| T x_n - P\| \cdot \| x_n - p\| \leq \| \, x_n - p\|^2 \Big) \, .$$

Letting now $a^2 = \lambda(1-\lambda) > 0$ and summing up the obtained inequality

$$a^2 \, \|x_n - T x_n\|^2 \leq \| \, x_n - p\|^2 - \| \, x_{n+1} - p\|^2$$

for $n = 0$ to $n = N$ we get

$$\lambda(1-\lambda) \sum_{n=0}^{N} \| \, x_n - T x_n\|^2 \leq \sum_{n=0}^{N} \left[\| \, x_n - p\|^2 - \| \, x_{n+1} - p\|^2 \right] = \\ = \| \, x_0 - p\|^2 - \| \, x_{N+1} - p\|^2 \leq \| \, x_0 - p\|^2 \, ,$$

which shows that $\sum\limits_{n=0}^{\infty} \| \, x_n - T x_n\|^2 < \infty$ and hence $\| \, x_n - T x_n\| \to 0$, as $n \to \infty$.

As T is demicompact, it results that there exists a strongly convergent subsequence $\{x_{n_i}\}$ such that $x_{n_i} \to p \in F_T$.

Since T is nonexpansive, $Tx_{n_i} \to Tp$ and $Tp = p$.

The convergence of the entire sequence $\{x_n\}_{n=0}^{\infty}$ to p now follows from the inequality $\| x_{n+1} - p\| \leq \| x_n - p\|$, which can be deduced from the nonexpansiveness of T and is valid for each n. □

Remarks.

1) The class of demicompact operators contains the compact operators, therefore by Theorem 3.2 we obtain, in particular, the result of Krasnoselskij [Kra55], and that of Schaefer [Sch57], established there in the more general context of uniformly convex Banach spaces;

2) From the proof of Theorem 3.2 it results that U_λ given by (2) is *asymptotically regular* , i.e., $\left\| U_\lambda^n x - U_\lambda^{n+1} x\right\| \to 0$, as $n \to \infty$, for any $x \in C$, that is,

$$x_n - x_{n+1} \to 0, \quad \text{as} \quad n \to \infty, \tag{3}$$

for any $x_0 \in C$.

The existence of the previous limit alone does not imply generally the convergence of the sequence $\{x_n\}_{n=0}^{\infty}$ to a fixed point of T (in Theorem 3.2 one additional assumption was the demicompactness of T). There are other possible additional assumptions to ensure the convergence of $\{x_n\}_{n=0}^{\infty}$ under the hypothesis of asymptotic regularity. For example, in the case of the real line, $C = [a, b]$ the closed bounded interval and $T : C \to C$ a continuous function, Hillam [Hil76] showed that the Picard iteration associated to T converges if and only if it is asymptotically regular;

3) Let us notice that the Krasnoselskij iteration is in fact the Picard iteration corresponding to the "averaged operator" U_λ associated to T and defined by (2);

4) The demicompactness on the whole D may be weakened to 0 by simultaneously adding an other assumption, to obtain the next result. A map T of $D \subset X$ into X is *demicompact at f* if, for any bounded sequence $\{x_n\}$ in D such that $x_n - T(x_n) \to f$ as $n \to \infty$, there exists a subsequence $\{x_{n_j}\}$ and an x in D such that $x_{n_j} \to x$ as $j \to \infty$ and $x - T(x) = f$. Clearly, when T is demicompact on D, it is demicompact at 0 but the converse is not true.

Corollary 3.1. *Let X be a uniformly convex Banach space, D a closed bounded convex set in X, and T a nonexpansive mapping of D into D such that T satisfies any one of the following two conditions:*
(i) (I-T) maps closed sets in D into closed sets in X;
(ii) T is demicompact at 0.

Then, for any given x_0 in C and any fixed number λ with $0 < \lambda < 1$, the Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty}$ given by (1) *converges (strongly) to a fixed point of T.*

Proof. It suffices to show that the averaged map T_λ satisfies all conditions $(a) - (e)$ in Exercise 2.14. □

Remarks.

1) Conditions (i) and (ii) in Corollary 3.1 are independent;

2) If in Theorem 3.2 we remove the assumption that T is demicompact, then the Krasnoselskij iteration does not longer converge strongly, in general, but it converges (at least) weakly to a fixed point, as shown by the next theorem.

Theorem 3.3. *Suppose T is a nonexpansive operator that maps a bounded closed convex set C of H into C and that $F_T = \{p\}$. Then the Krasnoselskij iteration converges weakly to p,*

$$U_\lambda^n x_0 \rightharpoonup p,$$

for any $x_0 \in C$.

Proof. It suffices to show that if $\{x_{n_j}\}_{j=0}^{\infty}$, $x_{n_j} = U_\lambda^{n_j} x$ converges weakly to a certain p_0, then p_0 is a fixed point of T or of U_λ and therefore $p_0 = p$. Suppose that $\{x_{n_j}\}_{j=0}^{\infty}$ does not converge weakly to p. Then

$$\| x_{n_j} - U_\lambda p_0\| \le \| U_\lambda x_{n_j} - U_\lambda p_0\| + \| x_{n_j} - U_\lambda x_{n_j}\| \le$$

$$\le \| x_{n_j} - p_0\| + \| x_{n_j} - U_\lambda x_{n_j}\|$$

and, using the arguments in the proof of Theorem 3.2, it results

$$\| x_{n_j} - U_\lambda x_{n_j}\| \to 0, \quad \text{as} \quad n \to \infty,$$

and so the last inequality implies that

$$\lim\ \sup \left(\| x_{n_j} - U_\lambda p_0\| - \| x_{n_j} - p_0\| \right) \le 0. \tag{4}$$

But, like in the proof of Theorem 3.2, we have

$$\| x_{n_j} - U_\lambda p_0\|^2 = \| (x_{n_j} - p_0) + (p_0 - U_\lambda p_0)\|^2 =$$

$$= \| x_{n_j} - p_0\|^2 + \|p_0 - U_\lambda p_0\|^2 + 2 \left\langle x_{n_j} - p_0, p_0 - U_\lambda p_0 \right\rangle,$$

which shows, together with $x_{n_j} \rightharpoonup p_0$ (as $j \to \infty$), that

$$\lim_{n\to\infty} \left[\| x_{n_j} - U_\lambda p_0\|^2 - \|x_{n_j} - p_0\|^2 \right] = \|p_0 - U_\lambda p_0\|^2 . \tag{5}$$

On the other hand, we have

$$\| x_{n_j} - U_\lambda p_0\|^2 - \|x_{n_j} - p_0\|^2 = \left(\| x_{n_j} - U_\lambda p_0\| - \| x_{n_j} - p_0\| \right) \cdot$$

$$\cdot \left(\| x_{n_j} - U_\lambda p_0\| + \| x_{n_j} - p_0\| \right) . \tag{6}$$

Since C is bounded, the sequence $\left\{ \| x_{n_j} - U_\lambda p_0\| + \| x_{n_j} - p_0\| \right\}$ is bounded, too, and by the relations (4)-(6) we get

$$\|p_0 - U_\lambda p_0\| \leq 0, \quad \text{i.e.} \quad U_\lambda p_0 = p_0 \Leftrightarrow p_0 \in F_T = \{p\},$$

which ends the proof. □

Remark. The assumption $F_T = \{p\}$ in Theorem 3.3 may be removed in order to obtain a more general result.

