2. The Stochastic Integral in General Hilbert Spaces (w.r.t. Brownian Motion)

This chapter is a slight modification of Chap. 1 in [FK01].

We fix two separable Hilbert spaces $\big(U, \langle\ ,\ \rangle_U\big)$ and $\big(H, \langle\ ,\ \rangle\big)$. The first part of this chapter is devoted to the construction of the stochastic Itô integral

$$\int_0^t \Phi(s)\, \mathrm{d}W(s), \quad t \in [0,T],$$

where $W(t)$, $t \in [0,T]$, is a Wiener process on U and Φ is a process with values that are linear but not necessarily bounded operators from U to H.

For that we first will have to introduce the notion of the standard Wiener process in infinite dimensions. Then there will be a short section about martingales in general Hilbert spaces. These two concepts are important for the construction of the stochastic integral which will be explained in the following section.

In the second part of this chapter we will present the Itô formula and the stochastic Fubini theorem and establish basic properties of the stochastic integral, including the Burkholder–Davis–Gundy inequality.

Finally, we will describe how to transmit the definition of the stochastic integral to the case that $W(t)$, $t \in [0,T]$, is a cylindrical Wiener process. For simplicity we assume that U and H are real Hilbert spaces.

2.1. Infinite-dimensional Wiener processes

For a topological space X we denote its Borel σ-algebra by $\mathcal{B}(X)$.

Definition 2.1.1. A probability measure μ on $\big(U, \mathcal{B}(U)\big)$ is called *Gaussian* if for all $v \in U$ the bounded linear mapping

$$v' : U \to \mathbb{R}$$

defined by

$$u \mapsto \langle u, v\rangle_U, \quad u \in U,$$

has a Gaussian law, i.e. for all $v \in U$ there exist $m := m(v) \in \mathbb{R}$ and $\sigma := \sigma(v) \in [0, \infty[$ such that, if $\sigma(v) > 0$,

$$\left(\mu \circ (v')^{-1}\right)(A) = \mu(v' \in A) = \frac{1}{\sqrt{2\pi\sigma^2}} \int_A e^{-\frac{(x-m)^2}{2\sigma^2}}\, \mathrm{d}x \quad \text{for all } A \in \mathcal{B}(\mathbb{R}),$$

and, if $\sigma(v) = 0$,

$$\mu \circ (v')^{-1} = \delta_{m(v)}.$$

Theorem 2.1.2. *A measure μ on $\big(U, \mathcal{B}(U)\big)$ is Gaussian if and only if*

$$\hat{\mu}(u) := \int_U e^{i\langle u,v\rangle_U}\, \mu(\mathrm{d}v) = e^{i\langle m,u\rangle_U - \frac{1}{2}\langle Qu,u\rangle_U}, \quad u \in U,$$

where $m \in U$ and $Q \in L(U)$ is nonnegative, symmetric, with finite trace (see Definition B.0.3; here $L(U)$ denotes the set of all bounded linear operators on U).

In this case μ will be denoted by $N(m, Q)$ where m is called mean and Q is called covariance (operator). The measure μ is uniquely determined by m and Q.

Furthermore, for all $h, g \in U$

$$\int \langle x, h\rangle_U\, \mu(\mathrm{d}x) = \langle m, h\rangle_U,$$

$$\int \big(\langle x, h\rangle_U - \langle m, h\rangle_U\big)\big(\langle x, g\rangle_U - \langle m, g\rangle_U\big)\, \mu(\mathrm{d}x) = \langle Qh, g\rangle_U,$$

$$\int \|x - m\|_U^2\, \mu(\mathrm{d}x) = \operatorname{tr} Q.$$

Proof. (cf. [DPZ92]) Obviously, a probability measure with this Fourier transform is Gaussian. Now let us conversely assume that μ is Gaussian. We need the following:

Lemma 2.1.3. *Let ν be a probability measure on $(U, \mathcal{B}(U))$. Let $k \in \mathbb{N}$ be such that*

$$\int_U \big|\langle z, x\rangle_U\big|^k\, \nu(\mathrm{d}x) < \infty \quad \forall\, z \in U.$$

Then there exists a constant $C = C(k, \nu) > 0$ such that for all $h_1, \ldots, h_k \in U$

$$\int_U \big|\langle h_1, x\rangle_U \cdots \langle h_k, x\rangle_U\big|\, \nu(\mathrm{d}x) \leqslant C\, \|h_1\|_U \cdots \|h_k\|_U.$$

In particular, the symmetric k-linear form

$$U^k \ni (h_1, \ldots, h_k) \;\mapsto\; \int \langle h_1, x\rangle_U \cdots \langle h_k, x\rangle_U\, \nu(\mathrm{d}x) \;\in \mathbb{R}$$

is continuous.

Proof. For $n \in \mathbb{N}$ define

$$U_n := \left\{ z \in U \;\middle|\; \int_U \left| \langle z, x\rangle_U \right|^k \nu(\mathrm{d}x) \leqslant n \right\}.$$

By assumption

$$U = \bigcup_{n=1}^{\infty} U_n.$$

Since U is a complete metric space, by the Baire category theorem, there exists $n_0 \in \mathbb{N}$ such that U_{n_0} has non-empty interior, so there exists a ball (with centre z_0 and radius r_0) $B(z_0, r_0) \subset U_{n_0}$. Hence

$$\int_U \left| \langle z_0 + y, x\rangle_U \right|^k \nu(\mathrm{d}x) \leqslant n_0 \quad \forall\, y \in B(0, r_0),$$

therefore for all $y \in B(0, r_0)$

$$\begin{aligned}
\int_U \left| \langle y, x\rangle_U \right|^k \nu(\mathrm{d}x) &= \int_U \left| \langle z_0 + y, x\rangle_U - \langle z_0, x\rangle_U \right|^k \nu(\mathrm{d}x) \\
&\leqslant 2^{k-1} \int_U \left| \langle z_0 + y, x\rangle_U \right|^k \nu(\mathrm{d}x) + 2^{k-1} \int_U \left| \langle z_0, x\rangle_U \right|^k \nu(\mathrm{d}x) \\
&\leqslant 2^k n_0.
\end{aligned}$$

Applying this for $y := r_0 z$, $z \in U$ with $|z|_U = 1$, we obtain

$$\int_U \left| \langle z, x\rangle_U \right|^k \nu(\mathrm{d}x) \leqslant 2^k n_0 r_0^{-k}.$$

Hence, if $h_1, \ldots, h_k \in U \setminus \{0\}$, then by the generalized Hölder inequality

$$\begin{aligned}
&\int_U \left| \left\langle \frac{h_1}{|h_1|_U}, x \right\rangle_U \cdots \left\langle \frac{h_k}{|h_k|_U}, x \right\rangle_U \right| \nu(\mathrm{d}x) \\
&\leqslant \left(\int_U \left| \left\langle \frac{h_1}{|h_1|_U}, x \right\rangle_U \right|^k \nu(\mathrm{d}x) \right)^{1/k} \cdots \left(\int_U \left| \left\langle \frac{h_k}{|h_k|_U}, x \right\rangle_U \right|^k \nu(\mathrm{d}x) \right)^{1/k} \\
&\leqslant 2^k n_0 r_0^{-k},
\end{aligned}$$

and the assertion follows. $\square$

Applying Lemma 2.1.3 for $k = 1$ and $\nu := \mu$ we obtain that

$$U \ni h \;\mapsto\; \int \langle h, x\rangle_U \; \mu(\mathrm{d}x) \;\in \mathbb{R}$$

is a continuous linear map, hence there exists $m \in U$ such that

$$\int_U \langle x, h\rangle_U \; \mu(\mathrm{d}x) = \langle m, h\rangle \quad \forall\, h \in H.$$

Applying Lemma 2.1.3 for $k = 2$ and $\nu := \mu$ we obtain that

$$U^2 \ni (h_1, h_2) \mapsto \int \langle x, h_1\rangle_U \langle x, h_2\rangle_U \; \mu(\mathrm{d}x) - \langle m, h_1\rangle_U \langle m, h_2\rangle_U$$

is a continuous symmetric bilinear map, hence there exists a symmetric $Q \in L(U)$ such that this map is equal to

$$U^2 \ni (h_1, h_2) \mapsto \langle Qh_1, h_2\rangle_U.$$

Since for all $h \in U$

$$\langle Qh, h\rangle_U = \int \langle x, h\rangle_U^2 \; \mu(\mathrm{d}x) - \left(\int \langle x, h\rangle_U \; \mu(\mathrm{d}x)\right)^2 \geqslant 0,$$

Q is positive definite. It remains to prove that Q is trace class (i.e.

$$\operatorname{tr} Q := \sum_{i=1}^{\infty} \langle Qe_i, e_i\rangle_U < \infty$$

for one (hence every) orthonormal basis $\{e_i \mid i \in \mathbb{N}\}$ of U, cf. Appendix B). We may assume without loss of generality that μ has mean zero, i.e. $m = 0$ ($\in U$), since the image measure of μ under the translation $U \ni x \mapsto x - m$ is again Gaussian with mean zero and the same covariance Q. Then we have for all $h \in U$ and all $c \in (0, \infty)$

$$\begin{aligned} 1 - e^{-\frac{1}{2}\langle Qh, h\rangle_U} &= \int_U \big(1 - \cos\langle h, x\rangle_U\big) \; \mu(\mathrm{d}x) \\ &\leqslant \int_{\{|x|_U \leqslant c\}} \big(1 - \cos\langle h, x\rangle_U\big) \; \mu(\mathrm{d}x) + 2\mu\big(\{x \in U \mid |x|_U > c\}\big) \\ &\leqslant \frac{1}{2} \int_{\{|x|_U \leqslant c\}} \big|\langle h, x\rangle_U\big|^2 \; \mu(\mathrm{d}x) + 2\mu\big(\{x \in U \mid |x|_U > c\}\big) \qquad (2.1.1) \end{aligned}$$

(since $1 - \cos x \leqslant \frac{1}{2}x^2$). Defining the positive definite symmetric linear operator Q_c on U by

$$\langle Q_c h_1, h_2\rangle_U := \int_{\{|x|_U \leqslant c\}} \langle h_1, x\rangle_U \cdot \langle h_2, x\rangle_U \; \mu(\mathrm{d}x), \quad h_1, h_2 \in U,$$

we even have that Q_c is trace class because for every orthonormal basis $\{e_k \mid k \in \mathbb{N}\}$ of U we have (by monotone convergence)

$$\begin{aligned} \sum_{k=1}^{\infty} \langle Q_c e_k, e_k\rangle_U &= \int_{\{|x|_U \leqslant c\}} \sum_{k=1}^{\infty} \langle e_k, x\rangle_U^2 \; \mu(\mathrm{d}x) = \int_{\{|x|_U \leqslant c\}} |x|_U^2 \; \mu(\mathrm{d}x) \\ &\leqslant c^2 < \infty. \end{aligned}$$

Claim: There exists $c_0 \in (0, \infty)$ (large enough) so that $Q \leqslant 2 \log 4 \, Q_{c_0}$ (meaning that $\langle Qh, h \rangle_U \leqslant 2 \log 4 \langle Q_{c_0} h, h \rangle_U$ for all $h \in U$).

