Itô's Integrated Formula for Strict Local Martingales with Jumps

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Summary. This note presents some properties of positive càdlàg local martingales which are not martingales – strict local martingales – extending the results from [MY06] to local martingales with jumps. Some new examples of strict local martingales are given. The construction relies on absolute continuity relationships between Dunkl processes and absolute continuity relationships between semi-stable Markov processes.

2000 AMS Subject Classification: 60G44, 60J75, 60J55, 60J25

Key words: Strict local martingales, Local time, Semi-stable Markov processes, Dunkl Markov processes

1 Main results

Let $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space. On $\Omega \times \mathbb{R}_+$ we denote by $\mathcal{O}$ and $\mathcal{P}$ respectively – the optional and predictable sigma fields and by $\mathcal{B}(\mathbb{R})$ the Borel sigma field. Consider $(S_t)_{t\geqslant 0}$ – an $\mathbb{R}_+$ valued local martingale with respect to the filtration $(\mathcal{F}_t)_{t\geqslant 0}$. For the definitions of local time for discontinuous local martingales we follow ([TL78], pages 17–22; see also [Mey76] and [Pro05]). For each $a \in \mathbb{R}$ there exists a continuous increasing process $(L_t^a,\ t \geqslant 0)$, such that Tanaka's formula holds:

$$
\begin{aligned}
(S_t - a)^+ = (S_0 - a)^+ &+ \int_{0+}^{t} \mathbf{1}_{\{S_{u-}>a\}} dS_u \\
&+ \sum_{0<u\leqslant t} \left[\mathbf{1}_{\{S_{u-}>a\}} (S_u - a)^- + \mathbf{1}_{\{S_{u-}\leqslant a\}} (S_u - a)^+ \right] + \frac{1}{2} L_t^a, \quad (1)
\end{aligned}
$$

which we write equivalently:

$$(S_t - a)^+ = (S_0 - a)^+ + \int_{0+}^{t} \mathbf{1}_{\{S_{u-}>a\}} dS_u + \frac{1}{2}\mathcal{L}_t^a.$$

Furthermore, there exists a $\mathcal{B}(\mathbb{R}) \times \mathcal{O}$ measurable version of $\mathcal{L}$, a.s. càdlàg in t, and a $\mathcal{B}(\mathbb{R}) \times \mathcal{P}$ measurable version of L, which is a.s. continuous in t. We will only consider such versions. Also note that for any $f \geqslant 0$, Borel,

$$\int_0^t f(S_u)\, d\langle S^c, S^c\rangle_u = \int_{-\infty}^{+\infty} f(a)\, L_t^a da.$$

We shall say that T is a $(\mathcal{F}_t)$ stopping time which reduces the local martingale S if $(S_{t\wedge T})$ is a uniformly integrable martingale. We shall say that a process X is in class (D) if the family $\{X_\tau,\ \tau$ - a.s. finite $(\mathcal{F}_t)$ stopping time$\}$ is uniformly integrable.

The following Theorem is a straightforward generalization of Theorem 1 in [MY06].

Theorem 1. *Let τ be an $(\mathcal{F}_t)$ stopping time such that $\tau < +\infty$ a.s. and $K \geqslant 0$. Then there is the following identity*

$$\mathbb{E}(S_\tau - K)^+ = \mathbb{E}(S_0 - K)^+ + \mathbb{E}J_\tau^K + \frac{1}{2}\mathbb{E}L_\tau^K - c_S(\tau), \tag{2}$$

where $c_S(\tau) := \mathbb{E}(S_0 - S_\tau)$,

$$\begin{aligned} J_\tau^K &:= \sum_{0<u\leqslant\tau} \mathbf{1}_{\{S_{u-}>K\}}(S_u - K)^- + \sum_{0<u\leqslant\tau} \mathbf{1}_{\{S_{u-}\leqslant K\}}(S_u - K)^+ \qquad (3)\\ &= \frac{1}{2}\left(\mathcal{L}_\tau^K - L_\tau^K\right) \end{aligned}$$

and $\left(L_t^K\right)_{t\geqslant 0}$ is the (continuous) local time at K of S.

Proof. Taking $a = K$ in Tanaka's formula (1), one has

$$\begin{aligned} (S_t - K)^+ - (S_0 - K)^+ = \int_{0+}^{t} \mathbf{1}_{\{S_{u-}>K\}} dS_u + \sum_{0<u\leqslant t} \mathbf{1}_{\{S_{u-}>K\}}(S_u - K)^- \\ + \sum_{0<u\leqslant t} \mathbf{1}_{\{S_{u-}\leqslant K\}}(S_u - K)^+ + \frac{1}{2}L_t^K, \end{aligned}$$

introducing J_t^K as in (3) one obtains

$$\begin{aligned} N_t^K &:= \left[(S_t - K)^+ - S_t\right] - \left[(S_0 - K)^+ - S_0 + J_t^K + \frac{1}{2}L_t^K\right] \\ &= \int_{0+}^{t} \mathbf{1}_{\{S_{u-}>K\}} dS_u + S_0 - S_t \end{aligned}$$

and $\left(N_t^K\right)_{t\geqslant 0}$ is a local martingale. Since $(S_t-K)^+ - S_t = -(S_t\wedge K)$, one has

$$-N_t^K = S_t\wedge K - S_0\wedge K + J_t^K + \frac{1}{2}L_t^K.$$

In order to get (2) it is enough to prove that N_t^K is in class (D), i.e., the family $N_\tau^K \mathbf{1}_{\{\tau<+\infty\}}$, where τ ranges all $(\mathcal{F}_t)$ stopping times, is uniformly integrable. Indeed, again from Tanaka's formula

$$(S_t-K)^+ - (S_0-K)^+ - J_t^K - \frac{1}{2}L_t^K = \int_{0+}^t \mathbf{1}_{\{S_{u-}>K\}} dS_u.$$

