3

Prerequisites

3.1 Linear Operators in Banach Spaces

In the following, we introduce basics of the language of Operator Theory that can also be found in most textbooks on Functional Analysis [179, 182, 186, 216, 225]. For the convenience of the reader, we also include associated proofs.

Lemma 3.1.1. (**Direct sums of Banach and Hilbert spaces**)

(i) Let $(X, \| \ \|_{\mathrm{X}})$ and $(Y, \| \ \|_{\mathrm{Y}})$ be Banach spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $\| \ \|_{\mathrm{X\times Y}} : X \times Y \to \mathbb{R}$ be defined by

$$\|(\xi, \eta)\|_{\mathrm{X\times Y}} := \sqrt{\|\xi\|_{\mathrm{X}}^2 + \|\eta\|_{\mathrm{Y}}^2}$$

for all $(\xi, \eta) \in \mathrm{X} \times \mathrm{Y}$. Then $(X \times Y, \| \ \|_{\mathrm{X\times Y}})$ is a Banach space.

(ii) Let $(X, \langle \, | \, \rangle_{\mathrm{X}})$ and $(Y, \langle \, | \, \rangle_{\mathrm{Y}})$ be Hilbert spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $\langle \, | \, \rangle_{\mathrm{X\times Y}} : (X \times Y)^2 \to \mathbb{K}$ be defined by

$$\langle (\xi, \eta) | (\xi', \eta') \rangle_{\mathrm{X\times Y}} := \langle \xi | \xi' \rangle_{\mathrm{X}} + \langle \eta | \eta' \rangle_{\mathrm{Y}}$$

for all $(\xi, \eta), (\xi', \eta') \in \mathrm{X} \times \mathrm{Y}$. Then $(X \times Y, \langle \, | \, \rangle_{\mathrm{X\times Y}})$ is a Hilbert space.

Proof. '(i)': Obviously, $\| \ \|_{\mathrm{X\times Y}}$ is positive definite and homogeneous. Further, it follows for $(\xi, \eta), (\xi', \eta') \in X \times Y$ by the Cauchy-Schwarz inequality for the Euclidean scalar product for $\mathbb{R}^2$ that

$$\begin{aligned}
&\|(\xi, \eta) + (\xi', \eta')\|_{\mathrm{X\times Y}}^2 \\
&\quad = \|\xi + \xi'\|_{\mathrm{X}}^2 + \|\eta + \eta'\|_{\mathrm{Y}}^2 \\
&\quad \leqslant (\, \|\xi\|_{\mathrm{X}} + \|\xi'\|_{\mathrm{X}})^2 + (\, \|\eta\|_{\mathrm{Y}} + \|\eta'\|_{\mathrm{Y}})^2 = (a + a')^2 + (b + b')^2 \\
&\quad = a^2 + b^2 + a'^2 + b'^2 + 2\,(a\,a' + b\,b') \\
&\quad \leqslant a^2 + b^2 + a'^2 + b'^2 + 2\,\sqrt{a^2 + b^2} \cdot \sqrt{a'^2 + b'^2} \\
&\quad = \left(\sqrt{a^2 + b^2} + \sqrt{a'^2 + b'^2}\right)^2 = (\, \|(\xi, \eta)\|_{\mathrm{X\times Y}} + \|(\xi', \eta')\|_{\mathrm{X\times Y}})^2,
\end{aligned}$$

where $a := \|\xi\|_X, a' := \|\xi'\|_X, b := \|\eta\|_Y, b' := \|\eta'\|_Y$, and hence that

$$\|(\xi,\eta) + (\xi',\eta')\|_{X\times Y} \leqslant \|(\xi,\eta)\|_{X\times Y} + \|(\xi',\eta')\|_{X\times Y}.$$

The completeness of $(X \times Y, \| \|_{X\times Y})$ is an obvious consequence of the completeness of X and Y.
'(ii)': Obviously, $\langle | \rangle_{X\times Y}$ is a positive definite symmetric bilinear, positive definite hermitian sesquilinear form, respectively. Further, the induced norm on $X \times Y$ coincides with the norm defined in (i). □

Definition 3.1.2. (**Linear Operators**) Let $(X, \| \|_X)$ and $(Y, \| \|_Y)$ be Banach spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Then we define

(i) A map A is called a *Y-valued linear operator in X* if its *domain* $D(A)$ is a subspace of X, $\text{Ran}\, A \subset Y$ and A is linear. If $(Y, \| \|_Y) = (X, \| \|_X)$ such a map is also called a *linear operator in X*.
(ii) If in addition A is a Y-valued linear operator in X:
   a) The *graph* $G(A)$ of A by

$$G(A) := \{(\xi, A\xi) \in X \times Y : \xi \in D(A)\}.$$

   Note that $G(A)$ is a subspace of $X \times Y$.
   b) A is *densely-defined* if $D(A)$ is in particular dense in X.
   c) A is *closed* if $G(A)$ is a closed subspace of $(X \times Y, \| \|_{X\times Y})$.
   d) A Y-valued linear operator B in X is said to be an *extension* of A, symbolically denoted by

$$A \subset B \quad \text{or} \quad B \supset A \ ,$$

   if $G(A) \subset G(B)$.
   e) A is *closable* if there is a closed extension. In this case,

$$\bigcap_{B\supset A, B\,\text{closed}} G(B)$$

   is a closed subspace of $X \times Y$ which, obviously, is the graph of a unique Y-valued closed linear extension $\bar{A}$ of A, called the *closure* of A. By definition, every closed extension B of A satisfies $B \supset \bar{A}$.
   f) If A is closed, a *core* of A is a subspace D of its domain such that the closure of $A|_D$ coincides with A, i.e., if

$$\overline{A|_D} = A \ .$$

Theorem 3.1.3. (**Elementary properties of linear operators**) Let $(X, \| \|_X)$, $(Y, \| \|_Y)$ be Banach spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, A a *Y-valued linear operator in X* and $B \in L(X, Y)$.

(i) $(D(A), \| \|_{\mathrm{A}})$, where $\| \|_{\mathrm{A}} : D(A) \to \mathbb{R}$ is defined by

$$\|\xi\|_{\mathrm{A}} := \|(\xi, A\xi)\|_{\mathrm{X}\times\mathrm{Y}} = \sqrt{\|\xi\|_{\mathrm{X}}^2 + \|A\xi\|_{\mathrm{Y}}^2}$$

for every $\xi \in D(A)$, is a normed vector space. Further, the inclusion ι_{A} : $(D(A), \| \|_{\mathrm{A}}) \hookrightarrow X$ is continuous and $A \in L((D(A), \| \|_{\mathrm{A}}), Y)$.

(ii) A is closed if and only if $(D(A), \| \|_{\mathrm{A}})$ is complete.

(iii) If A is closable, then $G(\bar{A}) = \overline{G(A)}$.

(iv) (*Inverse mapping theorem*) If A is closed and bijective, then $A^{-1} \in L(Y, X)$.

(v) (*Closed graph theorem*) In addition, let $D(A) = X$. Then A is bounded if and only if A is closed.

(vi) If A is closable, then $A + B$ is also closable and

$$\overline{A + B} = \bar{A} + B \ .$$

Proof. '(i)': Obviously, $(D(A), \| \|_{\mathrm{A}})$ is a normed vector space. Further, because of

$$\|\iota_{\mathrm{A}}\, \xi\|_{\mathrm{X}} = \|\xi\|_{\mathrm{X}} \leqslant \sqrt{\|\xi\|_{\mathrm{X}}^2 + \|A\xi\|_{\mathrm{Y}}^2} = \|\xi\|_{\mathrm{A}}$$

and

$$\|A\xi\|_{\mathrm{Y}} \leqslant \sqrt{\|\xi\|_{\mathrm{X}}^2 + \|A\xi\|_{\mathrm{Y}}^2} = \|\xi\|_{\mathrm{A}}$$

for every $\xi \in D(A)$, it follows that $\iota_{\mathrm{A}} \in L((D(A), \| \|_{\mathrm{A}}), X)$ and $A \in L((D(A), \| \|_{\mathrm{A}}), Y)$.

'(ii)': Let A be closed and $\xi_0, \xi_1, \dots$ a Cauchy sequence in $(D(A), \| \|_{\mathrm{A}})$. Then $(\xi_0, A\xi_0)$, $(\xi_1, A\xi_1), \dots$ is a Cauchy sequence in $G(A)$ and hence by Lemma 3.1.1 along with the closedness of $G(A)$ convergent to some $(\xi, A\xi) \in G(A)$. This implies that

$$\lim_{\nu\to\infty} \|\xi_\nu - \xi\|_{\mathrm{A}} = 0$$

and the convergence of $\xi_0, \xi_1, \dots$ in $(D(A), \| \|_{\mathrm{A}})$. Let $(D(A), \| \|_{\mathrm{A}})$ be complete and $(\xi, \eta) \in \overline{G(A)}$. Then there is a sequence $(\xi_0, A\xi_0), (\xi_1, A\xi_1), \dots$ in $G(A)$ which is convergent to (ξ, η). Hence $(\xi_0, A\xi_0), (\xi_1, A\xi_1), \dots$ is a Cauchy sequence in $X \times Y$. As a consequence, $\xi_0, \xi_1, \dots$ is a Cauchy sequence in $(D(A), \| \|_{\mathrm{A}})$ and therefore convergent to some $\xi' \in D(A)$. In particular,

$$\lim_{\nu\to\infty} \|(\xi_\nu, A\xi_\nu) - (\xi', A\xi')\|_{\mathrm{X}\times\mathrm{Y}} = 0$$

and hence $(\xi, \eta) = (\xi', A\xi') \in G(A)$.

'(iii)': Let A be closable. Then the closed graph of every closed extension of A contains $G(A)$ and hence also $\overline{G(A)}$. Therefore $G(\bar{A}) \supset \overline{G(A)}$. This implies in particular that $\overline{G(A)}$ is the graph of a map $\tilde{A}$. Further, $D(\tilde{A}) = \mathrm{pr}_1 \overline{G(A)}$, where $\mathrm{pr}_1 := (X \times Y \to X, (\xi, \eta) \mapsto \xi)$, is a subspace of X and $\tilde{A}$ is in particular a linear closed extension of A. Hence $\tilde{A} \supset \bar{A}$ and $\overline{G(A)} = G(\tilde{A}) \supset G(\bar{A})$.

'(iv)': Let A be closed and bijective. Then it follows by (ii) that $(D(A), \| \|_{\mathrm{A}})$ is a Banach space and that $A \in L((D(A), \| \|_{\mathrm{A}}), Y)$. Hence it follows by the 'inverse

mapping theorem', for e.g. see Theorem III.11 in Vol. I of [179], that $A^{-1} \in L(Y, (D(A), \| \|_{\mathrm{A}}))$ and by the continuity of ι_{A} that $A^{-1} \in L(Y, X)$.
'(v)': Let $D(A) = X$. If A is bounded and $\xi_0, \xi_1, \dots$ is some Cauchy sequence in $(X, \| \|_{\mathrm{A}})$, it follows by the continuity of ι_{A} that $\xi_0, \xi_1, \dots$ is a Cauchy sequence in X and hence convergent to some $\xi \in X$. Since A is continuous, it follows the convergence of $A\xi_0, A\xi_1, \dots$ to $A\xi$ and therefore also the convergence of $\xi_0, \xi_1, \dots$ in $(X, \| \|_{\mathrm{A}})$ to ξ. Hence $(X, \| \|_{\mathrm{A}})$ is complete and A is closed by (ii). If A is closed, it follows by (ii) that $(X, \| \|_{\mathrm{A}})$ is a Banach space and that the bijective X-valued linear operator ι_{A} is continuous. Hence ι_{A} is closed by the previous part of the proof. Therefore, the inverse of ι_{A} is continuous by (iv) and hence A is bounded.
'(vi)': Let A be closable. In a first step, we prove that $\bar{A} + B$ is closed. For this, let $(\xi, \eta) \in \overline{G(\bar{A} + B)}$. Then there is a sequence $\xi_0, \xi_1, \dots$ in $D(\bar{A})$ which is convergent to ξ and such that $(\bar{A} + B)\xi_0, (\bar{A} + B)\xi_1, \dots$ is convergent to η. Since B is continuous, it follows that $\bar{A}\xi_0, \bar{A}\xi_1, \dots$ is convergent to $\eta - B\xi$. Since $\bar{A}$ is closed, it follows that $\xi \in D(\bar{A})$ as well as $\bar{A}\xi = \eta - B\xi$ and hence that $\xi \in D(\bar{A} + B)$ as well as $(\bar{A} + B)\xi = \eta$. Hence $\bar{A} + B$ is closed, and therefore $A + B$ is closable such that $\bar{A} + B \supset \overline{A + B}$. Further, it follows by the previous part of the proof that $\overline{A + B} - B$ is a closed extension of A. Hence $\overline{A + B} - B \supset \bar{A}$ and therefore also $\overline{A + B} \supset \bar{A} + B$. Finally, it follows that $\overline{A + B} = \bar{A} + B$. □

Theorem 3.1.4. (**Existence of a discontinuous linear functional on every infinite dimensional normed vector space**) Let $(X, \| \|)$ be some infinite dimensional normed vector space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Then there is a discontinuous linear functional $\omega : X \to \mathbb{K}$.