To prove the claim let c_0 be so big that $\mu\big(\{x \in U \mid |x|_U > c_0\}\big) \leqslant \frac{1}{8}$. Let $h \in U$ such that $\langle Q_{c_0} h, h \rangle_U \leqslant 1$. Then (2.1.1) implies

$$1 - e^{-\frac{1}{2}\langle Qh, h \rangle_U} \leqslant \frac{1}{2} + \frac{1}{4} = \frac{3}{4},$$

hence $4 \geqslant e^{\frac{1}{2}\langle Qh, h \rangle_U}$, so $\langle Qh, h \rangle_U \leqslant 2 \log 4$. If $h \in U$ is arbitrary, but $\langle Q_{c_0} h, h \rangle_U \neq 0$, then we apply what we have just proved to $h / \langle Q_{c_0} h, h \rangle_U^{\frac{1}{2}}$ and the claim follows for such h. If, however, $\langle Q_{c_0} h, h \rangle = 0$, then for all $n \in \mathbb{N}$, $\langle Q_{c_0} nh, nh \rangle_U = 0 \leqslant 1$, hence by the above $\langle Qh, h \rangle_U \leqslant n^{-2} 2 \log 4$. Therefore, $\langle Q_{c_0} h, h \rangle_U = 0$ and the claim is proved, also for such h.

Since Q_{c_0} has finite trace, so has Q by the claim and the theorem is proved, since the uniqueness part follows from the fact that the Fourier transform is one-to-one. □

The following result is then obvious.

Proposition 2.1.4. *Let X be a U-valued Gaussian random variable on a probability space $(\Omega, \mathcal{F}, P)$, i.e. there exist $m \in U$ and $Q \in L(U)$ nonnegative, symmetric, with finite trace such that $P \circ X^{-1} = N(m, Q)$.*

Then $\langle X, u \rangle_U$ is normally distributed for all $u \in U$ and the following statements hold:

- $E\big(\langle X, u \rangle_U\big) = \langle m, u \rangle_U$ *for all* $u \in U$,
- $E\big(\langle X - m, u \rangle_U \cdot \langle X - m, v \rangle_U\big) = \langle Qu, v \rangle_U$ *for all* $u, v \in U$,
- $E\big(\|X - m\|_U^2\big) = \operatorname{tr} Q$.

The following proposition will lead to a representation of a U-valued Gaussian random variable in terms of real-valued Gaussian random variables.

Proposition 2.1.5. *If $Q \in L(U)$ is nonnegative, symmetric, with finite trace then there exists an orthonormal basis e_k, $k \in \mathbb{N}$, of U such that*

$$Qe_k = \lambda_k e_k, \quad \lambda_k \geqslant 0, \ k \in \mathbb{N},$$

and 0 is the only accumulation point of the sequence $(\lambda_k)_{k \in \mathbb{N}}$.

Proof. See [RS72, Theorem VI.21; Theorem VI.16 (Hilbert–Schmidt theorem)]. □

Proposition 2.1.6 (Representation of a Gaussian random variable). *Let $m \in U$ and $Q \in L(U)$ be nonnegative, symmetric, with $\operatorname{tr} Q < \infty$. In addition, we assume that e_k, $k \in \mathbb{N}$, is an orthonormal basis of U consisting of eigenvectors of Q with corresponding eigenvalues λ_k, $k \in \mathbb{N}$, as in Proposition 2.1.5, numbered in decreasing order.*

Then a U-valued random variable X on a probability space $(\Omega, \mathcal{F}, P)$ is Gaussian with $P \circ X^{-1} = N(m, Q)$ if and only if

$$X = \sum_{k \in \mathbb{N}} \sqrt{\lambda_k} \beta_k e_k + m \quad \textit{(as objects in } L^2(\Omega, \mathcal{F}, P; U)\textit{)},$$

where β_k, $k \in \mathbb{N}$, are independent real-valued random variables with $P \circ {\beta_k}^{-1} = N(0,1)$ for all $k \in \mathbb{N}$ with $\lambda_k > 0$. The series converges in $L^2(\Omega, \mathcal{F}, P; U)$.

Proof.

1. Let X be a Gaussian random variable with mean m and covariance Q. Below we set $\langle \ , \ \rangle := \langle \ , \ \rangle_U$.

   Then $X = \sum_{k \in \mathbb{N}} \langle X, e_k \rangle e_k$ in U where $\langle X, e_k \rangle$ is normally distributed with mean $\langle m, e_k \rangle$ and variance λ_k, $k \in \mathbb{N}$, by Proposition 2.1.4. If we now define

$$\beta_k : \begin{cases} = \frac{\langle X, e_k \rangle - \langle m, e_k \rangle}{\sqrt{\lambda_k}} & \text{if } k \in \mathbb{N} \text{ with } \lambda_k > 0 \\ \equiv 0 \ \in \mathbb{R} & \text{else,} \end{cases}$$

   then we get that $P \circ \beta_k^{-1} = N(0,1)$ and $X = \sum_{k \in \mathbb{N}} \sqrt{\lambda_k} \beta_k e_k + m$. To prove the independence of β_k, $k \in \mathbb{N}$, we take an arbitrary $n \in \mathbb{N}$ and $a_k \in \mathbb{R}$, $1 \leqslant k \leqslant n$, and obtain that for $c := -\sum_{k=1,\, \lambda_k \neq 0}^{n} \frac{a_k}{\sqrt{\lambda_k}} \langle m, e_k \rangle$

$$\sum_{k=1}^{n} a_k \beta_k = \sum_{\substack{k=1, \\ \lambda_k \neq 0}}^{n} \frac{a_k}{\sqrt{\lambda_k}} \langle X, e_k \rangle + c = \left\langle X, \sum_{\substack{k=1, \\ \lambda_k \neq 0}}^{n} \frac{a_k}{\sqrt{\lambda_k}} e_k \right\rangle + c$$

   which is normally distributed since X is a Gaussian random variable. Therefore we have that β_k, $k \in \mathbb{N}$, form a Gaussian family. Hence, to get the independence, we only have to check that the covariance of β_i and β_j, $i, j \in \mathbb{N}$, $i \neq j$, with $\lambda_i \neq 0 \neq \lambda_j$, is equal to zero. But this is clear since

$$\begin{aligned} E(\beta_i \beta_j) &= \frac{1}{\sqrt{\lambda_i \lambda_j}} E\big(\langle X - m, e_i \rangle \langle X - m, e_j \rangle\big) = \frac{1}{\sqrt{\lambda_i \lambda_j}} \langle Q e_i, e_j \rangle \\ &= \frac{\lambda_i}{\sqrt{\lambda_i \lambda_j}} \langle e_i, e_j \rangle = 0 \end{aligned}$$

   for $i \neq j$.

   Besides, the series $\sum_{k=1}^{n} \sqrt{\lambda_k} \beta_k e_k$, $n \in \mathbb{N}$, converges in $L^2(\Omega, \mathcal{F}, P; U)$ since the space is complete and

$$E\left(\Big\| \sum_{k=m}^{n} \sqrt{\lambda_k} \beta_k e_k \Big\|^2\right) = \sum_{k=m}^{n} \lambda_k E\big(|\beta_k|^2\big) = \sum_{k=m}^{n} \lambda_k.$$

   Since $\sum_{k \in \mathbb{N}} \lambda_k = \operatorname{tr} Q < \infty$ this expression becomes arbitrarily small for m and n large enough.

2. Let e_k, $k \in \mathbb{N}$, be an orthonormal basis of U such that $Qe_k = \lambda_k e_k$, $k \in \mathbb{N}$, and let β_k, $k \in \mathbb{N}$, be a family of independent real-valued Gaussian random variables with mean 0 and variance 1. Then it is clear that the series $\sum_{k=1}^n \sqrt{\lambda_k}\beta_k e_k + m$, $n \in \mathbb{N}$, converges to $X := \sum_{k\in\mathbb{N}} \sqrt{\lambda_k}\beta_k e_k + m$ in $L^2(\Omega, \mathcal{F}, P; U)$ (see part 1). Now we fix $u \in U$ and get that

$$\Big\langle \sum_{k=1}^n \sqrt{\lambda_k}\beta_k e_k + m, u \Big\rangle = \sum_{k=1}^n \sqrt{\lambda_k}\beta_k \langle e_k, u\rangle + \langle m, u\rangle$$

is normally distributed for all $n \in \mathbb{N}$ and the sequence converges in $L^2(\Omega, \mathcal{F}, P)$. This implies that the limit $\langle X, u\rangle$ is also normally distributed where

$$\begin{aligned} E\big(\langle X, u\rangle\big) &= E\Big(\sum_{k\in\mathbb{N}} \sqrt{\lambda_k}\beta_k \langle e_k, u\rangle + \langle m, u\rangle\Big) \\ &= \lim_{n\to\infty} E\Big(\sum_{k=1}^n \sqrt{\lambda_k}\beta_k \langle e_k, u\rangle\Big) + \langle m, u\rangle = \langle m, u\rangle \end{aligned}$$

and concerning the covariance we obtain that

$$\begin{aligned} &E\Big(\big(\langle X, u\rangle - \langle m, u\rangle\big)\big(\langle X, v\rangle - \langle m, v\rangle\big)\Big) \\ &= \lim_{n\to\infty} E\Big(\sum_{k=1}^n \sqrt{\lambda_k}\beta_k \langle e_k, u\rangle \sum_{k=1}^n \sqrt{\lambda_k}\beta_k \langle e_k, v\rangle\Big) \\ &= \sum_{k\in\mathbb{N}} \lambda_k \langle e_k, u\rangle\langle e_k, v\rangle = \sum_{k\in\mathbb{N}} \langle Qe_k, u\rangle\langle e_k, v\rangle \\ &= \sum_{k\in\mathbb{N}} \langle e_k, Qu\rangle\langle e_k, v\rangle = \langle Qu, v\rangle \end{aligned}$$

for all $u, v \in U$.

□

By part 2 of this proof we finally get the following existence result.

Corollary 2.1.7. *Let Q be a nonnegative and symmetric operator in $L(U)$ with finite trace and let $m \in U$. Then there exists a Gaussian measure $\mu = N(m, Q)$ on $\big(U, \mathcal{B}(U)\big)$.*

Let us give an alternative, more direct proof of Corollary 2.1.7 without using Proposition 2.1.6. For the proof we need the following exercise.