Let $(\tau_n)_{n\geqslant 1}$ $(\tau_n\to+\infty\ a.s.)$ be a sequence of $(\mathcal{F}_t)$ stopping times which reduces both $(S_t)_{t\geqslant 0}$ and $\left(\int_{0+}^t \mathbf{1}_{\{S_{u-}>K\}} dS_u\right)_{t\geqslant 0}$. Then one gets

$$\mathbb{E}J_{t\wedge\tau_n}^K + \frac{1}{2}\mathbb{E}L_{t\wedge\tau_n}^K = \mathbb{E}\left[(S_{t\wedge\tau_n}-K)^+ - (S_0-K)^+\right] \leqslant \mathbb{E}S_{t\wedge\tau_n} = \mathbb{E}S_0.$$

Finally, by Beppo-Levi:

$$\mathbb{E}J_t^K + \frac{1}{2}\mathbb{E}L_t^K \leqslant \mathbb{E}S_0$$

and

$$\mathbb{E}J_\infty^K + \frac{1}{2}\mathbb{E}L_\infty^K \leqslant \mathbb{E}S_0,$$

then for any $(\mathcal{F}_t)$ stopping time τ

$$\left|N_\tau^K \mathbf{1}_{\{\tau<+\infty\}}\right| \leqslant 2K + J_\infty^K + \frac{1}{2}L_\infty^K \text{ a.s.},$$

which ensures that $\left(N_t^K\right)_{t\geqslant 0}$ is in class (D). Therefore $\left(N_t^K\right)_{t\geqslant 0}$ is a uniformly integrable martingale and the result follows. □

Let τ be an $(\mathcal{F}_t)$ stopping time which is a.s. finite. With notations from [LN06] suppose that $S\in\mathcal{M}_{loc}^2$, and moreover that: $\langle S\rangle_\infty<\infty$ a.s., S^+ is in class (D), $|\varDelta S|\leqslant C$ and

$$\mathbb{E}e^{\varepsilon S_\tau}<\infty$$

for some positive constants C and ε. Then from Theorem 1.1 in [LN06] the term $c_S(\tau) := \mathbb{E}(S_0 - S_\tau)$ in (2) can be characterized as

$$c_S(\tau) = \lim_{\lambda\to\infty}\lambda\sqrt{\frac{\pi}{2}}\mathbb{P}\left(\langle S\rangle_\tau^{1/2}>\lambda\right) = \lim_{\lambda\to\infty}\lambda\sqrt{\frac{\pi}{2}}\mathbb{P}\left([S,S]_\tau^{1/2}>\lambda\right).$$

Besides as a consequence of (2) one obtains

$$c_S(\tau) = \lim_{K\to\infty}\left(\mathbb{E}J_\tau^K + \frac{1}{2}\mathbb{E}L_\tau^K\right).$$

Let τ be an a.s. finite $(\mathcal{F}_t)$ stopping time. Define

$$C^{\text{strict}}(K,\tau) := \lim_{n\to\infty} \mathbb{E}\left(S_{\tau\wedge T_n} - K\right)^+, \tag{4}$$

where $T_n \to \infty$ a.s., T_n reduces $(S_t)_{t\geqslant 0}$. The following proposition shows that this limit exists and does not depend on the reducing sequence T_n, $n \geqslant 1$.

Proposition 1. *Let τ be an a.s. finite $(\mathcal{F}_t)$ stopping time. Then*

$$C^{\text{strict}}(K,\tau) = \mathbb{E}\left(S_0 - K\right)^+ + \mathbb{E}J_\tau^K + \frac{1}{2}\mathbb{E}L_\tau^K. \tag{5}$$

Furthermore, if the process $(\Delta S_t)_{t\geqslant 0}$ is in class (D), then

$$C^{\text{strict}}(K,\tau) = \mathbb{E}\left[(S_\tau - K)^+\right] + \lim_{n\to\infty} n\mathbb{P}\left(S_\tau^* > n\right),$$

where $S_t^* := \sup_{0\leqslant u\leqslant t} S_u$.