Proof. For this, let B be a Hamel basis of X, i.e., a maximal linearly independent set, whose existence follows by an application of Zorn's lemma or the equivalent axiom of choice. Without restriction, it can be assumed that B contains only elements of norm 1. Since $(X, \| \|)$ is infinite dimensional, B contains infinitely many elements $e_0, e_1, e_2, \dots$. We define $\omega : B \to \mathbb{N}$ by $\omega(e_n) := n$ for all $n \in \mathbb{N}$ and $\omega(e) := 0$ for all other $e \in B$. Then there is a unique extension of ω to a linear functional on X. This functional is unbounded and hence discontinuous. □

Example 3.1.5. (**Example for a non-closable linear operator**) Let $a, b \in \mathbb{R}$ be such that $a < b$ and I the open interval between a and b. Define $\omega : C(\bar{I}, \mathbb{C}) \to \mathbb{C}$ by

$$\omega(f) := \lim_{x \to a} f(x)$$

for all $f \in C(\bar{I}, \mathbb{C})$. Obviously, ω is a densely-defined $\mathbb{C}$-valued linear operator in $L^2_{\mathbb{C}}(I)$. Further, let $g \in C(\bar{I}, \mathbb{C})$ be such that $g(a) \neq 0$. Then the sequence $g_0, g_1, \dots$ in $C(\bar{I}, \mathbb{C})$, defined by $g_\nu := g$ for all $\nu \in \mathbb{N}$, is converging in $L^2_{\mathbb{C}}(I)$ to g and

$$\lim_{\nu \to \infty} \omega(g_\nu) = g(a) \neq 0 \ .$$

Since $C_0(I, \mathbb{C})$ is dense in $L^2_{\mathbb{C}}(I)$, there is a sequence $h_0, h_1, \dots$ in $C_0(I, \mathbb{C})$ such that

$$\lim_{\nu \to \infty} \|h_\nu - g\|_2 = 0 \ .$$

For such a sequence

$$\lim_{\nu\to\infty} \omega(h_\nu) = 0 \ .$$

Hence $\overline{G(\omega)}$ contains the differing elements $(g,0),(g,g(a))$ and therefore ω is not closable. Note that the non-closability of ω is caused by its discontinuity in g.

Theorem 3.1.6. (An application of the closed graph theorem) Let $(X,\|\ \|_X)$, $(Y,\|\ \|_Y)$, $(Z,\|\ \|_Z)$ be Banach spaces over $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$, A a closed bijective Y-valued linear operator in X and B a *closable* Z-valued linear operator in X such that $D(B) \supset D(A)$. Then there is $C \in [0,\infty)$ such that

$$\|B\xi\|_Z \leqslant C\,\|A\xi\|_Y$$

for all $\xi \in D(A)$ and hence in particular $B|_{D(A)} \in L((D(A),\|\ \|_A),Z)$.

Proof. First, it follows by Theorem 3.1.3 (iv) that $A^{-1} \in L(Y,X)$. Further, $B \circ A^{-1}$ is a Z-valued linear operator on Y since A^{-1} maps into the domain of B. Let $(\eta,\zeta) \in G(\overline{B\circ A^{-1}})$. Then there is a sequence $(\eta_0, B(A^{-1}\eta_0)),(\eta_1,B(A^{-1}\eta_1)),\dots$ in $G(B\circ A^{-1})$ converging to (η,ζ). In particular,

$$\lim_{\nu\to\infty} \eta_\nu = \eta$$

and therefore also

$$\lim_{\nu\to\infty} A^{-1}\eta_\nu = A^{-1}\eta \ .$$

Since B is closable, it follows that $(A^{-1}\eta,\zeta) \in G(\bar{B})$ and hence because of $A^{-1}\eta \in D(A) \subset D(B)$ that $(A^{-1}\eta,\zeta) \in G(B)$. Therefore also $BA^{-1}\eta = \zeta$ and $(\eta,\zeta) \in G(B\circ A^{-1})$. Hence $B\circ A^{-1}$ is in addition closed and therefore by Theorem 3.1.3 (v) bounded. As a consequence, it follows

$$\|B\xi\|_Z = \|B\circ A^{-1}A\xi\|_Z \leqslant C\,\|A\xi\|_Y$$

for every $\xi \in D(A)$ where $C \in [0,\infty)$ is some bound for $B\circ A^{-1}$. □

Theorem 3.1.7. (Definition and elementary properties of the adjoint) Let $(X, \langle\,|\,\rangle_X)$ and $(Y,\langle\,|\,\rangle_Y)$ be Hilbert spaces over $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$, A a *densely-defined* Y-valued linear operator in X and $U : X\times Y \to Y\times X$ the Hilbert space isomorphism defined by $U(\xi,\eta) := (-\eta,\xi)$ for all $(\xi,\eta) \in X\times Y$.

(i) Then the closed subspace

$$[U(G(A))]^{\perp}$$

of $Y\times X$ is the graph of an uniquely determined X-valued linear operator A^* in Y which is in particular closed and called the *adjoint* of A. *If in addition* $(X,\langle\,|\,\rangle_X) = (Y,\langle\,|\,\rangle_Y)$, we call A *symmetric* if $A^* \supset A$ and *self-adjoint* if $A^* = A$.

(ii) If B is a Y-valued linear operator B in X such that $B \supset A$, then

$$B^* \subset A^* \ .$$

(iii) If A^* is densely-defined, then $A \subset A^{**} := (A^*)^*$ and hence A is in particular closable.
(iv) If A is closed, then A^* is densely-defined and $A^{**} = A$.
(v) If A is closable, then $\bar{A} = A^{**}$.

If in addition $(X, \langle\,|\,\rangle_{\mathrm{X}}) = (Y, \langle\,|\,\rangle_{\mathrm{Y}})$:

(vi) (*Maximality of self-adjoint operators*) If A is self-adjoint and $B \supset A$ is symmetric, then $B = A$.
(vii) If A is symmetric, then $\bar{A}$ is symmetric, too. Therefore, we call a symmetric A *essentially self-adjoint* if $\bar{A}$ is self-adjoint.
(viii) (*Hellinger-Toeplitz*) If $D(A) = X$ and A is self-adjoint, then $A \in L(X, X)$.

Proof. '(i)': First, it follows that

$$[U(G(A))]^{\perp} = \{(\eta, \xi) \in Y \times X : \langle(\eta, \xi)|U(\xi', A\xi')\rangle_{\mathrm{Y}\times\mathrm{X}} = 0 \text{ for all } \xi' \in D(A)\}$$

and hence that

$$[U(G(A))]^{\perp} = \{(\eta, \xi) \in Y \times X : \langle\eta|A\xi'\rangle_{\mathrm{Y}} = \langle\xi|\xi'\rangle_{\mathrm{X}} \text{ for all } \xi' \in D(A)\} .$$

In particular, it follows for $(\eta, \xi_1), (\eta, \xi_2) \in [U(G(A))]^{\perp}$ that

$$\langle\xi_1 - \xi_2|\xi'\rangle_{\mathrm{X}} = 0$$

for all $\xi' \in D(A)$ and hence that $\xi_1 = \xi_2$ since $D(A)$ is dense in X. As a consequence, by

$$A^* : \mathrm{pr}_1[U(G(A))]^{\perp} \to X ,$$

where $\mathrm{pr}_1 := (Y \times X \to Y, (\eta, \xi) \mapsto \eta)$, defined by

$$A^*\eta := \xi ,$$

for all $\eta \in \mathrm{pr}_1[U(G(A))]^{\perp}$, where $\xi \in X$ is the unique element such that $(\eta, \xi) \in [U(G(A))]^{\perp}$, there is defined a map such that

$$G(A^*) = [U(G(A))]^{\perp} .$$

Note that the domain of A^* is a subspace of Y. In particular, it follows for all $\eta, \eta' \in D(A^*)$ and $\lambda \in \mathbb{K}$

$$\begin{aligned}\langle\eta + \eta'|A\xi'\rangle_{\mathrm{Y}} &= \langle\eta|A\xi'\rangle_{\mathrm{Y}} + \langle\eta'|A\xi'\rangle_{\mathrm{Y}} = \langle A^*\eta|\xi'\rangle_{\mathrm{X}} + \langle A^*\eta'|\xi'\rangle_{\mathrm{X}} \\ &= \langle A^*\eta + A^*\eta'|\xi'\rangle_{\mathrm{X}} \\ \langle\lambda.\eta|A\xi'\rangle_{\mathrm{Y}} &= \lambda^{(*)} \cdot \langle\eta|A\xi'\rangle_{\mathrm{Y}} = \lambda^{(*)} \cdot \langle A^*\eta|\xi'\rangle_{\mathrm{X}} = \langle\lambda.A^*\eta|\xi'\rangle_{\mathrm{X}}\end{aligned}$$

for all $\xi' \in D(A)$ and hence also the linearity of A^*.
'(ii)': Obvious.
'(iii)': For this, let A^* be densely-defined. Then, it follows

$$\big(Y \times X \to X \times Y, (\eta, \xi) \mapsto (-\xi, \eta)\big) = -U^{-1}$$

and hence

$$\begin{aligned} G(A^{**}) &= \big[-U^{-1}(G(A^*))\big]^{\perp} = \big[U^{-1}(G(A^*))\big]^{\perp} = \big[U^{-1}[U(G(A))]^{\perp}\big]^{\perp} \\ &= \big[\,[U^{-1}U(G(A))]^{\perp}\big]^{\perp} = G(A)^{\perp\perp} = \overline{G(A)} \supset G(A)\ . \end{aligned} \tag{3.1.1}$$

'(iv)': For this, let A be closed. Then, it follows for $\eta \in [D(A^*)]^{\perp}$

$$\begin{aligned} (0, \eta) &\in \big[U^{-1}(G(A^*))\big]^{\perp} = \big[U^{-1}[U(G(A))]^{\perp}\big]^{\perp} = \big[\,[U^{-1}U(G(A))]^{\perp}\big]^{\perp} \\ &= G(A)^{\perp\perp} = \overline{G(A)} = G(A) \end{aligned}$$

and hence $\eta = 0$. Hence $D(A^*)$ is dense in X, and it follows by (3.1.1) that $G(A^{**}) = \overline{G(A)} = G(A)$.
'(v)': For this, let A be closable. Since $\bar{A}$ is densely defined and closed, it follows by (iv) that $\bar{A}^*$ is densely-defined. Because of $A \subset \bar{A}$, this implies that $A^* \supset \bar{A}^*$ and hence that A^* is densely-defined, too. Therefore, it follows by (iii) that $A \subset A^{**}$ and by (3.1.1) that $G(A^{**}) = \overline{G(A)} = G(\bar{A})$ and hence, finally, that $A^{**} = \bar{A}$. In the following, it is assumed that $(X, \langle\,|\,\rangle_{\mathrm{X}}) = (Y, \langle\,|\,\rangle_{\mathrm{Y}})$.
'(vi)': For this, let A be self-adjoint and B a symmetric extension of A. Then, it follows by using $G(B) \supset G(A)$ that

$$G(B) \subset G(B^*) = [U(G(B))]^{\perp} \subset [U(G(A))]^{\perp} = G(A^*) = G(A)$$

and hence $B \subset A \subset B$ and therefore, finally, that $B = A$.
'(vii)': For this, let A be symmetric. Then $A^* \supset A$ and hence also $A^* \supset \bar{A}$.

