A Note on a Change of Variable Formula with Local Time-Space for Lévy Processes of Bounded Variation

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Summary. We establish a mild variant of the change of variable formula with local time-space for "ripped" functions and Lévy processes of bounded variation.

1 Lévy processes of bounded variation and local time-space

In this short note we shall establish a change of variable formula for "ripped" time-space functions of Lévy processes of bounded variation at the cost of an additional integral with respect to local time-space in the formula. Roughly speaking, by a ripped function, we mean here a time-space function which is $C^{1,1}$ on both sides of a time dependent barrier and which may exhibit a discontinuity along the barrier itself. Such functions have appeared in the theory of optimal stopping problems for Markov processes of bounded variation (cf. [1, 3, 10, 11]). Our starting point is to give a brief review of the relevant features of Lévy processes of bounded variation and what is meant by local time-space for these processes.

Suppose that $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is a filtered probability space with filtration $\mathbb{F} = \{\mathcal{F}_t : t \geqslant 0\}$ satisfying the usual conditions of right continuity and completion. In this text, we take as our definition of a Lévy process on $(\Omega, \mathcal{F}, \mathbb{F}, P)$, the strong Markov, $\mathbb{F}$-adapted process $X = \{X(t) : t \geqslant 0\}$ with paths that are right continuous with left limits (càdlàg) having the properties that $P(X(0) = 0) = 1$ and for each $0 \leqslant s \leqslant t$, the increment $X(t) - X(s)$ is independent of $\mathcal{F}_s$ and has the same distribution as $X(t - s)$. On each finite time interval, X has paths of bounded variation (or just X has bounded variation for short) if and only if for each $t \geqslant 0$,

$$X(t) = \mathrm{d}t + \sum_{0<s\leqslant t} \Delta_s \tag{1}$$

where $\mathrm{d} \in \mathbb{R}$ and $\{\Delta_s : s \geqslant 0\}$ is a Poisson point process on $[0,\infty) \times (\mathbb{R}\backslash\{0\})$ with (time-space) intensity measure $dt \times \Pi(dx)$ satisfying

$$\int_{\mathbb{R}\backslash\{0\}} (1 \wedge |x|)\, \Pi(dx) < \infty.$$

Note that the latter integrability condition is necessary and sufficient for the convergence of $\sum_{0<s\leqslant t} |\Delta_s|$. The process X is further a compound Poisson process with drift if and only if $\Pi(\mathbb{R}\backslash\{0\}) < \infty$.

For any such Lévy process we say that 0 *is irregular for itself* if

$$P(T = 0) = 0$$

where T is the first visit of X to the origin,

$$T = \inf\{t > 0 : X_t = 0\}$$

with the usual definition $\inf \emptyset = \infty$ being understood in the present context as corresponding to the case that Y never visits the origin over the time interval $(0,\infty)$. Standard theory allows us to deduce that T is a stopping time. With the exception of a compound Poisson process, 0 is always irregular for itself within the class of Lévy processes of bounded variation. Further, again excluding the case of a compound Poisson process,

$$P(T < \infty) > 0 \Longleftrightarrow \mathrm{d} \neq 0. \tag{2}$$

We refer to [2] for a much deeper account of regularity properties of Lévy processes. For the purpose of this text we need to extend the idea of irregularity for points to irregularity of time-space curves.

Definition 1. Given a Lévy process X with finite variation, a measurable time-space curve $b : [0,\infty) \to \mathbb{R}$ is said to be *irregular for itself for* X if all $\infty > T \geqslant s \geqslant 0$,

$$P_{(s,b(s))}(\#\{t \in (s,T] : X(t) = b(t)\} < \infty) = 1$$

and $t \in \{s \geqslant 0 : X(s) = b(s)\}$ if and only if $\lim_{s\uparrow t} |X(s) - b(s)| = 0$.

A curve b which is irregular for itself for X allows for the construction of the almost surely finite counting measure

$$L^b : \mathcal{B}[0,\infty) \to \mathbb{N}$$

defined by

$$L^b[0,t] = 1 + \sum_{0<s\leqslant t} \delta_{(X(s)=b(s))}(s) \tag{3}$$

where $\delta_{\cdot}(s)$ is the Dirac unit mass at time s. Further, $L^b[0,\infty)$ is almost surely 1 if $\mathrm{d} = 0$. We call the right continuous process

$$L^b = \left\{L_t^b := L[0,t] : t \geqslant 0\right\}$$

local time-space for the curve b. Our choice of terminology here is motivated by [9] who gave the name *local time-space* for an analogous object defined for continuous semi-martingales.

Little seems to be known about local times of Lévy processes of bounded variation (see however [7]) and hence a full classification of all such curves b which are irregular for themselves for X remains an open question. The definition as given is not empty however as we shall now show with the following simple examples.

