Product of Harmonic Maps is Harmonic: A Stochastic Approach

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Summary. Let $\phi_j : M_j \to G$, $j = 1, 2, \ldots, n$, be harmonic mappings from Riemannian manifolds M_j to a Lie group G. Then the product $\phi_1\phi_2\cdots\phi_n$ is a harmonic mapping between $M_1 \times M_2 \times \cdots \times M_n$ and G. The proof is a combination of properties of Brownian motion in manifolds and Itô formulae for stochastic exponential and logarithm of product of semimartingales in Lie groups.

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1 Introduction

Let (M, g) and (N, h) be two compact Riemannian manifolds and consider $\phi : M \to N$ a C^∞-differentiable map. The *energy functional* (or *action integral*) of ϕ is defined as the integral of the *density energy function* $e(\phi)(x)$

$$E(\phi) = \int_M e(\phi)(x) dv_g,$$

where $e(\phi)(x) = 1/2\mathrm{tr}_g(\phi^* h)(x)$ and v_g is the Riemannian volume on M. A physical interpretation of the energy functional E could be the accumulated elastic energy on M, when this space is stretched on N.

We say that ϕ is a *harmonic mapping* if ϕ is a critical point of the functional E. The Euler–Lagrange formula in this case can be written in terms of the connections ∇^M and ∇^N on M and N, respectively: namely ϕ is harmonic

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if and only if the tension field $\tau(\phi)(x) = \mathrm{tr}(\widetilde{\nabla}\phi_* - \phi_*\nabla^M)$ vanishes everywhere in M, where $\widetilde{\nabla}$ is the induced connection on the induced bundle $\phi^{-1}TN$ (see, e.g., Urakawa [11]). The definition of harmonic mappings extends naturally to noncompact manifolds just considering the critical property of the energy functional locally on M. In particular, we recall that geodesics are harmonic mappings from the real line to a Riemannian manifold.

Variational problems have been, historically, a non trivial area of interest both for mathematicians and physicists. In the last couple of decades many important contributions on harmonic mappings were done, see, e.g., in one of the seminal papers, Calabi construction of harmonic mappings from two spheres into symmetric spaces [1]. For a classical text we refer to Eells and Lemaire [3]. For an approach of harmonic mappings into Lie groups see, e.g., Uhlenbeck [10]. This article is a contribution in the topic which comes as an application of stochastic tools in geometry.

Here we consider harmonic mappings with image in a Lie group G with a bi-invariant Riemannian metric. In this case the associated (Levi-Civita) left invariant connection on G, denoted by ∇^L, satisfies $\nabla^L_X Y = \frac{1}{2}[X,Y]$ for all X, Y left invariant vector fields in the Lie algebra $\mathcal{G}$, see, e.g., Cheeger and Ebin [2].

The aim of this paper is to present a direct stochastic proof that a product of harmonic mappings is a harmonic mapping. More precisely: given $\phi_j : M_j \to G$, $j = 1, 2, \ldots, n$, harmonic mappings between Riemannian manifolds M_j and a Lie group G with bi-invariant Riemannian metric, then the product $\phi_1\phi_2\cdots\phi_n$ is a harmonic mapping between $M_1 \times M_2 \times \cdots \times M_n$ and G. In the best of our knowledge, this is a new result in harmonic mappings theory.

Our proof is a combination of properties of Brownian motion in manifolds and Itô formulae for stochastic exponential and logarithm of products of semimartingales in Lie groups. Although we assume that the group has a bi-invariant metric, one can easily verify that our argument also holds for any Lie group considering the left connection $\nabla^L_X Y = \frac{1}{2}[X,Y]$ if $X, Y \in \mathcal{G}$.

We recall that a product of harmonic mappings appears also in other contexts: we mention here harmonic functions on Lie groups with respect to a Radon probability measure μ on G, see, e.g., Furstenberg [5]. A function $f : G \to \mathbb{R}$ is called *μ-harmonic* if

$$f(g) = \int_G f(gh)\ d\mu(h),$$

for every g in G. If $f_1 : (G_1, \mu_1) \to \mathbb{R}$ and $f_2 : (G_2, \mu_2) \to \mathbb{R}$ are harmonic functions, then Fubini theorem implies that $f_1 f_2 : G_1 \times G_2 \to \mathbb{R}$ is a $\mu_1 \times \mu_2$-harmonic function.