$$\begin{aligned} G(\bar{A}^*) &= [U(G(\bar{A}))]^{\perp} = [U\,\overline{G(A)}\,]^{\perp} = [\,\overline{U(G(A))}\,]^{\perp} = \big[[U(G(A))]^{\perp\perp}\big]^{\perp} \\ &= \overline{G(A^*)} = G(A^*) \supset G(\bar{A})\ . \end{aligned}$$

'(viii)': For this, let A be self-adjoint and $D(A) = X$. Then, $A = A^*$ is in particular closed and hence by Theorem 3.1.3 (v) bounded. □

Theorem 3.1.8. Let $(X, \langle\,|\,\rangle_{\mathrm{X}})$ be a Hilbert space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and Y a dense subspace of X. Further, let $\langle\,|\,\rangle_{\mathrm{Y}} : Y^2 \to \mathbb{K}$ be a scalar product for Y such that $(Y, \langle\,|\,\rangle_{\mathrm{Y}})$ is a Hilbert space over $\mathbb{K}$ and such that there is $\kappa > 0$ such that

$$\|\xi\|_{\mathrm{Y}}^2 \geqslant \kappa\,\|\xi\|^2$$

for all $\xi \in Y$ where $\|\ \|_{\mathrm{Y}} : Y \to \mathbb{R}$ is the norm induced on Y by $\langle\,|\,\rangle_{\mathrm{Y}}$. Then, there is a uniquely determined densely-defined, linear and self-adjoint operator in X such that $D(A)$ is a dense subspace of $(Y, \langle\,|\,\rangle_{\mathrm{Y}})$ and

$$\langle \xi | A\xi \rangle = \langle \xi | \xi \rangle_{\mathrm{Y}}$$

for all $\xi \in D(A)$. This A is given by

$$D(A) = \{\xi \in Y : \langle \xi|\cdot\rangle_{\mathrm{Y}} \in L((Y, \| \|), \mathbb{K})\}$$

and for every $\xi \in D(A)$

$$A\xi = \hat{\xi}$$

where $\hat{\xi} \in X$ is the, by the denseness of Y in X, the linear extension theorem and Riesz' lemma, uniquely determined element such that

$$\langle \xi|\cdot\rangle_{\mathrm{Y}} = \langle \hat{\xi}|\cdot\rangle\big|_{Y} \ . \tag{3.1.2}$$

In particular, A is semibounded from below with bound κ, i.e.,

$$\langle \xi|A\xi\rangle \geqslant \kappa \, \langle \xi|\xi\rangle$$

for all $\xi \in D(A)$.

Proof. For $\xi \in X$, it follows

$$|\,\langle \xi|\eta\rangle\,| \leqslant \|\xi\| \cdot \|\eta\| \leqslant \kappa^{-1/2} \, \|\xi\| \cdot \|\eta\|_{\mathrm{Y}}$$

for every $\eta \in Y$ and hence by Riesz' lemma the existence of a uniquely determined $\hat{\xi} \in Y$ such that

$$\langle \xi|\eta\rangle = \langle \hat{\xi}|\eta\rangle_{\mathrm{Y}}$$

for all $\eta \in Y$. Hence by $B\xi := \hat{\xi}$ for every $\xi \in X$ there is given map $B : X \to X$ such $\mathrm{Ran}B \subset Y$. Further, B is obviously linear and because of

$$\langle B\xi|\eta\rangle = \langle \eta|B\xi\rangle^{(*)} = \langle B\eta|B\xi\rangle_{\mathrm{Y}}^{(*)} = \langle B\xi|B\eta\rangle_{\mathrm{Y}} = \langle \xi|B\eta\rangle$$

for all $\xi, \eta \in X$ also symmetric. Further, for $\xi \in \ker B$, it follows that vanishing of the restriction of $\langle \xi|\cdot\rangle$ to Y and hence by the denseness of Y in X and Riesz' lemma that $\xi = 0$. Hence B is injective. In addition, $\mathrm{Ran}B$ is dense in $(Y, \| \|_{\mathrm{Y}})$ since for $\xi \in \mathrm{Ran}B^{\perp_{\mathrm{Y}}}$

$$0 = \langle \xi|B\eta\rangle_{\mathrm{Y}} = \langle B\eta|\xi\rangle_{\mathrm{Y}}^{(*)} = \langle \eta|\xi\rangle^{(*)} = \langle \xi|\eta\rangle$$

for every $\eta \in X$ and hence $\xi = 0$. As a consequence,

$$\overline{\mathrm{Ran\ B}} = \mathrm{Ran\ B}^{\perp_{\mathrm{Y}}\perp_{\mathrm{Y}}} = \{0\}^{\perp_{\mathrm{Y}}} = Y$$

where the closure is performed in $(Y, \| \|_{\mathrm{Y}})$. Therefore, since the inclusion $\iota_{\mathrm{Y} \hookrightarrow \mathrm{X}}$ of $(Y, \|\|_{\mathrm{Y}})$ into X is continuous and Y is dense in X, it follows also that $\mathrm{Ran}B$ is dense in X. In the following, we define A to be the inverse of the restriction of B in image to its range. Then A is a densely-defined, linear and symmetric operator in X with range X. In particular, it follows for $(\xi_1, \xi_2) \in G(A^*)$ and $\xi \in D(A)$

$$\begin{aligned}\langle \xi_1|A\xi\rangle &= \langle \xi_2|\xi\rangle = \langle B\xi_2|\xi\rangle_{\mathrm{Y}} = \langle B\xi_2|BA\xi\rangle_{\mathrm{Y}} = \langle BA\xi|B\xi_2\rangle_{\mathrm{Y}}^{(*)} = \langle A\xi|B\xi_2\rangle^{(*)} \\ &= \langle B\xi_2|A\xi\rangle\end{aligned}$$

and hence that $\xi_1 - B\xi_2$ is orthogonal to $\text{Ran}A = X$. As a consequence, it follows that $\xi_1 = B\xi_2$ and therefore that $\xi_1 \in D(A)$ and $A\xi_1 = \xi_2$. Hence $A^* \subset A$. Finally, since the symmetry of A implies that $A^* \supset A$, it follows that A is self-adjoint. In particular, it follows for $\xi \in D(A)$

$$\langle \xi | A\xi \rangle = \langle A\xi | \xi \rangle^{(*)} = \langle BA\xi | \xi \rangle_{\text{Y}}^{(*)} = \langle \xi | \xi \rangle_{\text{Y}}^{(*)} = \langle \xi | \xi \rangle_{\text{Y}} \geqslant \kappa \, \|\xi\|^2$$

and hence that A is in particular semibounded from below with bound κ. Further, if $\xi \in D(A)$, then

$$\langle \xi | \eta \rangle_{\text{Y}} = \langle BA\xi | \eta \rangle_{\text{Y}} = \langle A\xi | \eta \rangle$$

for all $\eta \in Y$ and hence $\langle \xi | \cdot \rangle_{\text{Y}} \in L((Y, \| \, \|), \mathbb{K})$. On the other hand, if $\xi \in Y$ is such that $\langle \xi | \cdot \rangle_{\text{Y}} \in L((Y, \| \, \|), \mathbb{K})$ and $\hat{\xi} \in X$ such that (3.1.2) is true, it follows that $B\hat{\xi} = \xi$ and hence $\xi \in D(A)$ and $A\xi = \hat{\xi}$. Finally, if A' is a densely-defined, linear and self-adjoint operator in X such that $D(A')$ is a dense subspace of $(Y, \langle \, | \, \rangle_{\text{Y}})$ and

$$\langle \xi | A'\xi \rangle = \langle \xi | \xi \rangle_{\text{Y}}$$

for all $\xi \in D(A')$, then it follows by polarization that

$$\langle \xi | \eta \rangle_{\text{Y}} = \langle \xi | A'\eta \rangle = \langle A'\xi | \eta \rangle$$

for all $\xi, \eta \in D(A')$. Hence it follows for $\xi \in D(A)$ by the denseness of $D(A)$ in $(Y, \| \, \|_{\text{Y}})$ and the continuity of $\iota_{\text{Y} \hookrightarrow \text{X}}$ that

$$\langle \xi | \eta \rangle_{\text{Y}} = \langle A'\xi | \eta \rangle$$

for all $\eta \in Y$. Hence $\langle \xi | \cdot \rangle_{\text{Y}} \in L((Y, \| \, \|), \mathbb{K})$ and by the foregoing $\xi \in D(A)$ and $A\xi = A'\xi$. As a consequence, it follows $A \supset A'$ and since A, A' are both self-adjoint that $A' = A$. □

Theorem 3.1.9. Let $(X, \langle \, | \, \rangle_{\text{X}})$ and $(Y, \langle \, | \, \rangle_{\text{Y}})$ be Hilbert spaces over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, A a densely-defined Y-valued, linear and closed operator in X. Then A^*A (as usual maximally defined) is a densely-defined linear, self-adjoint and positive operator in X. In particular, $D(A^*A)$ is a core for A and $\ker A = \ker(A^*A)$.

Proof. Since A is closed, it follows that $(D(A), \langle \; \rangle_{\text{A}})$, where $\langle \; \rangle_{\text{A}} : (D(A))^2 \to \mathbb{K}$ is defined by

$$\langle \xi | \eta \rangle_{\text{A}} := \langle \xi | \eta \rangle_{\text{X}} + \langle A\xi | A\eta \rangle_{\text{Y}}$$

for all $\xi, \eta \in D(A)$, is a Hilbert space. In particular, it follows

$$\|\xi\|_{\text{A}} = \sqrt{\|\xi\|_{\text{X}}^2 + \|A\xi\|_{\text{Y}}^2} \geqslant \|\xi\|_{\text{X}}$$

for all $\xi \in D(A)$ where $\| \, \|_{\text{A}}$ denotes the norm induced on $D(A)$ by $\langle \, | \, \rangle_{\text{A}}$. Hence by the previous theorem, $B : D(B) \to X$, defined by

$$D(B) = \{\xi \in D(A) : \langle A\xi | A \cdot \rangle_{\text{Y}} \in L((D(A), \| \, \|_{\text{X}}), \mathbb{K})\}$$

and for every $\xi \in D(B)$ by

$$B\xi = \hat{\xi} \ ,$$

where $\hat{\xi} \in X$ is the uniquely determined element such that

$$\langle A\xi | A\cdot \rangle_{\mathrm{Y}} = \langle \hat{\xi} - \xi | \cdot \rangle_{\mathrm{X}} \big|_{D(A)} \ ,$$

is densely-defined, linear, self-adjoint and semibounded from below with bound 1 and $D(B)$ is a dense subspace of $(D(A), \| \, \|_{\mathrm{A}})$. Hence it follows by the definition of A^*A that

$$A^*A = B - \mathrm{id}_{\mathrm{X}} \ .$$

As a consequence, A^*A is a densely-defined, linear, self-adjoint and positive operator in X. Further, since $D(B)$ is dense in $(D(A), \| \, \|_{\mathrm{A}})$, for $\xi \in D(A)$ there is a sequence $\xi_0, \xi_1, \ldots$ in $D(B)$ such that

$$\lim_{\nu \to \infty} \|\xi_\nu - \xi\|_{\mathrm{A}} = 0 \ .$$

Hence it follows also that

$$\lim_{\nu \to \infty} \|\xi_\nu - \xi\|_{\mathrm{X}} = 0 \ , \ \lim_{\nu \to \infty} \|A\xi_\nu - A\xi\|_{\mathrm{Y}} = 0$$

and therefore that $D(A^*A)$ is a core for A. Further, for $\xi \in \ker(A^*A)$ it follows

$$\langle A\xi | A\xi \rangle_{\mathrm{Y}} = \langle A^*A\xi | \xi \rangle_{\mathrm{X}} = 0$$

and hence $\xi \in \ker A$. Since $\ker A \subset \ker(A^*A)$, it follows finally that $\ker A = \ker(A^*A)$. □

Theorem 3.1.10. Let $(X, \langle \, | \, \rangle_{\mathrm{X}})$ and $(Y, \langle \, | \, \rangle_{\mathrm{Y}})$ be complex Hilbert spaces, A a densely-defined Y-valued, linear and closed operator in X. Then $\ker A$ is a closed subspace of X and the orthogonal projection $P_0 \in L(X, X)$ onto $\ker A$ is given by

$$P_0 = \mathrm{s-}\lim_{\nu \to \infty} (1 + \nu A^*A)^{-1}$$

where 's−lim' denotes the strong limit. Note in particular that, because of

$$(1 + \nu A^*A)\xi = \xi$$

for all $\xi \in \ker A$, the elements of $\ker A$ are *fixed points* of $(1 + \nu A^*A)^{-1}$ for every $\nu \in \mathbb{N}$.

