Generalized Itô Formulae and Space-Time Lebesgue–Stieltjes Integrals of Local Times

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Summary. Generalized Itô formulae are proved for time dependent functions of continuous real valued semi-martingales. The conditions involve left space and time first derivatives, with the left space derivative required to have locally bounded two-dimensional variation. In particular a class of functions with discontinuous first derivative is included. An estimate of Krylov allows further weakening of these conditions when the semi-martingale is a diffusion.

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1 Introduction

Extensions of Itô formula to less smooth functions are useful in studying many problems such as partial differential equations with some singularities, see below, and in the mathematics of finance. The first extension was obtained for $|X(t)|$ by Tanaka [24] with a beautiful use of local time. The generalized Itô formula in one-dimension for time independent convex functions was developed in [20] and for superharmonic functions in multidimensions in [5] and for distance functions in [15]. Extensions of Itô's formula have also been studied by [16], [12], [21], [11]. In [11], Itô's formula for $W^{1,2}_{loc}$ functions was studied using Lyons–Zheng's backward and forward stochastic integrals [18]. In [4], Itô's formula was extended to absolutely continuous functions with locally bounded derivative using the integral $\int_{-\infty}^{\infty} \nabla f(x)\, \mathrm{d}_x L_s(x)$.

This integral was defined through the existence of the expression $f(X(t)) - f(X(0)) - \int_0^t \frac{\partial}{\partial x} f(X(s))\,\mathrm{d}X(s)$; it was extended to $\int_0^t \int_{-\infty}^{\infty} \nabla f(s,x)\,\mathrm{d}_{s,x} L_s(x)$ for a time dependent function $f(s,x)$ using forward and backward integrals for Brownian motion in [6]. Recent activities in this direction have been to look for minimal assumptions on f to make this integral well defined for semi-martingales other than Brownian motion [7]. However, our motivation in establishing generalized Itô formulae was to use them to describe the asymptotics of the solution of heat equations in the presence of a caustic. Due to the appearance of caustics, the solution of the Hamilton–Jacobi equation, the leading term in the asymptotics, is no longer differentiable, but has a jump in the gradient across the shock wave front of the associated Burgers' equation. Therefore, the local time of continuous semi-martingales in a neighbourhood of the shock wave front of the Burgers equation and the jump of the derivatives of the Hamilton–Jacobi function (or equivalently the jump in the Burgers' velocity) appear naturally in the semi-classical representation of the corresponding solution to the heat equation [8]. None of the earlier versions of Itô's formula apply directly to this situation.

In this paper, we first generalize Itô's formula to the case of a continuous semi-martingale and a left continuous and locally bounded function $f(t,x)$ which satisfies (1) its left derivative $\frac{\partial^-}{\partial t} f(t,x)$ exists and is left continuous, (2) $f(t,x) = f_h(t,x) + f_v(t,x)$ with $f_h(t,x)$ being C^1 in x and $\nabla f_h(t,x)$ having left continuous and locally bounded left derivative $\Delta^- f_h(t,x)$, and f_v having left derivative $\nabla^- f_v(t,x)$ which is left continuous and of locally bounded variation in (t,x). Here we use the two-dimensional Lebesgue–Stieltjes integral of local time with respect to $\nabla^- f(t,x)$. The main result of this paper is formula (2.24). Formula (2.26) follows from (2.24) easily as a special case. These formulae appear to be new and in a good form for extensions to two dimensions [9]. Moreover, in [10], Feng and Zhao observed that the local time $L_t(x)$ can be considered as a rough path in x of finite 2-variation and therefore defined $\int_0^t \int_{-\infty}^{\infty} \nabla^- f(s,x)\,\mathrm{d}_{\mathrm{s,x}} L_s(x)$ pathwise by extending Young and Lyons' profound idea of rough path integration ([17], [25]) to two parameters. When this paper was nearly completed, we received two preprints concerning a generalized Itô's formula for a continuous function $f(t,x)$ with jump derivative $\nabla^- f(t,x)$, ([22], [13]). We remark that formula (2.26) was also observed by [22] independently.

In Section 3, we consider diffusion processes $X(t)$. We prove the generalized Itô formula for a function f with generalized derivative $\frac{\partial}{\partial t} f$ in $L^2_{loc}(dtdx)$ and generalized derivative $\nabla f(t,x)$ being of locally bounded variation in (t,x). We use an inequality from Krylov [16].

2 The continuous semimartingale case

We need the following definitions (see, e.g. [2], [19]): A two-variable function $f(s,x)$ is called monotonically increasing if whenever $s_2 \geqslant s_1$, $x_2 \geqslant x_1$,

$$f(s_2,x_2) - f(s_2,x_1) - f(s_1,x_2) + f(s_1,x_1) \geqslant 0.$$

It is called monotonically decreasing if $-f$ is monotonically increasing. The function f is called left continuous iff it is left continuous in both variables together, in other words, for any sequence $(s_1,x_1) \leqslant (s_2,x_2) \leqslant \cdots \leqslant (s_k,x_k) \to (s,x)$, we have $f(s_k,x_k) \to f(s,x)$ as $k \to \infty$. Here $(s,x) \leqslant (t,y)$ means $s \leqslant t$ and $x \leqslant y$. For a monotonically increasing and left continuous function $f(s,x)$, we can define a Lebesgue–Stieltjes measure by setting

$$\mu([s_1,s_2) \times [x_1,x_2)) = f(s_2,x_2) - f(s_2,x_1) - f(s_1,x_2) + f(s_1,x_1),$$

for $s_2 > s_1$ and $x_2 > x_1$. So for a measurable function $g(s,x)$, we can define the Lebesgue–Stieltjes integral by

$$\int_{t_1}^{t_2} \int_a^b g(s,x)\,\mathrm{d}_{s,x} f(s,x) = \int_{t_1}^{t_2} \int_a^b g(s,x)\,\mathrm{d}\mu.$$

Denote a partition $\mathcal{P}$ of $[t,s] \times [a,x]$ by $t = s_1 < s_2 < \cdots < s_m = s$, $a = x_1 < x_2 < \cdots < x_n = x$ and the variation of f associated with $\mathcal{P}$ by

$$\begin{aligned} &V_{\mathcal{P}}(f,[t,s] \times [a,x]) \\ &\quad = \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} |f(s_{i+1},x_{j+1}) - f(s_{i+1},x_j) - f(s_i,x_{j+1}) + f(s_i,x_j)| \end{aligned}$$

and the variation of f on $[t,s] \times [a,x]$ by

$$V_f([t,s] \times [a,x]) = \sup_{\mathcal{P}} V_{\mathcal{P}}(f,[t,s] \times [a,x]).$$

One can find Proposition 2.2, its proof and definition of the multi-dimensional Lebesgue–Stieltjes integral with respect to measures generated by functions of bounded variation in [19]. For the convenience of the reader, we include them here briefly.