In Section 2 we recall some basic facts and formulae on stochastic calculus in Lie groups. We refer mainly to Hakim-Dowek and Lépingle [6], nevertheless, it may happen that someone finds the proofs presented here simpler than in [6]. We shall use these formulae in Section 3, where we prove the main results.

By convention, all martingales, semimartingales, and local martingales are assumed to be continuous.

2 Preliminary results

Let M be a Riemannian manifold and consider $\theta_{X_t} \in T^*_{X_t}M$ an adapted stochastic one-form along X_t, an M-valued semimartingale. The integral of the form θ along X was proposed by Ikeda and Manabe [7] (see also, among others, Emery [4] or Meyer [9]). This integral is geometrically intrinsic, and it has a natural description in local charts: let $(U, x^1, \ldots, x^n)$ be a local system of coordinates in M, then θ can be written as $\theta_x = \theta^1(x)\, dx^1 + \cdots + \theta^n(x)\, dx^n$, where $\theta^i(x)$, $i = 1, 2, \ldots, n$, are (C^∞, say) functions in M. The Stratonovich integral of θ along X_t is given by:

$$\int \theta \circ dX_t = \sum_{i=1}^{n} \int \theta^i(X_t) \circ dX_t^i.$$

Let G be a Lie group with the corresponding Lie algebra $\mathcal{G}$. We denote by ω the (left) Maurer–Cartan form in G, i.e., if $v \in T_gG$, then $\omega_g(v) = L_{g^{-1}*}(v)$. It corresponds to the unique $\mathcal{G}$-valued left invariant one-form in G.

The logarithm of a semimartingale X_t on G (with $X_0 = e$) is the integral of the Maurer–Cartan form along X_t, namely, it is the following semimartingale in the Lie algebra:

$$(\log X)_t = \int_0^t \omega \circ dX_s.$$

Conversely, consider a semimartingale M_t in the Lie algebra $\mathcal{G}$. We recall that the (left) stochastic exponential $\epsilon(M)$ of M_t is the stochastic process X_t which is solution of the Stratonovich left invariant equation on G:

$$\begin{cases} dX_t = L_{X_t *} \circ dM_t, \\ X_0 = e. \end{cases}$$

An interesting geometric characterization of the exponential $\epsilon(M)$ is the fact that it corresponds to the stochastic development of $M_t \in T_eG$ to the group G with respect to the left invariant connection ∇^L. One easily checks that the logarithm is the inverse of the stochastic exponential ϵ.

Martingales in G (with respect to ∇^L-connection) and local martingales in the Lie algebra $\mathcal{G}$ are related by the following characterization, see Hakim-Dowek and Lépingle [6]:

Theorem 1. *A process X_t on G is a ∇^L-martingale if and only if $X_t = X_0 \cdot \epsilon(M)$ for some local martingale M in $\mathcal{G}$.*

The pull-back of Maurer–Cartan forms by homomorphisms of Lie groups is easily described by:

Lemma 1. *Let $\varphi : G \to H$ be a homomorphism of Lie groups. Then the pull-back $\varphi^* \omega_H$ satisfies, for $v \in T_g G$:*

$$(\varphi^* \omega_H) v = \varphi_* (\omega_G(v)).$$

In particular, if X is a semimartingale in G, then $\varphi_*(\log X) = \log(\varphi(X))$.