Exercise 2.1.8. *Consider $\mathbb{R}^\infty$ with the product topology. Let $\mathcal{B}(\mathbb{R}^\infty)$ denote its Borel σ-algebra. Prove:*

(i) $\mathcal{B}(\mathbb{R}^\infty) = \sigma(\pi_k \mid k \in \mathbb{N})$, where $\pi_k : \mathbb{R}^\infty \to \mathbb{R}$ denotes the projection on the k-th coordinate.

(ii) $l^2(\mathbb{R}) \ \Big(:= \Big\{(x_k)_{k\in\mathbb{N}} \in \mathbb{R}^\infty \ \Big| \ \sum_{k=1}^{\infty} x_k^2 < \infty\Big\}\Big) \quad \in \mathcal{B}(\mathbb{R}^\infty)$.

(iii) $\mathcal{B}(\mathbb{R}^\infty) \cap l^2(\mathbb{R}) = \sigma\big(\pi_k\,|_{l^2} \ \big|\ k \in \mathbb{N}\big)$.

(iv) Let $l^2(\mathbb{R})$ *be equipped with its natural norm*

$$\|x\|_{l^2} := \Big(\sum_{k=1}^{\infty} x_k^2\Big)^{\frac{1}{2}}, \quad x = (x_k)_{k\in\mathbb{N}} \in l^2(\mathbb{R}),$$

and let $\mathcal{B}\big(l^2(\mathbb{R})\big)$ *be the corresponding Borel* σ*-algebra. Then:*

$$\mathcal{B}\big(l^2(\mathbb{R})\big) = \mathcal{B}(\mathbb{R}^\infty) \cap l^2(\mathbb{R}).$$

Alternative Proof of Corollary 2.1.7. It suffices to construct $N(0,Q)$, since $N(m,Q)$ is the image measure of $N(0,Q)$ under translation with m. For $k \in \mathbb{N}$ consider the normal distribution $N(0,\lambda_k)$ on $\mathbb{R}$ and let ν be their product measure on $\big(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty)\big)$, i.e.

$$\nu = \prod_{k\in\mathbb{N}} N(0,\lambda_k) \quad \text{on } \big(\mathbb{R}^\infty, \mathcal{B}(\mathbb{R}^\infty)\big).$$

Here λ_k, $k \in \mathbb{N}$, are as in Proposition 2.1.5. Since the map $g : \mathbb{R}^\infty \to [0,\infty]$ defined by

$$g(x) := \sum_{k=1}^{\infty} x_k^2\,, \quad x = (x_k)_{k\in\mathbb{N}} \in \mathbb{R}^\infty,$$

is $\mathcal{B}(\mathbb{R}^\infty)$-measurable, we may calculate

$$\int_{\mathbb{R}^\infty} g(x)\,\nu(\mathrm{d}x) = \sum_{k=1}^{\infty} \int x_k^2\, N(0,\lambda_k)(\mathrm{d}x_k) = \sum_{k=1}^{\infty} \lambda_k < \infty.$$

Therefore, using Exercise 2.1.8(ii), we obtain $\nu\big(l^2(\mathbb{R})\big) = 1$. Restricting ν to $\mathcal{B}(\mathbb{R}^\infty) \cap l^2(\mathbb{R})$, by Exercise 2.1.8(iv) we get a probability measure, let us call it $\tilde{\mu}$, on $\big(l^2(\mathbb{R}), \mathcal{B}\big(l^2(\mathbb{R})\big)\big)$. Now take the orthonormal basis $\{e_k \mid k \in \mathbb{N}\}$ from Proposition 2.1.5 and consider the corresponding canonical isomorphism $I : l^2(\mathbb{R}) \to U$ defined by

$$I(x) = \sum_{k=1}^{\infty} x_k e_k, \quad x = (x_k)_{k\in\mathbb{N}} \in l^2(\mathbb{R}).$$

It is then easy to check that the image measure

$$\mu := \tilde{\mu} \circ I^{-1} \quad \text{on } \big(U, \mathcal{B}(U)\big)$$

is the desired measure, i.e. $\mu = N(0,Q)$. □

After these preparations we will give the definition of the standard Q-Wiener process. To this end we fix an element $Q \in L(U)$, nonnegative, symmetric and with finite trace and a positive real number T.

Definition 2.1.9. A U-valued stochastic process $W(t)$, $t \in [0,T]$, on a probability space $(\Omega, \mathcal{F}, P)$ is called a (standard) Q-Wiener process if:

- $W(0) = 0$,
- W has P-a.s. continuous trajectories,
- the increments of W are independent, i.e. the random variables

  $$W(t_1),\, W(t_2) - W(t_1), \dots, W(t_n) - W(t_{n-1})$$

  are independent for all $0 \leqslant t_1 < \cdots < t_n \leqslant T$, $n \in \mathbb{N}$,
- the increments have the following Gaussian laws:

  $$P \circ \big(W(t) - W(s)\big)^{-1} = N\big(0, (t-s)Q\big) \quad \text{for all } 0 \leqslant s \leqslant t \leqslant T.$$

Similarly to the existence of Gaussian measures the existence of a Q-Wiener process in U can be reduced to the real-valued case. This is the content of the following proposition.

Proposition 2.1.10 (Representation of the Q-Wiener process). *Let e_k, $k \in \mathbb{N}$, be an orthonormal basis of U consisting of eigenvectors of Q with corresponding eigenvalues λ_k, $k \in \mathbb{N}$. Then a U-valued stochastic process $W(t)$, $t \in [0,T]$, is a Q-Wiener process if and only if*

$$W(t) = \sum_{k \in \mathbb{N}} \sqrt{\lambda_k} \beta_k(t) e_k, \quad t \in [0,T], \tag{2.1.2}$$

where β_k, $k \in \{n \in \mathbb{N} \mid \lambda_n > 0\}$, are independent real-valued Brownian motions on a probability space $(\Omega, \mathcal{F}, P)$. The series even converges in $L^2\big(\Omega, \mathcal{F}, P; C([0,T], U)\big)$, and thus always has a P-a.s. continuous modification. (Here the space $C\big([0,T], U\big)$ is equipped with the sup *norm.) In particular, for any Q as above there exists a Q-Wiener process on U.*

Proof.

1. Let $W(t)$, $t \in [0,T]$, be a Q-Wiener process in U.

   Since $P \circ W(t)^{-1} = N(0, tQ)$, we see as in the proof of Proposition 2.1.6 that

   $$W(t) = \sum_{k \in \mathbb{N}} \sqrt{\lambda_k} \beta_k(t) e_k, \quad t \in [0,T],$$

   with

   $$\beta_k(t) : \begin{cases} = \frac{\langle W(t), e_k \rangle}{\sqrt{\lambda_k}} & \text{if } k \in \mathbb{N} \text{ with } \lambda_k > 0 \\ \equiv 0 & \text{else,} \end{cases}$$

for all $t \in [0,T]$. Furthermore, $P \circ \beta_k^{-1}(t) = N(0,t)$, $k \in \mathbb{N}$, and $\beta_k(t)$, $k \in \mathbb{N}$, are independent for each $t \in [0,T]$.

Now we fix $k \in \mathbb{N}$. First we show that $\beta_k(t)$, $t \in [0,T]$, is a Brownian motion:

If we take an arbitrary partition $0 = t_0 \leqslant t_1 < \cdots < t_n \leqslant T$, $n \in \mathbb{N}$, of $[0,T]$ we get that

$$\beta_k(t_1),\ \beta_k(t_2) - \beta_k(t_1), \ldots,\ \beta_k(t_n) - \beta_k(t_{n-1})$$

are independent for each $k \in \mathbb{N}$ since for $1 \leqslant j \leqslant n$

$$\beta_k(t_j) - \beta_k(t_{j-1}) = \begin{cases} \frac{1}{\sqrt{\lambda_k}} \big\langle W(t_j) - W(t_{j-1}), e_k \big\rangle & \text{if } \lambda_k > 0, \\ 0 & \text{else.} \end{cases}$$

Moreover, we obtain that for the same reason $P \circ \big(\beta_k(t) - \beta_k(s)\big)^{-1} = N(0, t-s)$ for $0 \leqslant s \leqslant t \leqslant T$.

In addition,

$$t \mapsto \frac{1}{\sqrt{\lambda_k}} \big\langle W(t), e_k \big\rangle = \beta_k(t)$$

is P-a.s. continuous for all $k \in \mathbb{N}$.

Secondly, it remains to prove that β_k, $k \in \mathbb{N}$, are independent.

We take $k_1, \ldots, k_n \in \mathbb{N}$, $n \in \mathbb{N}$, $k_i \neq k_j$ if $i \neq j$ and an arbitrary partition $0 = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_m \leqslant T$, $m \in \mathbb{N}$.

Then we have to show that

$$\sigma\big(\beta_{k_1}(t_1), \ldots, \beta_{k_1}(t_m)\big), \ldots,\ \sigma\big(\beta_{k_n}(t_1), \ldots, \beta_{k_n}(t_m)\big)$$

are independent.

We will prove this by induction with respect to m:

If $m = 1$ it is clear that $\beta_{k_1}(t_1), \ldots, \beta_{k_n}(t_1)$ are independent as observed above. Thus, we now take a partition $0 = t_0 \leqslant t_1 \leqslant \ldots \leqslant t_{m+1} \leqslant T$ and assume that

$$\sigma\big(\beta_{k_1}(t_1), \ldots, \beta_{k_1}(t_m)\big), \ldots,\ \sigma\big(\beta_{k_n}(t_1), \ldots, \beta_{k_n}(t_m)\big)$$

are independent. We note that

$$\begin{aligned} &\sigma\big(\beta_{k_i}(t_1), \ldots,\ \beta_{k_i}(t_m),\ \beta_{k_i}(t_{m+1})\big) \\ &\quad = \sigma\big(\beta_{k_i}(t_1), \ldots,\ \beta_{k_i}(t_m),\ \beta_{k_i}(t_{m+1}) - \beta_{k_i}(t_m)\big), \quad 1 \leqslant i \leqslant n, \end{aligned}$$

and that

$$\beta_{k_i}(t_{m+1}) - \beta_{k_i}(t_m) = \begin{cases} \frac{1}{\sqrt{\lambda_{k_i}}} \big\langle W(t_{m+1}) - W(t_m), e_{k_i} \big\rangle_U & \text{if } \lambda_k > 0, \\ 0 & \text{else,} \end{cases}$$

