
The Square Negative Correlation Property for Generalized Orlicz Balls

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Summary. Recently Anttila, Ball and Perissinaki proved that the squares of coordinate functions in l_p^n are negatively correlated. This paper extends their results to balls in generalized Orlicz norms on \mathbb{R}^n . From this, the concentration of the Euclidean norm and a form of the Central Limit Theorem for the generalized Orlicz balls is deduced. Also, a counterexample for the square negative correlation hypothesis for 1-symmetric bodies is given.

1 Introduction

Given a convex, central-symmetric body $K \subset \mathbb{R}^n$ of volume 1, consider the random variable $X = (X_1, X_2, \dots, X_n)$, uniformly distributed on K . We are interested in determining whether the vector has the *square negative correlation*, i.e. if

$$\text{cov}(X_i^2, X_j^2) := \mathbb{E}(X_i^2 X_j^2) - \mathbb{E}X_i^2 \mathbb{E}X_j^2 \leq 0.$$

We assume that K is in *isotropic position*, i.e. that

$$\mathbb{E}X_i = 0 \quad \text{and} \quad \mathbb{E}X_i \cdot X_j = L_K^2 \delta_{ij},$$

where δ_{ij} is the Kronecker delta and L_K is a positive constant. Since any convex body not supported on an affine subspace has an affine image which is in isotropic position, this is not a restrictive assumption.

The motivation in studying this problem comes from the so-called central limit problem for convex bodies, which is to show that most of the one-dimensional projections of the uniform measure on a convex body are approximately normal. It turns out that the bounds on the square correlation can be crucial to estimating the distance between the one-dimensional projections and the normal distribution (see for instance [ABP], [MM]). A related problem is to provide bounds for the quantity σ_K , defined by

$$\sigma_K^2 = \frac{\text{Var}(|X|^2)}{nL_K^4} = \frac{n\text{Var}(|X|^2)}{(\mathbb{E}|X|^2)^2},$$

where X is uniformly distributed on K . It is conjectured (see for instance [BK]) that σ_K is bounded by a universal constant for any convex symmetric isotropic body. Recently Anttila, Ball and Perissinaki (see [ABP]) observed that for $K = l_p^n$ the covariances of X_i^2 and X_j^2 are negative for $i \neq j$, and from this deduced a bound on σ_K in this class.

In this paper we shall study the covariances of X_i^2 and X_j^2 (or, more generally, of any functions depending on a single variable) on a convex, symmetric and isotropic body. We will show a general formula to calculate the covariance for given functions and K , and from this formula deduce the covariance of any increasing functions of different variables, in particular of the functions X_i^2 and X_j^2 , has to be negative on generalized Orlicz balls. Then we follow [ABP] to arrive at a concentration property and [MM] to get a Central Limit Theorem variant for generalized Orlicz balls.

The layout of this paper is as follows. First we define notations which will be used throughout the paper. In Section 2 we transform the formula for the square correlation into a form which will be used further on. In Section 3 we use the formula and the Brunn–Minkowski inequality to arrive at the square negative correlation property for generalized Orlicz balls. In Section 4 we show the corollaries, in particular a central-limit theorem for generalized Orlicz balls. Section 5 contains another application of the formula from Section 2, a simple counterexample for the square negative correlation hypothesis for 1-symmetric bodies.