Proposition 2.1 (Additivity of variation) *For $s_2 \geqslant s_1 \geqslant t$, and $a_2 \geqslant a_1 \geqslant a$,*

$$\begin{aligned} V_f([t,s_2] \times [a,a_2]) &= V_f([t,s_1] \times [a,a_2]) + V_f([t,s_2] \times [a,a_1]) \\ &\quad + V_f([s_1,s_2] \times [a_1,a_2]) - V_f([t,s_1] \times [a,a_1]). \quad (2.1) \end{aligned}$$

Proof. We only need to prove that for $a \leqslant a_1 < a_2$ and $t \leqslant s_1$,

$$V_f([t,s_1] \times [a,a_2]) = V_f([t,s_1] \times [a,a_1]) + V_f([t,s_1] \times [a_1,a_2]). \quad (2.2)$$

Our proof is similar to the case of one-dimension. We can always refine a partition $\mathcal{P}$ of $[t, s_1] \times [a, a_2]$ to include a_1. The refined partition is denoted by $\mathcal{P}'$. Then

$$V_{\mathcal{P}}(f, [t, s_1] \times [a, a_2]) \leqslant V_{\mathcal{P}'}(f, [t, s_1] \times [a, a_2]).$$

Then (2.2) follows easily. □

Proposition 2.2 *A function $f(s, x)$ of locally bounded variation can be decomposed as the difference of two increasing functions $f_1(s, x)$ and $f_2(s, x)$, in any quarter space $s \geqslant t, x \geqslant a$. Moreover, if f is also left continuous, then f_1 and f_2 can be taken left continuous.*

Proof. For any $(t, x) \in R^2$, define for $s \geqslant t$ and $x \geqslant a$,

$$2\tilde{f}_1(s, x) = V_f([t, s] \times [a, x]) + f(s, x),$$
$$2\tilde{f}_2(s, x) = V_f([t, s] \times [a, x]) - f(s, x).$$

Then $f(s, x) = \tilde{f}_1(s, x) - \tilde{f}_2(s, x)$. We need to prove that $\tilde{f}_1$ and $\tilde{f}_2$ are increasing functions. For this, let $s_2 \geqslant s_1 \geqslant t$, $a_2 \geqslant a_1 \geqslant a$, then use Proposition 2.1,

$$\begin{aligned} &2(\tilde{f}_1(s_2, a_2) - \tilde{f}_1(s_1, a_2) - \tilde{f}_1(s_2, a_1) + \tilde{f}_1(s_1, a_1)) \\ &\quad = V_f([t, s_2] \times [a, a_2]) - V_f([t, s_1] \times [a, a_2]) - V_f([t, s_2] \times [a, a_1]) \\ &\qquad + V_f([t, s_1] \times [a, a_1]) + f(s_2, a_2) - f(s_1, a_2) - f(s_2, a_1) + f(s_1, a_1) \\ &\quad = V_f([s_1, s_2] \times [a_1, a_2]) + f(s_2, a_2) - f(s_1, a_2) - f(s_2, a_1) + f(s_1, a_1) \\ &\quad \geqslant 0. \end{aligned}$$

So $\tilde{f}_1(s, x)$ is an increasing function. Similarly one can prove that $\tilde{f}_2(s, x)$ is an increasing function.

Define

$$f_i(s, x) = \lim_{t \uparrow s, y \uparrow x} \tilde{f}_i(t, y), \quad i = 1, 2.$$

Then since f is left continuous, so

$$f(s, x) = f_1(s, x) - f_2(s, x), \tag{2.3}$$

and f_1 and f_2 are as required. □

From Proposition 2.2, the two-dimensional Lebesgue–Stieltjes integral of a measurable function g with respect to the left continuous function f of bounded variation can be defined by

$$\begin{aligned} \int_{t_1}^{t_2} \int_a^b g(s, x)\, \mathrm{d}_{s,x} f(s, x) &= \int_{t_1}^{t_2} \int_a^b g(s, x)\, \mathrm{d}_{s,x} f_1(s, x) \\ &\quad - \int_{t_1}^{t_2} \int_a^b g(s, x)\, \mathrm{d}_{s,x} f_2(s, x) \text{ for } t_2 \geqslant t_1, b \geqslant a. \end{aligned}$$

Here f_1 and f_2 are taken to be left continuous.

It is worth pointing out that it is possible that a function $f(s,x)$ is of locally bounded variation in (s,x) but not of locally bounded variation in x for fixed s. For instance consider $f(s,x)=b(x)$, where $b(x)$ is not of locally bounded variation, then $V_f=0$. However it is easy to see that when a function $f(s,x)$ is of locally bounded variation in (s,x) and of locally bounded variation in x for a fixed $s=s_0$, then it is of locally bounded variation in x for all s. We denote by $V_{f(s)}[a,b]$ the variation of $f(s,x)$ on $[a,b]$ as a function of x for a fixed s.

Now we recall some well-known results of local time which will be used later in this paper. Let $X(s)$ be a continuous semi-martingale $X(s)=X(0)+M_s+V_s$ on a probability space $\{\Omega,\mathcal{F},P\}$. Here M_s is a continuous local martingale and V_s is a continuous process of bounded variation. Let $L_t(a)$ be the local time introduced by P. Lévy

$$L_t(a)=\lim_{\epsilon\downarrow 0}\frac{1}{2\epsilon}\int_0^t 1_{[a,a+\epsilon)}(X(s))\,\mathrm{d}\langle M,M\rangle_s \quad a.s., \tag{2.4}$$

for each t and a. Then it is well known that for each fixed $a\in R$, $L_t(a,\omega)$ is continuous, and nondecreasing in t and right continuous with left limit (cadlag) with respect to a ([14], [23]). Therefore we can consider the Lebesgue–Stieltjes integral $\int_0^\infty \phi(s)\,\mathrm{d}L_s(a,\omega)$ for each a for any Borel-measurable function ϕ. In particular

$$\int_0^\infty 1_{R-\{a\}}(X(s))dL_s(a,\omega)=0 \quad a.s. \tag{2.5}$$

Furthermore if ϕ is in $L^{1,1}_{loc}(ds)$, i.e. ϕ has locally integrable generalized derivative, then we have the following integration by parts formula

$$\int_0^t \phi(s)dL_s(a,\omega)=\phi(t)L_t(a,\omega)-\int_0^t \phi'(s)L_s(a,\omega)ds \quad a.s. \tag{2.6}$$

Moreover, if $g(s,x)$ is Borel measurable in s and x and bounded, by the occupation times formula (e.g. see [14], [23]),

$$\int_0^t g(s,X(s))d\langle M,M\rangle_s=2\int_{-\infty}^{\infty}\int_0^t g(s,a)dL_s(a,\omega)da \quad a.s.$$

If further $g(s,x)$ is in $L^{1,1}_{loc}(ds)$ for almost all x, then using the integration by parts formula, we have

$$\begin{aligned}\int_0^t g(s,X(s))d\langle M,M\rangle_s &= 2\int_{-\infty}^{\infty}\int_0^t g(s,a)dL_s(a,\omega)da\\ &= 2\int_{-\infty}^{\infty} g(t,a)L_t(a,\omega)da\\ &\quad -2\int_{-\infty}^{\infty}\int_0^t \frac{\partial}{\partial s}g(s,a)L_s(a,\omega)dsda \quad a.s.\end{aligned}$$

We first prove a theorem with $f_h = 0$. The result with a term f_h is a trivial generalization of Theorem 2.1.