Theorem 3.4. *Let C be a bounded closed convex subset of a Hilbert space and $T : C \to C$ be a nonexpansive operator. Then, for any x_0 in C, the Krasnoselskij iteration converges weakly to a fixed point of T.*

Proof. Let F_T be the set of all fixed points of T in C (which is nonempty, by Theorem 3.1, and convex, by Lemma 3.1). As T is nonexpansive, for each $p \in F_T$ and each n we have

$$\| x_{n+1} - p\| \leq \| x_n - p\| ,$$

which shows that the function $g(p) = \lim\limits_{n\to\infty} \| x_n - p\|$ is well defined and is a lower semicontinuous convex function on F_T. Let

$$d_0 = \inf\{g(p) : p \in F_T\}.$$

For each $\varepsilon > 0$, the set

$$F_\varepsilon = \{y : g(y) \leq d_0 + \varepsilon\}$$

is closed, convex, nonempty and bounded and, hence, weakly compact. Therefore $\bigcap\limits_{\varepsilon>0} F_\varepsilon \neq \emptyset$, and in fact

$$\bigcap_{\varepsilon>0} F_\varepsilon = \{y : g(y) = d_0\} \equiv F_0.$$

Moreover, F_0 contains exactly one point. Indeed, since F_0 is convex and closed, for $p_0, p_1 \in F_0$, and $p_\lambda = (1-\lambda)p_0 + \lambda p_1$,

$$g^2(p_\lambda) = \lim_{n\to\infty} \| p_\lambda - x_n\|^2 = \lim_{n\to\infty} (\|\lambda(p_1 - x_n) + (1-\lambda)(p_0 - x_n)\|^2) =$$

$$= \lim_{n\to\infty} (\lambda^2 \|p_1 - x_n\|^2 + (1-\lambda)^2 \| p_0 - x_n\|^2 +$$

$$+2\lambda(1-\lambda) \langle p_1 - x_n, p_0 - x_n\rangle) = \lim_{n\to\infty} (\lambda^2 \|p_1 - x_n\|^2 +$$

$$+(1-\lambda)^2 \| p_0 - x_n\|^2 + 2\lambda(1-\lambda) \| p_1 - x_n\| \cdot \| p_0 - x_n\|)+$$

$$+ \lim_{n\to\infty} \{2\lambda(1-\lambda) \, [\langle p_1 - x_n, p_0 - x_n\rangle - \| p_1 - x_n\| \cdot \| p_0 - x_n\|]\} =$$

$$= g^2(p) + \lim_{n\to\infty} \{2\lambda(1-\lambda) \; \langle p_1 - x_n, p_0 - x_n\rangle - \| p_1 - x_n\| \cdot \| p_0 - x_n\|\} .$$

Hence

$$\lim_{n\to\infty} \{2\lambda(1-\lambda) \, [\langle p_1 - x_n, p_0 - x_n\rangle - \| p_1 - x_n\| \cdot \| p_0 - x_n\|]\} = 0.$$

Since
$$\|p_1 - x_n\| \to d_0 \text{ and } \| p_0 - x_n\| \to d_0,$$
the latter relation implies that
$$\| p_1 - p_0\|^2 = \| (p_1 - x_n) + (x_n - p_0\|^2 = \| p_1 - x_n\|^2 +$$
$$+ \| x_n - p_0\|^2 - 2 < p_1 - x_n, p_0 - x_n > \to d_0^2 - d_0^2 - 2d_0^2 = 0,$$
giving a contradiction.

Now, in order to show that $x_n = U_\lambda^n x_0 \rightharpoonup p_0$, is suffices to assume that $x_{n_j} \rightharpoonup p$ for an infinite subsequence and then prove that $p = p_0$. By the arguments in Theorem 3.3, $p \in F_T$. Considering the definition of g and the fact that $x_{n_j} \to p$, we have
$$\left\| x_{n_j} - p_0\right\|^2 = \left\| x_{n_j} - p + p - p_0\right\|^2 = \left\| x_{n_j} - p\right\|^2 + \| p - p_0\|^2 -$$
$$-2 \left\langle x_{n_j} - p, p - p_0\right\rangle \to g^2(p) + \| p - p_0\|^2 = g^2(p_0) = d_0^2.$$
Since $g^2(p) \geq d_0^2$, the last inequality implies that
$$\| p - p_0\| \leq 0,$$
which means that $p = p_0$. □

3.2 Strictly Pseudocontractive Operators

In this section we present some convergence theorems for the Krasnoselskij iteration scheme in the class of pseudocontractive operators. The first of them is concerned with the computation of fixed points of strictly pseudocontractive operators.

Theorem 3.5. *Let C be a bounded closed convex subset of a Hilbert space and $T : C \to C$ be a strictly pseudocontractive operator, i.e., an operator for which there exists a constant $k < 1$ such that*
$$\|Tx - Ty\|^2 \leq \| x - y\|^2 + k \| (I - T) x - (I - T) y\|^2 , \quad x, y \in C. \tag{7}$$

Then, for any x_0 in C and any fixed μ such that $\mu < 1-k$ the Krasnoselskij iteration $\{x_n\}_{n=0}^\infty$, given by $x_0 \in C$ and
$$x_{n+1} = (1 - \mu) x_n + \mu T x_n , \qquad n = 0, 1, 2, \ldots , \tag{8}$$
converges weakly to a fixed point p of T.

If, additionally, we assume that T is demicompact, then $\{x_n\}_{n=0}^\infty$ converges strongly to p.

Proof. We denote as usually $T_t = (1-t)\,I + t\,T$ and show that T_t is nonexpansive. Indeed, by the pseudocontractiveness condition (7) it follows that $U = I - T$ is strongly monotone, i.e.,

$$< Ux - Uy, x - y > \geq \; m \parallel Ux - Uy \|^2 , \quad \text{with} \;\; m = \frac{1-k}{2} > 0.$$

Then, for any $t > 0$

$$\parallel T_t x - T_t y \|^2 = \parallel (I - tU)\,x - (I - tU)\,y \|^2 =$$

$$= \|x - y\|^2 + t^2 \parallel Ux - Uy \|^2 - 2t \;\; < Ux - Uy\,,\; x - y \, > \, \leq$$

$$\leq \parallel x - y \|^2 + (t^2 - 2t\,m) \;\parallel Ux - Uy \|^2 .$$

Now, if we take $t \leq 2m = 1 - k$, then from the preceding inequality we obtain

$$\parallel T_t x - \lambda_t y \| \leq \parallel x - y \| \;, \quad x, y \in C,$$

which shows that T_t is nonexpansive.

Now, by Theorem 3.4, T_t (and therefore T) has a fixed point p_0 in C and for any fixed λ with $0 < \lambda < 1$, the Krasnoselskij iteration $x_n = (T_t)^n_\lambda (x_0)$ associated to T_t converges weakly to some fixed point p of T in C.

But the iteration function $(T_t)_\lambda$ is in fact

$$(T_t)_\lambda = (1-\lambda)\,I + \lambda\,T_t = (1-\lambda)\,I + \lambda[(1-t)\,I + t\,T] = (1-\lambda t)\,I + \lambda\,t\,T = T\mu,$$

with $\mu = \lambda t < t \leq 1 - k$.

In order to prove the second part of the theorem, based on Theorem 3.3, it suffices to show that T_μ is demicompact. But this follows immediately from the demicompactness of T using the equality

$$T_\mu x - x = \mu\,(Tx - x),$$

valid for every x in C. □

3.3 Lipschitzian and Generalized Pseudocontractive Operators

Even though there is a rather strong connection between strictly pseudocontractive operators and generalized pseudocontractive operators, these two classes are however independent each other.