Example 2. Suppose simply that $b(t) = x$ for all $t \geqslant 0$ and some $x \in \mathbb{R}$ and that X is not a compound Poisson process. In this case, the local time process is nothing more than the number of visits to x plus one which is a similar definition to the one given in [7]. As can be deduced from the above introduction to Lévy processes of bounded variation, if $\mathrm{d} = 0$ then $L_t = 1$ for all $t > 0$. If on the other hand $\mathrm{d} \neq 0$ then since X has the property that $\{0\}$ is irregular for itself for X then the number of times X hits x in each finite time interval is almost surely finite. Further, X hits x by either creeping upwards over it or creeping downwards below according to the respective sign of d. (Creeping both upwards and downwards is not possible for Lévy processes which do not possess a Gaussian component). Creeping upwards above x occurs at first passage time T if and only if $\lim_{s\uparrow T} X(s) = x$. Since the same statement is true of downward creeping and X may only creep in at most one direction, it follows with the help of the Strong Markov Property that $t \in \{s > 0 : X(s) = x\}$ if and only if $\lim_{s\uparrow t} |X(s) - x| = 0$.

Example 3. More generally, if $\Pi(\mathbb{R}\backslash\{0\}) = \infty$ then an argument similar to the above shows that if b, satisfying $b(0+) = b(0)$ and $|b'(0+)| < \infty$, belongs to the class $C^1(0,\infty)$, then it is also irregular for itself for X. One needs to take advantage in this case of the fact that b has locally linear behaviour. Furthermore, one sees that points t for which $b'(t) = \mathrm{d}$ cannot be hit. We have excluded $\Pi(\mathbb{R}\backslash\{0\}) < \infty$ in order to avoid simple pathological examples such as the case of the compound Poisson process and $b(t) = 0$ for all $t \geqslant 0$.

2 A generalization of the change of variable formula

In this section we state our result. The idea is to take the change of variable formula and to weaken the assumption on the class of functions to which it applies. For clarity, let us first state the change of variable formula in the special form that it takes for bounded variation Lévy processes. See [13] or [12] for details of its proof.

Theorem 4. *Suppose that the time-space function f belongs to the class $C^{1,1}([0,\infty)\times\mathbb{R})$. Then for any Lévy process X of bounded variation,*

$$\begin{aligned} f(t,X(t)) - f(0,X(0)) = & \int_0^t \frac{\partial f}{\partial t}(s,X(s-))\,ds + \mathrm{d}\int_0^t \frac{\partial f}{\partial x}(s,X(s))ds \\ & + \sum_{0<s\leqslant t}\{f(s,X(s)) - f(s,X(s-))\} \end{aligned}$$

almost surely.

Remark 5. By inspection of the proof of the change of variable formula it is also clear that if for some random time T, $X_t \in D$ for all $t < T$ where D is an open set, then the change of variable formula as given above still holds on the event $\{t \leqslant T\}$ for functions $f \in C^{1,1}([0,\infty), D)$.

The generalization we are interested in consists in weakening the class $C^{1,1}([0,\infty)\times\mathbb{R})$ in the Change of Variable formula to the following class.

Definition 6. Suppose that $b : [0,\infty) \to \mathbb{R}$ is a measurable function. A function f is said to be $C^{1,1}([0,\infty)\times\mathbb{R})$ *ripped along* b if

$$f(t,x) = \begin{cases} f^{(1)}(t,x) & x > b(t),\ t \geqslant 0 \\ f^{(2)}(t,x) & x < b(t),\ t \geqslant 0 \end{cases}$$

where $f^{(1)}$ and $f^{(2)}$ each belong to the class $C^{1,1}([0,\infty)\times\mathbb{R})$.

We shall prove the following theorem.

Theorem 7. *Suppose that b is a measurable function which is irregular for itself for X and f is $C^{1,1}([0,\infty)\times\mathbb{R})$ ripped along b. Then for any Lévy process of bounded variation, X,*

$$\begin{aligned} f(t,X(t)) - f(0,X(0+)) = & \int_0^t \frac{\partial f}{\partial t}(s,X(s-))ds + \mathrm{d}\int_0^t \frac{\partial f}{\partial x}(s,X(s-))ds \\ & + \sum_{0<s\leqslant t}\{f(s,X(s)) - f(s,X(s-))\} \\ & + \int_0^t \{f(s,X(s+)) - f(s,X(s-))\}\,dL_s^b \end{aligned}$$

almost surely.

Note, the term $f(0,X(0+))$ is deliberate in place of $f(0,X(0))$ as, in the case that $X(0) = b(0)$, it is possible that the process $f(\cdot,X(\cdot))$ starts with a jump.