Proof. By Tanaka's formula

$$(S_t - K)^+ - (S_0 - K)^+ = \int_{0+}^t \mathbf{1}_{\{S_{u-}>K\}} dS_u + J_t^K + \frac{1}{2}L_t^K,$$

where J_t^K is defined by (3). Since $(S_t)_{t\geqslant 0}$ is a local martingale,

$$S_t^K := \int_{0+}^t \mathbf{1}_{\{S_{u-}>K\}} dS_u$$

is also a local martingale. We have seen in the proof of Theorem 1 that

$$N_t^K = \int_{0+}^t \mathbf{1}_{\{S_{u-}>K\}} dS_u + S_0 - S_t$$

is a uniformly integrable martingale, then

$$S_t^K := N_t^K + S_t - S_0$$

is the sum of a uniformly integrable martingale and a local martingale $(S_t)_{t\geqslant 0}$. Therefore a stopping time which reduces $(S_t)_{t\geqslant 0}$ reduces $\left(S_t^K\right)_{t\geqslant 0}$ as well.

Let T be an $(\mathcal{F}_t)$ stopping time which reduces $(S_t)_{t\geqslant 0}$. Then $S_{t\wedge T}$ and $S_{t\wedge T}^K$ are uniformly integrable martingales. For any τ – an a.s. finite $(\mathcal{F}_t)$ stopping time one gets

$$\mathbb{E}\left(S_{\tau\wedge T} - K\right)^+ = \mathbb{E}\left(S_0 - K\right)^+ + \mathbb{E}J_{\tau\wedge T}^K + \frac{1}{2}\mathbb{E}L_{\tau\wedge T}^K, \tag{6}$$

now taking for T a stopping time T_n, such that $T_n \to \infty$ a.s. and T_n reduces $(S_t)_{t\geqslant 0}$, one obtains that

$$C^{\text{strict}}(K,\tau) := \lim_{n\to\infty} \mathbb{E}\left(S_{\tau\wedge T_n} - K\right)^+$$

exists and does not depend on the sequence of stopping times reducing $(S_t)_{t\geqslant 0}$. Furthermore,

$$C^{\text{strict}}(K,\tau) = \mathbb{E}\left(S_0 - K\right)^+ + \mathbb{E}J_\tau^K + \frac{1}{2}\mathbb{E}L_\tau^K.$$

In order to move further suppose that the process $(\Delta S_t)_{t\geqslant 0}$ is in class (D). Take

$$T_n := \inf\left\{u > 0 \,|\, S_u > n\right\}.$$

Since $(S_t)_{t\geqslant 0}$ is an adapted càdlàg process, T_n is an $(\mathcal{F}_t)$ stopping time and $T_n \to \infty$ a.s. Since

$$|S_{t\wedge T_n}| \leqslant n + \Delta S_{T_n},$$

$(S_{t\wedge T_n})_{t\geqslant 0}$ is in class (D) and subsequently is a uniformly integrable martingale. In particular T_n is an $(\mathcal{F}_t)$ stopping time which reduces $(S_t)_{t\geqslant 0}$. Now one can get for any τ - an a.s. finite $(\mathcal{F}_t)$ stopping time

$$\mathbb{E}\left(S_{\tau\wedge T_n} - K\right)^+ = \mathbb{E}\left[\left(S_\tau - K\right)^+ \mathbf{1}_{\{\tau\leqslant T_n\}}\right] + \mathbb{E}\left[\left(S_{T_n} - K\right)^+ \mathbf{1}_{\{\tau > T_n\}}\right].$$

The left hand side converges and equals $C^{\text{strict}}(K,\tau)$. The first expression on the right hand side converges as well (by Beppo-Levi) to $\mathbb{E}\big[\left(S_\tau - K\right)^+\big]$. Hence $\mathbb{E}\big[\left(S_{T_n} - K\right)^+ \mathbf{1}_{\{\tau > T_n\}}\big]$ converges as well. Besides one has for $n > K$

$$\begin{aligned}(n-K)\,\mathbb{P}\left(\tau > T_n\right) &\leqslant \mathbb{E}\left[\left(S_{T_n} - K\right)^+ \mathbf{1}_{\{\tau > T_n\}}\right]\\ &\leqslant (n-K)\,\mathbb{P}\left(\tau > T_n\right) + \mathbb{E}\left[\Delta S_{T_n} \mathbf{1}_{\{\tau > T_n\}}\right]\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\left[\left(S_{T_n} - K\right)^+ \mathbf{1}_{\{\tau > T_n\}}\right] - \mathbb{E}\left[\Delta S_{T_n} \mathbf{1}_{\{\tau > T_n\}}\right] &\leqslant (n-K)\,\mathbb{P}\left(\tau > T_n\right)\\ &\leqslant \mathbb{E}\left[\left(S_{T_n} - K\right)^+ \mathbf{1}_{\{\tau > T_n\}}\right].\end{aligned}$$

Since $\left(\Delta S_{T_n} \mathbf{1}_{\{\tau > T_n\}}\right)_{n\geqslant 1}$ is a uniformly integrable family $\mathbb{E}\left[\Delta S_{T_n} \mathbf{1}_{\{\tau > T_n\}}\right] \to 0$, as $n \to \infty$, and

$$\lim_{n\to\infty} \mathbb{E}\left[\left(S_{T_n} - K\right)^+ \mathbf{1}_{\{\tau > T_n\}}\right] = \lim_{n\to\infty} (n-K)\,\mathbb{P}\left(\tau > T_n\right) = \lim_{n\to\infty} n\mathbb{P}\left(S_\tau^* > n\right).$$