Theorem 2.1 *Assume* $f : [0,\infty) \times R \to R$ *satisfies*

(i) f *is left continuous and locally bounded, with* $f(t,x)$ *jointly continuous from the right in* t *and left in* x *at each point* $(0,x)$,

(ii) the left derivatives $\frac{\partial^-}{\partial t} f$ *and* $\nabla^- f$ *exist at all points of* $(0,\infty) \times R$ *and* $[0,\infty) \times R$, *respectively,*

(iii) $\frac{\partial^-}{\partial t} f$ *and* $\nabla^- f$ *are left continuous and locally bounded,*

(iv) $\nabla^- f$ *is of locally bounded variation in* (t,x) *and* $\nabla^- f(0,x)$ *is of locally bounded variation in* x.

Then for any continuous semi-martingale $\{X(t), t \geqslant 0\}$

$$\begin{aligned} f(t,X(t)) &= f(0,X(0)) + \int_0^t \frac{\partial^-}{\partial s} f(s,X(s))ds + \int_0^t \nabla^- f(s,X(s))dX(s) \\ &\quad + \int_{-\infty}^{\infty} L_t(x)\, \mathrm{d}_x \nabla^- f(t,x) - \int_{-\infty}^{+\infty} \int_0^t L_s(x) \mathrm{d}_{s,x} \nabla^- f(s,x) \ a.s. \end{aligned} \tag{2.7}$$

Proof. By a standard localization argument we can assume that X and its quadratic variation are bounded processes and that f, $\frac{\partial^-}{\partial t} f$, $\nabla^- f$, $V_{\nabla^- f(t)}$ and $V_{\nabla^- f}$ are bounded (note here $V_{\nabla^- f(0)} < \infty$ and $V_{\nabla^- f} < \infty$ imply $V_{\nabla^- f(t)} < \infty$ for all $t \geqslant 0$). Note first that from (i) to (iii) the left partial derivatives of f agree with the distributional derivatives and so (iii) implies that f is absolutely continuous in each variable. We use standard regularizing mollifiers (e.g. see [14]). Define

$$\rho(x) = \begin{cases} c\mathrm{e}^{\frac{1}{(x-1)^2-1}}, & \text{if } x \in (0,2), \\ 0, & \text{otherwise.} \end{cases}$$

Here c is chosen such that $\int_0^2 \rho(x)dx = 1$. Take $\rho_n(x) = n\rho(nx)$ as mollifiers. Define

$$f_n(s,x) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \rho_n(x-y)\rho_n(s-\tau) f(\tau,y) d\tau dy, \quad n \geqslant 1, \tag{2.8}$$

where we set $f(\tau,y) = f(-\tau,y)$ if $\tau < 0$. Then $f_n(s,x)$ are smooth and

$$f_n(s,x) = \int_0^2 \int_0^2 \rho(\tau)\rho(z) f\left(s - \frac{\tau}{n}, x - \frac{z}{n}\right) d\tau dz, \quad n \geqslant 1. \tag{2.9}$$

Because of the absolutely continuity mentioned above, we can differentiate under the integral in (2.9) to see that $\frac{\partial}{\partial t} f_n(t,x)$, $\nabla f_n(t,x)$, $V_{\nabla f_n(t)}$ and $V_{\nabla f_n}$ are uniformly bounded. Moreover using Lebesgue's dominated convergence theorem, one can prove that as $n \to \infty$, for each (t,x) with $t \geqslant 0$,

$$f_n(t,x) \to f(t,x). \tag{2.10}$$

Also

$$\frac{\partial}{\partial t} f_n(t,x) \to \frac{\partial^-}{\partial t} f(t,x), \quad t > 0 \tag{2.11}$$

$$\nabla f_n(t,x) \to \nabla^- f(t,x), \quad t \geqslant 0. \tag{2.12}$$

Note the convergence in (2.10), (2.11), (2.12) is also in L^p_{loc}, $1 \leqslant p < \infty$.

It turns out for any $g(t,x)$ being continuous in t and C^1 in x and having a compact support, using the integration by parts formula and Lebesgue's dominated convergence theorem,

$$\begin{aligned}\lim_{n\to+\infty} \int_{-\infty}^{+\infty} g(t,x) \Delta f_n(t,x) dx &= -\lim_{n\to+\infty} \int_{-\infty}^{\infty} \nabla g(t,x) \nabla f_n(t,x) dx \\ &= -\int_{-\infty}^{\infty} \nabla g(t,x) \nabla^- f(t,x) dx. \end{aligned} \tag{2.13}$$

Note $\nabla^- f(t,x)$ is of bounded variation in x and $\nabla g(t,x)$ has a compact support, so

$$-\int_{-\infty}^{+\infty} \nabla g(t,x) \nabla^- f(t,x) dx = \int_{-\infty}^{+\infty} g(t,x) \, \mathrm{d}_x \nabla^- f(t,x). \tag{2.14}$$

Thus

$$\lim_{n\to+\infty} \int_{-\infty}^{+\infty} g(t,x) \Delta f_n(t,x) dx = \int_{-\infty}^{\infty} g(t,x) \, \mathrm{d}_x \nabla^- f(t,x). \tag{2.15}$$

Similarly, one can easily see from the integration by parts formula and Lebesgue's dominated convergence theorem, if $g(s,x)$ is C^1 in x with $\frac{\partial}{\partial s} \nabla g(s,x)$ being continuous and has a compact support in x,

$$\begin{aligned} &\lim_{n\to+\infty} \int_0^t \int_{-\infty}^{+\infty} g(s,x) \Delta \frac{\partial}{\partial s} f_n(s,x) dx ds \\ &= -\lim_{n\to+\infty} \int_0^t \int_{-\infty}^{+\infty} \nabla g(s,x) \nabla \frac{\partial}{\partial s} f_n(s,x) dx ds \\ &= -\lim_{n\to+\infty} \int_{-\infty}^{\infty} \left[\nabla g(s,x) \nabla f_n(s,x)\right]_0^t dx \\ &\quad + \lim_{n\to+\infty} \int_0^t \int_{-\infty}^{+\infty} \frac{\partial}{\partial s} \nabla g(s,x) \nabla f_n(s,x) dx ds \\ &= -\int_{-\infty}^{+\infty} \left[\nabla g(s,x) \nabla^- f(s,x)\right]_0^t dx \\ &\quad + \int_0^t \int_{-\infty}^{\infty} \frac{\partial}{\partial s} \nabla g(s,x) \nabla^- f(s,x) dx ds. \end{aligned} \tag{2.16}$$