Proof. From the linearity of A, it follows that $\ker A$ is a subspace of X. Further, for $\xi \in \overline{\ker A}$ there is a sequence $\xi_0, \xi_1, \ldots$ in $\ker A$ converging to ξ. The corresponding sequence $A\xi_0, A\xi_1, \ldots$ is converging to 0. Hence it follows by the closedness of A that $\xi \in \ker A$ and therefore that $\ker A = \overline{\ker A}$ is a closed subspace of X. According to the previous theorem, $\ker A$ equals the kernel of the densely-defined, linear, self-adjoint and positive operator $B := A^*A$ in X. In the next step, we prove for $\nu \in \mathbb{N}$ by using the functional calculus for B that

$$(1+\nu B)^{-1} = \frac{1}{1+\nu\,\mathrm{id}_{\sigma(B)}}(B)$$

where $\sigma(B)$ denotes the spectrum of B. For this, we first note that $1+\nu B$ is a densely-defined, linear and self-adjoint operator in X which is semibounded from below with bound 1. Hence the spectrum of $1+\nu B$ is contained in $[1,\infty)$ and therefore $1+\nu B$ is in particular bijective. Further, $1/(1+\nu\,\mathrm{id}_{\sigma(B)})$ is a bounded real-valued function on $\sigma(B)$ which is measurable with respect to every additive, monotone and regular interval function on $\mathbb{R}$. By the functional calculus for B, the Cayley transform $U_B = (B-i)(B+i)^{-1} = 1-2i(B+i)^{-1}$ of B is given by

$$U_B = \frac{\mathrm{id}_{\sigma(B)}-i}{\mathrm{id}_{\sigma(B)}+i}(B) = 1-2i\,\frac{1}{\mathrm{id}_{\sigma(B)}+i}(B)\ .$$

Hence it follows that

$$(B+i)^{-1} = \frac{1}{\mathrm{id}_{\sigma(B)}+i}(B)\ .$$

Further,

$$\begin{aligned}1+\nu B &= 1-\nu i+\nu(B+i) = \left[(1-\nu i)(B+i)^{-1}+\nu\right](B+i)\\ &= \left[\frac{1-\nu i}{\mathrm{id}_{\sigma(B)}+i}+\nu\right](B)\,(B+i) = \frac{1+\nu\,\mathrm{id}_{\sigma(B)}}{\mathrm{id}_{\sigma(B)}+i}(B)\,(B+i)\end{aligned}$$

and hence

$$(1+\nu B)^{-1} = \frac{1}{\mathrm{id}_{\sigma(B)}+i}(B)\,\frac{\mathrm{id}_{\sigma(B)}+i}{1+\nu\,\mathrm{id}_{\sigma(B)}}(B) = \frac{1}{1+\nu\,\mathrm{id}_{\sigma(B)}}(B)\ .$$

In a further step, we prove that the orthogonal projection $P_0 \in L(X,X)$ onto $\ker B$ is given by

$$\left(\chi_{\{0\}}|_{\sigma(B)}\right)(B)$$

where $\chi_{\{0\}}$ denotes the characteristic function of $\{0\}$. First, it follows by the functional calculus for B that $\left(\chi_{\{0\}}|_{\sigma(B)}\right)(B)$ is an idempotent, self-adjoint, bounded and linear on X and hence an orthogonal projection. For $\xi \in X$, it follows that

$$\begin{aligned}(B+i)^{-1}\left(\chi_{\{0\}}|_{\sigma(B)}\right)(B)\xi &= \frac{1}{\mathrm{id}_{\sigma(B)}+i}(B)\left(\chi_{\{0\}}|_{\sigma(B)}\right)(B)\xi\\ &= \frac{1}{i}\left(\chi_{\{0\}}|_{\sigma(B)}\right)(B)\xi\end{aligned}$$

and hence that

$$\left(\chi_{\{0\}}|_{\sigma(B)}\right)(B)\xi \in \ker B\ .$$

In particular, it follows in the case that 0 is no eigenvalue of B that $\left(\chi_{\{0\}}|_{\sigma(B)}\right)(B)$ is the zero operator which projects onto $\ker B = \{0\}$. Further, if 0 is an eigenvalue of B, it follows for $\xi \in \ker B$ by the functional calculus for B that

$$\left(\chi_{\{0\}}|_{\sigma(B)}\right)(B)\xi = \left(\chi_{\{0\}}|_{\sigma(B)}\right)(0)\xi = \xi$$

and hence also in this case that $\text{Ran}\left(\chi_{\{0\}}|_{\sigma(B)}\right)(B) = \ker B$. Finally,

$$\left(\frac{1}{1+\nu\,\text{id}_{\sigma(B)}}\right)_{\nu\in\mathbb{N}}$$

is a sequence which is uniformly bounded by 1 and everywhere on $\sigma(B)$ pointwise convergent to $\chi_{\{0\}}|_{\sigma(B)}$. Hence it follows by the functional calculus for B that

$$\text{s}-\lim_{\nu\to\infty}\frac{1}{1+\nu\,\text{id}_{\sigma(B)}}(B) = \left(\chi_{\{0\}}|_{\sigma(B)}\right)(B)\ .$$

□

Theorem 3.1.11. (Elementary properties of the resolvent) Let $(X, \|\ \|_X)$ be a non-trivial Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and A a densely-defined closed linear operator in X.

(i) We define the *resolvent set* $\rho(A) \subset \mathbb{K}$ of A by

$$\rho(A) := \{\lambda \in \mathbb{K} : A - \lambda \text{ is bijective}\}\ .$$

Then $\rho(A)$ is an *open* subset of $\mathbb{K}$. Therefore, its complement $\sigma(A) := \mathbb{K}\backslash\rho(A)$, which is called the *spectrum* of A, is a *closed* subset of $\mathbb{K}$.

(ii) We define the *resolvent* $R_A : \rho(A) \to L(X, X)$ of A by

$$R_A(\lambda) := (A - \lambda)^{-1}$$

for every $\lambda \in \rho(A)$. Then R_A satisfies the *first resolvent formula*

$$R_A(\mu) - R_A(\lambda) = (\mu - \lambda)\, R_A(\mu) R_A(\lambda) \tag{3.1.3}$$

for every $\lambda, \mu \in \rho(A)$ and the *second resolvent formula*

$$R_A(\lambda) - R_B(\lambda) = R_A(\lambda)(B - A) R_B(\lambda) \tag{3.1.4}$$

for every $\lambda \in \rho(A) \cap \rho(B)$ where B is some closed linear operator in X having the *same domain* as A, i.e., $D(B) = D(A)$.

(iii) For every $\xi \in X$, $\omega \in L(X, \mathbb{K})$ is the corresponding function

$$\omega \circ R_A\xi$$

real-analytic/holomorphic. Here $R_A\xi : \rho(A) \to X$ is defined by $(R_A\xi)(\lambda) := R_A(\lambda)\xi$.

Proof. '(i), (iii)': For this, let $\lambda_0 \in \rho(A)$. Then $A - \lambda_0$ is a closed densely-defined bijective linear operator in X and hence $R_A(\lambda_0) \in L(X, X)\backslash\{0\}$. Then it follows for every $\lambda \in U_{1/\|R_A(\lambda_0)\|}(\lambda_0)$

$$A - \lambda = \left[\text{id}_X - (\lambda - \lambda_0)\,.\,R_A(\lambda_0)\right](A - \lambda_0)$$

and therefore, since

$$\mathrm{id}_X - (\lambda - \lambda_0).R_A(\lambda_0)$$

is bijective as a consequence of

$$\|(\lambda - \lambda_0).R_A(\lambda_0)\| < 1 \ ,$$

see e.g. [128] Chapter IV, §2, Theorem 2, that $A - \lambda$ is bijective as a composition of bijective maps and

$$R_A(\lambda) = \sum_{k=0}^{\infty} (\lambda - \lambda_0)^k \, [R_A(\lambda_0)]^{k+1} \ . \tag{3.1.5}$$

Hence $\lambda \in \rho(A)$ and in particular for every $\xi \in X$, $\omega \in L(X, \mathbb{K})$

$$(\omega \circ R_A \xi)(\lambda) = \sum_{k=0}^{\infty} \omega \left([R_A(\lambda_0)]^{k+1} \xi \right) (\lambda - \lambda_0)^k \ . \tag{3.1.6}$$

'(ii)': For $\lambda, \mu \in \rho(A)$ and every $\xi \in D(A)$, it follows

$$(A - \mu)\xi = (A - \lambda)\xi + (\lambda - \mu)\xi$$

and hence for every $\eta \in X$

$$(A - \mu)R_A(\lambda)\eta = \eta + (\lambda - \mu)R_A(\lambda)\eta \ .$$

The last implies

$$R_A(\lambda) = R_A(\mu) + (\lambda - \mu)R_A(\mu)R_A(\lambda)$$

and hence (3.1.3). Finally, let $B : D(A) \to X$ be some closed linear operator in X. Then it follows for every $\mu \in \rho(A)$, $\lambda \in \rho(B)$ and every $\xi \in D(A)$

$$(A - \mu)\xi = (A - B)\xi + (B - \lambda)\xi + (\lambda - \mu)\xi$$

and hence for every $\eta \in X$

$$(A - \mu)R_B(\lambda)\eta = (A - B)R_B(\lambda)\eta + \eta + (\lambda - \mu)R_B(\lambda)\eta \ .$$

The last implies

$$R_B(\lambda) = R_A(\mu)(A - B)R_B(\lambda) + R_A(\mu) + (\lambda - \mu)R_A(\mu)R_B(\lambda)$$

and hence (3.1.4). □

3.2 Weak Integration of Banach Space-Valued Maps

The integration of Banach space-valued maps [49, 52, 225] is an essential tool in the study of semigroups of operators. Most authors use for this the Bochner integral.

Instead, the so called weak (or Pettis) integral is developed in the following up to the level needed for the remainder of the course. The use of the more general weak integral is mainly due to the validity of Theorem 3.2.2 below which seems to favour the approach via weak integration in the important special case of Hilbert space-valued maps. On the other hand, in the following only integrals of maps are needed which are a.e. defined and a.e. continuous on open subsets of $\mathbb{R}^n$ for some $n \in \mathbb{N}^*$. For this class of functions, it can easily be seen that the weak integral and the Bochner integral coincide if existent.