Finally

$$\lim_{n\to\infty} \mathbb{E}\left(S_{\tau\wedge T_n} - K\right)^+ = \mathbb{E}\left[\left(S_\tau - K\right)^+\right] + \lim_{n\to\infty} n\mathbb{P}\left(S_\tau^* > n\right),$$

where $S_t^* := \sup_{0\leqslant u\leqslant t} S_u$. □

Remark 1. Note that from Theorem 1 and Proposition 1 for any positive local martingale S, such that $(\Delta S_t)_{t\geqslant 0}$ is in class (D), and for any a.s. finite $(\mathcal{F}_t)$ stopping time τ

$$c_S(\tau) = \lim_{n\to\infty} n\mathbb{P}(S_\tau^* > n).$$

Remark 2. Under the conditions of Proposition 1

$$C^{\text{strict}}(K,\ \tau) = \sup_{\sigma\text{-}(\mathcal{F}_t)\text{ stopping time}} \mathbb{E}(S_{\sigma\wedge\tau} - K)^+. \tag{7}$$

Proof. From (6) one obtains that for any pair of $(\mathcal{F}_t)$ stopping times τ, σ and a sequence of stopping times R_n, such that $R_n \to \infty$ a.s. and R_n reduces $(S_t)_{t\geqslant 0}$

$$\mathbb{E}(S_{\tau\wedge\sigma\wedge R_n} - K)^+ = \mathbb{E}(S_0 - K)^+ + \mathbb{E}J^K_{\tau\wedge\sigma\wedge R_n} + \frac{1}{2}\mathbb{E}L^K_{\tau\wedge\sigma\wedge R_n}.$$

Then by Fatou's Lemma

$$\begin{aligned}\mathbb{E}(S_{\tau\wedge\sigma} - K)^+ &\leqslant \mathbb{E}(S_0 - K)^+ + \liminf_{n\to\infty}\left[\mathbb{E}J^K_{\tau\wedge\sigma\wedge R_n} + \frac{1}{2}\mathbb{E}L^K_{\tau\wedge\sigma\wedge R_n}\right]\\ &= \mathbb{E}(S_0 - K)^+ + \mathbb{E}J^K_{\tau\wedge\sigma} + \frac{1}{2}\mathbb{E}L^K_{\tau\wedge\sigma}\\ &\leqslant \mathbb{E}(S_0 - K)^+ + \mathbb{E}J^K_{\tau} + \frac{1}{2}\mathbb{E}L^K_{\tau}.\end{aligned}$$

Now (7) follows from (5) and (4). □

Remark 3. The original proof of Proposition 2 in [MY06] differs a little from ours. In order to obtain (6), the fact that the stopping time which reduces $(S_t)_{t\geqslant 0}$ reduces as well $\left(S_t^K\right)_{t\geqslant 0}$, is not used. Let us go through this other proof and see that there is no contradiction.

Proof. Let T be an $(\mathcal{F}_t)$ stopping time which reduces $(S_t)_{t\geqslant 0}$ and T_n^K, $n \geqslant 1$, $T_n^K \to \infty$ be a sequence of stopping times that reduce $\left(S_t^K\right)_{t\geqslant 0}$. Then for any τ – an a.s. finite $(\mathcal{F}_t)$ stopping time – one gets

$$\mathbb{E}\left(S_{\tau\wedge T\wedge T_n^K} - K\right)^+ = \mathbb{E}(S_0 - K)^+ + \mathbb{E}J^K_{\tau\wedge T\wedge T_n^K} + \frac{1}{2}\mathbb{E}L^K_{\tau\wedge T\wedge T_n^K}.$$

On the right hand side one can pass to the limit as $T_n^K \to \infty$ by Beppo-Levi and get a finite limit as soon as we already know from the proof of Theorem 1 that

$$\mathbb{E}J^K_\infty + \frac{1}{2}\mathbb{E}L^K_\infty \leqslant \mathbb{E}S_0.$$

On the left hand side, $\left(S_{\tau\wedge T\wedge T_n^K}\right)_{n\geqslant 1}$ is a uniformly integrable martingale, thus it converges in L^1 to $S_{\tau\wedge T}$. Finally one gets

$$\mathbb{E}(S_{\tau\wedge T} - K)^+ = \mathbb{E}(S_0 - K)^+ + \mathbb{E}J^K_{\tau\wedge T} + \frac{1}{2}\mathbb{E}L^K_{\tau\wedge T}, \tag{8}$$

which is the same as (6). □

For any μ – a finite measure on $\mathbb{R}_+$ define

$$F_\mu(x) := \int_0^{+\infty} \mu(dK)(x-K)^+$$

and $\bar{\mu} := \int_0^{+\infty} \mu(dK)$. As in [MY06] we have the following Proposition and Corollary (the proofs are the same as in the continuous case).