Thus

$$\lim_{n\to+\infty}\int_0^t\int_{-\infty}^{+\infty} g(s,x)\Delta\frac{\partial}{\partial s}f_n(s,x)dxds$$
$$=\int_0^t\int_{-\infty}^{\infty} g(s,x)\mathrm{d}_{s,x}\nabla^- f(s,x). \qquad (2.17)$$

Now suppose $g(s,x)$ is continuous in s and cadlag in x jointly, and has compact support. (In particular, g is bounded). We claim (2.15) and (2.17) still valid. For this define

$$g_m(t,x)=\int_{-\infty}^{\infty}\rho_m(y-x)g(t,y)dy=\int_0^2\rho(z)g\left(x+\frac{z}{m}\right)dz.$$

To see (2.15), using Lebesgue's dominated convergence theorem, note that there is a compact set $G_t\subset R^1$ such that

$$\max_{x\in G_t}|g_m(t,x)-g(t,x)|\to 0 \text{ as } m\to+\infty,$$
$$g_m(t,x)=g(t,x)=0 \qquad \text{for } x\notin G_t.$$

Note

$$\int_{-\infty}^{+\infty} g(t,x)\Delta f_n(t,x)dx = \int_{-\infty}^{+\infty} g_m(t,x)\Delta f_n(t,x)dx$$
$$+\int_{-\infty}^{+\infty}(g(t,x)-g_m(t,x))\Delta f_n(t,x)dx. \quad (2.18)$$

It is easy to see from (2.15) and using Lebesgue's dominated convergence theorem, that

$$\lim_{m\to\infty}\lim_{n\to\infty}\int_{-\infty}^{+\infty} g_m(t,x)\Delta f_n(t,x)dx=\lim_{m\to\infty}\int_{-\infty}^{\infty} g_m(t,x)\,\mathrm{d}_x\nabla^- f(t,x)$$
$$=\int_{-\infty}^{\infty} g(t,x)\,\mathrm{d}_x\nabla^- f(t,x). \qquad (2.19)$$

Moreover,

$$\left|\int_{-\infty}^{+\infty}(g(t,x)-g_m(t,x))\Delta f_n(t,x)dx\right|$$
$$\leqslant\left|\int_{-\infty}^{+\infty}(g(t,x)-g_m(t,x))\,\mathrm{d}_x\nabla f_n(t,x)\right|$$
$$\leqslant \max_{x\in G}|g(t,x)-g_m(t,x)|V_{\nabla f_n(t)}(G). \qquad (2.20)$$

This leads easily to

$$\lim_{m\to\infty}\limsup_{n\to\infty}\left|\int_{-\infty}^{+\infty}(g(t,x)-g_m(t,x))\Delta f_n(t,x)dx\right|=0. \qquad (2.21)$$

Now we use (2.18), (2.19), (2.21)

$$\begin{aligned}
&\limsup_{n\to\infty}\int_{-\infty}^{+\infty} g(t,x)\Delta f_n(t,x)dx\\
&\quad=\lim_{m\to\infty}\limsup_{n\to\infty}\int_{-\infty}^{+\infty} g_m(t,x)\Delta f_n(t,x)dx\\
&\qquad+\lim_{m\to\infty}\limsup_{n\to\infty}\int_{-\infty}^{+\infty}(g(t,x)-g_m(t,x))\Delta f_n(t,x)dx\\
&\quad=\int_{-\infty}^{\infty} g(t,x)\mathrm{d}_x\nabla^- f(t,x).
\end{aligned}$$

Similarly we also have

$$\liminf_{n\to\infty}\int_{-\infty}^{+\infty} g(t,x)\Delta f_n(t,x)dx=\int_{-\infty}^{\infty} g(t,x)\,\mathrm{d}_x\nabla^- f(t,x).$$

So (2.15) holds for a cadlag function g with a compact support.

Similarly we can prove (2.17) holds for a cadlag function g with a compact support. That is there exists a compact $G\subset R^1$ such that $g(s,x)=0$ for $x\notin G$ and $s\in[0,t]$.

To complete the proof of (2.7), use Itô's formula for the smooth function $f_n(s,X(s))$, then a.s.

$$\begin{aligned}
f_n(t,X(t))-f_n(0,X(0))=\int_0^t\frac{\partial}{\partial s}f_n(s,X(s))ds+\int_0^t\nabla f_n(s,X(s))dX(s)\\
+\frac{1}{2}\int_0^t\Delta f_n(s,X(s))d\langle M,M\rangle_s.
\end{aligned}\tag{2.22}$$

As $n\to\infty$, for all $t\geqslant 0$,

$$f_n(t,X(t))-f_n(0,X(0))\to f(t,X(t))-f(0,X(0))\quad a.s.,$$

and

$$\int_0^t\frac{\partial}{\partial s}f_n(s,X(s))ds\to\int_0^t\frac{\partial^-}{\partial s}f(s,X(s))ds\quad a.s.,$$

$$\int_0^t\nabla f_n(s,X(s))dV_s\to\int_0^t\nabla^- f(s,X(s))dV_s\quad a.s.$$

and

$$E\int_0^t\big(\nabla f_n(s,X(s))\big)^2d\langle M,M\rangle_s\to E\int_0^t\big(\nabla^- f(s,X(s))\big)^2d\langle M,M\rangle_s.$$