Notation

Throughout the paper $K \subset \mathbb{R}^n$ will be a convex central-symmetric body of volume 1 in isotropic position. Recall that by isotropic position we mean that for any vector $\theta \in S^{n-1}$ we have $\int_K \langle \theta, x \rangle^2 dx = L_K^2$ for some constant L_K . For $A \subset \mathbb{R}^n$ by $|A|$ we will denote the Lebesgue volume of A . For $x \in \mathbb{R}^n$, $|x|$ will mean the Euclidean norm of x . We assume that \mathbb{R}^n is equipped with the standard Euclidean structure and with the canonic orthonormal base (e_1, \dots, e_n) . For $x \in \mathbb{R}^n$ by x_i we shall denote the i th coordinate of x , i.e. $\langle e_i, x \rangle$. We will consider K as a probability space with the Lebesgue measure restricted to K as the probability measure. If there is any danger of confusion, then \mathbb{P}_K will denote the probability with respect to this measure, \mathbb{E}_K will denote the expected value with respect to \mathbb{P}_K , and so on. By X we will usually denote the n -dimensional random vector equidistributed on K , while X_i will denote its i th coordinate. By the covariance $\text{cov}(Y, Z)$ for real random variables Y, Z we mean $\mathbb{E}(YZ) - \mathbb{E}Y\mathbb{E}Z$. By an 1-symmetric body K we mean one that is invariant under reflections in the coordinate hyperplanes, or equivalently, such a body that $(x_1, x_2, \dots, x_n) \in X \iff (\varepsilon_1 x_1, \varepsilon_2 x_2, \dots, \varepsilon_n x_n \in X)$ for any choice of $\varepsilon_i \in \{-1, 1\}$. The parameter σ_K , as in [BK], will be defined by

$$\sigma_K^2 = \frac{\text{Var}(|X|^2)}{nL_K^4} = \frac{n\text{Var}(|X|^2)}{(\mathbb{E}|X|^2)^2}.$$

For any $n \geq 1$ and convex increasing functions $f_i : [0, \infty) \rightarrow [0, \infty)$, $i = 1, \dots, n$ satisfying $f_i(0) = 0$ (called the Young functions) we define the generalized Orlicz ball $K \subset \mathbb{R}^n$ to be the set of points $x = (x_1, \dots, x_n)$ satisfying

$$\sum_{i=1}^n f_i(|x_i|) \leq 1.$$

This is easily proven to be convex, symmetric and bounded, thus

$$\|x\| = \inf\{\lambda : x \in \lambda K\}$$

defines a norm on \mathbb{R}^n . In the case of equal functions f_i the norm is called an *Orlicz norm*, in the general case a *generalized Orlicz norm*. Examples of Orlicz norms include the l_p norms for any $p \geq 1$ with $f(t) = |t|^p$ being the Young functions. The generalized Orlicz spaces are also referred to as modular sequence spaces (I thank the referee for pointing this out to me).

2 The General Formula

We wish to calculate $\text{cov}(f(X_i), g(X_j))$, where f and g are univariate functions, $i \neq j$ and X_i, X_j are the coordinates of the random vector X , equidistributed on a convex, symmetric and isotropic body K . For simplicity we will assume $i = 1, j = 2$ and denote X_1 by Y and X_2 by Z . For any $(y, z) \in \mathbb{R}^2$ let $m(y, z)$ be equal to the $n - 2$ -dimensional Lebesgue measure of the set $(\{(y, z)\} \times \mathbb{R}^{n-2}) \cap K$. We set out to prove:

Theorem 2.1. *For any symmetric, convex body K in isotropic position and any functions f, g we have*

$$\text{cov}(f(Y), g(Z)) = \int_{\mathbb{R}^4, |y| > |\bar{y}|, |z| > |\bar{z}|} (m(y, z)m(\bar{y}, \bar{z}) - m(y, \bar{z})m(\bar{y}, z)) (f(y) - f(\bar{y})) (g(z) - g(\bar{z})) .$$

Furthermore, for 1-symmetric bodies and symmetric functions we will have the following corollary:

Corollary 2.2. *For any symmetric, convex, unconditional body K in isotropic position and symmetric functions f, g we have*

$$\text{cov}(f(Y), g(Z)) = 16 \int_{\mathbb{R}^4, y > \bar{y} > 0, z > \bar{z} > 0} (m(y, z)m(\bar{y}, \bar{z}) - m(y, \bar{z})m(\bar{y}, z)) (f(y) - f(\bar{y})) (g(z) - g(\bar{z})) .$$

The corollary is a simple consequence of the fact that for symmetric functions f and g and an 1-symmetric body K the integrand is invariant under

the change of the sign of any of the variables, so we may assume all of them are positive.