$1 \leqslant i \leqslant n$, are independent since they are pairwise orthogonal in $L^2(\Omega, \mathcal{F}, P; \mathbb{R})$ and since $W(t_{m+1}) - W(t_m)$ is a Gaussian random variable. If we take $A_{i,j} \in \mathcal{B}(\mathbb{R})$, $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m+1$, then because of the independence of $\sigma\big(W(s) \mid s \leqslant t_m\big)$ and $\sigma\big(W(t_{m+1}) - W(t_m)\big)$ we get that

$$
\begin{aligned}
&P\Big(\bigcap_{i=1}^{n}\{\beta_{k_i}(t_1) \in A_{i,1}, \dots, \beta_{k_i}(t_m) \in A_{i,m}, \\
&\qquad \beta_{k_i}(t_{m+1}) - \beta_{k_i}(t_m) \in A_{i,m+1}\}\Big) \\
=&P\Big(\underbrace{\bigcap_{i=1}^{n}\bigcap_{j=1}^{m}\{\beta_{k_i}(t_j) \in A_{i,j}\}}_{\in\,\sigma(W(s)\,\mid\, s \leqslant t_m)} \cap \underbrace{\bigcap_{i=1}^{n}\{\beta_{k_i}(t_{m+1}) - \beta_{k_i}(t_m) \in A_{i,m+1}\}}_{\in\,\sigma(W(t_{m+1}) - W(t_m))}\Big) \\
=&P\Big(\bigcap_{i=1}^{n}\bigcap_{j=1}^{m}\{\beta_{k_i}(t_j) \in A_{i,j}\}\Big) \cdot P\Big(\bigcap_{i=1}^{n}\{\beta_{k_i}(t_{m+1}) - \beta_{k_i}(t_m) \in A_{i,m+1}\}\Big) \\
=&\Big(\prod_{i=1}^{n} P\Big(\bigcap_{j=1}^{m}\{\beta_{k_i}(t_j) \in A_{i,j}\}\Big)\Big) \\
&\cdot \Big(\prod_{i=1}^{n} P\{\beta_{k_i}(t_{m+1}) - \beta_{k_i}(t_m) \in A_{i,m+1}\}\Big) \\
=&\prod_{i=1}^{n} P\Big(\bigcap_{j=1}^{m}\{\beta_{k_i}(t_j) \in A_{i,j}\} \cap \{\beta_{k_i}(t_{m+1}) - \beta_{k_i}(t_m) \in A_{i,m+1}\}\Big)
\end{aligned}
$$

and therefore the assertion follows.

2. If we define
$$W(t) := \sum_{k\in\mathbb{N}} \sqrt{\lambda_k}\beta_k(t)e_k, \quad t \in [0, T],$$
where β_k, $k \in \mathbb{N}$, are independent real-valued continuous Brownian motions then it is clear that $W(t)$, $t \in [0, T]$, is well-defined in $L^2(\Omega, \mathcal{F}, P; U)$. Besides, it is obvious that the process $W(t)$, $t \in [0, T]$, starts at zero and that
$$P \circ \big(W(t) - W(s)\big)^{-1} = N\big(0, (t-s)Q\big), \quad 0 \leqslant s < t \leqslant T,$$
by Proposition 2.1.6. It is also clear that the increments are independent.

Thus it remains to show that the above series converges in $L^2\big(\Omega, \mathcal{F}, P; C([0, T], U)\big)$. To this end we set
$$W^N(t, \omega) := \sum_{k=1}^{N} \sqrt{\lambda_k}\beta_k(t, \omega)e_k$$

for all $(t, \omega) \in \Omega_T := [0,T] \times \Omega$ and $N \in \mathbb{N}$. Then W^N, $N \in \mathbb{N}$, is P-a.s. continuous and we have that for $M < N$

$$E\Big(\sup_{t\in[0,T]} \big\|W^N(t) - W^M(t)\big\|_U^2\Big) = E\Big(\sup_{t\in[0,T]} \sum_{k=M+1}^{N} \lambda_k \beta_k^2(t)\Big)$$
$$\leqslant \sum_{k=M+1}^{N} \lambda_k E\big(\sup_{t\in[0,T]} \beta_k^2(t)\big) \leqslant c \sum_{k=M+1}^{N} \lambda_k$$

where $c_i = E\big(\sup_{t\in[0,T]} \beta_1^2(t)\big) < \infty$ because of Doob's maximal inequality for real-valued submartingales. As $\sum_{k\in\mathbb{N}} \lambda_k = \operatorname{tr} Q < \infty$, the assertion follows. □

Definition 2.1.11 (Normal filtration). A filtration $\mathcal{F}_t$, $t \in [0,T]$, on a probability space $(\Omega, \mathcal{F}, P)$ is called normal if:

- $\mathcal{F}_0$ contains all elements $A \in \mathcal{F}$ with $P(A) = 0$ and
- $\mathcal{F}_t = \mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s$ for all $t \in [0,T]$.

Definition 2.1.12 (Q-Wiener process with respect to a filtration). A Q-Wiener process $W(t)$, $t \in [0,T]$, is called a Q-Wiener process with respect to a filtration $\mathcal{F}_t$, $t \in [0,T]$, if:

- $W(t)$, $t \in [0,T]$, is adapted to $\mathcal{F}_t$, $t \in [0,T]$, and
- $W(t) - W(s)$ is independent of $\mathcal{F}_s$ for all $0 \leqslant s \leqslant t \leqslant T$.

In fact it is possible to show that any U-valued Q-Wiener process $W(t)$, $t \in [0,T]$, is a Q-Wiener process with respect to a normal filtration:
We define

$$\mathcal{N} := \big\{A \in \mathcal{F} \,\big|\, P(A) = 0\big\}\,, \quad \tilde{\mathcal{F}}_t := \sigma\big(W(s) \,\big|\, s \leqslant t\big)$$
$$\text{and} \quad \tilde{\mathcal{F}}_t^0 := \sigma(\tilde{\mathcal{F}}_t \cup \mathcal{N}).$$

Then it is clear that

$$\mathcal{F}_t := \bigcap_{s>t} \tilde{\mathcal{F}}_s^0, \quad t \in [0,T], \tag{2.1.3}$$

is a normal filtration and we get:

Proposition 2.1.13. *Let $W(t)$, $t \in [0,T]$, be an arbitrary U-valued Q-Wiener process on a probability space $(\Omega, \mathcal{F}, P)$. Then it is a Q-Wiener process with respect to the normal filtration $\mathcal{F}_t$, $t \in [0,T]$, given by* (2.1.3).

Proof. It is clear that $W(t)$, $t \in [0,T]$, is adapted to $\mathcal{F}_t$, $t \in [0,T]$. Hence we only have to verify that $W(t) - W(s)$ is independent of $\mathcal{F}_s$, $0 \leqslant s < t \leqslant T$. But if we fix $0 \leqslant s < t \leqslant T$ it is clear that $W(t) - W(s)$ is independent of $\tilde{\mathcal{F}}_s$ since

$$\begin{aligned}&\sigma\big(W(t_1), W(t_2), \dots, W(t_n)\big)\\&\quad= \sigma\big(W(t_1), W(t_2) - W(t_1), \dots, W(t_n) - W(t_{n-1})\big)\end{aligned}$$

for all $0 \leqslant t_1 < t_2 < \cdots < t_n \leqslant s$. Of course, $W(t) - W(s)$ is then also independent of $\tilde{\mathcal{F}}_s^0$. To prove now that $W(t) - W(s)$ is independent of $\mathcal{F}_s$ it is enough to show that

$$P\Big(\big\{W(t) - W(s) \in A\big\} \cap B\Big) = P\big(W(t) - W(s) \in A\big) \cdot P(B)$$

for any $B \in \mathcal{F}_s$ and any closed subset $A \subset U$ as $\mathcal{E} := \{A \subset U \mid A \text{ closed}\}$ generates $\mathcal{B}(U)$ and is stable under finite intersections. But we have

$$\begin{aligned}&P\Big(\big\{W(t) - W(s) \in A\big\} \cap B\Big)\\&\quad= E\Big(1_A \circ \big(W(t) - W(s)\big) \cdot 1_B\Big)\\&\quad= \lim_{n\to\infty} E\bigg(\Big[\Big(1 - n\,\mathrm{dist}\big(W(t) - W(s), A\big)\Big) \vee 0\Big] 1_B\bigg)\\&\quad= \lim_{n\to\infty}\lim_{m\to\infty} E\bigg(\Big[\Big(1 - n\,\mathrm{dist}\big(W(t) - W(s + \tfrac{1}{m}), A\big)\Big) \vee 0\Big] 1_B\bigg)\\&\quad= \lim_{n\to\infty}\lim_{m\to\infty} E\bigg(\Big(1 - n\,\mathrm{dist}\big(W(t) - W(s + \tfrac{1}{m}), A\big)\Big) \vee 0\bigg) \cdot P(B)\\&\quad= P\big(W(t) - W(s) \in A\big) \cdot P(B),\end{aligned}$$

since $W(t) - W(s + \frac{1}{m})$ is independent of $\tilde{\mathcal{F}}^0_{s+\frac{1}{m}} \supset \mathcal{F}_s$ if m is large enough. □

2.2. Martingales in general Banach spaces

Analogously to the real-valued case it is possible to define the conditional expectation of any Bochner integrable random variable with values in an arbitrary separable Banach space $\big(E, \| \ \|\big)$. This result is formulated in the following proposition.

Proposition 2.2.1 (Existence of the conditional expectation). *Assume that E is a separable real Banach space. Let X be a Bochner integrable E-valued random variable defined on a probability space $(\Omega, \mathcal{F}, P)$ and let $\mathcal{G}$ be a σ-field contained in $\mathcal{F}$.*

Then there exists a unique, up to a set of P-probability zero, Bochner integrable E-valued random variable Z, measurable with respect to $\mathcal{G}$ such that

$$\int_A X \, \mathrm{d}P = \int_A Z \, \mathrm{d}P \quad \textit{for all } A \in \mathcal{G}. \tag{2.2.1}$$

The random variable Z is denoted by $E(X \mid \mathcal{G})$ and is called the conditional expectation *of X given $\mathcal{G}$. Furthermore,*

$$\big\|E(X \mid \mathcal{G})\big\| \leqslant E\big(\|X\| \bigm| \mathcal{G}\big).$$

Proof. (cf. [DPZ92, Proposition 1.10, p. 27]) Let us first show uniqueness.

Since E is a separable Banach space, there exist $l_n \in E^*$, $n \in \mathbb{N}$, separating the points of E. Suppose that Z_1, Z_2 are Bochner integrable, $\mathcal{G}$-measurable mappings from Ω to E such that

$$\int_A X \, \mathrm{d}P = \int_A Z_1 \, \mathrm{d}P = \int_A Z_2 \, \mathrm{d}P \quad \text{for all } A \in \mathcal{G}.$$

Then for $n \in \mathbb{N}$ by Proposition A.2.2

$$\int_A \big(l_n(Z_1) - l_n(Z_2)\big) \, \mathrm{d}P = 0 \quad \text{for all } A \in \mathcal{G}.$$

Applying this with $A := \big\{l_n(Z_1) > l_n(Z_2)\big\}$ and $A := \big\{l_n(Z_1) < l_n(Z_2)\big\}$ it follows that $l_n(Z_1) = l_n(Z_2)$ P-a.s., so

$$\Omega_0 := \bigcap_{n \in \mathbb{N}} \big\{l_n(Z_1) = l_n(Z_2)\big\}$$

has P-measure one. Since l_n, $n \in \mathbb{N}$, separate the points of E; it follows that $Z_1 = Z_2$ on Ω_0.