Therefore in $L^2(\Omega, P)$,

$$\int_0^t \nabla f_n(s, X(s))dM_s \to \int_0^t \nabla^- f(s, X(s))dM_s.$$

To see the convergence of the last term, we recall the well-known result that the local time $L_s(x)$ is jointly continuous in s and cadlag with respect to x and has a compact support in space x for each s. As $L_s(x)$ is an increasing function of s for each x , so if $G \subset R^1$ is the support of L_t, then $L_s(x) = 0$ for all $x \notin G$ and $s \leqslant t$. Now we use the occupation times formula, the integration by parts formula and (2.15), (2.17) for the case when g is cadlag with compact support in x,

$$\begin{aligned}
&\frac{1}{2}\int_0^t \Delta f_n(s, X(s))d\langle M, M\rangle_s \\
&\quad = \int_{-\infty}^{+\infty}\int_0^t \Delta f_n(s, x)d_s L_s(x)dx \\
&\quad = \int_{-\infty}^{+\infty} \Delta f_n(t, x)L_t(x)dx - \int_{-\infty}^{+\infty}\int_0^t \frac{d}{ds}\Delta f_n(s, x)L_s(x)dsdx \\
&\quad \to \int_{-\infty}^{\infty} L_t(x)\,\mathrm{d}_x\nabla^- f(t, x) - \int_{-\infty}^{+\infty}\int_0^t L_s(x)\mathrm{d}_{s,x}\nabla^- f(s, x) \quad a.s.,
\end{aligned}$$

as $n \to \infty$. This proves the desired formula. □

The smoothing procedure can easily be modified to prove that if $f : R^+ \times R \to R$ satisfies (i), (ii) and (iii) of Theorem 2.1, is also C^1 in x and the left derivative $\Delta^- f(t, x)$ exists at all points of $[0, \infty) \times R$ and is jointly left continuous and locally bounded, then $\Delta f_n(t, x) \to \Delta^- f(t, x)$ as $n \to \infty$, $t > 0$. Thus

$$\begin{aligned}
f(t, X(t)) = f(0, X(0)) &+ \int_0^t \frac{\partial^-}{\partial s} f(s, X(s))ds + \int_0^t \nabla f(s, X(s))dX(s) \\
&+ \frac{1}{2}\int_0^t \Delta^- f(s, X(s))d\langle X\rangle_s \quad a.s. \qquad (2.23)
\end{aligned}$$

The next theorem is an easy extension of Theorem 2.1 and formula (2.23).

Theorem 2.2 *Assume $f : R^+ \times R \to R$ satisfies conditions (i), (ii) and (iii) of Theorem 2.1. Further suppose*

$$f(t, x) = f_h(t, x) + f_v(t, x)$$

where

(i) $f_h(t, x)$ is C^1 in x with $\nabla f_h(t, x)$ having left partial derivative $\Delta^- f_h(t, x)$, (with respect to x), which is left continuous and locally bounded,

(ii) $f_v(t,x)$ has a left continuous derivative $\nabla^- f_v(t,x)$ at all points (t,x) $[0,\infty)\times R$, which is of locally bounded variation in (t,x) and of locally bounded in x for $t=0$.
Then for any continuous semi-martingale $\{X(t), t\geqslant 0\}$,

$$\begin{aligned} f(t,X(t)) = f(0,X(0)) &+ \int_0^t \frac{\partial^-}{\partial s} f(s,X(s))ds + \int_0^t \nabla^- f(s,X(s))dX(s) \\ &+\frac{1}{2}\int_0^t \Delta^- f_h(s,X(s))d\langle X\rangle_s + \int_{-\infty}^{\infty} L_t(x)\,\mathrm{d}_x \nabla^- f_v(t,x) \\ &- \int_{-\infty}^{+\infty}\int_0^t L_s(x)\mathrm{d}_{s,x}\nabla^- f_v(s,x) \quad a.s. \end{aligned} \tag{2.24}$$

Proof. Mollify f_h and f_v, and so f, as in the proof of Theorem 2.1. Apply Itô's formula to the mollification of f and take the limits as in the proofs of Theorem 2.1 and (2.23). □

If f has discontinuity of first and second order derivatives across a curve $x = l(t)$, where $l(t)$ is a continuous function of locally bounded variation, it will be convenient to consider the continuous semi-martingale

$$X^*(s) = X(s) - l(s),$$

and let $L_s^*(a)$ be its local time. We can prove the following version of our main results:

Theorem 2.3 *Assume $f : R^+ \times R \to R$ satisfies conditions (i), (ii) and (iii) of Theorem 2.1. Moreover, suppose $f(t,x) = f_h(t,x) + f_v(t,x)$, where $f_h(t,x)$ is C^1 in x and $\nabla f_h(t,x)$ has left derivative $\Delta^- f_h(t,x)$ which is left continuous and locally bounded, and there exists a curve $x = l(t)$, $t \geqslant 0$, a continuous function of locally bounded variation such that $\nabla^- f_v(t, x+l(t))$ as a function of (t,x) is of locally bounded variation in (t,x) and of locally bounded in x for $t=0$. Then*

$$\begin{aligned} f(t,X(t)) = \ & f(0,z) + \int_0^t \frac{\partial}{\partial s} f(s,X(s))ds + \int_0^t \nabla^- f(s,X(s))dX(s) \\ &+ \frac{1}{2}\int_0^t \Delta^- f_h(s,X(s))d\langle X\rangle_s + \int_{-\infty}^{\infty} L_t^*(x)\,\mathrm{d}_x \nabla^- f_v(t,x+l(t)) \\ &- \int_{-\infty}^{+\infty}\int_0^t L_s^*(x)\mathrm{d}_{s,x}\nabla^- f_v(s,x+l(s)) \quad a.s. \end{aligned} \tag{2.25}$$

Proof. We only need to consider the case when $f_h = 0$ as the general case will follow easily. We basically follow the proof of Theorem 2.1 and apply Itô's formula to f_n and $X(s)$. We still have (2.22). But by the occupation times formula, a.s.

$$\begin{aligned}
&\frac{1}{2}\int_0^t \Delta f_n(s,X(s))d\langle M,M\rangle_s \\
&= \frac{1}{2}\int_0^t \Delta f_n(s,X^*(s)+l(s))d\langle M,M\rangle_s \\
&= \int_{-\infty}^{+\infty}\int_0^t \Delta f_n(s,x+l(s))d_sL_s^*(x)dx \\
&= \int_{-\infty}^{+\infty} \Delta f_n(t,x+l(t))L_t^*(x)dx - \int_{-\infty}^{+\infty}\int_0^t \frac{d}{ds}\Delta f_n(s,x+l(s))L_s^*(x)dsdx \\
&\to \int_{-\infty}^{\infty} L_t^*(x)\,\mathrm{d}_x\nabla^- f(t,x+l(t)) - \int_{-\infty}^{+\infty}\int_0^t L_s^*(x)\mathrm{d}_{s,x}\nabla^- f(s,x+l(s)),
\end{aligned}$$

as $n\to\infty$ as in the proof of Theorem 2.1. This proves the desired formula.