Proof. Once $\varphi(L_{g^{-1}}(h)) = L_{\varphi(g)^{-1}}(\varphi(h))$, the chain rule implies that

$$L_{\varphi(g)^{-1}*}(\varphi_*(v)) = \varphi_*(L_{g^{-1}*}(v)).$$

For the last formula, by definition: $\log \varphi(X) = \int \varphi^* \omega_H \circ dX$. The result follows directly by the first part of the Lemma and the very definition of $\varphi_* \log X$. □

Denote by $I_g : G \to G$ the adjoint in the group G given by $h \mapsto ghg^{-1}$. The map I_g is an automorphism of G and its derivative corresponds to the isomorphism of the Lie algebra called adjoint in $\mathcal{G}$ denoted by $Ad(g) = I_{g*} : \mathcal{G} \to \mathcal{G}$. We have that $R_g^* \, \omega = Ad(g^{-1})\omega$ (see, e.g., Kobayashi and Nomizu [8]). The pull-back of the canonical form by multiplication and inverse is given by:

Proposition 1. *Let $m : G \times G \to G$ be the multiplication and $i : G \to G$ be the inverse in the group. Then the pull-backs satisfy:*

a) $m^* \omega = Ad^{-1}(\pi_2)(\pi_1^* \omega) + \pi_2^* \omega$.
b) $i^* \omega = -Ad \ \omega$.

Proof. Let $w = (u, v) \in T_{(g,h)} G \times G \simeq T_g G \times T_h G$. Then

$$\begin{aligned} m^* \omega(w) &= \omega(m_* w) = \omega(R_{h*} u + L_{g*} v) \\ &= L_{(gh)^{-1}*}(R_{h*} u + L_{g*} v) \\ &= L_{h^{-1}*} R_{h*} L_{g^{-1}*} u + L_{h^{-1}*} L_{g^{-1}*} L_{g*} v \\ &= Ad(h^{-1}) \omega(u) + \omega(v). \end{aligned}$$

For the inverse function, consider the diagonal map $\Delta : G \to G \times G$ given by $\Delta(g) = (g, g)$. We have that $m \circ (Id \times i) \circ \Delta = e$, then the pull-back $(m \circ (Id \times i) \circ \Delta)^* \omega = 0$ which implies, using the formula of item (a), that

$$Ad\omega + i^* \omega = 0. \qquad \square$$

Lemma 2. *Given semimartingales X and Y in G, we have the following Itô formulas:*

a) $\log(XY) = \int Ad(Y^{-1}) \circ d(\log X) + \log Y$.
b) $\log(X^{-1}) = -\int Ad(X) \circ d(\log X)$.

Proof. The first formula follows from the calculation:

$$\begin{aligned}\log(XY) = \int \omega \circ dm(X,Y) &= \int m^*\omega \circ d(X,Y)\\ &= \int \left(Ad^{-1}(\pi_2)\pi_1^*\omega + \pi_2^*\omega\right) \circ d(X,Y)\\ &= \int Ad(Y^{-1}) \circ d\left(\int \omega \circ dX\right) + \int \omega \circ dY\\ &= \int Ad(Y^{-1}) \circ d\log X + \log Y.\end{aligned}$$

For the second formula, apply the identity (a) with $Y = X^{-1}$. □

We have now a direct way to prove the stochastic Campbell–Hausdorff formula (cf. Hakim-Dowek and Lépingle [6]).

Theorem 2. *We have that:*

a) $\epsilon(M+N) = \epsilon\left(\int Ad(\epsilon(N)) \circ dM\right)\epsilon(N)$.

b) $\epsilon(M)^{-1} = \epsilon\left(-\int Ad(\epsilon(M)) \circ dM\right)$.

3 Harmonic mappings

Consider M and N two Riemannian manifolds and let $f : M \to N$ be a C^∞-differentiable map. The key point in stochastic geometry which matters in the question addressed in this article is the following result, due originally to Bismut:

Theorem 3. *A mapping $f : M \to N$ is a harmonic mapping if and only if for all Brownian motion B_t in M, $f(B_t)$ is a ∇^N-martingale in N.*

See, e.g., Emery [4].

Theorem 4 (Main result). *Let $\phi_j : M_j \to G$, $j = 1, 2, \ldots, n$, be harmonic mappings from Riemannian manifolds M_j to a Lie group G (with respect to the connection ∇^L). Then the product $\phi_1\phi_2\cdots\phi_n$ is a harmonic mapping between $M_1 \times M_2 \times \cdots \times M_n$ and G.*

Proof. It is enough to take $n = 2$. Consider $f_1 : M_1 \to G$ and $f_2 : M_2 \to G$ two harmonic mappings. Let B^1 and B^2 be independent Brownian motions in M_1 and M_2, respectively. Then (B^1, B^2) is a Brownian motion in the product space $M_1 \times M_2$. We have to prove that the product $f_1(B^1)f_2(B^2)$ is a martingale in the group G. By Theorem 1 this product is a martingale if and only if its logarithm is a local martingale.