Proposition 2. *Under the notations and assumptions of Theorem 1*

$$\mathbb{E}[F_\mu(S_\tau)] = F_\mu(S_0) + \mathbb{E}\left[\int_0^{+\infty} \mu(dK)\left(J_\tau^K + \frac{1}{2}L_\tau^K\right)\right] - \bar{\mu} c_S(\tau).$$

Corollary 1. *The process*

$$F_\mu(S_t) - F_\mu(S_0) - \int_0^{+\infty} \mu(dK)\left(J_t^K + \frac{1}{2}L_t^K\right) - \bar{\mu}(S_t - S_0),\ t \geqslant 0$$

is a martingale.

2 Examples

One can trivially construct strict local martingales from continuous strict local martingales: indeed, $M_t := M_t^{(c)} + M_t^{(d)}$ and $\left(M_t^{(c)}\right)$ is a strict local martingale and $\left(M_t^{(d)}\right)$ is a uniformly integrable martingale, then (M_t) is a strict local martingale.

We now obtain strict local martingales with jumps which are generalizations of the strict local martingale $\left(1/R_t^{(3)}\right)$, where $\left(R_t^{(3)}\right)$ is a Bessel process of dimension 3. As in the case of $\left(1/R_t^{(3)}\right)$, such strict local martingales can be obtained from absolute continuity relationships between two Dunkl Markov processes instead of Bessel processes. For simplicity, we consider here only one dimensional Dunkl Markov processes (see [GY06]).

The Dunkl Markov process (X_t) with parameter k is a Feller process with extended generator given for $f \in C^2(\mathbb{R})$ by

$$\mathcal{L}_k f(x) = \frac{1}{2} f''(x) + k\left(\frac{1}{x} f'(x) - \frac{f(x) - f(-x)}{2x^2}\right),$$

where $k \geqslant 0$. Note that $|X|$ is a Bessel process with index $\nu := k - \frac{1}{2}$. Denote by $P_x^{(k)}$ the law of (X_t) started at $x \in \mathbb{R}$, and by $\left(\mathcal{F}_t^X\right)$ the natural filtration of X.

Proposition 3. *Let* $0 \leqslant k < \frac{1}{2} \leqslant k'$ *and* $x > 0$. *Define*

$$T_0 := \inf\{s \geqslant 0 \,|\, X_{s-} = 0 \ \text{or}\ X_s = 0\}.$$

Then $P_x^{(k)}(T_0 < +\infty) = 1$ *and there is the following absolute continuity relationship:*

$$P_x^{(k')}\Big|_{\mathcal{F}_t^X}$$

$$= \left(\frac{|X_{t\wedge T_0}|}{|x|}\right)^{k'-k}\left(\frac{k'}{k}\right)^{N_{t\wedge T_0}}\exp\left(-\frac{(k')^2-k^2}{2}\int_0^{t\wedge T_0}\frac{ds}{X_s^2}\right)P_x^{(k)}\Big|_{\mathcal{F}_t^X}, \quad (9)$$

where N_t *denotes the number of jumps of* X *on* $[0,\ t]$. *Furthermore*

$$M_t := \left(\frac{|x|}{|X_t|}\right)^{k'-k}\left(\frac{k}{k'}\right)^{N_t}\exp\left(\frac{(k')^2-k^2}{2}\int_0^t\frac{ds}{X_s^2}\right) \quad (10)$$

is a strict local martingale under $P_x^{(k')}$, *and*

$$P_x^{(k)}(T_0 > t) = \mathbb{E}_x^{(k')}M_t, \quad (11)$$

where $\mathbb{E}_x^{(k')}$ *is the expectation under* $P_x^{(k')}$.

Remark 4. Note that the law of T_0 under $P_x^{(k)}$ is that of $x^2/\left(2Z_{\left(\frac{1}{2}-k\right)}\right)$, where $Z_{\left(\frac{1}{2}-k\right)}$ is a gamma variable of a parameter $\frac{1}{2}-k$ (see page 98 in [Yor01]).

Proof. Let X be a Dunkl Markov process. Note that $\Delta X_s = X_s - X_{s-} = -2X_{s-}$, when $\Delta X_s \neq 0$. Hence if $X_s = 0$, then $X_{s-} = 0$ and

$$T_0 = \inf\{s \geqslant 0 \,|X_{s-} = 0\} = \inf\{s \geqslant 0 \,||X_s| = 0\}.$$

In order to prove (9) we proceed as in the proof of Proposition 4 in [GY06]. First we need to extend Theorem 3 in [GY06] for $k < \frac{1}{2}$. Since $|X|$ is a Bessel process with index $\left(k - \frac{1}{2}\right)$, for $k < \frac{1}{2}$, $T_0 < +\infty$ a.s., and, for $k \geqslant \frac{1}{2}$, $T_0 = +\infty$ a.s. Denote

$$\tau_t := \inf\left\{s \geqslant 0 \,\middle|\, \int_0^s \frac{du}{X_u^2} = t\right\},$$

then τ_t is a continuous strictly increasing time change and $\tau_\infty = T_0$. Denote $Y_u := X_{\tau_u}$. Since for any $f \in C^2(\mathbb{R})$