□

Corollary 2.1 *Assume $f: R^+\times R\to R$ satisfies condition (i) of Theorem 2.1 and its left derivative $\frac{\partial^-}{\partial t}f$ exists on $(0,\infty)\times R$ and is left continuous. Further suppose that there exists a curve $x=l(t)$ of locally bounded variation such that f is C^1 in x off the curve with ∇f having left and right limits in x at each point (t,x) and a left continuous and locally bounded left derivative $\Delta^- f$ on x not equal to $l(t)$. Also assume $\nabla f(t,l(t)+y-)$ as a function of t and y is locally bounded and jointly left continuous if $y\leqslant 0$, and $\nabla f(t,l(t)+y+)$ is locally bounded and jointly left continuous in t and right continuous in y if $y\geqslant 0$. Then for any continuous semi-martingale $\{X(t),t\geqslant 0\}$,*

$$\begin{aligned}
f(t,X(t)) &= f(0,X(0)) + \int_0^t \frac{\partial^-}{\partial s} f(s,X(s))ds \\
&\quad + \int_0^t \nabla^- f(s,X(s))dX(s) + \frac{1}{2}\int_0^t \Delta^- f(s,X(s))d\langle X,X\rangle_s \\
&\quad + \int_0^t (\nabla f(s,l(s)+) - \nabla f(s,l(s)-))d_sL_s^*(0) \quad a.s. \qquad (2.26)
\end{aligned}$$

Proof. At first we assume temporarily that $(\nabla f(t,l(t)+)-\nabla f(t,l(t)-))$ is of bounded variation. This condition will be dropped later. Formula (2.26) can be read from (2.25) by considering

$$\begin{aligned}
f_h(t,x) &= f(t,x) + (\nabla f(t,l(t)-) - \nabla f(t,l(t)+))(x-l(t))^+, \\
f_v(t,x) &= (\nabla f(t,l(t)+) - \nabla f(t,l(t)-))(x-l(t))^+,
\end{aligned}$$

and integration by parts formula and noticing $\nabla^- f_v(t,x+l(t))$ is of locally bounded variation in (t,x). Let $g(t,y)=f(t,y+l(t))$. In terms of X^*, (2.26) can be rewritten as

$$\begin{aligned} g(t, X^*(t)) &= g(0, X^*(0)) + \int_0^t g(ds, X^*(s)) + \int_0^t \nabla^- g(s, X^*(s)) dX^*(s) \\ &+ \frac{1}{2} \int_0^t \Delta^- g(s, X^*(s)) d\langle X^*, X^* \rangle_s \\ &+ \int_0^t (\nabla g(s, 0+) - \nabla g(s, 0-)) d_s L_s^*(0) \quad a.s. \end{aligned} \tag{2.27}$$

Here

$$g(ds, y) = d_s g(s, y) = \frac{\partial^-}{\partial s} f(s, y + l(s)) ds + \nabla^- f(s, y + l(s)) dl(s).$$

Now without assuming that $(\nabla f(t, l(t)+) - \nabla f(t, l(t)-))$ is of bounded variation, we can prove the formula by a smoothing procedure in the variable t. To see this, let

$$g_n(t, y) = \int_0^2 \rho(\tau) g\left(t - \frac{\tau}{n}, y\right) d\tau = \int_0^2 \rho(\tau) f\left(t - \frac{\tau}{n}, y + l\left(t - \frac{\tau}{n}\right)\right) d\tau,$$

with $l(s) = l(0)$ if $s < 0$ and $f(s, x) = f(-s, x)$ for $s < 0$ as usual. Then as $n \to \infty$,

$$\begin{aligned} \int_0^t g_n(ds, X^*(s)) &= \int_0^t \int_0^2 \rho(\tau) \frac{\partial^-}{\partial s} f\left(s - \frac{\tau}{n}, X^*(s) + l\left(s - \frac{\tau}{n}\right)\right) d\tau ds \\ &\quad + \int_0^t \int_0^2 \rho(\tau) \nabla^- f\left(s - \frac{\tau}{n}, X^*(s) + l\left(s - \frac{\tau}{n}\right)\right) \\ &\qquad \times dl\left(s - \frac{\tau}{n}\right) d\tau \\ &\to \int_0^t \frac{\partial^-}{\partial s} f\left(s, X^*(s) + l(s)\right) ds + \nabla^- f(s, X^*(s) + l(s)) dl(s) \\ &= \int_0^t g(ds, X^*(s)) \quad a.s. \end{aligned} \tag{2.28}$$

It is easy to see that for all (t, y)

$$g_n(t, y) \to g(t, y) \tag{2.29}$$

and for all $y \neq 0$,

$$\nabla g_n(t, y) \to \nabla g(t, y), \qquad \Delta^- g_n(t, y) \to \Delta^- g(t, y), \tag{2.30}$$

with uniform local bounds. Moreover, we can see that as $y \to 0\pm$ and $n \to \infty$,

$$\nabla^{\pm} g_n(t, y) = \int_0^2 \rho(\tau) \nabla^{\pm} g\left(t - \frac{\tau}{n}, y\right) d\tau \to \nabla g(t, 0\pm). \tag{2.31}$$

Since $\nabla g_n(t, 0\pm)$ are smooth in t they are of locally bounded variation. From (2.27),

$$
\begin{aligned}
g_n(t, X^*(t)) = g_n(0, X^*(0)) &+ \int_0^t g_n(ds, X^*(s)) + \int_0^t \nabla^- g_n(s, X^*(s)) dX^*(s) \\
&+ \frac{1}{2}\int_0^t \Delta^- g_n(s, X^*(s)) d\langle X^*, X^*\rangle_s \\
&+ \int_0^t (\nabla g_n(s, 0+) - \nabla g_n(s, 0-)) d_s L_s^*(0) \quad a.s. \qquad (2.32)
\end{aligned}
$$

We obtain the desired formula by passing to the limits using (2.28), (2.29), (2.30) and (2.31). □

Remark 2.1 *(i) Formula (2.26) was also observed by Peskir in [22] and [13] independently.*

(ii) From the proof of Theorem 2.1, one can take different mollifications, e.g. one can take (2.9) as

$$
f_n(s, x) = \int_0^2 \int_0^2 \rho(\tau)\rho(z) f\left(s + \frac{\tau}{n}, x + \frac{z}{n}\right) d\tau dz, \quad n \geqslant 1.
$$

This will lead to as $n \to \infty$,

$$
\frac{\partial}{\partial s} f_n(s, x) \to \frac{\partial^+}{\partial s} f(s, x)
$$

instead of (2.11), if $\frac{\partial^+}{\partial s} f(s, x)$ is jointly right continuous. Therefore we have the following more general Itô's formula

$$
\begin{aligned}
f(t, X(t)) = f(0, z) &+ \int_0^t \frac{\partial^{s_1}}{\partial s} f(s, X(s)) ds + \int_0^t \nabla^{s_2} f(s, X(s)) dX(s) \\
&+ \frac{1}{2}\int_0^t \Delta^{s_2} f_h(s, X(s)) d\langle X\rangle_s \\
&+ \int_{-\infty}^{\infty} L_t(x)\, \mathrm{d}_x \nabla^{s_2} f_v(t, x) - \int_{-\infty}^{+\infty} \int_0^t L_s(x) \mathrm{d}_{s,x} \nabla^{s_2} f_v(s, x) \ a.s.,
\end{aligned}
$$

where $s_1 = \pm$ and $s_2 = \pm$.