As concerns the sign of $\text{cov}(f, g)$, which is what we set out to determine, we have the following simple corollary:

Corollary 2.3. *For any central-symmetric, convex, 1-symmetric body K in isotropic position and symmetric functions f, g that are non-decreasing on $[0, \infty)$ if for all $y > \bar{y} > 0, z > \bar{z} > 0$ we have*

$$m(y, \bar{z})m(\bar{y}, z) \geq m(y, z)m(\bar{y}, \bar{z}), \quad (1)$$

then

$$\text{cov}(f, g) \leq 0.$$

Similarly, if the opposite inequality is satisfied for all $y > \bar{y} > 0$ and $z > \bar{z} > 0$, then the covariance is non-negative.

Proof. The second and third bracket of the integrand in Corollary 2.2 is positive under the assumptions of Corollary 2.3. Thus if we assume the first bracket is negative, then the whole integrand is negative, which implies the integral is negative, and vice-versa. \square

Proof of Theorem 2.1. We have

$$\text{cov}(f(Y), g(Z)) = \mathbb{E}f(Y)g(Z) - \mathbb{E}f(Y)\mathbb{E}g(Z).$$

From the Fubini theorem we have

$$\mathbb{E}f(Y)g(Z) = \int_{\mathbb{R}^2} m(y, z)f(y)g(z),$$

and similar equations for $\mathbb{E}f(Y)$ and $\mathbb{E}g(Z)$.

For any function h of two variables $a, b \in A$ we can write $\int_{A^2} h(a, b) = \int_{A^2} h(b, a) = \frac{1}{2} \int_{A^2} h(a, b) + h(b, a)$. We shall repeatedly use this trick to transform the formula for the covariance of f and g into the required form:

$$\begin{aligned} \mathbb{E}f(Y)\mathbb{E}g(Z) &= \int_{\mathbb{R}^2} m(y, z)f(y) \int_{\mathbb{R}^2} m(\bar{y}, \bar{z})g(\bar{z}) \\ &= \int_{\mathbb{R}^4} m(y, z)m(\bar{y}, \bar{z})f(y)g(\bar{z}) = \int_{\mathbb{R}^4} m(\bar{y}, \bar{z})m(y, z)f(\bar{y})g(z) \\ &= \frac{1}{2} \int_{\mathbb{R}^4} m(\bar{y}, \bar{z})m(y, z)(f(\bar{y})g(z) + f(y)g(\bar{z})). \end{aligned}$$

We repeat this trick, exchanging z and \bar{z} (and leaving y and \bar{y} unchanged):

$$\begin{aligned} \mathbb{E}f(Y)\mathbb{E}g(Z) &= \frac{1}{4} \int_{\mathbb{R}^4} m(\bar{y}, \bar{z})m(y, z)(f(y)g(\bar{z}) + f(\bar{y})g(z)) \\ &\quad + m(\bar{y}, z)m(y, \bar{z})(f(y)g(\bar{z}) + f(\bar{y})g(z)). \end{aligned}$$

We perform the same operations on the second part of the covariance. To get a integral over \mathbb{R}^4 we multiply by an $\mathbb{E}1$ factor (this in effect will free us from the assumption that the body's volume is 1):

$$\begin{aligned} \mathbb{E}f(Y)g(Z)\mathbb{E}1 &= \int_{\mathbb{R}^4} m(y, z)m(\bar{y}, \bar{z})f(y)g(z) \\ &= \frac{1}{4} \int_{\mathbb{R}^4} m(y, z)m(\bar{y}, \bar{z})(f(y)g(z) \\ &\quad + f(\bar{y})g(\bar{z})) + m(y, \bar{z})m(\bar{y}, z)(f(y)g(\bar{z}) + f(\bar{y})g(z)). \end{aligned}$$