Definition 3.2.1. (**Weak Integral/Pettis' integral**) Let $n \in \mathbb{N}^*$ and $(X, \| \ \|)$ be a Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We define for every X-valued map f which is a.e. defined on $\mathbb{R}^n$:

(i) f is *weakly measurable* if $\omega \circ f$ is measurable for all $\omega \in L(X, \mathbb{K})$,
(ii) f is *weakly summable* if $\omega \circ f$ is summable for every $\omega \in L(X, \mathbb{K})$ and if there is $\xi \in X$ such that

$$\omega(\xi) = \int_{\mathbb{R}^n} \omega \circ f \, dv^n$$

for every $\omega \in L(X, \mathbb{K})$. Such ξ, if existent, is unique since $L(X, \mathbb{K})$ separates points on X.[1] For this reason, we define in that case the *weak* (or *Pettis*) *integral* of f by

$$\int_{\mathbb{R}^n} f \, dv^n := \xi \ .$$

Theorem 3.2.2. (**Existence of the weak integral for reflexive Banach spaces**) Let $n \in \mathbb{N}^*$, $(X, \| \ \|)$ be a reflexive Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and f a X-valued map which is a.e. defined on $\mathbb{R}^n$. Then f is weakly summable if and only if $\omega \circ f$ is summable for every $\omega \in L(X, \mathbb{K})$.

Proof. If f is weakly summable, by definition, $\omega \circ f$ is summable for every $\omega \in L(X, \mathbb{K})$. If, on the other hand, $\omega \circ f$ is summable for every $\omega \in L(X, \mathbb{K})$, we define $A : L(X, \mathbb{K}) \to L^1_{\mathbb{K}}(\mathbb{R}^n)$ by

$$A\omega := \omega \circ f$$

for every $\omega \in L(X, \mathbb{K})$. Obviously, A is linear. A is in addition closed. For this, let $\omega \in L(X, \mathbb{K})$, $\omega_1, \omega_2, \dots$ be a sequence in $L(X, \mathbb{K})$ such that $\omega_1 \circ f, \omega_2 \circ f, \dots$ is convergent to some $g \in L(X, \mathbb{K})$. Then a subsequence of $\omega_1 \circ f, \omega_2 \circ f, \dots$ is converging a.e. pointwise on $\mathbb{R}^n$ to g. Hence $\omega \circ f$ is a.e. equal to g on $\mathbb{R}^n$ and therefore $A\omega = g$. Hence $A \in L(L(X, \mathbb{K}), L^1_{\mathbb{K}}(\mathbb{R}^n))$ by the closed graph theorem, Theorem 3.1.3 (v). As a consequence, $I_A : L(X, \mathbb{K}) \to \mathbb{K}$, defined by

$$I_A(\omega) := \int_{\mathbb{R}^n} \omega \circ f \, dv^n \ ,$$

is an element of $L(L(X, \mathbb{K}), \mathbb{K})$. Since X is reflexive, it follows the existence of $\xi \in X$ such that $I_A(\omega) = \omega(\xi)$ for all $\omega \in L(X, \mathbb{K})$ and therefore, finally, the weak summability of f. □

[1] See, e.g., [186] Theorem 3.4.

Remark 3.2.3. For an example of an actual calculation of a weak integral, compare the proof of Lemma 10.2.1 (v).

Theorem 3.2.4. **(Elementary properties of the weak integral)** Let $n \in \mathbb{N}^*$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $(X, \| \|_X)$, $(Y, \| \|_Y)$ be Banach spaces over $\mathbb{K}$, f, g be X-valued maps which are a.e. defined on $\mathbb{R}^n$ and weakly summable, $\lambda \in \mathbb{K}$ and $T \in L(X, Y)$.

(i) If f is weakly integrable and g is a.e. equal to f, then g is weakly integrable and

$$\int_{\mathbb{R}^n} g \, dv^n = \int_{\mathbb{R}^n} f \, dv^n \; .$$

(ii) Then $f + g$, λf and $T \circ f$ are weakly integrable and

$$\int_{\mathbb{R}^n} f + g \; dv^n = \int_{\mathbb{R}^n} f \, dv^n + \int_{\mathbb{R}^n} g \, dv^n \quad , \quad \int_{\mathbb{R}^n} \lambda f \, dv^n = \lambda \int_{\mathbb{R}^n} f \, dv^n \quad ,$$

$$\int_{\mathbb{R}^n} T \circ f \, dv^n = T \int_{\mathbb{R}^n} f \, dv^n \; .$$

(iii) For every $f \in L^1_{\mathbb{K}}(\mathbb{R}^n)$ and every $\xi \in X$:

$$\int_{\mathbb{R}^n} f . \xi \, dv^n = \left(\int_{\mathbb{R}^n} f \, dv^n \right) . \xi$$

where $f . \xi$ is defined by $(f . \xi)(x) := f(x) . \xi$ for all x in the domain of f.

Proof. '(i)': Obvious.
'(ii)': For every $\omega \in L(X, \mathbb{K})$, $\omega \circ (f + g) = \omega \circ f + \omega \circ g$, $\omega \circ (\lambda f) = \lambda \, \omega \circ f$ is summable and

$$\int_{\mathbb{R}^n} \omega \circ (f + g) \, dv^n = \omega \left(\int_{\mathbb{R}^n} f \, dv^n + \int_{\mathbb{R}^n} g \, dv^n \right) \; ,$$

$$\int_{\mathbb{R}^n} \omega \circ (\lambda f) \, dv^n = \omega \left(\lambda \int_{\mathbb{R}^n} f \, dv^n \right) \; .$$

Further, it follows for every $\omega \in L(Y, \mathbb{K})$ that $\omega \circ T \in L(X, \mathbb{K})$ and hence the summability of $\omega \circ (T \circ f) = (\omega \circ T) \circ f$ and

$$\int_{\mathbb{R}^n} \omega \circ (T \circ f) \, dv^n = (\omega \circ T) \left(\int_{\mathbb{R}^n} f \, dv^n \right) = \omega \left(T \int_{\mathbb{R}^n} f \, dv^n \right) \; .$$

'(iii)': For this, let $f \in L^1_{\mathbb{K}}(\mathbb{R}^n)$ and $\xi \in X$. Then it follows for every $\omega \in L(X, \mathbb{K})$ that $\omega \circ (f . \xi) = \omega(\xi) \, . \, f$ is summable and that

$$\int_{\mathbb{R}^n} \omega \circ (f . \xi) \, dv^n = \omega \left(\left(\int_{\mathbb{R}^n} f \, dv^n \right) . \xi \right) .$$

□

Theorem 3.2.5. (**Existence of the weak integral**) Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $(X, \| \ \|)$ be a $\mathbb{K}$-Banach space, $n \in \mathbb{N}^*$, Ω a non-empty open subset of $\mathbb{R}^n$ and $f : \Omega \to X$ almost everywhere continuous.

(i) There is a sequence $(s_\nu)_{\nu\in\mathbb{N}}$ of step functions such that $\text{supp}(s_\nu) \subset \Omega, \text{Ran}(s_\nu) \subset \text{Ran}(f) \cup \{0_X\}$ for all $\nu \in \mathbb{N}$ and for almost all $x \in \mathbb{R}^n$

$$\lim_{\nu\to\infty} s_\nu(x) = \hat{f}(x)$$

where $\hat{f} : \mathbb{R}^n \to X$ is defined by $\hat{f}(x) := f(x)$ for all $x \in \Omega$ and $\hat{f}(x) := 0_X$ for all $x \in \mathbb{R}^n \backslash \Omega$. (As a consequence, $\hat{f}$ is '*strongly measurable*'.)

(ii) $\hat{f}$ is *essentially separably-valued*, i.e., there is a zero set $M \subset \mathbb{R}^n$ along with an at most countable subset D of X such that $\hat{f}(\mathbb{R}^n \backslash M) \subset \overline{D}$.

(iii) If $\|\hat{f}(x)\| \leqslant h(x)$ for almost all $x \in \mathbb{R}^n$ and some a.e. on $\mathbb{R}^n$ defined summable function h, then $\hat{f}$ is weakly-summable, $\|\hat{f}\|$ is summable and

$$\left\| \int_{\mathbb{R}^n} \hat{f} \, dv^n \right\| \leqslant \int_{\mathbb{R}^n} \|\hat{f}\| \, dv^n \ . \tag{3.2.1}$$

Proof. '(i)': For this, we define for every $\nu \in \mathbb{N}^*$, $k \in \mathbb{Z}^n$ the interval I^ν_k of side length $1/\nu$ by

$$I^\nu_k := \left[\frac{k_1}{\nu}, \frac{k_1+1}{\nu}\right) \times \cdots \times \left[\frac{k_n}{\nu}, \frac{k_n+1}{\nu}\right) \ .$$

The family $\left(I^\nu_k\right)_{k\in\mathbb{Z}^n}$ gives a decomposition of $\mathbb{R}^n$ into pairwise disjoint bounded intervals of length $1/\nu$. We define for every $\nu \in \mathbb{N}^*$ a corresponding step function $s_\nu : \mathbb{R}^n \to X$ by

$$s_\nu(x) := f(x^\nu_k) \ , \ x \in I^\nu_k$$

for all $I^\nu_k \subset U_\nu(0) \cap \Omega$ where x^ν_k is some chosen element of I^ν_k. For all other $x \in \mathbb{R}^n$, we define $s_\nu(x) := 0_X$. Note that $\text{Ran}(s_\nu) \subset \text{Ran}(f) \cup \{0_X\}$. Then it follows for every point $x \in \Omega$ of continuity of f that $\lim_{\nu\to\infty} s_\nu(x) = f(x)$: Since f is continuous in x and Ω is open, for given $\varepsilon > 0$, there is $\delta > 0$ such that $U_\delta(x) \subset \Omega$ and at the same time such that $f(y) \in U_\varepsilon(f(x))$ for all $y \in U_\delta(x)$. Hence for $\nu > \max\{|x| + \delta, \sqrt{n}/\delta\}$ it follows that $x \in U_\nu(0) \cap \Omega$,

$$x \in I^\nu_{([\nu x_1],\ldots,[\nu x_n])} \subset B_{\sqrt{n}/\nu}(x) \subset U_\delta(x) \subset U_\nu(0) \cap \Omega$$

where $[\,] : \mathbb{R} \to \mathbb{Z}$ is the floor function defined by $[y] := \max\{k \in \mathbb{Z} : k \leqslant y\}$, and hence $\|s_\nu(x) - f(x)\| = \|f(x^\nu_k) - f(x)\| < \varepsilon$ where $k := ([\nu x_1], \ldots, [\nu x_n])$. Finally, for $x \notin \Omega$, it follows that $\lim_{\nu\to\infty} s_\nu(x) = 0_X$ because $s_\nu(x) = 0_X$ for all $\nu \in \mathbb{N}^*$.
'(ii)': Let M consist of those $x \in \mathbb{R}^n$ for which $(s_\nu(x))_{\nu\in\mathbb{N}^*}$ fails to converge to $\hat{f}(x)$. By (i) M is a zero set. In addition, let D be the union of the ranges of all s_ν, $\nu \in \mathbb{N}^*$. Then D is at most countable, and $f(\mathbb{R}^n \backslash M)$ is contained in the closure of D.
'(iii)': For this, let h be as described in (iii) and $(s_\nu)_{\nu\in\mathbb{N}^*}$ be as defined defined in the proof of (i). Then it follows that $\|\hat{f}\|$ is measurable since a.e. on $\mathbb{R}^n$ pointwise limit of a sequence of measurable functions and hence also summable since a.e. on $\mathbb{R}^n$

majorized by the summable function h. In the following, let $\varepsilon > 0$. Then we define for every $\nu \in \mathbb{N}^*$ the step function

$$t_\nu(x) := \begin{cases} s_\nu(x) & \text{if } \|s_\nu(x)\| \leqslant (1+\varepsilon)\,\|\hat{f}(x)\| \\ 0 & \text{if } \|s_\nu(x)\| > (1+\varepsilon)\,\|\hat{f}(x)\| \end{cases}$$

for every $x \in \mathbb{R}^n$. Then also

$$\lim_{\nu\to\infty} t_\nu(x) = \hat{f}(x) \ ,$$

for almost all $x \in \mathbb{R}^n$. Further, $\|t_\nu - \hat{f}\|$ is Lebesgue summable for every $\nu \in \mathbb{N}^*$. To prove this, we notice that for any $\mu \in \mathbb{N}^*$ the corresponding function $\|t_\nu - t_\mu\|$ is a step function, and that $(\|t_\nu - t_\mu\|)_{\mu\in\mathbb{N}^*}$ converges almost everywhere on $\mathbb{R}^n$ pointwise to $\|t_\nu - \hat{f}\|$. Hence $\|t_\nu - \hat{f}\|$ is measurable. In addition, $(2+\varepsilon)h$ is a summable majorant for $\|t_\nu - \hat{f}\|$ and hence $\|t_\nu - \hat{f}\|$ is also summable. Further, $(\|t_\nu - \hat{f}\|)_{\nu\in\mathbb{N}^*}$ is almost everywhere on $\mathbb{R}^n$ convergent to 0 and is majorized by the summable function $(2+\varepsilon)h$. Hence it follows by Lebesgue's dominated convergence theorem that

$$\lim_{\nu\to\infty} \int_{\mathbb{R}^n} \|t_\nu - \hat{f}\|\,dv^n = 0 \ . \tag{3.2.2}$$