Formula (2.24) is in a very general form. It includes the classical Itô formula, Tanaka's formula, Meyer's formula for convex functions, the formula given by Azéma, Jeulin, Knight and Yor [3] and formula (2.26). In the following we will give some examples for which (2.26) and some known generalized Itô formulae do not immediately apply, but formula (2.24) can be applied. These examples can be presented in different forms to include local times on curves.

Example 2.1 *Consider the function*

$$f(t,x) = (\sin \pi x \sin \pi t)^+.$$

Then

$$\nabla^- f(t,x) = \pi \cos \pi x \sin \pi t \; 1_{\sin \pi x \sin \pi t > 0}.$$

One can verify that $\nabla^- f(t,x)$ is of locally bounded variation in (t,x). This can be easily seen from Proposition 2.1 and the simple fact that

$$\cos \pi x \sin \pi t \; 1_{\sin \pi x \sin \pi t > 0} = \begin{cases} \cos \pi x \sin \pi t, & \text{if } i \leqslant t < i+1, j \leqslant x < j+1, i+j \text{ is even} \\ 0, & \text{otherwise}. \end{cases}$$

Therefore

$$\begin{aligned} (\sin \pi X(t) \sin \pi t)^+ = & \pi \int_0^t \cos \pi s \sin \pi X(s) \; 1_{\sin \pi X(s) \sin \pi s > 0} ds \\ & + \pi \int_0^t \cos \pi X(s) \sin \pi s \; 1_{\sin \pi X(s) \sin \pi s > 0} dX(s) \\ & + \pi \sin \pi t \int_{-\infty}^{\infty} L_t(a) \, \mathrm{d_a} (\cos \pi a \; 1_{\sin \pi a \sin \pi t > 0}) \\ & - \pi \int_0^t \int_{-\infty}^{\infty} L_s(a) \mathrm{d_{s,a}} (\cos \pi a \sin \pi s \; 1_{\sin \pi a \sin \pi s > 0}). \end{aligned}$$

One can expand the last two integrals to see the jump of

$$\cos \pi a \; \sin \pi s \; 1_{\sin \pi a \; \sin \pi s > 0}.$$

Note in Example 2.1, $\nabla^- f(t,x)$ has jump on the boundary of each interval $i \leqslant t < i+1$, $j \leqslant x < j+1$. One can use this example as a prototype to construct many other examples with other types of derivative jumps.

Example 2.2 *Consider the function*

$$f(t,x) = (\sin \pi x)^{\frac{1}{3}} (\sin \pi x \sin \pi t)^+.$$

Then

$$\begin{aligned} \nabla^- f(t,x) = & \frac{1}{3} \pi \cos \pi x (\sin \pi x)^{-\frac{2}{3}} (\sin \pi x \sin \pi t)^+ \\ & + \pi (\sin \pi x)^{\frac{1}{3}} \cos \pi x \sin \pi t \; 1_{\sin \pi x \sin \pi t > 0}. \end{aligned}$$

One can verify that $\nabla^- f(t,x)$ is of locally bounded variation in (t,x) and continuous. In fact,

$$\cos \pi x (\sin \pi x)^{-\frac{2}{3}} (\sin \pi x \sin \pi t)^+ = \begin{cases} \cos \pi x (\sin \pi x)^{\frac{1}{3}} \sin \pi t, & \text{if } i \leqslant t < i+1, \\ & j \leqslant x < j+1, i+j \text{ is even} \\ 0, & \text{otherwise} \end{cases}$$

then it is easy to see that $\cos \pi x (\sin \pi x)^{-\frac{2}{3}} (\sin \pi x \sin \pi t)^+$ *is of locally bounded variation in* (t, x) *using Proposition 2.1. Similarly one can see that* $(\sin \pi x)^{\frac{1}{3}} \cos \pi x \sin \pi t \ 1_{\sin \pi x \sin \pi t > 0}$ *is of locally bounded variation in* (t, x) *as well.*

Note $\Delta^- f(t, x)$ *blows up when* x *is near an integer value, and their left and right limits also blow up. However one can apply our generalized Itô's formula (2.24) to this function so that*

$$\begin{aligned} &(\sin \pi X(t))^{\frac{1}{3}} (\sin \pi X(t) \sin \pi t)^+ \\ &= \pi \int_0^t (\sin \pi X(s))^{\frac{4}{3}} \cos \pi s \ 1_{\sin \pi X(s) \sin \pi s > 0} \mathrm{ds} \\ &\quad + \int_{-\infty}^{\infty} L_t(a) \, \mathrm{d_a} \bigg(\frac{1}{3} \pi \cos \pi a (\sin \pi a)^{-\frac{2}{3}} (\sin \pi a \sin \pi t)^+ \\ &\qquad + \pi (\sin \pi a)^{\frac{1}{3}} \cos \pi a \sin \pi t \ 1_{\sin \pi a \sin \pi t > 0} \bigg) \\ &\quad - \int_0^t \int_{-\infty}^{\infty} L_s(a) \, \mathrm{d_{s,a}} \bigg(\frac{1}{3} \pi \cos \pi a (\sin \pi a)^{-\frac{2}{3}} (\sin \pi a \sin \pi s)^+ \\ &\qquad + \pi (\sin \pi a)^{\frac{1}{3}} \cos \pi a \sin \pi s \ 1_{\sin \pi a \sin \pi s > 0} \bigg). \end{aligned}$$

3 The case for Itô processes

For Itô processes, we can allow some of the generalized derivatives of f to be only in $L^2_{loc}(dtdx)$. Consider

$$X(t) = X(0) + \int_0^t \sigma_r dW_r + \int_0^t b_r dr. \tag{3.1}$$

Here W_r is a one-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_r\}_{r \geqslant 0}, P)$ and σ_r and b_r are progressively measurable with respect to $\{\mathcal{F}_r\}$ and satisfy the following conditions: for all $t > 0$

$$\int_0^t |\sigma_r|^2 dr < \infty, \quad \int_0^t |b_r| dr < \infty \quad a.s. \tag{3.2}$$

Under condition (3.2), the process (3.1) is well defined. For any $N > 0$, define $\tau_N = \inf\{s : |X(s)| \geqslant N\}$. Assume there exist constants $\delta > 0$ and $K > 0$ such that

$$\sigma_t(\omega) \geqslant \delta > 0, \quad |\sigma_t(\omega)| + |b_t(\omega)| \leqslant K, \quad \text{for all } (t, \omega) \text{ with } t \leqslant \tau_N. \tag{3.3}$$

The following inequality due to Krylov [16] plays an important role.