This is the motivation why, in addition to the short previous section, we consider here generalized pseudocontractions which are also Lipschitzian, a class for which we can use the Krasnoselskij iteration in order to approximate their fixed points.

Definition 3.2. Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. An operator $T : H \to H$ is said to be a *generalized pseudo-contraction* if there exists a constant $r > 0$ such that, for all x, y in H,

$$\| Tx - Ty\|^2 \leq r^2 \ \| x - y \|^2 + \| Tx - Ty - r(x - y) \|^2 . \tag{9}$$

Remarks.

1) Condition (9) is equivalent to

$$\langle Tx - Ty, \ x - y \rangle \ \leq \ r \ \| \ x - y\|^2 , \quad \text{for all} \quad x, y \in H, \tag{10}$$

or to

$$\langle (I - T)\, x - (I - T)\, y \rangle \ \geq \ (1 - r) \ \| \ x - y\|^2 . \tag{11}$$

Relation (11) implies that $U = I - T$ is strongly monotone for $r < 1$.

2) If $r = 1$, then a generalized pseudo-contraction reduces to a pseudo-contraction;

3) By the Cauchy-Schwarz inequality

$$| \ \langle Tx - Ty, \, x - y \, \rangle | \leq \| \ Tx - Ty\| \cdot \| \ x - y\| ,$$

we obtain that any Lipschitzian operator T, that is, any operator for which there exists $s > 0$ such that

$$\| \ Tx - Ty\| \leq s \cdot \| \ x - y\| \ , \quad x, y \in H, \tag{12}$$

is also a generalized pseudo-contractive operator, with $r = s$.

This, however, does not exclude the possibility that a certain operator T be simultaneously Lipschitzian with constant s, and generalized pseudo-contractive with constant r, and $r < s$. The existence of the last inequality is, in fact, the only reason of considering together Lipschitzian and generalized pseudo-contractive operators.

4) On the other hand, Theorem 3.6 below is obtained under the essential assumptions $r < 1$ and $s \geq 1$. Consequently, in the following, we shall assume that the Lipschitzian constant s and the generalized pseudo-contractivity constant r fulfill the conditions

$$0 < r < 1 \quad \text{and} \quad r \leq s. \tag{13}$$

Example 3.1. Let H be the real line $\mathbb{R}$ endowed with the Euclidean inner product and norm, $K = \left[\frac{1}{2}, 2\right]$ and $T : K \to K$ a function given by $Tx = \dfrac{1}{x}$, for all x in K.

Then T is Lipschitzian with constant $s = 4$ (so T is also generalized pseudo-contractive with constant $r = 4$).

Moreover, T is generalized pseudocontractive with any constant $r > 0$. It is easy to see that T has a unique fixed point, $F_T = \{1\}$, and that, for any initial choice $x_0 = a \neq 1$, the Picard iteration yields the oscillatory sequence

$$a, \frac{1}{a}, a, \frac{1}{a}, \dots$$

Theorem 3.6. *Let K be a non-empty closed convex subset of a real Hilbert space and $T : K \to K$ a generalized pseudocontractive and Lipschitzian operator with the corresponding constants r and s fulfilling (13). Then*

(i) T has an unique fixed point p;

(ii) for each x_0 in K, the Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty}$, given by

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n\,, \quad n = 0, 1, 2, \dots, \tag{14}$$

converges (strongly) to p, for all $\lambda \in (0,1)$ satisfying

$$0 < \lambda < 2(1-r)/(1-2r+s^2). \tag{15}$$

(iii) Both the a priori

$$\| x_n - p\| \le \frac{\theta^n}{1-\theta} \cdot \| x_1 - x_0\|\ ,\ n = 1, 2, \dots \tag{16}$$

and a posteriori

$$\| x_n - p\| \le \frac{\theta}{1-\theta} \cdot \| x_n - x_{n-1}\|\ , \quad n = 1, 2, \dots \tag{17}$$

estimates hold, with

$$\theta = \left((1-\lambda)^2 + 2\lambda(1-\lambda)\, r + \lambda^2 s^2\right)^{1/2}. \tag{18}$$

Proof. We consider the averaged operator F associated to T,

$$Fx = (1-\lambda)x + \lambda \cdot Tx\,, \quad x \in K, \tag{19}$$

for all $\lambda \in [0,1]$. Since K is convex, we have that $F(K) \subset K$ for each $\lambda \in [0,1]$.

As a closed subset of a Hilbert space, K is a complete metric space. We claim that F is a $\theta-$contraction with θ given by (18).

Indeed, since T is generalized pseudo-contractive and Lipschitzian, we have

$$\|Fx - Fy\|^2 = \|\,(1-\lambda)\,x + \lambda\, Tx - (1-\lambda)\,y - \lambda Ty\|^2 =$$

$$= \|\,(1-\lambda)(x-y) + \lambda(Tx - Ty)\|^2 = (1-\lambda)^2 \cdot \|\,x-y\|^2 +$$

$$+2\lambda(1-\lambda)\cdot\langle Tx - Ty,\, x-y\rangle + \lambda^2 \cdot \|Tx - Ty\|^2 \le$$

$$\le \left((1-\lambda)^2 + 2\lambda(1-\lambda)r + \lambda^2 s^2\right)\cdot \|\,x-y\|^2\,,$$

which yields

$$\|Fx - Fy\| \le \theta \cdot \|\,x-y\|\,, \quad \text{for all} \quad x, y \in K.$$

In view of condition (15), it results that $0 < \theta < 1$, so the mapping F is a $\theta-$contraction. In order to obtain the conclusion we now apply the contraction mapping principle (Theorem 2.1) for the operator F and the complete metric space K. □

Remarks.

1) The *a priori* estimate (16) in Theorem 3.6 shows that the Krasnoselskij iteration converges to p at least as fast as the geometric series of ratio θ;

2) The Krasnoselskij iteration solves several situations when the Picard iteration does not converge.

Example 3.2. Let K be as in Example 3.1. Here $s = 4$ and $r > 0$ arbitrary. Taking, for example, $r = 0.5$ we get

$$2(1-r)/(1-2r+s^2) = 1/16,$$

and so, by Theorem 3.6, the sequence $\{x_n\}_{n=0}^{\infty}$ given by

$$x_{n+1} = (1-\lambda)\cdot x_n + \lambda \cdot \frac{1}{x_n}, \qquad n = 0, 1, 2, \ldots \tag{20}$$

converges strongly to the fixed point $p = 1$ of T, for all values of λ in the interval $\left(0, \dfrac{1}{16}\right)$.

Remark. It is of interest to answer the following question: amongst all the Krasnoselskij iterations $\{x_n\}_{n=0}^{\infty}$ in the family (14), obtained when λ ranges the interval $(0, a)$, with

$$a = \frac{2(1-r)}{(1-2r+s^2)},$$

is there a certain iteration to be the fastest one (in that family) ?

To answer this question, we shall adopt a suitable concept of convergence rate.

Let $\{x_n\}$ and $\{y_n\}$ be two sequences that converge to p (as $n \to \infty$), satisfying the estimate (16) with $\theta = \theta_1$ and $\theta = \theta_2$, respectively, and such that $\theta_1, \theta_2 \in (0, 1)$. We shall say that $\{x_n\}$ *converges faster than* $\{y_n\}$ if

$$\theta_1 < \theta_2.$$

Equipped now with this concept of rate of convergence, Theorem 3.7 below answers in the affirmative the previous question.