To show existence we first assume that X is a simple function. So, there exist $x_1, \dots, x_N \in E$ and pairwise disjoint sets $A_1, \dots, A_N \in \mathcal{F}$ such that

$$X = \sum_{k=1}^{N} x_k 1_{A_k}.$$

Define

$$Z := \sum_{k=1}^{N} x_k E(1_{A_k} \mid \mathcal{G}).$$

Then obviously Z is $\mathcal{G}$-measurable and satisfies (2.2.1). Furthermore,

$$\|Z\| \leqslant \sum_{k=1}^{N} \|x_k\| E(1_{A_k} \mid \mathcal{G}) = E\Big(\sum_{k=1}^{N} \|x_k\| 1_{A_k} \Bigm| \mathcal{G}\Big) = E\big(\|X\| \bigm| \mathcal{G}\big). \tag{2.2.2}$$

Taking expectation we get

$$E\big(\|Z\|\big) \leqslant E\big(\|X\|\big). \tag{2.2.3}$$

For general X take simple functions X_n, $n \in \mathbb{N}$, as in Lemma A.1.4 and define Z_n as above with X_n replacing X. Then by (2.2.3) for all $n, m \in \mathbb{N}$

$$E\big(\|Z_n - Z_m\|\big) \leqslant E\big(\|X_n - X_m\|\big),$$

so $Z := \lim_{n\to\infty} Z_n$ exists in $L^1(\Omega, \mathcal{F}, P; E)$. Therefore, for all $A \in \mathcal{G}$

$$\int_A X \,\mathrm{d}P = \lim_{n\to\infty} \int_A X_n \,\mathrm{d}P = \lim_{n\to\infty} \int_A Z_n \,\mathrm{d}P = \int_A Z \,\mathrm{d}P.$$

Clearly, Z can be chosen $\mathcal{G}$-measurable, since so are the Z_n. Furthermore, by (2.2.2)

$$\big\|E(X \mid \mathcal{G})\big\| = \|Z\| = \lim_{n\to\infty} \|Z_n\| \leqslant \lim_{n\to\infty} E\big(\|X_n\| \bigm| \mathcal{G}\big) = E\big(\|X\| \bigm| \mathcal{G}\big),$$

where the limits are taken in $L^1(P)$. □

Later we will need the following result:

Proposition 2.2.2. *Let $(E_1, \mathcal{E}_1)$ and $(E_2, \mathcal{E}_2)$ be two measurable spaces and $\Psi : E_1 \times E_2 \to \mathbb{R}$ a bounded measurable function. Let X_1 and X_2 be two random variables on $(\Omega, \mathcal{F}, P)$ with values in $(E_1, \mathcal{E}_1)$ and $(E_2, \mathcal{E}_2)$ respectively, and let $\mathcal{G} \subset \mathcal{F}$ be a fixed σ-field.*

Assume that X_1 is $\mathcal{G}$-measurable and X_2 is independent of $\mathcal{G}$, then

$$E\big(\Psi(X_1, X_2) \bigm| \mathcal{G}\big) = \hat{\Psi}(X_1)$$

where

$$\hat{\Psi}(x_1) = E\big(\Psi(x_1, X_2)\big), \quad x_1 \in E_1.$$

Proof. A simple exercise or see [DPZ92, Proposition 1.12, p. 29]. □

Remark 2.2.3. *The previous proposition can be easily extended to the case where the function Ψ is not necessarily bounded but nonnegative.*

Definition 2.2.4. Let $M(t)$, $t \geqslant 0$, be a stochastic process on $(\Omega, \mathcal{F}, P)$ with values in a separable Banach space E, and let $\mathcal{F}_t$, $t \geqslant 0$, be a filtration on $(\Omega, \mathcal{F}, P)$.

The process M is called an *$\mathcal{F}_t$-martingale*, if:

- $E\big(\|M(t)\|\big) < \infty$ for all $t \geqslant 0$,
- $M(t)$ is $\mathcal{F}_t$-measurable for all $t \geqslant 0$,
- $E\big(M(t) \bigm| \mathcal{F}_s\big) = M(s)$ P-a.s. for all $0 \leqslant s \leqslant t < \infty$.

Remark 2.2.5. *Let M be as above such that $E(\|M(t)\|) < \infty$ for all $t \in [0,T]$. Then M is an $\mathcal{F}_t$-martingale if and only if $l(M)$ is an $\mathcal{F}_t$-martingale for all $l \in E^*$. In particular, results like optional stopping etc. extend to E-valued martingales.*

There is the following connection to real-valued submartingales.

Proposition 2.2.6. *If $M(t)$, $t \geqslant 0$, is an E-valued $\mathcal{F}_t$-martingale and $p \in [1,\infty)$, then $\|M(t)\|^p$, $t \geqslant 0$, is a real-valued $\mathcal{F}_t$-submartingale.*

Proof. Since E is separable there exist $l_k \in E^*$, $k \in \mathbb{N}$, such that $\|z\| = \sup l_k(z)$ for all $z \in E$. Then for $s < t$

$$\begin{aligned} E\big(\|M_t\| \bigm| \mathcal{F}_s\big) &\geqslant \sup_k E\big(l_k(M_t) \bigm| \mathcal{F}_s\big) \\ &= \sup_k l_k\big(E(M_t \mid \mathcal{F}_s)\big) \\ &= \sup_k l_k(M_s) = \|M_s\|. \end{aligned}$$

This proves the assertion for $p = 1$. Then Jensen's inequality implies the assertion for all $p \in [1,\infty)$. □

Theorem 2.2.7 (Maximal inequality). *Let $p > 1$ and let E be a separable Banach space.*

If $M(t)$, $t \in [0,T]$, is a right-continuous E-valued $\mathcal{F}_t$-martingale, then

$$\begin{aligned} \Big(E\Big(\sup_{t\in[0,T]} \|M(t)\|^p\Big)\Big)^{\frac{1}{p}} &\leqslant \frac{p}{p-1} \sup_{t\in[0,T]} \Big(E\big(\|M(t)\|^p\big)\Big)^{\frac{1}{p}} \\ &= \frac{p}{p-1}\Big(E\big(\|M(T)\|^p\big)\Big)^{\frac{1}{p}}. \end{aligned}$$

Proof. The inequality is a consequence of the previous proposition and Doob's maximal inequality for real-valued submartingales. □

Remark 2.2.8. *We note that in the inequality in Theorem 2.2.7 the first norm is the standard norm on $L^p\big(\Omega, \mathcal{F}, P; C([0,T];E)\big)$, whereas the second is the standard norm on $C\big([0,T]; L^p(\Omega,\mathcal{F},P;E)\big)$. So, for right-continuous E-valued $\mathcal{F}_t$-martingales these two norms are equivalent.*

Now we fix $0 < T < \infty$ and denote by $\mathcal{M}_T^2(E)$ the space of all E-valued continuous, square integrable martingales $M(t)$, $t \in [0,T]$. This space will play an important role with regard to the definition of the stochastic integral. We will use especially the following fact.

Proposition 2.2.9. *The space $\mathcal{M}_T^2(E)$ equipped with the norm*

$$\|M\|_{\mathcal{M}_T^2} := \sup_{t\in[0,T]} \Big(E\big(\|M(t)\|^2\big)\Big)^{\frac{1}{2}} = \Big(E\big(\|M(T)\|^2\big)\Big)^{\frac{1}{2}}$$
$$\leqslant \Big(E\big(\sup_{t\in[0,T]} \|M(t)\|^2\big)\Big)^{\frac{1}{2}} \leqslant 2 \cdot E\big(\|M(T)\|^2\big)^{\frac{1}{2}}.$$

is a Banach space.

Proof. By the Riesz–Fischer theorem the space $L^2\big(\Omega, \mathcal{F}, P; C\big([0,T], E\big)\big)$ is complete. So, we only have to show that $\mathcal{M}_T^2$ is closed. But this is obvious since even $L^1(\Omega, \mathcal{F}, P; E)$-limits of martingales are martingales. □

Proposition 2.2.10. *Let $T > 0$ and $W(t)$, $t \in [0,T]$, be a U-valued Q-Wiener process with respect to a normal filtration $\mathcal{F}_t$, $t \in [0,T]$, on a probability space $(\Omega, \mathcal{F}, P)$. Then $W(t)$, $t \in [0,T]$, is a continuous square integrable $\mathcal{F}_t$-martingale, i.e. $W \in \mathcal{M}_T^2(U)$.*

Proof. The continuity is clear by definition and for each $t \in [0,T]$ we have that $E\big(\|W(t)\|_U^2\big) = t \operatorname{tr} Q < \infty$ (see Proposition 2.1.4). Hence let $0 \leqslant s \leqslant t \leqslant T$ and $A \in \mathcal{F}_s$. Then we get by Proposition A.2.2 that

$$\Big\langle \int_A W(t) - W(s) \,\mathrm{d}P, u \Big\rangle_U = \int_A \big\langle W(t) - W(s), u \big\rangle_U \,\mathrm{d}P$$
$$= P(A) \int \big\langle W(t) - W(s), u \big\rangle_U \,\mathrm{d}P = 0$$

for all $u \in U$ as $\mathcal{F}_s$ is independent of $W(t) - W(s)$ and $E\big(\langle W(t) - W(s), u\rangle_U\big) = 0$ for all $u \in U$. Therefore,

$$\int_A W(t) \,\mathrm{d}P = \int_A W(s) + \big(W(t) - W(s)\big) \,\mathrm{d}P$$
$$= \int_A W(s) \,\mathrm{d}P + \int_A W(t) - W(s) \,\mathrm{d}P$$
$$= \int_A W(s) \,\mathrm{d}P, \qquad \text{for all } A \in \mathcal{F}_s.$$

□

2.3. The definition of the stochastic integral

For the whole section we fix a positive real number T and a probability space $(\Omega, \mathcal{F}, P)$ and we define $\Omega_T := [0,T] \times \Omega$ and $P_T := \mathrm{d}x \otimes P$ where $\mathrm{d}x$ is the Lebesgue measure.

Moreover, let $Q \in L(U)$ be symmetric, nonnegative and with finite trace and we consider a Q-Wiener process $W(t)$, $t \in [0,T]$, with respect to a normal filtration $\mathcal{F}_t$, $t \in [0,T]$.