This result complements the recent results of [9] which concern an extension of Itô's formula for continuous semi-martingales. Peskir accommodates for the case that the time-space function, f, to which Itô's formula is applied has a disruption in its smoothness along a *continuous* space time barrier of *bounded variation.* In particular, on either side of the barrier, the function is equal to a $C^{1,2}(\mathbb{R} \times [0, \infty))$ time-space function but, unlike the case here, it is assumed that there is continuity in f across the barrier. The formula that Peskir obtained has an additional integral with respect to the semi-martingale local time at zero of the distance of the underlying semi-martingale from the boundary (this is again a semi-martingale) which he calls *local time-space.* As mentioned above, we have chosen for obvious reasons to refer to the integrator in the additional term obtained in Theorem 7 as local time-space also. Peskir's results build further on those of [8] and [4] for Brownian motion and in this sense our results now bring the discussion into the particular and somewhat simpler class of bounded variation semi-martingales that we study here. [5,6,9] all have further results for general and special types of semi-martingales. However, the present study is currently the only one which considers discontinuous functions. We have introduced local time-space as a counting measure rather than an occupation density at zero of the semi-martingale $X - b$ as one normally sees. In the current context, the latter is in fact identically zero (cf. [12]). Other definitions of local time-space may be possible in order to work with more general classes of curves than those given in Definition 1 and hence the current presentation merely scratches the surface of the problem considered.

3 Proofs

Proof (of Theorem 7). The essence of the proof is based around a telescopic sum which we shall now describe. Define the inverse local time process $\tau = \{\tau_t : t \geqslant 0\}$ where

$$\tau_t = \inf\{s > 0 : L^b_s > t\}$$

for each $t \geqslant 0$. Note the second strict inequality in the definition ensures that τ is a càdlàg process and since $L^b_0 = 1$ by definition, it follows that $\tau_0 = 0$. The process τ is nothing more than a step function which increases on the integers $k = 1, 2, 3, \ldots$ by an amount corresponding to the length of the excursion of X from b whose right end point corresponds to the k-th crossing of b by X. Note that even when $X_0 \neq b(0)$ we count the section of the path of X until it first meets b as an (incomplete) excursion.

The increment in $\{f(s, X(s)) : s \geqslant 0\}$ between $s = 0+$ and $s = t$ can be seen as the accumulation of the increments incurred by X crossing the boundary b, the excursions of X from b and the final increment between the last time of contact of X with b and time t. We have

$$\begin{aligned} f(t,X(t)) - f(0,X(0+)) &= \int_0^t \{f(s,X(s+)) - f(s,X(s-))\}\, dL_s^b \\ &+ \sum_{s\leqslant L_t^b} \left\{f(\tau_s, X_{\tau_s}) - f(\tau_{s-}, X_{\tau_{s-}})\right\} \mathbf{1}_{(|\Delta\tau_s|>0)} \\ &+ \left\{f(t,X(t)) - f\left(\tau_{L_t^b}, X_{\tau_{L_t^b}-}\right)\right\} \end{aligned} \tag{4}$$

The proof is then completed once we know that the increments in the curly brackets of the second and third term on the right hand side of (4) observe the same development as the change of variable formula. Indeed, taking account of the Strong Markov Property, it would suffice to prove that under the given assumptions on f we have that for all $t \in (0,\infty)$

$$\begin{aligned} & f(t\wedge\eta, X(t\wedge\eta)) - f(0,X(0+)) \\ &= \int_0^{t\wedge\eta} \frac{\partial f}{\partial t}(s,X(s-))ds + \mathrm{d}\int_0^{t\wedge\eta} \frac{\partial f}{\partial x}(s,X(s-))ds \\ &+ \sum_{0<s\leqslant t\wedge\eta} \{f(s,X(s)) - f(s,X(s-))\}\,. \end{aligned} \tag{5}$$

Note that η is the first strictly positive time that $X - b = 0$.

The statement in (5) is intuitively appealing since up to the stopping time η the process X does not intersect with the boundary b and hence the discontinuity in f should not appear in a development of the function $f(\cdot, X(\cdot))$. The result is proved in the lemma below and thus concludes the proof of the main result. □

Lemma 8. *Under the assumptions of Theorem 7, the identity (5) holds for all $t \in (0,\infty)$.*