Theorem 3.7. *Let all assumptions in Theorem 3.6 be satisfied. Then the fastest iteration* $\{x_n\}_{n=0}^{\infty}$ *in the family (14), with* $\lambda \in (0, a)$, *is the one obtained for*

$$\lambda_{\min} = (1-r)\,/\,(1-2r+s^2). \tag{21}$$

Proof. We have to find the minimum of the quadratic function

$$f(x) = (1-x)^2 + 2x(1-x)\,r + x^2 s^2,$$

with respect to x, that is to minimize the function

$$f(x) = (1-2r+s^2)\,x^2 - 2(1-r)\,x + 1\,, \quad x \in (0,a),$$

with a given by

$$a = 2(1-r)/(1-2r+s^2). \tag{22}$$

This is an elementary task. Indeed from (13) we have that

$$1-2r+s^2 \geq (1-r)^2 > 0,$$

and hence f does admit a minimum, which is attained for

$$x = \lambda_{\min},$$

with $\lambda_{\min}$ given by (21). The minimum value of $f(x)$ is then

$$f_{\min} = (s^2-r^2)/(1-2r+s^2),$$

which shows that the minimum value of θ given by (18) is

$$\theta_{\min} = \left((s^2-r^2)\,/\,(1-2r+s^2)\right)^{1/2},$$

that completes the proof. □

Remarks.

1) It is important to notice that if $s < 1$, that is, T is actually a $s-$contraction, then $a > 1$ and hence $\lambda = 1 \in (0,a)$. This shows that among all Krasnoselskij iterations (14) that converge to the fixed point of T, we also find the Picard iteration associated to T, which is obtained from (14) for $\lambda = 1$. (This of course does not happen if $s \geq 1$);

2) As for the Picard iteration we have a similar a priori estimation, we can compare the Picard iteration to the fastest Krasnoselskij iteration in the family (14), with $\lambda \in (0,a)$:

a) If $r = s^2 < 1$, then we have

$$\theta_{\min} = s,$$

which means that the fastest Krasnoselskij iteration in the family (14) coincides with the Picard iteration itself;

b) If $r \neq s^2$, then it is easy to check that we have

$$\theta_{\min} < s,$$

(since $s < 1$), which shows that the Krasnoselskij iteration (14) with $\lambda = \lambda_{\min}$ is *faster* than the Picard iteration associated to T.

In this case, the fastest iteration from (14) may be regarded as an *accelerating* procedure of the Picard iteration.

Example 3.3. For T and K as in Examples 3.1 and 3.2, and for a certain $r \in (0,1)$, we obtain the fastest Krasnoselskij iteration for

$$\lambda = (1-r)\,/\,(1-2r+16).$$

If we take $r = 0.5$, then (14) converges for each $\lambda \in \left(0, \dfrac{1}{16}\right)$. The fastest Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty}$ in this family is then obtained for $\lambda = \dfrac{1}{32}$, and is given by

$$x_{n+1} = \frac{1}{32}\left(31\,x_n + \frac{1}{x_n}\right), \quad n = 0, 1, 2, \ldots .$$

The averaged operator F,

$$F(x) = \frac{1}{32}\left(31\,x + \frac{1}{x}\right),$$

associated to T is a contraction and has the contraction coefficient

$$\theta_{\min} = \frac{\sqrt{63}}{8} = 0.992,$$

which is very close to 1.

The fastest Krasnoselskij iteration obtained in this way, converges very slowly to $p = 1$, the fixed point of T, as shown by the next Example.

Example 3.4. Starting with $x_0 = 1.5$, and $x_0 = 1.25$, respectively, the first 32 iterations are the following:

n	x_n	n	x_n	n	x_n	n	x_n
0	1.5	16	1.203	0	1.25	16	1.0960
1	1.473	17	1.191	1	1.2359	17	1.0902
2	1.449	18	1.180	2	1.2226	18	1.0848
3	1.425	19	1.170	3	1.2100	19	1.0797
4	1.402	20	1.160	4	1.1980	20	1.0749
5	1.381	21	1.151	5	1.1866	21	1.0704
6	1.360	22	1.142	6	1.1759	22	1.0662
7	1.341	23	1.133	7	1.1657	23	1.0584
8	1.322	24	1.126	8	1.1561	24	1.0515
9	1.304	25	1.118	9	1.1470	25	1.0484
10	1.287	26	1.111	10	1.1384	26	1.0454
11	1.271	27	1.105	11	1.1303	27	1.0426
12	1.256	28	1.098	12	1.1226	28	1.0400
13	1.242	29	1.087	13	1.1153	29	1.0376
14	1.228	30	1.082	14	1.1085	30	1.0353
15	1.215	31	1.077	15	1.1021	31	1.0331

3.4 Pseudo φ-Contractive Operators

In this section we want to show how we can unify in a single concept various notions as nonexpansive, Lipschitzian, pseudo-contractive type operators etc. For this new class of operators, called pseudo φ-contractive, we shall prove a convergence theorem for the Krasnoselskij fixed point procedure.

Let H be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\|\cdot\|$. For the operators $T : H \to H$, let us denote by

1) C_0, the class of $a-$contractions, $0 \leq a < 1$;
2) C_1, the class of nonexpansive operators;
3) C_2, the class of strictly pseudo-contractive operators;
4) C_3, the class of pseudo-contractive operators;
5) C_4, the class of generalized pseudo-contractive operators.

The next lemmas are immediate consequences of the results given in the previous sections and chapters.

Lemma 3.2.
1) $T \in C_3$ *if and only if*

$$\langle Tx - Ty,\, x - y \rangle \ \leq \ \| x - y\|^2 , \quad \text{for all} \quad x, y \in H;$$

2) $T \in C_3$ *if and only if*

$$\langle (I - T)\, x - (I - T)\, y,\ x - y \rangle \ \geq \ 0, \text{ for all } \ x, y \in H.$$

Lemma 3.3.
1) $T \in C_4$ *if and only if there exists* $r > 0$ *such that*

$$\langle Tx - Ty,\, x - y \rangle \ \leq \ r \cdot \| x - y\|^2 , \quad \text{for all} \ \ x, y \in H;$$

2) $T \in C_4$ *if and only if there exists* $r > 0$ *such that*

$$\langle (I - T)\, x - (I - T)\, y,\ x - y \rangle \ \geq \ (1 - r) \cdot \| x - y\|^2 , \quad \text{for all} \ \ x, y \in H.$$

Lemma 3.4. $T \in C_2$ *if and only if there exists* $k > 0$ *such that*

$$\langle (I - T)\, x - (I - T)\, y,\ x - y \rangle \ \geq \ k \cdot \| x - y\|^2 , \quad \text{for all} \ \ x, y \in H.$$

Remark. It is also easy to prove the following inclusions

$$C_0 \subset C_1 \subset C_2 \subset C_3 \subset C_4.$$

Definition 3.3. An operator $T : H \to H$ is said to be (*strictly) pseudo φ-contractive* if, for any $a, b, c \in \mathbb{R}$ with $a+b+c = 1$, there exists a (comparison) function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, such that

$$a \cdot \| x-y\|^2 + b \cdot \langle Tx - Ty,\, x-y\rangle + c \cdot \| Tx - Ty\|^2 \leq \varphi^2 \left(\| x-y\| \right), \tag{23}$$

holds, for all x, y in H.

Example 3.4.