In addition, it follows for $\mu, \nu \in \mathbb{N}^*$ that

$$\begin{aligned} \left\| \int_{\mathbb{R}^n} t_\mu\,dv^n - \int_{\mathbb{R}^n} t_\nu\,dv^n \right\| &= \left\| \int_{\mathbb{R}^n} (t_\mu - t_\nu)\,dv^n \right\| \leqslant \int_{\mathbb{R}^n} \|t_\mu - t_\nu\|\,dv^n \\ &\leqslant \int_{\mathbb{R}^n} \|t_\mu - \hat{f}\|\,dv^n + \int_{\mathbb{R}^n} \|t_\nu - \hat{f}\|\,dv^n \end{aligned}$$

and hence by (3.2.2) and the completeness of X that

$$\lim_{\nu\to\infty} \int_{\mathbb{R}^n} t_\nu\,dv^n = \xi$$

for some $\xi \in X$. Note in particular that

$$\left\| \int_{\mathbb{R}^n} t_\nu\,dv^n \right\| \leqslant \int_{\mathbb{R}^n} \|t_\nu\|\,dv^n \leqslant (1+\varepsilon) \int_{\mathbb{R}^n} \|\hat{f}\|\,dv^n$$

and hence that

$$\|\xi\| \leqslant (1+\varepsilon) \int_{\mathbb{R}^n} \|\hat{f}\|\,dv^n \ . \tag{3.2.3}$$

Further, it follows by Lebesgue's dominated convergence theorem for every $\omega \in L(X, \mathbb{C})$

$$\int_{\mathbb{R}^n} \omega \circ \hat{f}\,dv^n = \lim_{\nu\to\infty} \int_{\mathbb{R}^n} \omega \circ t_\nu\,dv^n = \omega\left(\lim_{\nu\to\infty} \int_{\mathbb{R}^n} t_\nu\,dv^n\right) = \omega(\xi) \ .$$

Hence $\hat{f}$ is weakly-summable and

$$\int_{\mathbb{R}^n} \hat{f}\,dv^n = \xi \ .$$

Finally, (3.2.1) follows by (3.2.3). □

Remark 3.2.6. It is not difficult to see that a function f satisfying the assumptions of Theorem 3.2.5 and the additional assumption of Theorem 3.2.5 (iii) is Bochner integrable and that its Bochner integral and its weak integral coincide.

Corollary 3.2.7. (**Fubini's theorem for a class of weakly integrable functions**) Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $(X, \| \,\|)$ a $\mathbb{K}$-Banach space, $m, n \in \mathbb{N}^*$, Ω be a non-empty open subset of $\mathbb{R}^{m+n}$, $f : \Omega \to X$ be almost everywhere continuous and such that $\|\hat{f}\|$ is a.e. on $\mathbb{R}^{m+n}$ majorized by a summable function h where $\hat{f} : \mathbb{R}^{m+n} \to X$ is defined by $\hat{f}(x) := f(x)$ for all $x \in \Omega$ and $\hat{f}(x) := 0_X$ for all $x \in \mathbb{R}^{m+n} \backslash \Omega$. Then there is a zero set $N_1 \subset \mathbb{R}^m$ such that

(i) $\hat{f}(x, \cdot)$ is weakly summable for all $x \in \mathbb{R}^m \backslash N_1$.
(ii)
$$\left(\mathbb{R}^m \backslash N_1 \to X, x \mapsto \int_{\mathbb{R}^n} \hat{f}(x, \cdot)\, dv^n \right)$$
is weakly summable and
$$\int_{\mathbb{R}^{m+n}} \hat{f}\, dv^{m+n} = \int_{\mathbb{R}^m} \left(\mathbb{R}^m \backslash N_1 \to X, x \mapsto \int_{\mathbb{R}^n} \hat{f}(x, \cdot)\, dv^n \right) dv^m \ .$$

Proof. '(i)': First, we note that by integration theory for any zero set $N \subset \mathbb{R}^{m+n}$, there is a zero set $N_1 \subset \mathbb{R}^m$ such that
$$N_x := \{ y \in \mathbb{R}^n : (x, y) \in N \}$$
is a zero set for all $x \in \mathbb{R}^m \backslash N_1$. Further, by Theorem 3.2.5 it follows the weak summability of $\hat{f}$ and the summability of $\|\hat{f}\|$. Also, according to the proof of Theorem 3.2.5 (iii), there is a sequence $(s_\nu)_{\nu \in \mathbb{N}}$ of step functions on $\mathbb{R}^{m+n}$ such that $\text{supp}(s_\nu) \subset \Omega$, $\text{Ran}(s_\nu) \subset \text{Ran}(f) \cup \{0_X\}$ for all $\nu \in \mathbb{N}$,
$$\lim_{\nu \to \infty} s_\nu(x) = \hat{f}(x)$$
for almost all $x \in \mathbb{R}^{m+n}$,
$$\|s_\nu(x)\| \leqslant 2\, \|\hat{f}(x)\|$$
for all $\nu \in \mathbb{N}$, $x \in \mathbb{R}^{m+n}$ and
$$\lim_{\nu \to \infty} \int_{\mathbb{R}^{m+n}} s_\nu\, dv^{m+n} = \int_{\mathbb{R}^{m+n}} \hat{f}\, dv^{m+n} \ .$$
Hence there is a zero set $N_1 \subset \mathbb{R}^m$ such that for all $x \in \mathbb{R}^m \backslash N_1$ the corresponding sequence of step functions $(s_\nu(x, \cdot))_{\nu \in \mathbb{N}}$ satisfies
$$\lim_{\nu \to \infty} s_\nu(x, \cdot) = \hat{f}(x, \cdot)$$
almost everywhere on $\mathbb{R}^n$ and at the same time such that $\|\hat{f}(x, \cdot)\|$ is summable. In particular, it follows for such x that $\|s_\nu(x, \cdot) - \hat{f}(x, \cdot)\|$ is Lebesgue summable for

every $\nu \in \mathbb{N}^*$. To prove this, we notice that for any $\mu \in \mathbb{N}^*$ the corresponding function $\|s_\nu(x,\cdot) - s_\mu(x,\cdot)\|$ is a step function and that $(\|s_\nu(x,\cdot) - s_\mu(x,\cdot)\|)_{\mu \in \mathbb{N}^*}$ converges almost everywhere on $\mathbb{R}^n$ pointwise to $\|s_\nu(x,\cdot) - \hat{f}(x,\cdot)\|$. Hence $\|s_\nu(x,\cdot) - \hat{f}(x,\cdot)\|$ is measurable. In addition, $2\|\hat{f}(x,\cdot)\|$ is a summable majorant for $\|s_\nu(x,\cdot) - \hat{f}(x,\cdot)\|$ and hence $\|s_\nu(x,\cdot) - \hat{f}(x,\cdot)\|$ is also summable. Further, $(\|s_\nu(x,\cdot) - \hat{f}(x,\cdot)\|)_{\nu \in \mathbb{N}^*}$ is almost everywhere on $\mathbb{R}^n$ convergent to 0 and is majorized by the summable function $2\|\hat{f}(x,\cdot)\|$. Hence it follows by Lebesgue's dominated convergence theorem that

$$\lim_{\nu \to \infty} \int_{\mathbb{R}^n} \|s_\nu(x,\cdot) - \hat{f}(x,\cdot)\| \, dv^n = 0 \ .$$

Further,

$$\left\| \int_{\mathbb{R}^n} s_\mu(x,\cdot)\, dv^n - \int_{\mathbb{R}^n} s_\nu(x,\cdot)\, dv^n \right\| \leqslant \int_{\mathbb{R}^n} \|s_\mu(x,\cdot) - s_\nu(x,\cdot)\| \, dv^n$$
$$\leqslant \int_{\mathbb{R}^n} \|s_\mu(x,\cdot) - \hat{f}(x,\cdot)\| \, dv^n + \int_{\mathbb{R}^n} \|s_\nu(x,\cdot) - \hat{f}(x,\cdot)\| \, dv^n$$

for all $\mu, \nu \in \mathbb{N}$, and hence it follows by the completeness of $(X, \| \ \|)$ the existence of $\xi_{\rm x} \in X$ such that

$$\lim_{\nu \to \infty} \int_{\mathbb{R}^n} s_\nu(x,\cdot)\, dv^n = \xi_{\rm x} \ .$$

In particular

$$\|\xi_{\rm x}\| \leqslant \int_{\mathbb{R}^n} \|\hat{f}(x,\cdot)\| \, dv^n$$

since

$$\left\| \int_{\mathbb{R}^n} s_\nu(x,\cdot)\, dv^n \right\| \leqslant \int_{\mathbb{R}^n} \|s_\nu(x,\cdot)\| \, dv^n \leqslant \int_{\mathbb{R}^n} \|\hat{f}(x,\cdot)\| \, dv^n$$

for every $\nu \in \mathbb{N}$. Since

$$\omega(\xi_{\rm x}) = \lim_{\nu \to \infty} \int_{\mathbb{R}^n} \omega \circ s_\nu(x,\cdot)\, dv^n = \int_{\mathbb{R}^n} \omega \circ \hat{f}(x,\cdot)\, dv^n$$

for all $\omega \in L(X, \mathbb{K})$, it follows the weak integrability of $\hat{f}(x,\cdot)$ and

$$\lim_{\nu \to \infty} \int_{\mathbb{R}^n} s_\nu(x,\cdot)\, dv^n = \int_{\mathbb{R}^n} \hat{f}(x,\cdot)\, dv^n \ .$$

'(ii)': Further, we define for every $\nu \in \mathbb{N}$ the corresponding step function t_ν on $\mathbb{R}^m$ by

$$t_\nu(x) := \int_{\mathbb{R}^n} s_\nu(x,\cdot)\, dv^n$$

for all $x \in \mathbb{R}^m$ and $F : \mathbb{R}^m \backslash N_1 \to X$ by

$$F(x) := \int_{\mathbb{R}^n} \hat{f}(x,\cdot)\, dv^n$$

for all $x \in \mathbb{R}^m \backslash N_1$. Then

$$\lim_{\nu \to \infty} t_\nu(x) = F(x)$$

and

$$\|t_\nu(x)\| \leqslant \int_{\mathbb{R}^n} \|\hat{f}(x,\cdot)\| \, dv^n \ , \ \|F(x)\| \leqslant \int_{\mathbb{R}^n} \|\hat{f}(x,\cdot)\| \, dv^n$$

for all $x \in \mathbb{R}^m \backslash N_1$. Note that

$$\left(\mathbb{R}^m \backslash N_1 \to \mathbb{R}, x \mapsto \int_{\mathbb{R}^n} \|\hat{f}(x,\cdot)\| \, dv^n \right)$$

is summable by Fubini's theorem. Also, it follows by Fubini's theorem that

$$\int_{\mathbb{R}^m} t_\nu \, dv^m = \int_{\mathbb{R}^{m+n}} s_\nu \, dv^{m+n}$$

for every $\nu \in \mathbb{N}$ and hence that

$$\lim_{\nu \to \infty} \int_{\mathbb{R}^m} t_\nu \, dv^m = \int_{\mathbb{R}^{m+n}} \hat{f} \, dv^{m+n} \ .$$