Proof. First fix some $\kappa > 0$, define

$$\sigma_{\kappa,0} = \inf\{t \geqslant 0 : |X(t) - b(t)| > \kappa\}.$$

and $\Omega_\kappa = \{\omega \in \Omega : \sigma_{\kappa,0} < \eta\}$. Next define for each $j \geqslant 1$ the stopping times

$$\sigma_{\kappa,j} = \inf\left\{t > \sigma_{\kappa,j-1} : |X(t) - b(t)| < \frac{1}{2}|X\left(\sigma_{\kappa,j-1}\right) - b(\sigma_{\kappa,j-1})|\right\}$$

where we again work with the usual definition $\inf\emptyset = \infty$. On the set $\Omega_\kappa \cap \{\eta < \infty\}$ we have that

$$\limsup_{j\uparrow\infty} |X(\sigma_{\kappa,j}) - b(\sigma_{\kappa,j})| \leqslant \lim_{j\uparrow\infty}\left(\frac{1}{2}\right)^j |X_{\sigma_{\kappa,0}}| = 0$$

and hence by the definition of irregularity of b for itself for X,

$$\lim_{j\uparrow\infty} \sigma_{\kappa,j} = \eta \tag{6}$$

where the limit is interpreted to be infinite on the set $\{\eta = \infty\}$. It is also clear that, since X has right continuous paths,

$$\lim_{\kappa \downarrow 0} P(\Omega_\kappa) = 1. \tag{7}$$

Over the time interval $[\sigma_{\kappa,j-1}, \sigma_{\kappa,j})$ the process X does not enter a tube of positive, $\mathcal{F}_{\sigma_{\kappa,j-1}}$-measurable radius around the curve b, we may appeal to then the standard Change of Variable Formula to deduce that on Ω_κ

$$\begin{aligned} & f\left(\sigma_{\kappa,j} \wedge t, X_{\sigma_{\kappa,j}\wedge t}\right) - f\left(\sigma_{\kappa,j-1} \wedge t, X_{\sigma_{\kappa,j-1}\wedge t}\right) \\ & \quad = \int_{\sigma_{\kappa,j-1}\wedge t}^{\sigma_{\kappa,j}\wedge t} \frac{\partial f}{\partial t}(s, X(s-))ds + \mathrm{d} \int_{\sigma_{\kappa,j-1}\wedge t}^{\sigma_{\kappa,j}\wedge t} \frac{\partial f}{\partial x}(s, X(s-))ds \\ & \qquad + \sum_{\sigma_{\kappa,j-1}\wedge t < s \leqslant \sigma_{\kappa,j}\wedge t} \left\{ f(s, X(s)) - f(s, X(s-)) \right\}. \end{aligned} \tag{8}$$

Hence on Ω_κ we have

$$\begin{aligned} & f(\eta \wedge t, X(\eta \wedge t)) - f(\sigma_{\kappa,0}, X(\sigma_{\kappa,0})) \\ & \quad = \sum_{j \geqslant 1} \left\{ f\left(\sigma_{\kappa,j} \wedge t, X(\sigma_{\kappa,j} \wedge t)\right) - f\left(\sigma_{\kappa,j-1} \wedge t, X(\sigma_{\kappa,j-1} \wedge t)\right) \right\} \\ & \quad = \sum_{j \geqslant 1} \int_0^{\eta \wedge t} \left\{ \frac{\partial f}{\partial t}(s, X(s-))ds + \mathrm{d}\frac{\partial f}{\partial x}(s, X(s-)) \right\} \mathbf{1}_{(\sigma_{\kappa,j-1}\wedge t < s \leqslant \sigma_{\kappa,j}\wedge t)} ds \\ & \qquad + \sum_{j \geqslant 1} \sum_{0 < s \leqslant \eta \wedge t} \left\{ f(s, X(s)) - f(s, X(s-)) \right\} \mathbf{1}_{(\sigma_{\kappa,j-1}\wedge t < s \leqslant \sigma_{\kappa,j}\wedge t)} \\ & \quad = \int_0^{\eta \wedge t} \frac{\partial f}{\partial t}(s, X(s-))ds + \mathrm{d} \int_0^{\eta \wedge t} \frac{\partial f}{\partial x}(s, X(s-))ds \\ & \qquad + \sum_{0 < s \leqslant \eta \wedge t} \left\{ f(s, X(s)) - f(s, X(s-)) \right\}. \end{aligned}$$

where the final equality follows from (an almost sure version of) Fubini's theorem which in turn appeals to the assumption that the limits of f, $\partial f / \partial t$ and $\partial f / \partial x$ all exist and are finite when approaching any point on the curve b. In particular, to deal with the final term in the second equality, note that an almost sure uniform bound of the form

$$|f(s, X(s)) - f(s, X(s-))| \leqslant C |\triangle X(s)|$$

holds (for random C) because of the assumptions on $\partial f / \partial x$ and hence the double sum converges (as X is a process of bounded variation). Since κ may be chosen arbitrarily small, (7) shows that (5) is true almost surely on Ω. □

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