$$f(X_t) - f(X_0) - \int_0^t \mathcal{L}_k f(X_s)\,ds$$

is a local martingale,

$$f(Y_u) - f(Y_0) - \int_0^t Y_s^2 \mathcal{L}_k f(Y_s)\,ds$$

is a local martingale. Then, as in the proof of Theorem 4 in [GY06], one obtains that Y is of the form

$$Y_u = \exp\left(\beta_u^{(\nu)} + i\pi N_u^{(k/2)}\right),$$

where $\nu := k - \frac{1}{2}$, $\left(\beta_u^{(\nu)}\right)$ is a Brownian motion with drift ν, $\left(N_u^{(k/2)}\right)$ is a Poisson process with parameter $k/2$ independent from $\left(\beta_u^{(\nu)}\right)$. Denote

$$A_t := \int_0^t \frac{du}{X_u^2},$$

then $\tau_{A_t} = t$, for $t < T_0$. Hence

$$X_t = Y_{A_t},\ t < T_0. \tag{12}$$

Note also that differentiating the equality $A_{\tau_t} = t$ with respect to time one gets

$$\frac{d}{dt}\tau_t = Y_t^2$$

and $A_t = \inf\left\{s \geqslant 0 \,\middle|\, \int_0^s Y_u^2 du = t\right\}$, $t < T_0$. Note that (9) is equivalent to

$$P_x^{(k)}\Big|_{\mathcal{F}_t^X \cap \{t<T_0\}} = \left(\frac{|x|}{|X_t|}\right)^{k'-k} \left(\frac{k}{k'}\right)^{N_t} \exp\left(\frac{(k')^2 - k^2}{2}\int_0^t \frac{ds}{X_s^2}\right) P_x^{(k')}\Big|_{\mathcal{F}_t^X}. \tag{13}$$

Indeed (9) is equivalent to

$$\begin{aligned}\mathbb{E}_x^{(k')}\left(F\left(X_s,\ s \leqslant t\right)\right) &= \mathbb{E}_x^{(k)}\left(F\left(X_s,\ s \leqslant t\right)\frac{1}{M_{t\wedge T_0}}\right)\\ &= \mathbb{E}_x^{(k)}\left(F\left(X_s,\ s \leqslant t\right)\frac{1}{M_t}\mathbf{1}_{\{t<T_0\}}\right),\end{aligned}$$

for any bounded measurable F, (M_t) is given by (10). Then

$$\mathbb{E}_x^{(k')}\left(M_t \mathbf{1}_{\{t<T_0\}}\hat{F}\left(X_s,\ s \leqslant t\right)\right) = \mathbb{E}_x^{(k)}\left(\hat{F}\left(X_s,\ s \leqslant t\right)\mathbf{1}_{\{t<T_0\}}\right),$$

which is equivalent to (13). By (12) X is associated to the pair $\left(\beta^{(\nu)}, N^{(k/2)}\right)$ under $P^{(k)}$, and to the pair $\left(\beta^{(\nu')}, N^{(k'/2)}\right)$ under $P^{(k')}$. Both pairs consist of a Brownian motion with drift and a Poisson process which are mutually independent, and $\nu := k - \frac{1}{2}$, $\nu' := k' - \frac{1}{2}$. Now in the same way as in the proof of Proposition 4 in [GY06], for any bounded measurable F,

$$\mathbb{E}_x^{(k)}\left(F\left(\beta_s^{(\nu)}, N_s^{(k)},\ s \leqslant t\right)\right) = \mathbb{E}_x^{(k')}\left(D_t F\left(\beta_s^{(\nu')}, N_s^{(k')},\ s \leqslant t\right)\right),$$

where

$$D_t := \exp\left((\nu - \nu')\, \beta_t^{(\nu')} - \frac{1}{2}\left(\nu^2 - (\nu')^2\right) t \right) \left(\frac{k}{k'}\right)^{N_t^{(k'/2)}} \exp\left(-\frac{1}{2}(k - k')\, t \right)$$

$$= \exp\left((k - k')\, \beta_t^{(\nu')} - \frac{1}{2}\left(k^2 - (k')^2\right) t \right) \left(\frac{k}{k'}\right)^{N_t^{(k'/2)}}$$

and $D_{A_t} = M_t$, $t < T_0$. Denote $\mathcal{G}_t := \sigma\left\{ \beta_s^{(\nu')}, N_s^{(k'/2)},\ s \leqslant t \right\}$, then

$$F\left(\beta_{A_s}^{(\nu')}, N_{A_s}^{(k'/2)} \right) \mathbf{1}_{\{A_s \leqslant u\}}$$

is $\mathcal{G}_{A_s \wedge u}$ measurable and

$$\begin{aligned} &\mathbb{E}_x^{(k)}\left(F\left(\beta_{A_s}^{(\nu)}, N_{A_s}^{(k/2)} \right) \mathbf{1}_{\{A_s \leqslant u\}} \right) \\ &= \mathbb{E}_x^{(k')}\left(\mathbb{E}^{(k')}\left(D_t \,|\mathcal{G}_{A_s \wedge u} \right) F\left(\beta_{A_s}^{(\nu')}, N_{A_s}^{(k'/2)} \right) \mathbf{1}_{\{A_s \leqslant u\}} \right) \\ &= \mathbb{E}_x^{(k')}\left(D_{A_s} F\left(\beta_{A_s}^{(\nu')}, N_{A_s}^{(k'/2)} \right) \mathbf{1}_{\{A_s \leqslant u\}} \right). \end{aligned}$$