By the Itô formula (a) of Lemma 2 we have that:

$$\log(f_1(B^1)f_2(B^2)) = \int Ad(f_2(B^2)^{-1}) \circ d(\log f_1(B^1)) + \log f_2(B^2). \tag{1}$$

By hypothesis, the integrator $\log f_1(B^1)$ and the last term $\log f_2(B^2)$ are local martingales. Moreover, the Stratonovich integral reduces to an Itô integral, since the correction term vanishes by independence of the Brownian motions. Hence $\log(f_1(B^1)f_2(B^2))$ is a local martingale in the Lie algebra $\mathcal{G}$, hence the product $f_1 \cdot f_2$ is harmonic. □

If the group G is Abelian, the proof above is straightforward since the adjoint is the identity.

Example 1 (Product of geodesics is harmonic). Let G be a Lie group with a bi-invariant metric. Consider $X_1, \ldots, X_n$ elements of the Lie algebra $\mathcal{G}$. Then the map $f : (\mathbb{R}^n, <,>) \to G$ defined by

$$f(t_1, \ldots, t_n) = \exp(t_1 X_1) \cdot \ldots \cdot \exp(t_n X_n)$$

is harmonic. Again, with $n = 2$, given (B_t^1, B_t^2) a Brownian motion on the plane $\mathbb{R}^2$, then:

$$\log\left(\exp(B_t^1 X_1)\exp(B_t^2 X_2)\right) = \int_0^t Ad\left((\exp(B_s^1 X_1))^{-1}\right) \circ d(B_s^2 X_2) + B_t^2 X_2,$$

where a direct calculation shows that the correction term of the Stratonovich integral is $[X_2, X_1]\, d[B_t^1, B_t^2] = 0$. Note that this example also holds for general geodesics (not only starting at the identity).

A corollary of the proof of the theorem shows a partial converse of the theorem.

Corollary 1. *Let $f_1 : M \to G$ and $f_2 : N \to G$ be two C^∞-differentiable map. If the product $f_1 \cdot f_2$ is harmonic and one of the two mappings, f_1 or f_2, is harmonic, then the other map is also harmonic.*

Proof. The proof follows from (1), where the Stratonovich integral reduces to an Itô integral. The left-hand side is a local martingale by hypothesis and Theorem 1. If f_1 is harmonic then the integrator is a local martingale, hence $\log f_2(B^2)$ is also a local martingale and f_2 is harmonic.

On the other hand, if f_2 is harmonic, then the (Itô) integral is a local martingale. By Doob-Meyer decomposition, and the fact that the adjoint Ad is an isomorphism it follows that $\log(f_1(B^1))$ is a local martingale, hence f_1 is harmonic. □

The factorization result of the corollary cannot be improved: a harmonic product may not be product of harmonic components. Consider, for example, the harmonic function in the Abelian group $(\mathbb{R}^2, +)$ given by $f(x, y) = x^2 - y^2$.

Example 2 (Invariance by geodesic translations). Let $f : M \to G$ be a C^∞-differentiable map, and let $X_1, \ldots, X_n$ be elements of the Lie algebra $\mathcal{G}$. The map f is harmonic if and only if $\exp(t_1 X_1) \cdot \ldots \cdot \exp(t_n X_n) \cdot f : \mathbb{R}^n \times M \to G$ is harmonic. Yet, f is harmonic if and only if $f \cdot \exp(t_1 X_1) \cdot \ldots \cdot \exp(t_n X_n) : M \times \mathbb{R}^n \to G$ is harmonic.

Finally, we remark that if the group G is endowed with a bi-invariant metric then the inverse $i : g \mapsto g^{-1}$ is an isometry, hence a harmonic morphism. Therefore, a map $f : M \to G$ is harmonic if and only if f^{-1} is also harmonic.

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