1) Any Lipschitzian operator T is pseudo $\varphi-$contractive with $a = 0, b = 0, c = 1$ and $\varphi(t) = t$;

2) Any pseudo-contractive operator is also of pseudo $\varphi-$contractive type with $a = 0, b = 1, c = 0$ and $\varphi(t) = t$;

3) Any generalized pseudo-contractive operator is a (strictly, if $r < 1$) pseudo $\varphi-$contractive operator, with $a = 0, b = 1,\ c = 0$ and $\varphi(t) = r{\cdot}t, r > 0$;

4) Any strictly pseudocontractive operator is a pseudo $\varphi-$contractive operator, with $a = \dfrac{k-1}{2k},\ b = 1,\ c = \dfrac{1-k}{2k}$ and $\varphi(t) = t$;

5) Any strongly pseudocontractive operator is a pseudo $\varphi-$contractive operator, with $a = \dfrac{r\,t}{2(1+r)}\,,\ b = 1\,,\ c = -\dfrac{r\,t}{2(1+r)}, \varphi(u) = \dfrac{rt^2 + 2r + 2}{2t(r+1)} \cdot u$.

There are many convergence theorems concerning the approximation of fixed points for several classes of pseudocontractive type operators. The next theorem shows that the Krasnoselskij iteration converges to a fixed point of any strictly pseudo $\varphi-$contraction.

Theorem 3.8. *Let K be a nonempty closed convex subset of a real Hilbert space H and $T : K \to K$ a strictly pseudo $\varphi-$contractive operator. Then*

(i) *T has an unique fixed point p in K;*

(ii) *For each $x_0 \in K$, the Krasnoselskij iteration $\{x_n\}_{n=0}^{\infty}$ given by (14) converges strongly to p, for all $\lambda \in (0, 1)$;*

(iii) *If, additionally, φ is a $(c)-$comparison function, then*

$$\| x_n - p\| \leq s\left(\| x_n - x_{n+1}\|\right), \qquad n = 1, 2, \ldots$$

(*where $s(t) = \sum\limits_{k=0}^{\infty} \varphi^k(t)$ denotes the sum of the comparison series*).

Proof. The proof is similar to that of Theorem 3.6. We consider the associated operator

$$Fx = (1-\lambda)\, x + \lambda\, Tx\,, \quad x \in K$$

and show that $F : K \to K$ is a $\varphi-$contraction. Indeed, by (23) we get

$$\| Fx - Fy\|^2 \leq \varphi^2 \left(\| x-y\|\right), \quad \text{for all } \ x, y \in K,$$

which shows that F is a $\varphi-$contraction.

Now, by Theorems 2.7 and 2.8, the conclusion immediately follows. $\square$

Remarks.

1) If T is not a strictly pseudo $\varphi-$contraction, then Theorem 3.8 is no longer valid;

2) We can obtain a result similar to the one given by Theorem 2.10 by considering in the right hand side of (23) the expression

$$\varphi^2 \left(\| x-y\| , \| x-Tx\| , \| y-Ty\| , \| x-Ty\| , \| y-Tx\| \right),$$

given by a 5-dimensional comparison function rather than a one-dimensional function;

3) If T is Lipschitzian and generalized pseudocontractive (with $r<1$), then by Theorem 3.8 we obtain exactly Theorem 3.6, by taking the most used comparison function, i.e.,

$$\varphi(t) = r \cdot t;$$

4) The next two examples illustrate why we needed to consider special classes of pseudocontractive operators and not simply pseudocontractive operators in some of the convergence theorems stated in this chapter.

Example 3.5. Let $\mathbb{R}$ denote the reals with the usual norm, $K=[0,1]$ and define $T: K \to \mathbb{R}$ by $Tx = \dfrac{1}{2}x+1$. Then T is a $\dfrac{1}{2}$-contraction and hence is strongly pseudocontractive, but T has no fixed points in K.

Example 3.6. Let $\mathbb{R}$ denote the reals with the usual norm, $K=\{1,2\}$ and define $T: K \to K$ by $T(1)=2, T(2)=1$. Then T is strongly pseudocontractive, but T has no fixed point in K.

3.5 Quasi Nonexpansive Operators

The convergence of Picard iteration for two classes of particular quasi nonexpansive operators was studied in Section 2.3, see also Exercise 2.14, which gives a convergence theorem for the whole class of quasi nonexpansive operators, when some additional assumptions are satisfied.

In the case of Hilbert spaces, see Exercise 3.5, it is known that nonexpansive operators are asymptotically regular. Since quasi nonexpansive operators strictly include the nonexpansive ones, even though a quasi nonexpansive operator is generally not asymptotically regular, however, its averaged operator is asymptotically regular in the case of uniformly Banach spaces, as the next Lemma shows.

Lemma 3.5. *Let X be a uniformly convex Banach space, D a subset of X, and T a mapping of D into X such that $F_T \neq \emptyset$ and T is quasi nonexpansive. Let T_λ be the averaged operator associated to T, i.e.,*

$$T_\lambda(x) = (1-\lambda)x + \lambda Tx,\ x \in D.$$

If there exists $x_0 \in D$ and $\lambda \in (0,1)$ such that the Krasnoselskij iteration $\{T_\lambda^n(x_0)\}$ is defined and lies in D for each $n \geq 1$, then T_λ is asymptotically regular at x_0, that is,

$$\lim_{n\to\infty}[T_\lambda^n(x_0) - T_\lambda^{n+1}(x_0)] = 0.$$

Proof. Let p be any element in F_T and let x_0 be a point in D satisfying the conditions above. T_λ is also quasi nonexpansive since $F_{T_\lambda} = F_T \neq \emptyset$ and for all x in D we have

$$\|T_\lambda(x) - p\| = \|\lambda x - \lambda p + (1-\lambda)(Tx - p)\| \leq \lambda \|x - p\| + (1-\lambda) \|x - p\| =$$

$$= \|x - p\| .$$

This implies

$$\|x_{n+1} - p\| = \|T_\lambda x_n - p\| \leq \|x_n - p\| , \text{ for each } n \geq 1,$$

and therefore $\{\|x_n - p\|\}$ converges to some $d_0 \geq 0$.

If $d_0 = 0$, then $\lim\limits_{n\to\infty} x_n = p$ and so in this case $x_n - x_{n+1} = T_\lambda^n(x_0) - T_\lambda^{n+1}(x_0) \to 0$ as $n \to \infty$, as required. In the case $d_0 > 0$, since $\|x_n - p\| \to d_0$, $\|T_\lambda x_n - p\| \leq \|x_n - p\|$ for each n, and

$$\lim_{n\to\infty} \|T_\lambda x_n - p\| = \lim_{n\to\infty} \|x_n - p\| = d_0,$$

it follows from the uniform convexity of X that

$$\lim_{n\to\infty} \|(x_n - p) - (T_\lambda x_n - p)\| = 0,$$

i.e.,

$$\lim_{n\to\infty} \|(x_n - T_\lambda x_n\| = \lim_{n\to\infty} \left\|T_\lambda^n(x_0) - T_\lambda^{n+1}(x_0)\right\| = 0. \qquad \square$$

The following Lemma will be also useful to prove the main result of this section and is important by itself.

Lemma 3.6 *Let X be a strictly convex Banach space and D a closed convex subset of X. If T is a continuous mapping of D into X such that $F_T \neq \emptyset$ and*

$$\|Tx - p\| \leq \|x - p\| , \textit{ for } x \in D \setminus F_T \textit{ and } p \in F_T, \tag{24}$$

then F_T is a convex set.