Thus:

$$\begin{aligned} \text{cov}(f(Y), g(Z)) &= \mathbb{E}(f(Y)g(Z))\mathbb{E}1 - \mathbb{E}f(Y)\mathbb{E}g(Z) \\ &= \frac{1}{4} \left(\int_{\mathbb{R}^4} m(y, z)m(\bar{y}, \bar{z})(f(y)g(z) + f(\bar{y})g(\bar{z})) \right. \\ &\quad + m(y, \bar{z})m(\bar{y}, z)(f(y)g(\bar{z}) + f(\bar{y})g(z)) \\ &\quad - m(\bar{y}, \bar{z})m(y, z)(f(y)g(\bar{z}) + f(\bar{y})g(z)) \\ &\quad \left. - m(\bar{y}, z)m(y, \bar{z})(f(y)g(z) + f(\bar{y})g(\bar{z})) \right) \\ &= \frac{1}{4} \int_{\mathbb{R}^4} \left((m(y, \bar{z})m(\bar{y}, z) - m(y, z)m(\bar{y}, \bar{z}))(f(y)g(\bar{z}) + f(\bar{y})g(z)) \right. \\ &\quad \left. + (m(y, z)m(\bar{y}, \bar{z}) - m(\bar{y}, z)m(y, \bar{z}))(f(y)g(z) + f(\bar{y})g(\bar{z})) \right) \\ &= \frac{1}{4} \int_{\mathbb{R}^4} (m(y, \bar{z})m(\bar{y}, z) - m(y, z)m(\bar{y}, \bar{z})) \\ &\quad \cdot (f(y)g(\bar{z}) + f(\bar{y})g(z) - f(y)g(z) - f(\bar{y})g(\bar{z})) \\ &= \frac{1}{4} \int_{\mathbb{R}^4} (m(y, \bar{z})m(\bar{y}, z) - m(y, z)m(\bar{y}, \bar{z}))(f(y) - f(\bar{y}))(g(\bar{z}) - g(z)) . \end{aligned}$$

Finally, notice that if we exchange y and \bar{y} in the above formula, then the formula's value will not change — the first and second bracket will change signs, and the third will remain unchanged. The same applies to exchanging z and \bar{z} . Thus

$$\begin{aligned} \text{cov}(f, g) &= \\ &\int_{\mathbb{R}^4, |y|>|\bar{y}|, |z|>|\bar{z}|} (m(y, z)m(\bar{y}, \bar{z}) - m(y, \bar{z})m(\bar{y}, z))(f(y) - f(\bar{y}))(g(z) - g(\bar{z})) . \end{aligned}$$

□

3 Generalized Orlicz Spaces

Now we will concentrate on the case of symmetric, non-decreasing functions on generalized Orlicz spaces. We will prove the inequality (1):

Theorem 3.1. *If K is a ball in an generalized Orlicz norm on \mathbb{R}^n , then for any $y > \bar{y} > 0$ and $z > \bar{z} > 0$ we have*

$$m(y, \bar{z})m(\bar{y}, z) \geq m(y, z)m(\bar{y}, \bar{z}). \tag{2}$$

From this Theorem and Corollary 2.3 we get

Corollary 3.2. *If K is a ball in an generalized Orlicz norm on \mathbb{R}^n and f, g are symmetric functions that are non-decreasing on $[0, \infty)$, then $\text{cov}_K(f, g) \leq 0$.*

It now remains to prove the inequality (2).

Proof of Theorem 3.1. Let f_i denote the Young functions of K . Let us consider the ball $K' \subset \mathbb{R}^{n-1}$, being an generalized Orlicz ball defined by the Young functions $\Phi_1, \Phi_2, \dots, \Phi_{n-1}$, where $\Phi_i(t) = f_{i+1}(t)$ for $i > 1$ and $\Phi_1(t) = t$ — that is, we replace the first two Young functions of K by a single identity function.

For any $x \in \mathbb{R}$ let P_x be the set $(\{x\} \times \mathbb{R}^{n-2}) \cap K'$, and $|P_x|$ be its $n - 2$ -dimensional Lebesgue measure. K' is a convex set, thus, by the Brunn–Minkowski inequality (see for instance [G]) the function $x \mapsto |P_x|$ is a logarithmically concave function. This means that $x \mapsto \log |P_x|$ is a concave function, or equivalently that

$$|P_{tx+(1-t)y}| \geq |P_x|^t \cdot |P_y|^{1-t}.$$

In particular, for given real positive numbers a, b, c we have

$$\begin{aligned} |P_{a+c}| &\geq |P_a|^{b/(b+c)} |P_{a+b+c}|^{c/(b+c)}, \\ |P_{a+b}| &\geq |P_a|^{c/(b+c)} |P_{a+b+c}|^{b/(b+c)}, \end{aligned}$$