As $u \to +\infty$ one gets

$$\begin{aligned} &\mathbb{E}_x^{(k)}\left(F\left(\beta_{A_s}^{(\nu)}, N_{A_s}^{(k/2)} \right) \mathbf{1}_{\{A_s < +\infty\}} \right) \\ &= \mathbb{E}_x^{(k')}\left(D_{A_s} F\left(\beta_{A_s}^{(\nu')}, N_{A_s}^{(k'/2)} \right) \mathbf{1}_{\{A_s < +\infty\}} \right). \end{aligned} \tag{14}$$

Noting that $(A_s < +\infty) = (s < T_0)$, (14) leads to (13). From (13) one easily obtains (11). Suppose that (M_t) is a martingale then from (11) for any $t \geqslant 0$ $P_x^{(k)}(T_0 > t) = 1$ and $P_x^{(k)}(T_0 = +\infty) = 1$, which is impossible because $k < \frac{1}{2}$. Hence (M_t) is a strict local martingale. □

Other examples of strict local martingales with jumps can be obtained from absolute continuity relationships between two non-negative semi-stable Markov processes. We shortly recall the definition of a semi-stable Markov process (see [Lam72]):

A semi-stable Markov process (with index of stability $\alpha = 1$) on $\mathbb{R}_+ := [0, +\infty)$ is a Markov process (X_t) with the following scaling property: for any $c > 0$

$$\left(\frac{1}{c} X_{ct}^{(x)} \right)_{t \geqslant 0} \overset{(d)}{=} \left(X_t^{(xc^{-1})} \right)_{t \geqslant 0},$$

where $\left(X_t^{(x)}\right)$ denotes a semi-stable Markov process started at $x > 0$. Denote

$$T_0 := \inf\{s \geqslant 0\,|X_{s-} = 0 \text{ or } X_s = 0\}, \tag{15}$$

then Lamperti in [Lam72] showed that: either $T_0 = +\infty$ a.s., or $T_0 < +\infty$ a.s. and $X_{T_0-} = 0$ a.s., or $T_0 < +\infty$ a.s. and $X_{T_0-} > 0$ a.s. Furthermore this does not depend on the starting point $x > 0$.

Note that for a semi-stable Markov process the following Lamperti relation is true. We suppose that there is no killing inside $(0,\ \infty)$.

Proposition 4. *Let (ξ_t) be a one-dimensional Lévy process, starting at 0. Define*

$$A_t^{(x)} := \int_0^t x \exp(\xi_s)\, ds,$$

for any $x > 0$. Then the process (X_u), defined implicitly by

$$x \exp \xi_t = X_{A_t^{(x)}},\ t < T_0, \tag{16}$$

is a semi-stable Markov process, starting at x, and

$$A_\infty^{(x)} = T_0, \tag{17}$$

where T_0 is defined by (15). *The converse is also true.*

Denote

$$\tau_t^{(x)} := \inf\left\{s \geqslant 0 \,\middle|\, A_s^{(x)} = t\right\}.$$

Let $\left(\mathcal{F}_t^\xi\right)$ be the natural filtration of (ξ_t) and $\left(\mathcal{F}_t^X\right)$ be the natural filtration of (X_t). As in [CPY94], using Proposition 4, one obtains the following absolute continuity relationship between two semi-stable Markov processes.

Proposition 5. *Suppose that (X_t) is a semi-stable Markov process associated with Lévy process (ξ_t) via Lamperti relation* (16) *and $\mathbb{E}_P e^{b\xi_t} = e^{t\rho(b)} < \infty$. Define Q by*

$$Q|_{\mathcal{F}_t^X \cap \{t<T_0\}} = \left(\frac{X_t}{x}\right)^b \exp\left(-\rho(b)\int_0^t \frac{ds}{X_s^2}\right) P|_{\mathcal{F}_t^X \cap \{t<T_0\}},$$

where T_0 is defined by (15). *Then, under Q, (X_t) is still a semi-stable Markov process associated with Lévy process (ξ_t) via Lamperti relation* (16) *and*

$$\tilde{\Psi}(u) = \Psi(u - ib) - \Psi(-ib),$$

where Ψ, $\tilde{\Psi}$ are the characteristic exponents of (ξ_t) under P and Q respectively.