2.3.1. Scheme of the construction of the stochastic integral

Step 1: First we consider a certain class $\mathcal{E}$ of elementary $L(U,H)$-valued processes and define the mapping

$$\begin{array}{rccl} \mathrm{Int}: & \mathcal{E} & \to & \mathcal{M}_T^2(H) =: \mathcal{M}_T^2 \\ & \Phi & \mapsto & \int_0^t \Phi(s)\,\mathrm{d}W(s), \quad t \in [0,T]. \end{array}$$

Step 2: We prove that there is a certain norm on $\mathcal{E}$ such that

$$\mathrm{Int}: \mathcal{E} \to \mathcal{M}_T^2$$

is an isometry. Since $\mathcal{M}_T^2$ is a Banach space this implies that Int can be extended to the abstract completion $\bar{\mathcal{E}}$ of $\mathcal{E}$. This extension remains isometric and it is unique.

Step 3: We give an explicit representation of $\bar{\mathcal{E}}$.

Step 4: We show how the definition of the stochastic integral can be extended by localization.

2.3.2. The construction of the stochastic integral in detail

Step 1: First we define the class $\mathcal{E}$ of all elementary processes as follows.

Definition 2.3.1 (Elementary process). An $L = L(U,H)$-valued process $\Phi(t)$, $t \in [0,T]$, on $(\Omega, \mathcal{F}, P)$ with normal filtration $\mathcal{F}_t$, $t \in [0,T]$, is said to be *elementary* if there exist $0 = t_0 < \cdots < t_k = T$, $k \in \mathbb{N}$, such that

$$\Phi(t) = \sum_{m=0}^{k-1} \Phi_m 1_{]t_m, t_{m+1}]}(t), \quad t \in [0,T],$$

where:

- $\Phi_m : \Omega \to L(U,H)$ is $\mathcal{F}_{t_m}$-measurable, w.r.t. strong Borel σ-algebra on $L(U,H)$, $0 \leqslant m \leqslant k-1$,
- Φ_m takes only a finite number of values in $L(U,H)$, $1 \leqslant m \leqslant k-1$.

If we define now

$$\mathrm{Int}(\Phi)(t) := \int_0^t \Phi(s)\,\mathrm{d}W(s) := \sum_{m=0}^{k-1} \Phi_m \big(W(t_{m+1} \wedge t) - W(t_m \wedge t)\big), \quad t \in [0,T],$$

(this is obviously independent of the representation) for all $\Phi \in \mathcal{E}$, we have the following important result.

Proposition 2.3.2. *Let $\Phi \in \mathcal{E}$. Then the stochastic integral* $\int_0^t \Phi(s)\, \mathrm{d}W(s)$, $t \in [0,T]$, *defined in the previous way, is a continuous square integrable martingale with respect to* $\mathcal{F}_t$, $t \in [0,T]$, *i.e.*

$$\mathrm{Int} : \mathcal{E} \to \mathcal{M}_T^2.$$

Proof. Let $\Phi \in \mathcal{E}$ be given by

$$\Phi(t) = \sum_{m=0}^{k-1} \Phi_m 1_{]t_m, t_{m+1}]}(t), \quad t \in [0,T],$$

as in Definition 2.3.1. Then it is clear that

$$t \mapsto \int_0^t \Phi(s)\, \mathrm{d}W(s) = \sum_{m=0}^{k-1} \Phi_m \big(W(t_{m+1} \wedge t) - W(t_m \wedge t)\big)$$

is P-a.s. continuous because of the continuity of the Wiener process and the continuity of $\Phi_m(\omega) : U \to H$, $0 \leqslant m \leqslant k-1$, $\omega \in \Omega$. In addition, we get for each summand that

$$\begin{aligned} &\Big\| \Phi_m \big(W(t_{m+1} \wedge t) - W(t_m \wedge t)\big) \Big\| \\ \leqslant &\|\Phi_m\|_{L(U,H)} \big\| W(t_{m+1} \wedge t) - W(t_m \wedge t) \big\|_U . \end{aligned}$$

Since $W(t)$, $t \in [0,T]$, is square integrable this implies that $\int_0^t \Phi(s)\, \mathrm{d}W(s)$ is square integrable for each $t \in [0,T]$.

To prove the martingale property we take $0 \leqslant s \leqslant t \leqslant T$ and a set A from $\mathcal{F}_s$. If $\big\{\Phi_m(\omega) \bigm| \omega \in \Omega\big\} := \{L_1^m, \dots, L_{k_m}^m\}$ we obtain by Proposition A.2.2 and the martingale property of the Wiener process (more precisely using optional stopping) that

$$
\begin{aligned}
&\int_A \sum_{m=0}^{k-1} \Phi_m\big(W(t_{m+1}\wedge t) - W(t_m \wedge t)\big)\,\mathrm{d}P \\
&= \sum_{\substack{0\leqslant m\leqslant k-1,\\ t_{m+1}<s}} \int_A \Phi_m\big(W(t_{m+1}\wedge s) - W(t_m\wedge s)\big)\,\mathrm{d}P \\
&\quad + \sum_{\substack{0\leqslant m\leqslant k-1,\\ s\leqslant t_{m+1}}} \sum_{j=1}^{k_m} \int_{A\cap\{\Phi_m=L_j^m\}} L_j^m\big(W(t_{m+1}\wedge t) - W(t_m\wedge t)\big)\,\mathrm{d}P \\
&= \sum_{\substack{0\leqslant m\leqslant k-1,\\ t_{m+1}<s}} \int_A \Phi_m\big(W(t_{m+1}\wedge s) - W(t_m\wedge s)\big)\,\mathrm{d}P \\
&\quad + \sum_{\substack{0\leqslant m\leqslant k-1,\\ s\leqslant t_{m+1}}} \sum_{j=1}^{k_m} L_j^m \int_{\underbrace{A\cap\{\Phi_m=L_j^m\}}_{\in\mathcal{F}_{s\vee t_m}}} W(t_{m+1}\wedge t) - W(t_m\wedge t)\,\mathrm{d}P \\
&= \sum_{\substack{0\leqslant m\leqslant k-1,\\ t_{m+1}<s}} \int_A \Phi_m\big(W(t_{m+1}\wedge s) - W(t_m\wedge s)\big)\,\mathrm{d}P \\
&\quad + \sum_{\substack{0\leqslant m\leqslant k-1,\\ t_m<s\leqslant t_{m+1}}} \sum_{j=1}^{k_m} L_j^m \int_{A\cap\{\Phi_m=L_j^m\}} W(t_{m+1}\wedge s) - W(t_m\wedge s)\,\mathrm{d}P \\
&= \int_A \sum_{m=0}^{k-1} \Phi_m\big(W(t_{m+1}\wedge s) - W(t_m\wedge s)\big)\,\mathrm{d}P. \qquad \square
\end{aligned}
$$

Step 2: To verify the assertion that there is a norm on $\mathcal{E}$ such that $\mathrm{Int}\colon \mathcal{E}\to\mathcal{M}_T^2$ is an isometry, we have to introduce the following notion.

Definition 2.3.3 (Hilbert–Schmidt operator). Let e_k, $k\in\mathbb{N}$, be an orthonormal basis of U. An operator $A\in L(U,H)$ is called Hilbert-Schmidt if

$$\sum_{k\in\mathbb{N}} \langle Ae_k, Ae_k\rangle < \infty.$$

In Appendix B we take a close look at this notion. So here we only summarize the results which are important for the construction of the stochastic integral.

The definition of a Hilbert–Schmidt operator and the number

$$\|A\|_{L_2} := \Big(\sum_{k\in\mathbb{N}} \|Ae_k\|^2\Big)^{\frac{1}{2}}$$

are independent of the choice of the basis (see Remark B.0.6(i)). Moreover, the space $L_2(U,H)$ of all Hilbert–Schmidt operators from U to H equipped with the inner product

$$\langle A,B\rangle_{L_2} := \sum_{k\in\mathbb{N}} \langle Ae_k, Be_k\rangle$$

is a separable Hilbert space (see Proposition B.0.7). Later, we will use the fact that $\|A\|_{L^2(U,H)} = \|A^*\|_{L^2(H,U)}$, where A^* is the adjoint operator of A (see Remark B.0.6(i)). Furthermore, compositions of Hilbert–Schmidt with bounded linear operators are again Hilbert–Schmidt.

Besides we recall the following fact.

Proposition 2.3.4. *If $Q \in L(U)$ is nonnegative and symmetric then there exists exactly one element $Q^{\frac{1}{2}} \in L(U)$ nonnegative and symmetric such that $Q^{\frac{1}{2}} \circ Q^{\frac{1}{2}} = Q$.*

If, in addition, $\operatorname{tr} Q < \infty$ we have that $Q^{\frac{1}{2}} \in L_2(U)$ where $\|Q^{\frac{1}{2}}\|_{L_2}^2 = \operatorname{tr} Q$ and of course $L \circ Q^{\frac{1}{2}} \in L_2(U,H)$ for all $L \in L(U,H)$.

Proof. [RS72, Theorem VI.9, p. 196] □

After these preparations we simply calculate the $\mathcal{M}_T^2$-norm of

$$\int_0^t \Phi(s)\,\mathrm{d}W(s),\ t \in [0,T],$$

and get the following result.

Proposition 2.3.5. *If $\Phi = \sum_{m=0}^{k-1} \Phi_m 1_{]t_m,t_{m+1}]}$ is an elementary $L(U,H)$-valued process then*

$$\Big\|\int_0^{\cdot} \Phi(s)\,\mathrm{d}W(s)\Big\|_{\mathcal{M}_T^2}^2 = E\Big(\int_0^T \big\|\Phi(s)\circ Q^{\frac{1}{2}}\big\|_{L_2}^2\,\mathrm{d}s\Big) =: \|\Phi\|_T^2 \ \ \textit{(“Itô-isometry”)}.$$

Proof. If we set $\Delta_m := W(t_{m+1}) - W(t_m)$ then we get that

$$\begin{aligned}\Big\|\int_0^{\cdot} \Phi(s)\,\mathrm{d}W(s)\Big\|_{\mathcal{M}_T^2}^2 &= E\bigg(\Big\|\int_0^T \Phi(s)\,\mathrm{d}W(s)\Big\|_H^2\bigg) = E\bigg(\Big\|\sum_{m=0}^{k-1}\Phi_m\Delta_m\Big\|_H^2\bigg)\\ &= E\Big(\sum_{m=0}^{k-1}\|\Phi_m\Delta_m\|_H^2\Big) + 2E\Big(\sum_{0\leqslant m<n\leqslant k-1}\langle\Phi_m\Delta_m,\Phi_n\Delta_n\rangle_H\Big).\end{aligned}$$

Claim 1:

$$\begin{aligned}E\Big(\sum_{m=0}^{k-1}\|\Phi_m\Delta_m\|_H^2\Big) &= \sum_{m=0}^{k-1}(t_{m+1}-t_m)E\big(\|\Phi_m\circ Q^{\frac{1}{2}}\|_{L_2}^2\big)\\ &= \int_0^T E\Big(\big\|\Phi(s)\circ Q^{\frac{1}{2}}\big\|_{L^2}^2\Big)\,\mathrm{d}s.\end{aligned}$$