Proof. Let x and y be any two distinct points of F_T and, for $t \in (0,1)$, denote $z_t = tx + (1-t)y$. Since D is convex, $z_t \in D$. Suppose, contrary to our assertion, that $z_t \notin F_T$ for some $t \in (0,1)$. This means $z_t \in D \setminus F_T$. Then, it follows by (24) that

$$\|x-y\| \leq \|x-T(z_t)\| + \|T(z_t)-y\| \leq \|x-z_t\| + \|z_t-y\|.$$

Since X is strictly convex, we have that

$$x - T(z_t) = a(T(z_t) - y), \text{ for some } a > 0,$$

from which we obtain

$$T(z_t) = \frac{1}{1+a}x + \frac{a}{1+a}y,$$

which shows that $T(z_t)$ lies on the line determined by x and y. On the other hand,

$$\|x-T(z_t)\| \leq \|x-z_t\| \text{ and } \|T(z_t)-y\| \leq \|z_t-y\|.$$

Thus $T(z_t)$ must coincide with z_t. □

In the last part of this section we are interested to obtain convergence theorems for Krasnoselskij iteration under the basic assumption that T or T_λ is strictly quasi nonexpansive and that T satisfies the so-called Frum-Ketkov contractive condition. To this end we also need the following lemma.

Lemma 3.7. *Let D be a closed convex subset of X and T a selfmap of D such that*

$$d(T(x), K) \leq kd(x, K), \text{ for all } x \in D \tag{25}$$

for some convex compact set K in X and constant $k < 1$. If $T_\lambda = \lambda I + (1-\lambda)T$ is the averaged mapping and $\lambda \in (0,1)$, then

$$d(T_\lambda(x), K) \leq k_\lambda d(x, K), \text{ for each } x \in D, \tag{26}$$

where $k_\lambda = \lambda + (1-\lambda)k < 1$.

Proof. Let λ be fixed in $(0,1)$, and $x \in D$, fixed. Since clearly $0 < k_\lambda < 1$, it suffices to prove (26).

For a given $\delta > 0$, there exist $y_\delta \in K$ and $z_\delta \in K$ such that

$$\|x-y_\delta\| \leq d(x,K) + \delta/(2\lambda), \quad \|Tx - z_\delta\| \leq d(Tx, K) + \delta/(2(1-\lambda)).$$

Let $w_\lambda = \lambda y_\delta + (1-\lambda)z_\delta$. Since K is convex, we have $w_\lambda \in K$. Then

$$\begin{aligned} d(T_\lambda x, K) &\leq \|T_\lambda x - w_\lambda\| = \|\lambda(x-y_\delta) + (1-\lambda)(Tx - z_\delta)\| \leq \\ &\leq \lambda \|x-y_\delta\| + (1-\lambda)\|Tx-z_\delta\| \leq k_\lambda d(x,K) + \delta, \end{aligned}$$

and since $\delta > 0$ was chosen arbitrarily, the conclusion follows. □

The main result of this section is given by the next Theorem.

Theorem 3.9. *Let D be a closed convex set in a strictly convex Banach space X and let $T : D \to D$ be a conditionally quasi-nonexpansive operator. Suppose further that there exists a convex compact set K in X and a number $k < 1$ such that (25) holds.*

Then, for any $x_0 \in D$ *and any* $\lambda \in (0,1)$*, the Krasnoselskij iteration* $\{T_\lambda^n(x_0)\}$ *converges to a fixed point of* T.

Proof. By the convexity of D it follows that T_λ maps D into itself. Since T satisfies (25), by Lemma 3.7, T_λ satisfies (26) and hence, in view of Frum-Ketkov fixed point theorem, see Exercise 3.20, $Fix\,(T_\lambda) \neq \emptyset$. Moreover, since X is strictly convex and T is conditionally quasi-nonexpansive, it results that T_λ is conditionally strictly quasi nonexpansive, i.e.,

$$\|T_\lambda x - T_\lambda\| < \|x - y\|$$

for all $x \neq y$ in D, whenever $Fix\,(T_\lambda) \neq \emptyset$.

In fact, as $Fix\,(T_\lambda) \neq \emptyset$, T_λ is strictly nonexpansive.

On the other hand, by the same Frum-Ketkov contractive condition, it results

$$d(T_\lambda^n(x_0), K) \leq k_\lambda^n d(x_0, K)$$

and since $k_\lambda < 1$, this implies $\lim\limits_{n\to\infty} d(T_\lambda^n(x_0), K) = 0$, and since K is compact, this forces $\{x_n \equiv T_\lambda^n(x_0)\}$ to contain a convergent subsequence $\{x_{n_j}\}_{j\geq 1}$ with $\lim\limits_{j\to\infty} = x^*$.

The quasi nonexpansiveness condition implies that

$$\lim_{n\to\infty} d(x_n, Fix\,(T_\lambda)) = d \geq 0$$

exists. Therefore, it suffices to prove that $d = 0$. If $x^* \in Fix\,(T_\lambda)$, then $d = 0$. If $x^* \notin Fix\,(T_\lambda)$, then by the strictly quasi nonexpansiveness property, for every $x \in D \setminus Fix\,(T_\lambda)$, there exists $p = p_x \in Fix\,(T_\lambda)$ such that

$$\|T_\lambda x - T_\lambda\| < \|x - y\|\,.$$

This implies that T_λ is continuous at x^*, and hence

$$\|T_\lambda x^* - p\| = \left\|T_\lambda\left(\lim_{j\to\infty} x_{n_j}\right) - p\right\| = \lim_{n\to\infty} \|T_\lambda^n(x_0) - p\| =$$

$$\lim_{j\to\infty} \left\|T_\lambda^{n_j}(x_0) - p\right\| = \lim_{j\to\infty} \left\|x_{n_j} - p\right\| = \left\|\lim_{j\to\infty} x_{n_j} - p\right\| = \|x^* - p\|\,, \quad (27)$$

(where the middle equalities hold since, T_λ quasi nonexpansive implies that $\lim\limits_{n\to\infty} \|T_\lambda^n(x_0) - p\|$ exists).

But the equality (27) is a contradiction, hence always $d = 0$.

Now, by $\lim\limits_{n\to\infty} d(x_n, Fix\,(T_\lambda)) = 0$ we can prove that $\{x_n\}$ is a Cauchy sequence and, as it contains a convergent subsequence, it is convergent in the whole and $x^* \in Fix\,(T_\lambda)$. $\square$

3.6 Bibliographical Comments

§3.1.

The first result on the convergence of averaged sequences involving two successive terms of the Picard iteration, i.e., the expression

$$\frac{1}{2}(x_n + Tx_n),$$

has been obtained by Krasnoselskij [Kra55]. There, it was shown that if K is a closed bounded convex subset of a uniformly convex Banach space and $T : K \to K$ is a nonexpansive and compact operator (i.e., T is continuous and $T(K)$ is relatively compact), then the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = \frac{1}{2}(x_n + Tx_n), n \geq 0$$

converges strongly to a fixed point of T.

Krasnoselskij gave no estimation of the rate of convergence of $\{x_n\}_{n=0}^{\infty}$ and, in fact, it is typical of iteration methods involving nonexpansive mappings that their convergence may be arbitrarily slow. Actually, Oblomskaja [Obl68] gave a linear example where convergence is slower that $n^{-\alpha}$ for all $\alpha \in (0, 1)$. In this context, we also mention the monograph Patterson [Pat74, Chapter 4] which contains a thorough discussion of successive approximation method for linear operators, and an extensive bibliography.