Proof. Let us consider the change of measure given by the Esscher transform:

$$Q|_{\mathcal{F}_t^\xi} = \exp\left(b\xi_t - \rho(b)\, t\right) P|_{\mathcal{F}_t^\xi}.$$

Since $\mathbb{E}e^{b\xi_t} = e^{t\rho(b)} < \infty$

$$M_t := \exp\left(b\xi_t - \rho(b)\, t\right)$$

is a martingale. Furthermore (ξ_t) is still a Lévy process under Q. Note that $\left\{\tau_t^{(x)} < +\infty\right\} = \{t < T_0\}$ and for any $t < T_0$

$$A^{(x)}_{\tau_t^{(x)}} = t. \tag{18}$$

Denote $\mathcal{G}_t := \mathcal{F}^{\xi}_{\tau_t^{(x)}}$, then for any $A \in \mathcal{G}_t$

$$Q\left(A \cap \left\{\tau_t^{(x)} \leqslant u\right\}\right) = \mathbb{E}_P\left(\mathbf{1}_{A\cap\left\{\tau_t^{(x)}\leqslant u\right\}} \exp\left(b\xi_{\tau_t^{(x)}} - \rho(b)\, \tau_t^{(x)}\right)\right). \tag{19}$$

Note that $(X_t/x)^b = \exp\left(b\xi_{\tau_t^{(x)}}\right)$ on $\{t < T_0\}$. Differentiating (18) one gets that

$$\frac{d}{dt}\tau_t^{(x)} = \frac{1}{X_t^2}.$$

Letting u tend to infinity, from (19) one gets

$$\begin{aligned} &Q\left(A \cap \left\{\tau_t^{(x)} < +\infty\right\}\right) \\ &\quad = \mathbb{E}_P\left(\mathbf{1}_{A\cap\left\{\tau_t^{(x)}<+\infty\right\}} \left(\frac{X_t}{x}\right)^b \exp\left(-\rho(b)\int_0^t \frac{ds}{X_s^2}\right)\right). \end{aligned}$$

But from (17) $\left\{\tau_t^{(x)} < +\infty\right\} = \{t < T_0\}$. Hence

$$Q\left(A \cap \{t < T_0\}\right) = \mathbb{E}_P\left(\mathbf{1}_{A\cap\{t<T_0\}} \left(\frac{X_t}{x}\right)^b \exp\left(-\rho(b)\int_0^t \frac{ds}{X_s^2}\right)\right). \quad \square$$

Let us find the range of the parameter b such that

$$M_t := \left(\frac{X_t}{x}\right)^b \exp\left(-\rho(b)\int_0^t \frac{ds}{X_s^2}\right)$$

is a strict local martingale. Note that it is sufficient to find b such that $Q(T_0 < +\infty) = 1$ and $P(T_0 = +\infty) = 1$. Indeed, given such a parameter

$$Q(t < T_0) = \mathbb{E}_P(M_t)$$

and as in the proof of Proposition 3 (M_t) is a strict local P-martingale.

Now let ξ be a Lévy process under P with the characteristic exponent Ψ, given by Lévy–Khintchine formula

$$\Psi(\lambda) = ia\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{\mathbb{R}} \left(1 - e^{i\lambda x} + i\lambda x \mathbf{1}_{\{|x|<1\}}\right) \pi(dx),$$

where $a \in \mathbb{R}$, $\sigma^2 \geqslant 0$ and π is a positive measure on $\mathbb{R}\setminus\{0\}$ such that

$$\int \left(1 \wedge |x|^2\right) \pi(dx) < \infty.$$

Let us suppose that π has compact support. Then $\mathbb{E}_P e^{b\xi_t} = e^{t\rho(b)} < \infty$ for any t and b. Let the semi-stable Markov process X be associated to ξ via Lamperti relation. Conditions for $P(T_0 = +\infty) = 1$ or $P(T_0 < +\infty) = 1$ bearing on $\left(a,\ \sigma^2,\ \pi\right)$ can be deduced from Theorem 1 in [BY05]. Note that $T_0 < +\infty$ if and only if $\xi_t \to -\infty$. Since π has compact support, from the Central Limit Theorem for a Lévy process, $\xi_t \to -\infty$ if and only if $\mathbb{E}_P\xi_1 < 0$ i.e.,

$$-a + \int_{|x|>1} x\pi(dx) < 0.$$

Let Q be given by Proposition 5. Denote by $\tilde{\Psi}$ the characteristic exponent of ξ under Q, then

$$\tilde{\Psi}(\lambda) = i\lambda\left[a - b\sigma^2 + \int_{|x|<1} x\left(1 - e^{bx}\right)\pi(dx)\right] + \frac{1}{2}\sigma^2\lambda^2 + \int_{\mathbb{R}} \left(1 - e^{i\lambda x} + i\lambda x \mathbf{1}_{\{|x|<1\}}\right) \tilde{\pi}(dx),$$

where $\tilde{\pi}(dx) = e^{bx}\pi(dx)$. Hence, in order to have $Q(T_0 < +\infty) = 1$ and $P(T_0 = +\infty) = 1$ one can choose b such that

$$-a + \int_{|x|>1} x\pi(dx) \geqslant 0 \tag{20}$$

and

$$-a + b\sigma^2 - \int_{|x|<1} x\left(1 - e^{bx}\right)\pi(dx) + \int_{|x|>1} xe^{bx}\pi(dx) < 0. \tag{21}$$

It is easy to see that (20) and (21) imply that $b < 0$. For example, for any given a and π, such that (20) is true, one can always choose $b < 0$ such that

$$b\sigma^2 - e^{-b}\int_{x<-1} |x|\,\pi(dx) < a - \int_{x>1} x\pi(dx), \tag{22}$$

which implies (21). Note that condition (22) is more restrictive than (21).

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