To prove this we take an orthonormal basis f_k, $k \in \mathbb{N}$, of H and get by the Parseval identity and Levi's monotone convergence theorem that

$$E\big(\|\Phi_m \Delta_m\|_H^2\big) = \sum_{l\in\mathbb{N}} E\big(\langle \Phi_m \Delta_m, f_l\rangle_H^2\big) = \sum_{l\in\mathbb{N}} E\Big(E\big(\langle \Delta_m, \Phi_m^* f_l\rangle_U^2 \mid \mathcal{F}_{t_m}\big)\Big).$$

Taking an orthonormal basis e_k, $k \in \mathbb{N}$, of U we obtain that

$$\Phi_m^* f_l = \sum_{k\in\mathbb{N}} \langle f_l, \Phi_m e_k\rangle_H e_k.$$

Since $\langle f_l, \Phi_m e_k\rangle_H$ is $\mathcal{F}_{t_m}$-measurable, this implies that $\Phi_m^* f_l$ is $\mathcal{F}_{t_m}$-measurable by Proposition A.1.3. Using the fact that $\sigma(\Delta_m)$ is independent of $\mathcal{F}_{t_m}$ we obtain by Lemma 2.2.2 that for P-a.e. $\omega \in \Omega$

$$\begin{aligned} E\big(\langle \Delta_m, \Phi_m^* f_l\rangle_U^2 \mid \mathcal{F}_{t_m}\big)(\omega) &= E\Big(\big\langle \Delta_m, \Phi_m^*(\omega) f_l\big\rangle_U^2\Big) \\ &= (t_{m+1} - t_m)\Big\langle Q\big(\Phi_m^*(\omega) f_l\big), \Phi_m^*(\omega) f_l\Big\rangle_U, \end{aligned}$$

since $E\big(\langle \Delta_m, u\rangle_U^2\big) = (t_{m+1} - t_m)\langle Qu, u\rangle_U$ for all $u \in U$. Thus, the symmetry of $Q^{\frac{1}{2}}$ finally implies that

$$\begin{aligned} E\big(\|\Phi_m \Delta_m\|_H^2\big) &= \sum_{l\in\mathbb{N}} E\Big(E\big(\langle \Delta_m, \Phi_m^* f_l\rangle_U^2 \mid \mathcal{F}_{t_m}\big)\Big) \\ &= (t_{m+1} - t_m) \sum_{l\in\mathbb{N}} E\big(\langle Q\Phi_m^* f_l, \Phi_m^* f_l\rangle_U\big) \\ &= (t_{m+1} - t_m) \sum_{l\in\mathbb{N}} E\Big(\big\|Q^{\frac{1}{2}} \Phi_m^* f_l\big\|_U^2\Big) \\ &= (t_{m+1} - t_m) E\Big(\Big\|\big(\Phi_m \circ Q^{\frac{1}{2}}\big)^*\Big\|_{L_2(H,U)}^2\Big) \\ &= (t_{m+1} - t_m) E\Big(\big\|\Phi_m \circ Q^{\frac{1}{2}}\big\|_{L_2(U,H)}^2\Big). \end{aligned}$$

Hence the first assertion is proved and it only remains to verify the following claim.

Claim 2:

$$E\big(\langle \Phi_m \Delta_m, \Phi_n \Delta_n\rangle_H\big) = 0\,, \quad 0 \leqslant m < n \leqslant k-1.$$

But this can be proved in a similar way to Claim 1:

$$\begin{aligned} E\big(\langle \Phi_m \Delta_m, \Phi_n \Delta_n\rangle_H\big) &= E\Big(E\big(\langle \Phi_n^* \Phi_m \Delta_m, \Delta_n\rangle_U \mid \mathcal{F}_{t_n}\big)\Big) \\ &= \int E\Big(\big\langle \Phi_n^*(\omega)\Phi_m(\omega)\Delta_m(\omega), \Delta_n\big\rangle_U\Big)\, P(\mathrm{d}\omega) = 0, \end{aligned}$$

since $E\big(\langle u, \Delta_n\rangle_U\big) = 0$ for all $u \in U$ (see Proposition 2.2.2). Hence the assertion follows. $\square$

Hence the right norm on $\mathcal{E}$ has been identified. But strictly speaking $\| \ \|_T$ is only a seminorm on $\mathcal{E}$. Therefore, we have to consider equivalence classes of elementary processes with respect to $\| \ \|_T$ to get a norm on $\mathcal{E}$. For simplicity we will not change the notation but stress the following fact.

Remark 2.3.6. *If two elementary processes Φ and $\tilde{\Phi}$ belong to one equivalence class with respect to $\| \ \|_T$ it does not follow that they are equal P_T-a.e. because their values only have to correspond on $Q^{\frac{1}{2}}(U)$ P_T-a.e.*

Thus we finally have shown that

$$\mathrm{Int} : \big(\mathcal{E}, \| \ \|_T\big) \to \big(\mathcal{M}_T^2, \| \ \|_{\mathcal{M}_T^2}\big)$$

is an isometric transformation. Since $\mathcal{E}$ is dense in the abstract completion $\bar{\mathcal{E}}$ of $\mathcal{E}$ with respect to $\| \ \|_T$ it is clear that there is a unique isometric extension of Int to $\bar{\mathcal{E}}$.

Step 3: To give an explicit representation of $\bar{\mathcal{E}}$ it is useful, at this moment, to introduce the subspace $U_0 := Q^{\frac{1}{2}}(U)$ with the inner product given by

$$\langle u_0, v_0\rangle_0 := \big\langle Q^{-\frac{1}{2}}u_0, Q^{-\frac{1}{2}}v_0\big\rangle_U,$$

$u_0, v_0 \in U_0$, where $Q^{-\frac{1}{2}}$ is the pseudo inverse of $Q^{\frac{1}{2}}$ in the case that Q is not one-to-one. Then we get by Proposition C.0.3(i) that $(U_0, \langle \ , \ \rangle_0)$ is again a separable Hilbert space.

The separable Hilbert space $L_2(U_0, H)$ is called L_2^0. By Proposition C.0.3(ii) we know that $Q^{\frac{1}{2}}g_k$, $k \in \mathbb{N}$, is an orthonormal basis of $\big(U_0, \langle \ , \ \rangle_0\big)$ if g_k, $k \in \mathbb{N}$, is an orthonormal basis of $\big(\operatorname{Ker} Q^{\frac{1}{2}}\big)^{\perp}$. This basis can be supplemented to a basis of U by elements of $\operatorname{Ker} Q^{\frac{1}{2}}$. Thus we obtain that

$$\|L\|_{L_2^0} = \big\|L \circ Q^{\frac{1}{2}}\big\|_{L_2} \quad \text{for each } L \in L_2^0.$$

Define $L(U,H)_0 := \big\{T\,|_{U_0} \ \big| \ T \in L(U,H)\big\}$. Since $Q^{\frac{1}{2}} \in L_2(U)$ it is clear that $L(U,H)_0 \subset L_2^0$ and that the $\| \ \|_T$-norm of $\Phi \in \mathcal{E}$ can be written in the following way:

$$\|\Phi\|_T = \left(E\bigg(\int_0^T \|\Phi(s)\|_{L_2^0}^2 \, \mathrm{d}s\bigg)\right)^{\frac{1}{2}}$$

Besides we need the following σ-field:

$$\begin{aligned}\mathcal{P}_T &:= \sigma\Big(\big\{]s,t] \times F_s \ \big| \ 0 \leqslant s < t \leqslant T,\, F_s \in \mathcal{F}_s\big\} \cup \big\{\{0\} \times F_0 \ \big| \ F_0 \in \mathcal{F}_0\big\}\Big) \\ &= \sigma\big(Y : \Omega_T \to \mathbb{R} \ \big| \ Y \text{ is left-continuous and adapted to} \\ &\qquad\qquad \mathcal{F}_t,\, t \in [0,T]\big).\end{aligned}$$

Let $\tilde{H}$ be an arbitrary separable Hilbert space. If $Y : \Omega_T \to \tilde{H}$ is $\mathcal{P}_T/\mathcal{B}(\tilde{H})$-measurable it is called ($\tilde{H}$-)predictable.

If, for example, the process Y itself is continuous and adapted to $\mathcal{F}_t$, $t \in [0,T]$, then it is predictable.

So, we are now able to characterize $\bar{\mathcal{E}}$.

Claim: There is an explicit representation of $\bar{\mathcal{E}}$ and it is given by

$$\begin{aligned}\mathcal{N}_W^2(0,T;H) &:= \big\{\Phi : [0,T] \times \Omega \to L_2^0 \;\big|\; \Phi \text{ is predictable and } \|\Phi\|_T < \infty\big\} \\ &= L^2\big([0,T] \times \Omega, \mathcal{P}_T, dt \otimes P; L_2^0\big).\end{aligned}$$

For simplicity we also write $\mathcal{N}_W^2(0,T)$ or $\mathcal{N}_W^2$ instead of $\mathcal{N}_W^2(0,T;H)$.

To prove this claim we first notice the following facts:

1. Since $L(U,H)_0 \subset L_2^0$ and since any $\Phi \in \mathcal{E}$ is L_2^0-predictable by construction we have that $\mathcal{E} \subset \mathcal{N}_W^2$.

2. Because of the completeness of L_2^0 we get by Appendix A that

$$\mathcal{N}_W^2 = L^2(\Omega_T, \mathcal{P}_T, P_T; L_2^0)$$

is also complete.

Therefore $\mathcal{N}_W^2$ is at least a candidate for a representation of $\bar{\mathcal{E}}$. Thus there only remains to show that $\mathcal{E}$ is a dense subset of $\mathcal{N}_W^2$. But this is formulated in Proposition 2.3.8 below, which can be proved with the help of the following lemma.

Lemma 2.3.7. *There is an orthonormal basis of L_2^0 consisting of elements of $L(U,H)_0$. This implies especially that $L(U,H)_0$ is a dense subset of L_2^0.*

Proof. Since Q is symmetric, nonnegative and $\operatorname{tr} Q < \infty$ we know by Lemma 2.1.5 that there exists an orthonormal basis e_k, $k \in \mathbb{N}$, of U such that $Qe_k = \lambda_k e_k$, $\lambda_k \geqslant 0$, $k \in \mathbb{N}$. In this case $Q^{\frac{1}{2}} e_k = \sqrt{\lambda_k} e_k$, $k \in \mathbb{N}$ with $\lambda_k > 0$, is an orthonormal basis of U_0 (see Proposition C.0.3(ii)).