Schaefer [Sch57] extended Krasnoselskij's result to the case when the constant $1/2$ is replaced by a $\lambda \in (0, 1)$, obtaining in this way the first result for the general Krasnoselskij iteration, defined by (1). Then, Edelstein [Ede66] extended the previous result to the case when E is strictly convex.

Petryshyn [Pt66a] extended the results of Krasnoselskij and Schaefer to demicompact nonexpansive mappings $T : K \to E$ that satisfy a Leray-Schauder condition on the boundary ∂K of K, using the so-called iteration-retraction method, that can work only in Hilbert spaces, while the results of Krasnoselskij and Schaefer were derived in the more general setting of a uniformly convex Banach space.

A new technique, based on a generalization of the projection method to Banach spaces was recently developed by Alber [Alb96] and his collaborators.

Browder and Petryshyn [BrP66], [BrP67] carried further the results of Krasnoselskij and Schaefer, investigating the convergence of the Krasnoselskij (and Picard) iterations for nonexpansive operators $T : E \to E$ which are asymptotically regular and for which $I - T$ maps bounded closed sets into closed sets. Further extensions were obtained by Diaz and Metcalf [DiM67], [DiM69], Dotson [Dot70], Outlaw [Out69] and Petryshyn [Pet67], [Pet71].

The weak convergence of the Krasnoselskij iteration process was first proved by Schaefer [Sch57], for the class of continuous nonexpansive operators. The extension of this result to general nonexpansive operators was carried out in two stages by Browder and Petryshyn [BrP66] and Opial [Op67a], respectively.

The results included in this Section are taken from the following sources: Theorem 3.1, which is the well known Browder-Gohde-Kirk fixed point theorem in a Hilbert space setting, is Theorem 4 in Browder and Petryshyn [BrP67]; Theorem 3.2 is Theorem 6 of Petryshyn [Pt66a], reformulated in Browder and Petryshyn [BrP67], while Theorem 3.3 is Theorem 7 and Theorem 3.4 is Theorem 8, both taken from the same paper by Browder and Petryshyn [BrP67], where many other interesting results for approximating fixed points are given. Corollary 3.1 is Corollary 2.1 in Petryshyn and Williamson [PWi73], where several results from Browder and Petryshyn [BrP67] are extended and improved.

§3.2.

Theorem 3.5 in this Section rewrites Theorem 12 in Browder and Petryshyn [BrP67]. Theorem 14 in the same paper concerns the convergence of a modified Krasnoselskij iteration, obtained by fixing the first term of the linear convex combination, i.e., the iterative sequence is defined by means of the iteration function $F_\lambda x := \lambda Tx + (1-\lambda)u_{0,}\ \lambda \in (0,1)$, where u_0 is fixed.

Several other results for this iteration procedure have been also obtained independently by Browder [Br67b] and respectively by Halpern [Hal67], in a Hilbert space setting. Their results say that: if x_λ is the fixed point of F_λ (which is a λ-contraction), then the sequence $\{x_\lambda\}$ converges strongly to a fixed point of T as $\lambda \to 1$. Later, Reich [Rei80] extended this result to uniformly smooth Banach spaces. Thereafter, Singh, S.P. and Watson, B. [SWa93] extended the result of Browder and Halpern to nonexpansive nonself operators satisfying Rothe's boundary condition.

Recently Xu, H.K. and Yin [XYi95] proved the convergence in the case of nonexpansive nonself operators defined on a nonempty closed convex (not necessarily bounded) subset of a Hilbert space. By adding the inwardness condition, Xu, H.K. [XuH97] extended the latter to uniformly smooth Banach spaces. For other related results, see also Jaggi [Ja77a], [Ja77b], Rhoades, B.E., Sessa, S., Khan, M.S., Swaleh, M. [RSK87], Jung and Kim, S.S. [JKS95], [JK98a] and [JK98b] and Section 6.5.

§3.3.

The content of Section 3.3 is taken from Berinde [Be02e], [Be02a]. Theorem 3.6, without part (iii) regarding error estimates, has been proved by Verma, R.U. [Ve97a], but the proof given here is at least formally different.

Theorem 3.7 has the merit to find the fastest Krasnoselskij iteration, under the assumptions of Theorem 3.6. The argument we exploited in order to do this was mentioned in passing in Browder and Petryshyn [BrP67].

§3.4.

The results in Section 3.4 are taken from Berinde [Be03a]. Various parts of them were communicated, in different stages of evolution, at some international conferences. Examples 3.5 and 3.6 are taken from Osilike [Os97c].

§3.5.

All results in this section are taken from Petryshyn and Williamson [PWi73]: Lemmas 3.5, 3.6 and 3.7 are respectively Lemma 2.1, Lemma 2.2 and Lemma 3.1, while Theorem 3.9 is Theorem 3.3 there. Exercise 3.21 is Example 3.1. Condition (25) was first used in Frum-Ketkov [FrK67], see Exercise 3.20, but a correct proof of this result was given by Nussbaum [Nus72]. For a recent result involving a Frum-Ketkov condition see Binh [Bin04].

Exercises and Miscellaneous Results

3.1. (a) Prove that if H is a Hilbert space then for any $u, v \in H$ we have

$$\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right). \qquad (*)$$

(b) Show that a Banach space X is a Hilbert space if and only if the identity $(*)$ is satisfied for all $u, v \in X$.

3.2. Let H be a Hilbert space, $C \subset H$ a closed bounded convex subset. For a fixed element v_0 in C and a number $s \in (0,1)$, define U_s by
$U_s(x) = (1-s)v_0 + sTx, \quad x \in C.$
Show that: (a) U_s maps C into C; (b) U_s is a s-contraction.

3.3. Let H be a Hilbert space, $C \subset H$ a closed bounded convex subset, $T : C \to C$ and for $\lambda \in (0,1)$, define the averaged map
$T_\lambda(x) = (1-\lambda)x + \lambda Tx, \quad x \in C$. Show that:
(a) T_λ maps C into C;
(b) If T is nonexpansive then T_λ is nonexpansive as well;
(c) T and T_λ have the same fixed point set, i.e., $Fix\,(T) = Fix\,(T_\lambda)$.

3.4. Browder and Petryshyn (1967)
Let H be a Hilbert space, $C \subset H$ a closed bounded convex subset, $T : C \to C$ nonexpansive and, for $\lambda \in (0,1)$, define the averaged map

$$T_\lambda(x) = (1-\lambda)x + \lambda Tx, \quad x \in C.$$

Show that if $\{x_n\}$ is the Picard iteration associated to T_λ and $x_0 \in C$, that is, the Krasnoselskij iteration associated to T and x_0, then

$$\sum_{n=0}^{\infty} \|x_{n+1} - x_n\|^2 < \infty.$$

Deduce from the above result that T_λ is asymptotically regular.

3.5. Let H be a Hilbert space, $C \subset H$ a closed bounded convex subset. If $T : C \to C$ is nonexpansive, then T is asymptotically regular, i.e., for any $x \in C$,

$$\left\|T^{n+1}x - T^n x\right\| \to 0 \text{ as } n \to \infty.$$

3.6. Let H be a Hilbert space and $C \subset H$ be a closed bounded convex subset. For each $x \in H$ define $R_C x$ as the nearest point to x in C.
(a) If $C = B(x_0, r)$, show that $R_C : H \to C$ is given by

$$R_C x = \begin{cases} x, & \text{if } \|x - x_0\| \leq r \\ \dfrac{r(x - x_0)}{\|x - x_0\|}, & \text{if } \|x - x_0\| \geq r; \end{cases}$$

(b) Show that R_C is nonexpansive.