and as a consequence when we multiply the two inequalities,

$$|P_{a+b}| \cdot |P_{a+c}| \geq |P_a| \cdot |P_{a+b+c}|. \tag{3}$$

Now let us consider the ball K . Let us take any $y > \bar{y} > 0$ and $z > \bar{z} > 0$. Let $a = f_1(\bar{y}) + f_2(\bar{z})$, $b = f_1(y) - f_1(\bar{y})$, and $c = f_2(z) - f_2(\bar{z})$. The numbers a, b and c are positive from the assumptions on y, z, \bar{y} and \bar{z} and because the Young functions are increasing. Then $m(\bar{y}, \bar{z})$ is equal to the measure of the set

$$\begin{aligned} &\left\{ x_3, x_4, \dots, x_n : f_1(\bar{y}) + f_2(\bar{z}) + \sum_{i=3}^n f_i(x_i) \leq 1 \right\} \\ &= \left\{ x_3, x_4, \dots, x_n : a + \sum_{i=2}^n \Phi_i(x_i) \leq 1 \right\} = P_a. \end{aligned}$$

Similarly $m(y, \bar{z}) = |P_{a+b}|$, $m(\bar{y}, z) = |P_{a+c}|$ i $m(y, z) = |P_{a+b+c}|$.

Substituting those values into the inequality (3) we get the thesis:

$$m(y, \bar{z})m(\bar{y}, z) \geq m(y, z)m(\bar{y}, \bar{z}). \quad \square$$

4 The Consequences

For the consequences we will take $f(t) = g(t) = t^2$. The first simple consequence is the concentration property for generalized Orlicz balls. Here, we follow the argument of [ABP] for l_p balls.

Theorem 4.1. *For every generalized Orlicz ball $K \subset \mathbb{R}^n$ we have*

$$\sigma_K \leq \sqrt{5}.$$

Proof. From the Cauchy-Schwartz inequality we have

$$n^2 L_K^4 = \left(\sum_{i=1}^n \mathbb{E}_K X_i^2 \right)^2 = (\mathbb{E}_K |X|^2)^2 \leq \mathbb{E}_K |X|^4.$$

On the other hand from Corollary 3.2 we have

$$\begin{aligned} \mathbb{E}_K |X|^4 &= \mathbb{E}_K \left(\sum_{i=1}^n X_i^2 \right)^2 = \sum_{i=1}^n \mathbb{E}_K X_i^4 + \sum_{i \neq j} \mathbb{E}_K X_i^2 X_j^2 \\ &\leq \sum_{i=1}^n \mathbb{E}_K X_i^4 + \sum_{i \neq j} \mathbb{E}_K X_i^2 \mathbb{E}_K X_j^2 \\ &= \sum_{i=1}^n \mathbb{E}_K X_i^4 + n(n-1)L_K^4. \end{aligned}$$

As for 1-symmetric bodies the density of X_i is symmetric and log-concave, we know (see e.g. [KLO], Section 2, Remark 5)

$$\mathbb{E}_K X_i^4 \leq 6(\mathbb{E}_K X_i^2)^2 = 6L_K^4,$$

thence

$$n^2 L_K^4 \leq \mathbb{E}_K |X|^4 \leq (n^2 + 5n)L_K^4.$$

This gives us

$$\text{Var}(|X|^2) = \mathbb{E}_K |X|^4 - n^2 L_K^4 \leq 5nL_K^4,$$

and thus

$$\sigma_K^2 = \frac{\text{Var}|X|^2}{nL_K^4} \leq 5. \quad \square$$

Corollary 4.2. *For every generalized Orlicz ball $K \subset \mathbb{R}^n$ and for every $t > 0$ we have*

$$\mathbb{P}_K \left(\left| \frac{|X|^2}{n} - L_K^2 \right| \geq t \right) \leq \frac{5L_K^4}{nt^2}$$

and

$$\mathbb{P}_K \left(\left| \frac{|X|}{\sqrt{n}} - L_K \right| \geq t \right) \leq \frac{5L_K^2}{nt^2}.$$