In particular, it follows that $\|t_\nu - F\|$ is Lebesgue summable for every $\nu \in \mathbb{N}^*$. To prove this, we notice that for any $\mu \in \mathbb{N}^*$ the corresponding function $\|t_\nu - t_\mu\|$ is a step function and that $(\|t_\nu - t_\mu\|)_{\mu \in \mathbb{N}^*}$ converges almost everywhere on $\mathbb{R}^n$ pointwise to $\|t_\nu - F\|$. Hence $\|t_\nu - F\|$ is measurable. In addition,

$$\left(\mathbb{R}^m \backslash N_1 \to \mathbb{R}, x \mapsto \int_{\mathbb{R}^n} 2 \, \|\hat{f}(x,\cdot)\| \, dv^n \right) \tag{3.2.4}$$

is a summable majorant for $\|t_\nu - F\|$ and hence $\|t_\nu - F\|$ is also summable. Further, $(\|t_\nu - F\|)_{\nu \in \mathbb{N}^*}$ is almost everywhere on $\mathbb{R}^m$ convergent to 0 and is majorized by the summable function (3.2.4). Hence it follows by Lebesgue's dominated convergence theorem that

$$\lim_{\nu \to \infty} \int_{\mathbb{R}^m} \|t_\nu - F\| \, dv^m = 0 \ .$$

As a consequence,

$$\omega \left(\int_{\mathbb{R}^{m+n}} \hat{f} \, dv^{m+n} \right) = \lim_{\nu \to \infty} \int_{\mathbb{R}^m} \omega \circ t_\nu \, dv^m = \int_{\mathbb{R}^m} \omega \circ F \, dv^m$$

for all $\omega \in L(X, \mathbb{K})$. Finally, this implies the weak integrability of F and that

$$\int_{\mathbb{R}^m} F \, dv^m = \int_{\mathbb{R}^{m+n}} \hat{f} \, dv^{m+n} \ .$$

□

Theorem 3.2.8. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $(X, \| \, \|)$ a $\mathbb{K}$-Banach space and $f : [a, b] \to X$ be bounded and almost everywhere continuous. Then $F : [a, b] \to X$ defined by

$$F(x) := \int_a^x f(t) \, dt$$

for every $x \in [a, b]$ is continuous. Furthermore, if f is continuous in $x \in (a, b)$, then F is differentiable in x and

$$F'(x) = f(x) \ .$$

Proof. Obviously, by Theorem 3.2.5, it follows the weak integrability of $\chi_{[a,x]} . \hat{f}$ for all $x \in [a, b]$. Further, it follows for $x, y \in [a, b]$ that

$$\|F(y) - F(x)\| = \left\| \int_x^y f(t) \, dt \right\| \leqslant \int_x^y \|f(t)\| \, dt \leqslant M \cdot |y - x|$$

if $y \geqslant x$ as well as

$$\|F(y) - F(x)\| = \left\| \int_y^x f(t) \, dt \right\| \leqslant \int_y^x \|f(t)\| \, dt \leqslant M \cdot |y - x|$$

if $y < x$, where $M \geqslant 0$ is such that $\|f(t)\| \leqslant M$ for all $t \in [a, b]$, and hence the continuity of F. Further, let f be continuous in $x \in (a, b)$. Hence for given $\varepsilon > 0$, there is $\delta > 0$ such that

$$\|f(t) - f(x)\| < \varepsilon$$

for all $t \in [a, b]$ satisfying $|t - x| < \delta$. Now let $h \in \mathbb{R}^*$ be such that $|h| < \delta$ and small enough such that $x + h \in (a, b)$. We consider the cases $h > 0$ and $h < 0$. In the first case, it follows that

$$\begin{aligned} \left\| \frac{1}{h} . \big(F(x+h) - F(x)\big) - f(x) \right\| &= \left\| \frac{1}{h} \left[\int_a^{x+h} f(t) \, dt - \int_a^x f(t) \, dt \right] - f(x) \right\| \\ &= \left\| \frac{1}{h} \int_x^{x+h} [f(t) - f(x)] \, dt \right\| \leqslant \frac{1}{h} \int_x^{x+h} \|f(t) - f(x)\| \, dt \leqslant \varepsilon \ . \end{aligned}$$

Analogously, in the second case,

$$\begin{aligned} \left\| \frac{1}{h} . \big(F(x+h) - F(x)\big) - f(x) \right\| &= \left\| \frac{1}{h} \left[\int_a^{x+h} f(t) \, dt - \int_a^x f(t) \, dt \right] - f(x) \right\| \\ &= \left\| -\frac{1}{h} \int_{x+h}^{x} [f(t) - f(x)] \, dt \right\| \leqslant \frac{1}{|h|} \int_{x+h}^{x} \|f(t) - f(x)\| \, dt \leqslant \varepsilon \ . \end{aligned}$$

Hence it follows

$$\lim_{h \to 0, h \neq 0} \frac{1}{h} . \big(F(x+h) - F(x)\big) = f(x) \ .$$

□

Theorem 3.2.9. Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $(X, \| \, \|)$ be a $\mathbb{K}$-Banach space and $f : [a,b] \to X$ continuous where a and b are some elements of $\mathbb{R}$ such that $a < b$. Further, let $F : [a,b] \to X$ be continuous and differentiable on (a,b) such that $F'(x) = f(x)$ for all $x \in (a,b)$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a) \; . \tag{3.2.5}$$

Proof. For this, let $\omega \in L(X, \mathbb{K})$. Then $\omega \circ f$, $\omega \circ F$ are continuous, and $\omega \circ F$ is differentiable on (a,b) with derivative $\omega \circ f|_{(a,b)}$. Hence it follows by the fundamental theorem of calculus that

$$\omega\left(\int_a^b f(x)\,dx\right) = \int_a^b (\omega \circ f)(x)\,dx = (\omega \circ F)(b) - (\omega \circ F)(a) = \omega(F(b) - F(a))$$

and hence (3.2.5) since $L(X, \mathbb{K})$ separates points on X. □

Theorem 3.2.10. (**Substitution rule for weak integrals**) Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $(X, \| \, \|)$ a $\mathbb{K}$-Banach space, $n \in \mathbb{N}^*$, Ω_1, Ω_2 non-empty open subsets of $\mathbb{R}^n$, $f : \Omega_2 \to X$ almost everywhere continuous and such that $\|f\|$ is summable. Finally, let $h : \Omega_1 \to \Omega_2$ be continuously differentiable such that $h'(x) \neq 0$ for all $x \in \Omega_1$ and bijective. Then $|\det(h')|.(f \circ h)$ is weakly summable and

$$\int_{\Omega_2} f\,dv^n = \int_{\Omega_1} |\det(h')|.(f \circ h)\,dv^n \; . \tag{3.2.6}$$

Proof. First, it follows by the inverse mapping theorem that $h^{-1} : \Omega_2 \to \Omega_1$ is continuously differentiable. Hence it follows by the substitution rule for Lebesgue integrals that $h^{-1}(N_f) \subset \Omega_1$ is a zero set where $N_f \subset \Omega_2$ denotes the set of discontinuities of f. In particular, $|\det(h')|.(f \circ h)$ is a.e. continuous and

$$\| |\det(h')|.(f \circ h) \| \leqslant |\det(h')| \cdot (\|f\| \circ h) \; .$$

Since $|\det(h')| \cdot (\|f\| \circ h)$ is summable, it follows that $|\det(h')|.(f \circ h)$ is weakly summable. Further, it follows by the substitution rule for Lebesgue integrals that

$$\begin{aligned} \omega\left(\int_{\Omega_2} f\,dv^n\right) &= \int_{\Omega_2} \omega \circ f\,dv^n = \int_{\Omega_1} |\det(h')|.[(\omega \circ f) \circ h]\,dv^n \\ &= \omega\left(\int_{\Omega_1} |\det(h')|.(f \circ h)\,dv^n\right) \end{aligned}$$

for every $\omega \in L(X, \mathbb{K})$ and hence (3.2.6). □

Theorem 3.2.11. (**Integration of strongly continuous maps**) Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $(X, \| \, \|_X)$, $(Y, \| \, \|_Y)$ be $\mathbb{K}$-Banach spaces, $n \in \mathbb{N}^*$ and Ω a non-empty open subset of $\mathbb{R}^n$. Further, let $f : \Omega \to L(X,Y)$ be such that for every $\xi \in X$ the corresponding map $f\xi := (\Omega \to Y, x \mapsto f(x)\xi)$ is almost everywhere continuous and for which there is some a.e. on $\mathbb{R}^n$ defined summable function h such that

$$\|\hat{f}(x)\| \leqslant h(x)$$

for almost all $x \in \mathbb{R}^n$. Then by

$$\int_{\mathbb{R}^n} \hat{f}\, dv^n := \left(X \to Y, \xi \mapsto \int_{\mathbb{R}^n} \hat{f}\xi\, dv^n \right) , \tag{3.2.7}$$

there is defined a bounded linear operator on X satisfying

$$\left\| \int_{\mathbb{R}^n} \hat{f}\, dv^n \right\| \leqslant \|h\|_1 . \tag{3.2.8}$$

Proof. For this, let $\xi \in X$. Then $\hat{f}\xi$ is almost everywhere continuous and

$$\|\hat{f}\xi\|_{\mathrm{Y}} \leqslant \|\xi\|_{\mathrm{X}} . h .$$

Hence it follows by Theorem 3.2.5 that $\hat{f}\xi$ is weakly integrable, that $\|\hat{f}\xi\|_{\mathrm{Y}}$ is integrable as well as

$$\left\| \int_{\mathbb{R}^n} \hat{f}\xi\, dv^n \right\|_{\mathrm{Y}} \leqslant \int_{\mathbb{R}^n} \|\hat{f}\xi\|_{\mathrm{Y}}\, dv^n \leqslant \|h\|_1 \cdot \|\xi\|_{\mathrm{X}} . \tag{3.2.9}$$

Hence it follows that by (3.2.7) it is defined a map from X to Y which is linear by the linearity of the weak integral. Finally, the boundedness of that operator and (3.2.8) follows from (3.2.9). □

3.3 Exponentials of Bounded Linear Operators

This section defines the exponential function exp on $L(X, X)$ where X is a Banach space. The Theorems 3.3.1 and 4.1.1 at the beginning of the next section give a complete characterization of all semigroups which are continuous in the topology induced on $L(X, X)$ by the operator norm. For every such semigroup $T : [0, \infty) \to L(X, X)$, there is a uniquely determined $A \in L(X, X)$ such that $T(t) = \exp(tA)$ for every $t \in [0, \infty)$. Hence there is a unique extension of T to a homomorphism of $(\mathbb{R}, +)$ into $(L(X, X), \circ)$ given by $(\mathbb{R} \to L(X, X), t \mapsto \exp(tA))$. As a consequence of Theorem 3.3.1 (i), for every $\xi \in X$ the corresponding $u := (\mathbb{R} \to X, t \mapsto \exp(tA)\xi)$ satisfies $u(0) = \xi$ and

$$u'(t) = -Au(t) \tag{3.3.1}$$

for every $t \in \mathbb{R}$. Here $'$ denotes the ordinary derivative of functions with values in X. Applications of (3.3.1) with $A \in L(X, X)$ are usually restricted to finite dimensional X, i.e., to systems of ordinary differential equations of the first order. An exception to this is given in Chapter 5.2. Equations of the type (3.3.1) in infinite dimensions usually involve partial differential operators. In general, such operators induce unbounded linear operators in Banach spaces.