3.7. Figueiredo-Karlovitz
If the mapping R_C defined in Exercise 3.6 for $C = B(0, 1)$ is nonexpansive for a Banach space X of dimension > 2, then X is a Hilbert space.

3.8. Let H be a Hilbert space, $C \subset H$ a closed bounded convex subset and $T : C \to C$ a strictly pseudo-contractive operator. Show that there exist values of $\lambda \in (0, 1)$ such that the averaged operator

$$T_\lambda(x) = (1 - \lambda)x + \lambda T x, \quad x \in C,$$

is nonexpansive.

3.9. Let H be a Hilbert space, $K \subset H$ a closed bounded convex subset. Show that any Lipschitzian operator $T : K \to K$ is also generalized pseudo-contractive with the same constant but the reverse is not true.

3.10. If K is a closed convex subset of a strictly convex Banach space X and $T : K \to K$ is nonexpansive, then F_T is closed and convex.

3.11. Let $X = \mathbb{R}^2$ be endowed with the norm $\|(x, y)\|_\infty = \max\{|x|, |y|\}$ and define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(x, y) = (x, |x|)$. Then
(a) T is nonexpansive;
(b) F_T is not convex.

3.12. Consider the unit ball in the space C_0 of all sequences of real numbers with limit 0 endowed with the sup norm and define $T : C_0 \to C_0$ by

$$Tx = (x_1, 1 - |x_1|, x_2, x_3, \dots), \quad x = (x_1, x_2, x_3, \dots).$$

Show that
(a) T is nonexpansive;
(b) $F_T = \{u, -u\}$, where $u = (1, 0, 0, 0, \dots)$ (hence F_T is disconnected).

3.13. Let $C[0, 1]$ be endowed with the Chebyshev's norm and let B be given by

$$B = \{x : [0, 1] \to \mathbb{R} \mid x(0) = 0, x(1) = 1 \text{ and } 0 \leq x(t) \leq 1, t \in (0, 1)\}.$$

Define T on B by $Tx(t) = tx(t), t \in [0, 1]$. Then
(a) T has no fixed points in B;
(b) If $\{x_n(t)\}$ is the Krasnoselskij iteration with $x_0(t) = 0$, we have

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0.$$

3.14. Alspach (1981)
Let $X = L^1[0,1]$ and $K = \left\{ f \in X | \int\limits_0^1 f = 1, 0 \le f \le 2 \text{ a.e.} \right\}$. Then
(a) K is a closed convex subset of $[0,2]$ (and hence it is weakly compact);
(b) The mapping $T : K \to K$ given by

$$Tf(t) = \begin{cases} \min\{2f(2t), 2\}, & \text{if } 0 \le t \le \dfrac{1}{2} \\ \max\{2f(2t-1) - 2, 0\}, & \text{if } \dfrac{1}{2} < t < 1 \end{cases}$$

is isometric on K but has no fixed points. (This shows that a weakly compact convex set in a Banach space does not have the fixed point property for nonexpansive operators)

3.15. Let K be a subset of a Banach space X and $T : K \to K$ be nonexpansive and $x_0 \in K$. Show that

$$\lim_{n\to\infty} \left\| T^n x_0 - T^{n+1} x_0 \right\|$$

always exists but this limit may be nonzero.

3.16. Baillon, Bruck and Reich (1978)
Let X be a Banach space, K a bounded, closed and convex subset of X, $T : K \to K$ nonexpansive and T_λ the averaged operator, i.e.,

$$T_\lambda(x) = (1-\lambda)x + \lambda Tx, \quad x \in K \text{ and } \lambda \in (0,1).$$

Then, for any $x \in K$,

$$\lim_{n\to\infty} \left\| T_\lambda^{n+1} x - T_\lambda^n x \right\| = \frac{1}{k} \lim_{n\to\infty} \left\| T_\lambda^{n+k} x - T_\lambda^n x \right\| = \lim_{n\to\infty} \frac{1}{n} \left\| T_\lambda^n x \right\|.$$

3.17. Ishikawa (1976)
Let X be a Banach space, K a bounded, closed and convex subset of X and $T : K \to K$ be nonexpansive. For $\lambda \in (0,1)$, let T_λ be the averaged operator associated to T, i.e.,

$$T_\lambda(x) = (1-\lambda)x + \lambda Tx, \quad x \in K$$

and define the sequences $\{x_n\}$ and $\{y_n\}$ as follows

$$x_{n+1} = T_\lambda x_n; \quad y_n = Ty_n, \ n = 0, 1, 2, \ldots$$

Then
(a) For each $i, n \in \mathbb{N}$,

$$\|y_{i+n} - x_i\| \ge (1-\lambda)^{-n} \left[\|y_{i+n} - x_{i+n}\| - \|y_i - x_i\| \right] + (1 + n\lambda) \|y_i - x_i\|;$$

and
(b) $\lim\limits_{n\to\infty} \|x_n - Tx_n\| = 0.$

3.18. Opial (1967)
Let X be a uniformly Banach space having a weakly continuous duality map and let x^* be the weak limit of a weakly convergent sequence $\{x_n\}$. Then

$$\limsup_{n\to\infty} \|x_n - x^*\| < \limsup_{n\to\infty} \|x_n - x\|, \text{ for all } x \neq x^*.$$

(Opial's condition)

3.19. Browder and Petryshyn (1967)
If X is uniformly convex, C is bounded and $T : C \to C$ is asymptotically regular, then the weak sequential limits of $\{T^n x\}$ are fixed points of T, i.e., $\omega_w(x) \subset F_T$.

3.20. Frum-Ketkov (1967)
Let D be a closed convex subset of a Banach space X and $T : D \to D$ a continuous map. Assume that there exist a compact set $K \subset X$ and a constant $k < 1$ such that

$$d(Tx, K) \leq kd(x, K), \text{ for each } x \in D.$$

Then T has a fixed point.

3.21. Petryshyn and Williamson (1973)
Let $X = l^p, 1 < p < \infty$ the space of infinite sequences of real numbers $x = (x_1, x_2, \dots)$ whose norm, $\|x\| \equiv \left(\sum_{i\geq 1} |x_i|^p\right)^{1/p}$ is finite. Show that
(a) l^p is uniformly convex;
(b) The collection $\{e_i | i \geq 1\}$ forms a Schauder basis for l_p, where e_i are the unit vectors in l^p of the form $e_j = \{\delta_{ij}\}_{j\geq 1}$, that is, each $x \in l^p$ has a unique representation in terms of this collection;
Let B be the unit ball in l_p with center 0 and let $\{f_i\}_{i\geq 1}$ be a family of nonexpansive self-mappings of the interval $[-1, 1]$ with $f_i(0) = 0,\ i \geq 1$. Define T for $x \in B$ by

$$Tx \equiv f_1(x_1)e_1 + \frac{1}{2}\sum_{i>1} f_i(x_i)e_i,\ x = (x_1, x_2, \dots) \in B.$$

(c) Show that T is well defined, $T(B) \subset B$ and T is nonexpansive;
(d) Show that $K \equiv \{x \in l^p | x_i = 0, i > 1; |x_1| \leq 1\}$ is convex and compact and for any $x \in B$, T satisfies the Frum-Ketkov contractive condition:

$$d(Tx, K) \leq \frac{1}{2} d(x, K);$$

(e) Apply Theorem 3.8 to show that the Krasnoselskij iteration associated to T converges for any $x_0 \in B$ and any $\lambda \in (0, 1)$ to a fixed point of T in B.