Proof. From the estimate on the variance of $|X|^2$ and Chebyshev's inequality we get

$$t^2 \mathbb{P}_K \left(\left| \frac{|X|^2}{n} - L_K^2 \right| \geq t \right) \leq \mathbb{E}_K \left(\frac{|X|^2}{n} - L_K^2 \right)^2 \leq \frac{1}{n^2} \text{Var}(|X|^2) \leq \frac{5}{n} L_K^4.$$

For the second part let $t > 0$. We have

$$\begin{aligned} \mathbb{P}_K(|X| - \sqrt{n}L_K \geq t\sqrt{n}) &\leq \mathbb{P}_K(|X|^2 - nL_K^2 \geq tnL_K) \\ &\leq \frac{5L_K^4}{t^2 n L_K^2} = \frac{5L_K^2}{t^2 n}. \quad \square \end{aligned}$$

This result confirms the so-called *concentration hypothesis* for generalized Orlicz balls. The hypothesis, see e.g. [BK], states that the Euclidean norm concentrates near the value $\sqrt{n}L_K$ as a function on K . More precisely, for a given $\varepsilon > 0$ we say that K satisfies the ε -concentration hypothesis if

$$\mathbb{P}_K \left(\left| \frac{|X|}{\sqrt{n}} - L_K \right| \geq \varepsilon L_K \right) \leq \varepsilon.$$

From Corollary 4.2 we get that the class of generalized Orlicz balls satisfies the ε -concentration hypothesis with $\varepsilon = \sqrt{5}n^{-1/3}$.

A more complex consequence is the Central Limit Property for generalized Orlicz balls. For $\theta \in S^{n-1}$ let $g_\theta(t)$ be the density of the random variable $\langle X, \theta \rangle$. Let g be the density of $\mathcal{N}(0, L_K^2)$. Then for most θ the density g_θ is very close to g . More precisely, by part 2 of Corollary 4 in [MM] we get

Corollary 4.3. *There exists an absolute constant c such that*

$$\sup_{t \in \mathbb{R}} \left| \int_{-\infty}^t (g_\theta(s) - g(s)) ds \right| \leq c \|\theta\|_3^{3/2}.$$

5 The Counterexample for 1-Symmetric Bodies

It is generally known that the negative square correlation hypothesis does not hold in general in the class of 1-symmetric bodies. However, the formula from section 2 allows us to give a counterexample without any tedious calculations. Let $K \subset \mathbb{R}^3$ be the ball of the norm defined by

$$\|(x, y, z)\| = |x| + \max\{|y|, |z|\}.$$

The quantity $m(y, z)$ considered in Corollary 2.3, defined as the volume of the cross-section $(\mathbb{R} \times \{y, z\}) \cap K$ is equal to $2(1 - \max\{|y|, |z|\})$ for $|y|, |z| \leq 1$ and 0 for greater $|y|$ or $|z|$. To check the inequality (1) for $y > \bar{y} > 0$ and $z > \bar{z} > 0$ we may assume without loss of generality that $y \geq z$ (as K is invariant under the exchange of y and z). We have

$$\begin{aligned}
& m(y, \bar{z})m(\bar{y}, z) - m(y, z)m(\bar{y}, \bar{z}) \\
&= 4(1 - \max\{y, \bar{z}\})(1 - \max\{\bar{y}, z\}) - 4(1 - \max\{y, z\})(1 - \max\{\bar{y}, \bar{z}\}) \\
&= 4(1 - y)(1 - \max\{\bar{y}, z\}) - 4(1 - y)(1 - \max\{\bar{y}, \bar{z}\}) \\
&= 4(1 - y)(\max\{\bar{y}, \bar{z}\} - \max\{\bar{y}, z\}) .
\end{aligned}$$

As $y \leq 1$ all we have to consider is the sign of the third bracket. However, as $z > \bar{z}$, the third bracket is never positive, and is negative when $z > \bar{y}$. Thus from Corollary 2.3 the covariance $\text{cov}(f, g)$ is positive for any increasing symmetric functions $f(Y)$ and $g(Z)$, in particular for $f(Y) = Y^2$ and $g(Z) = Z^2$.

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