Theorem 3.3.1. (**Definition and properties of the exponential function**) Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $(X, \| \, \|)$ a $\mathbb{K}$-Banach space. Then we define the exponential function $\exp : L(X,X) \to L(X,X)$ by

$$\exp(A) := \sum_{k=0}^{\infty} \frac{1}{k!} \, . \, A^k$$

where $A^0 := \mathrm{id}_X$ and $A^{k+1} := A \circ A^k$ for all $k \in \mathbb{N}$. Note that this series is absolutely convergent since $\|A^k\| \leqslant \|A\|^k$ for all $k \in \mathbb{N}$.

(i) The map $u_A : \mathbb{K} \to L(X,X)$, defined by

$$u_A(t) := \exp(t.A)$$

for every $t \in \mathbb{K}$, is differentiable with derivative

$$u_A'(t) = A \circ u_A(t)$$

for all $t \in \mathbb{K}$.

(ii) For all $A, B \in L(X,X)$ satisfying $A \circ B = B \circ A$

$$\exp(A + B) = \exp(A) \circ \exp(B) \ . \tag{3.3.2}$$

(iii) For all $A \in L(X,X)$ satisfying $\|A\| \leqslant 1$, $n \in \mathbb{N}$ and $\xi \in X$,

$$\left\| \exp\big(n.(A - \mathrm{id}_X)\big)\xi - A^n \xi \right\| \leqslant \sqrt{n} \cdot \|(A - \mathrm{id}_X)\xi\| \ . \tag{3.3.3}$$

Proof. '(i)': For this, let $A \in L(X,X)$. Then it follows for $t \in \mathbb{K}$, $h \in \mathbb{K}^*$, by using the bilinearity and continuity of the composition map on $((L(X,X))^2$, that

$$\begin{aligned}
&\left\| \frac{1}{h} . \left[\exp((t+h).A) - \exp(t.A) \right] - A \circ \exp(t.A) \right\| \\
&= \left\| \sum_{k=2}^{\infty} \frac{1}{k!} \left[\frac{(t+h)^k - t^k}{h} - kt^{k-1} \right] . A^k \right\| \\
&= \lim_{n \to \infty} \left\| \sum_{k=2}^{n} \frac{1}{k!} \left[\frac{(t+h)^k - t^k}{h} - kt^{k-1} \right] . A^k \right\| \ .
\end{aligned} \tag{3.3.4}$$

Further, for any $n \in \mathbb{N}, n \geqslant 2$:

$$\left\| \sum_{k=2}^{n} \frac{1}{k!} \left[\frac{(t+h)^k - t^k}{h} - kt^{k-1} \right] . A^k \right\| \leqslant \sum_{k=2}^{n} \frac{1}{k!} \left| \frac{(t+h)^k - t^k}{h} - kt^{k-1} \right| \|A\|^k \ , \tag{3.3.5}$$

and for any $k \in \mathbb{N}, k \geqslant 2$:

$$\begin{aligned}
\left| \frac{(t+h)^k - t^k}{h} - kt^{k-1} \right| &= \left| \frac{t+h-t}{h} \cdot \left[\sum_{l=0}^{k-1} (t+h)^l \cdot t^{k-(l+1)} \right] - kt^{k-1} \right| \\
&= \left| \sum_{l=1}^{k-1} \left[(t+h)^l \cdot t^{k-(l+1)} - t^{k-1} \right] \right| = \left| \sum_{l=1}^{k-1} t^{k-(l+1)} \left[(t+h)^l - t^l \right] \right|
\end{aligned}$$

$$= \left| \sum_{l=1}^{k-1} \sum_{m=0}^{l-1} (t+h)^m \cdot t^{k-(m+2)} \right| \cdot |h| \leqslant |h| \cdot \sum_{l=1}^{k-1} \sum_{m=0}^{l-1} (\,|t| + |h|\,)^{k-2}$$
$$= \frac{|h|}{2} \cdot k(k-1) \cdot (\,|t| + |h|\,)^{k-2} \ .$$

Inserting the last into (3.3.5) gives

$$\left\| \sum_{k=2}^{n} \frac{1}{k!} \left[\frac{(t+h)^k - t^k}{h} - kt^{k-1} \right] . A^k \right\| \leqslant \frac{|h|}{2} \sum_{k=2}^{n} \frac{1}{(k-2)!} \cdot (\,|t| + |h|\,)^{k-2} \, \|A\|^k$$
$$\leqslant \frac{|h| \cdot \|A\|^2}{2} \, \exp\big((|t| + |h|) \cdot \|A\| \big) \ .$$

Finally, inserting the last into (3.3.4) gives

$$\left\| \frac{1}{h} . \left[\exp((t+h).A) - \exp(t.A) \right] - A \circ \exp(t.A) \right\|$$
$$\leqslant \frac{|h| \cdot \|A\|^2}{2} \, \exp\big((|t| + |h|) \cdot \|A\| \big)$$

and hence

$$\lim_{h \to 0, h \neq 0} \left\| \frac{1}{h} . \left[\exp((t+h).A) - \exp(t.A) \right] - A \circ \exp(t.A) \right\| = 0 \ .$$

‘(ii)’: For this, let $A, B \in L(X, X)$ be such that $A \circ B = B \circ A$ and $t \in \mathbb{K}, h \in \mathbb{K}^*$. Then

$$\Big\| \frac{1}{h} . \big(u_A(t+h) \circ u_B(t+h)$$
$$- u_A(t) \circ u_B(t) \big) - \big(u_A'(t) \circ u_B(t) + u_A(t) \circ u_B'(t) \big) \Big\|$$
$$= \Big\| \left[\frac{1}{h} . \big(u_A(t+h) - u_A(t) \big) - u_A'(t) \right] \circ u_B(t)$$
$$+ u_A(t) \circ \left[\frac{1}{h} . \big(u_B(t+h) - u_B(t) \big) - u_B'(t) \right]$$
$$+ \frac{1}{h} . \big(u_A(t+h) - u_A(t) \big) \circ \big(u_B(t+h) - u_B(t) \big) \Big\|$$
$$\leqslant \left\| \left[\frac{1}{h} . \big(u_A(t+h) - u_A(t) \big) - u_A'(t) \right] \right\| \cdot \|u_B(t)\|$$
$$+ \|u_A(t)\| \cdot \left\| \left[\frac{1}{h} . \big(u_B(t+h) - u_B(t) \big) - u_B'(t) \right] \right\|$$
$$+ \left\| \frac{1}{h} . \big(u_A(t+h) - u_A(t) \big) \right\| \cdot \left\| \big(u_B(t+h) - u_B(t) \big) \right\| \ .$$

Hence it follows by (i) the differentiability of $g_{A,B} : \mathbb{K} \to L(X, X)$ defined by $h_{A,B}(t) := u_{A+B}(t) - u_A(t) \circ u_B(t)$ for every $t \in \mathbb{K}$ and

$$\begin{aligned} h'_{A,B}(t) &= (A + B) \circ u_{A+B}(t) - A \circ u_A(t) \circ u_B(t) - u_A(t) \circ B \circ u_B(t) \\ &= (A + B) \circ u_{A+B}(t) - A \circ u_A(t) \circ u_B(t) - B \circ u_A(t) \circ u_B(t) \\ &= (A + B) \circ h_{A,B}(t) \end{aligned}$$

for all $t \in \mathbb{K}$ where the bilinearity and continuity of the composition map on $(L(X, X))^2$ has been used as well as that $A \circ B = B \circ A$ by assumption. Hence it follows by $h_{A,B}(0) = u_{A+B}(0) - u_A(0) \circ u_B(0) = 0$ along with Theorem 3.2.9, Theorem 3.2.5 that

$$\|h_{A,B}(t)\| \leqslant \|A + B\| \cdot \int_0^t \| h_{A,B}(s)\| \, ds$$

for all $t \in [0, \infty)$. As a consequence, it follows for $\varepsilon > 0$ that

$$\|h_{A,B}(t)\| < \varepsilon \, e^{t\|A+B\|} \tag{3.3.6}$$

for all $t \in [0, \infty)$. Because otherwise there is $t_0 \in (0, \infty)$ such that

$$\|h_{A,B}(t_0)\| \geqslant \varepsilon \, e^{t_0\|A+B\|}$$

and such that (3.3.6) is valid for all $t \in [0, t_0)$. Then

$$\begin{aligned} \|h_{A,B}(t_0)\| &\leqslant \|A + B\| \cdot \int_0^{t_0} \| h_{A,B}(s)\| \, ds \leqslant \|A + B\| \cdot \int_0^{t_0} \varepsilon \, e^{t\|A+B\|} \, ds \\ &= \varepsilon \cdot \left(e^{t_0\|A+B\|} - 1\right) < \varepsilon \cdot e^{t_0\|A+B\|} \ . \lightning \end{aligned}$$

From (3.3.6) it follows that $h_{A,B}(t) = 0$ for all $t \geqslant 0$ and hence (3.3.2).
'(iii)': For this, let $A \in L(X, X)$ be such that $\|A\| \leqslant 1$, $n \in \mathbb{N}$ and $\xi \in X$. Then

$$\begin{aligned} \left\| \exp\left(n.(A - \mathrm{id}_X)\right)\xi - A^n\xi \right\| &= e^{-n} \cdot \| \exp(n.A)\xi - e^n . A^n\xi\| \\ &= e^{-n} \cdot \lim_{m\to\infty} \left\| \sum_{k=0}^{m} \frac{n^k}{k!} (A^k - A^n)\xi \right\| \ . \end{aligned} \tag{3.3.7}$$

Further, it follows for $m \in \mathbb{N}$ by using the Cauchy-Schwarz inequality for the Euclidean scalar product on $\mathbb{R}^{m+1}$:

$$\begin{aligned} \left\| \sum_{k=0}^{m} \frac{n^k}{k!} (A^k - A^n)\xi \right\| &\leqslant \sum_{k=0}^{m} \frac{n^k}{k!} \left\|(A^k - A^n)\xi\right\| \leqslant \sum_{k=0}^{m} \frac{n^k}{k!} \left\|(A^{|k-n|} - \mathrm{id}_X)\xi\right\| \\ &= \sum_{k=0}^{m} \frac{n^k}{k!} \left\| \sum_{l=0}^{|k-n|-1} A^l \circ (A - \mathrm{id}_X)\xi \right\| \leqslant \|(A - \mathrm{id}_X)\xi\| \cdot \sum_{k=0}^{m} |k - n| \, \frac{n^k}{k!} \end{aligned}$$

$$
\begin{aligned}
&\leqslant \|(A-\mathrm{id}_X)\xi\| \cdot \left(\sum_{k=0}^{m}(k-n)^2\,\frac{n^k}{k!}\right)^{1/2} \cdot \left(\sum_{k=0}^{m}\frac{n^k}{k!}\right)^{1/2} &(3.3.8)\\
&\leqslant \|(A-\mathrm{id}_X)\xi\| \cdot e^{n/2} \cdot \left(\sum_{k=0}^{\infty}(k-n)^2\,\frac{n^k}{k!}\right)^{1/2}\\
&= \|(A-\mathrm{id}_X)\xi\| \cdot e^{n/2} \cdot \left(\sum_{k=0}^{\infty}\left[k(k-1)-(2n-1)k+n^2\right]\frac{n^k}{k!}\right)^{1/2}\\
&= \|(A-\mathrm{id}_X)\xi\| \cdot e^{n/2}\left(\left[n^2-(2n-1)n+n^2\right]e^n\right)^{1/2} = \sqrt{n}\,e^n\,\|(A-\mathrm{id}_X)\xi\| \ .
\end{aligned}
$$

Finally, (3.3.3) follows from (3.3.7) and (3.3.8). □