Lemma 3.1 *Assume condition (3.2) and (3.3). Then there exists a constant $M > 0$, depending only on δ and K such that*

$$E\int_0^{t\wedge\tau_N} |f(r, X(r))|dr \leqslant M \left(\int_0^t \int_{-N}^{+N} (f(r,x))^2 drdx\right)^{\frac{1}{2}}. \tag{3.4}$$

Denote again by $L_t(x)$ the local time of the diffusion process $X(t)$ at level x. We can prove the following theorem.

Theorem 3.1 *Assume $f(t,x)$ is continuous with generalized derivative $\frac{\partial}{\partial t}f$ in $L^2_{loc}(dtdx)$ and generalized derivative ∇f of locally bounded variation in (t,x) and of locally bounded variation in x for $t=0$. Consider an Itô process $X(t)$ given by (3.1) with σ and b satisfying (3.2) and (3.3). Then a.s.*

$$\begin{aligned} f(t, X(t)) = f(0, X(0)) &+ \int_0^t \frac{\partial}{\partial s} f(s, X(s))ds + \int_0^t \nabla f(s, X(s))dX(s) \\ &+ \int_{-\infty}^{\infty} L_t(x)\, \mathrm{d}_x \nabla f(t,x) - \int_{-\infty}^{+\infty}\int_0^t L_s(x)\mathrm{d}_{s,x}\nabla f(s,x). \end{aligned} \tag{3.5}$$

Proof. Define f_n by (2.8). From a well-known result on Sobolev spaces (see Theorem 3.16, p.52 in [1]), we know that as $n \to \infty$,

$$f_n(t,x) \to f(t,x),$$

for all (t,x) and for any $N > 0$

$$\begin{aligned} &\frac{\partial}{\partial t} f_n \to \frac{\partial}{\partial t} f, \text{ in } L^2([0,t]\times[-N,N]) \\ &\nabla f_n \to \nabla f, \text{ in } L^4([0,t]\times[-N,N]). \end{aligned}$$

As in the proof of Theorem 2.1, we have the Itô formula (2.22) for $f_n(t\wedge\tau_N, X(t\wedge\tau_N))$. The convergence of the terms $f_n(t\wedge\tau_N, X(t\wedge\tau_N))$, and $\frac{1}{2}\int_0^{t\wedge\tau_N} \sigma_s^2 \Delta f_n(s, X(s))ds$ is the same as before. Now by using Lemma 3.1,

$$\begin{aligned} &E\left|\int_0^{t\wedge\tau_N} \frac{\partial}{\partial s} f_n(s, X(s))ds - \int_0^{t\wedge\tau_N} \frac{\partial}{\partial s} f(s, X(s))ds\right| \\ &\leqslant E\int_0^{t\wedge\tau_N} \left|\frac{\partial}{\partial s} f_n(s, X(s)) - \frac{\partial}{\partial s} f(s, X(s))\right| ds \\ &\leqslant M\left(\int_0^t \int_{-N}^{N} \left(\frac{\partial}{\partial s} f_n(s,x) - \frac{\partial}{\partial s} f(s,x)\right)^2 dsdx\right)^{\frac{1}{2}} \to 0 \end{aligned}$$

as $n \to \infty$. Similarly one can prove

$$\int_0^{t\wedge\tau_N} b_s \nabla f_n(s, X(s))ds \to \int_0^{t\wedge\tau_N} b_s \nabla f(s, X(s))ds \quad in \quad L^1(dP).$$

Moreover, there exists a constant $M > 0$ such that

$$\begin{aligned} &E\left(\int_0^{t\wedge\tau_N} \sigma_s \nabla f_n(s, X(s))dW_s - \int_0^t \sigma_s \nabla f(s, X(s))dW_s\right)^2 \\ &= E\left(\int_0^{t\wedge\tau_N} \sigma_s^2(\nabla f_n(s, X(s)) - \nabla f(s, X(s))\right)^2 ds \\ &\leqslant M\left(\int_0^t \int_{-N}^N (\nabla f_n(s, x) - \nabla f(s, x)\right)^4 dsdx \to 0 \end{aligned}$$

as $n \to \infty$. Therefore we have proved that

$$\begin{aligned} &f(t\wedge\tau_N, X(t\wedge\tau_N)) \\ &= f(0, X(0)) + \int_0^{t\wedge\tau_N} \frac{\partial}{\partial s} f(s, X(s))ds + \int_0^{t\wedge\tau_N} \nabla f(s, X(s))dX(s) \\ &+ \int_{-\infty}^{\infty} L_{t\wedge\tau_N}(x)\, \mathrm{d}_x \nabla f(t\wedge\tau_N, x) - \int_{-\infty}^{+\infty}\int_0^{t\wedge\tau_N} L_s(x)\mathrm{d}_{s,x}\nabla f(s, x). \end{aligned}$$

The desired formula follows. □

Recall the following extension of Itô's formula due to Krylov [16]: if $f : R^+ \times R$ is C^1 in x and ∇f is absolutely continuous with respect to x for each t and the generalized derivatives $\frac{\partial}{\partial s} f(s, x)$ and Δf are in $L^2_{loc}(dtdx)$, then

$$\begin{aligned} f(t, X(t)) = f(0, z) &+ \int_0^t \frac{\partial}{\partial s} f(s, X(s))ds + \int_0^t \nabla f(s, X(s))dX(s) \\ &+ \frac{1}{2}\int_0^t \sigma_s^2 \Delta f(s, X(s))ds \quad a.s. \end{aligned} \tag{3.6}$$

The next theorem is an easy consequence of the method of proof of Theorem 3.1 and of formula (3.6).

Theorem 3.2 *Assume $f(t, x)$ is continuous and its generalized derivative $\frac{\partial}{\partial t} f$ is in $L^2_{loc}(dtdx)$. Moreover $f(t, x) = f_h(t, x) + f_v(t, x)$ with $f_h(t, x)$ being C^1 in x and $\nabla f_h(t, x)$ having generalized derivative $\Delta f_h(t, x)$ in $L^2_{loc}(dtdx)$, and f_v having generalized derivative $\nabla f_v(t, x)$ being of locally bounded variation in (t, x) and of locally bounded variation in x for $t = 0$. Suppose $X(t)$ is an Itô process given by (3.1) with σ and b satisfying (3.2) and (3.3). Then,*

$$
\begin{aligned}
f(t,X(t)) = f(0,X(0)) &+ \int_0^t \frac{\partial}{\partial s} f(s,X(s))ds + \int_0^t \nabla f(s,X(s))dX(s) \\
&+ \frac{1}{2}\int_0^t \Delta f_h(s,X(s))d\langle X\rangle_s + \int_{-\infty}^{\infty} L_t(x)\,\mathrm{d}_x \nabla f_v(t,x) \\
&- \int_{-\infty}^{+\infty}\int_0^t L_s(x)\mathrm{d}_{s,x}\nabla f_v(s,x) \quad a.s.
\end{aligned} \tag{3.7}
$$

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