If f_k, $k \in \mathbb{N}$, is an orthonormal basis of H then by Proposition B.0.7 we know that

$$f_j \otimes \sqrt{\lambda_k} e_k = f_j \langle \sqrt{\lambda_k} e_k, \cdot \rangle_{U_0} = \frac{1}{\lambda_k} f_j \langle e_k, \cdot \rangle_U, \quad j,k \in \mathbb{N},\ \lambda_k > 0,$$

form an orthonormal basis of L_0^2 consisting of operators in $L(U,H)$. But, of course,

$$\overline{\operatorname{span}\left(\frac{1}{\sqrt{\lambda_k}} f_j \otimes e_k \;\middle|\; j,k \in \mathbb{N} \text{ with } \lambda_k > 0\right)} = L_2^0.$$

$\square$

Proposition 2.3.8. *If Φ is a L_2^0-predictable process such that $\|\Phi\|_T < \infty$ then there exists a sequence Φ_n, $n \in \mathbb{N}$, of $L(U,H)_0$-valued elementary processes such that*

$$\|\Phi - \Phi_n\|_T \longrightarrow 0 \quad \text{as } n \to \infty.$$

Proof. **Step 1:** If $\Phi \in \mathcal{N}_W^2$ there exists a sequence of simple random variables $\Phi_n = \sum_{k=1}^{M_n} L_k^n 1_{A_k^n}$, $A_k^n \in \mathcal{P}_T$ and $L_k^n \in L_2^0$, $n \in \mathbb{N}$, such that

$$\|\Phi - \Phi_n\|_T \longrightarrow 0 \quad \text{as } n \to \infty.$$

As L_2^0 is a Hilbert space this is a simple consequence of Lemma A.1.4 and Lebesgue's dominated convergence theorem.

Thus the assertion is reduced to the case that $\Phi = L1_A$ where $L \in L_2^0$ and $A \in \mathcal{P}_T$.

Step 2: Let $A \in \mathcal{P}_T$ and $L \in L_2^0$. Then there exists a sequence L_n, $n \in \mathbb{N}$, in $L(U,H)_0$ such that

$$\|L1_A - L_n 1_A\|_T \longrightarrow 0 \quad \text{as } n \to \infty.$$

This result is obvious by Lemma 2.3.7 and thus now we only have to consider the case that $\Phi = L1_A$, $L \in L(U,H)_0$ and $A \in \mathcal{P}_T$.

Step 3: If $\Phi = L1_A$, $L \in L(U,H)_0$, $A \in \mathcal{P}_T$, then there is a sequence Φ_n, $n \in \mathbb{N}$, of elementary $L(U,H)_0$-valued processes in the sense of Definition 2.3.1 such that

$$\|L1_A - \Phi_n\|_T \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

To show this it is sufficient to prove that for any $\varepsilon > 0$ there is a finite union $\Lambda = \bigcup\limits_{n=1}^{N} A_n$ of pairwise disjoint predictable rectangles

$$A_n \in \big\{]s,t] \times F_s \;\big|\; 0 \leqslant s < t \leqslant T,\, F_s \in \mathcal{F}_s\big\} \cup \big\{\{0\} \times F_0 \;\big|\; F_0 \in \mathcal{F}_0\big\} =: \mathcal{A}$$

such that

$$P_T\big((A \setminus \Lambda) \cup (\Lambda \setminus A)\big) < \varepsilon.$$

For then we get that $\sum_{n=1}^{N} L1_{A_n}$ differs from an elementary process by a function of type $1_{\{0\}\times F_0}$ with $F_0 \in \mathcal{F}_0$, which has $\|\cdot\|_T$-norm zero and

$$\Big\| L1_A - \sum_{n=1}^{N} L1_{A_n} \Big\|_T^2 = E\bigg(\int_0^T \Big\| L\Big(1_A - \sum_{n=1}^{N} 1_{A_n}\Big) \Big\|_{L_2^0}^2 \,\mathrm{d}s \bigg) \leqslant \varepsilon \|L\|_{L_2^0}^2.$$

Hence we define

$$\mathcal{K} := \Big\{ \bigcup_{i \in I} A_i \;\Big|\; I \text{ is finite and } A_i \in \mathcal{A},\, i \in I \Big\}.$$

Then $\mathcal{K}$ is an algebra and any element in $\mathcal{K}$ can be written as a finite disjoint union of elements in $\mathcal{A}$. Now let $\mathcal{G}$ be the family of all $A \in \mathcal{P}_T$ which can be approximated by elements of $\mathcal{K}$ in the above sense. Then $\mathcal{G}$ is a Dynkin system and therefore $\mathcal{P}_T = \sigma(\mathcal{K}) = \mathcal{D}(\mathcal{K}) \subset \mathcal{G}$ as $\mathcal{K} \subset \mathcal{G}$. □

Step 4: Finally the so-called localization procedure provides the possibility to extend the definition of the stochastic integral even to the linear space

$$\mathcal{N}_W(0,T;H) := \left\{ \Phi : \Omega_T \to L_2^0 \;\middle|\; \Phi \text{ is predictable with } P\left(\int_0^T \|\Phi(s)\|_{L_2^0}^2 \, \mathrm{d}s < \infty \right) = 1 \right\}.$$

For simplicity we also write $\mathcal{N}_W(0,T)$ or $\mathcal{N}_W$ instead of $\mathcal{N}_W(0,T;H)$ and $\mathcal{N}_W$ is called the class of *stochastically integrable* processes on $[0,T]$.

The extension is done in the following way:

For $\Phi \in \mathcal{N}_W$ we define

$$\tau_n := \inf\left\{ t \in [0,T] \;\middle|\; \int_0^t \|\Phi(s)\|_{L_2^0}^2 \, \mathrm{d}s > n \right\} \wedge T. \tag{2.3.1}$$

Then by the right-continuity of the filtration $\mathcal{F}_t$, $t \in [0,T]$, we get that

$$\begin{aligned}
\{\tau_n \leqslant t\} &= \bigcap_{m \in \mathbb{N}} \left\{ \tau_n < t + \frac{1}{m} \right\} \\
&= \underbrace{\bigcap_{m \in \mathbb{N}} \bigcup_{q \in [0, t+\frac{1}{m}[\cap \mathbb{Q}} \underbrace{\left\{ \int_0^q \|\Phi(s)\|_{L_2^0}^2 \, \mathrm{d}s > n \right\}}_{\in \mathcal{F}_q \text{ by the real Fubini theorem}}}_{\in \mathcal{F}_{t+\frac{1}{m}} \text{ and decreasing in } m} \quad \in \mathcal{F}_t.
\end{aligned}$$

Therefore τ_n, $n \in \mathbb{N}$, is an increasing sequence of stopping times with respect to $\mathcal{F}_t$, $t \in [0,T]$, such that

$$E\left(\int_0^T \|1_{]0,\tau_n]}(s)\Phi(s)\|_{L_2^0}^2 \, \mathrm{d}s \right) \leqslant n < \infty.$$

In addition, the processes $1_{]0,\tau_n]}\Phi$, $n \in \mathbb{N}$, are still L_2^0-predictable since $1_{]0,\tau_n]}$ is left-continuous and $(\mathcal{F}_t)$-adapted or since

$$\begin{aligned}
]0,\tau_n] &:= \big\{ (s,\omega) \in \Omega_T \;\big|\; 0 < s \leqslant \tau_n(\omega) \big\} \\
&= \Big(\big\{ (s,\omega) \in \Omega_T \;\big|\; \tau_n(\omega) < s \leqslant T \big\} \cup \{0\} \times \Omega \Big)^c \\
&= \Big(\underbrace{\bigcup_{q \in \mathbb{Q}} \big(]q,T] \times \underbrace{\{\tau_n \leqslant q\}}_{\in \mathcal{F}_q}\big)}_{\in \mathcal{P}_T} \cup \{0\} \times \Omega \Big)^c \in \mathcal{P}_T.
\end{aligned}$$

Thus we get that the stochastic integrals

$$\int_0^t 1_{]0,\tau_n]}(s)\Phi(s) \, \mathrm{d}W(s), \quad t \in [0,T],$$

are well-defined for all $n \in \mathbb{N}$. For arbitrary $t \in [0,T]$ we set

$$\int_0^t \Phi(s)\,\mathrm{d}W(s) := \int_0^t 1_{]0,\tau_n]}(s)\Phi(s)\,\mathrm{d}W(s), \tag{2.3.2}$$

where n is an arbitrary natural number such that $\tau_n \geqslant t$. (Note that the sequence τ_n, $n \in \mathbb{N}$, even reaches T P-a.s., in the sense that for P-a.e. $\omega \in \Omega$ there exists $n(\omega) \in \mathbb{N}$ such that $\tau_n(\omega) = T$ for all $n \geqslant n(\omega)$.)

To show that this definition is consistent we have to prove that for arbitrary natural numbers $m < n$ and $t \in [0,T]$

$$\int_0^t 1_{]0,\tau_m]}(s)\Phi(s)\,\mathrm{d}W(s) = \int_0^t 1_{]0,\tau_n]}(s)\Phi(s)\,\mathrm{d}W(s) \quad P\text{-a.s.}$$

on $\{\tau_m \geqslant t\} \subset \{\tau_n \geqslant t\}$. This result follows from the following lemma, which implies that the process in (2.3.2) is a continuous H-valued local martingale.

Lemma 2.3.9. *Assume that $\Phi \in \mathcal{N}_W^2$ and that τ is an $\mathcal{F}_t$-stopping time such that $P(\tau \leqslant T) = 1$. Then there exists a P-null set $N \in \mathcal{F}$ independent of $t \in [0,T]$ such that*

$$\begin{aligned}\int_0^t 1_{]0,\tau]}(s)\Phi(s)\,\mathrm{d}W(s) &= \mathrm{Int}\big(1_{]0,\tau]}\Phi\big)(t) = \mathrm{Int}(\Phi)(\tau\wedge t)\\ &= \int_0^{\tau\wedge t}\Phi(s)\,\mathrm{d}W(s) \quad \textit{on } N^c \textit{ for all } t \in [0,T].\end{aligned}$$

Proof. Since both integrals which appear in the equation are P-a.s. continuous we only have to prove that they are equal P-a.s. at any fixed time $t \in [0,T]$.
Step 1: We first consider the case that $\Phi \in \mathcal{E}$ and that τ is a simple stopping time which means that it takes only a finite number of values.

Let $0 = t_0 < t_1 < \cdots < t_k \leqslant T$, $k \in \mathbb{N}$, and

$$\Phi = \sum_{m=0}^{k-1} \Phi_m 1_{]t_m,t_{m+1}]}$$

where $\Phi_m : \Omega \to L(U,H)$ is $\mathcal{F}_{t_m}$-measurable and only takes a finite number of values for all $0 \leqslant m \leqslant k-1$.

If τ is a simple stopping time there exists $n \in \mathbb{N}$ such that $\tau(\Omega) = \{a_0,\ldots,a_n\}$ and

$$\tau = \sum_{j=0}^{n} a_j 1_{A_j}$$

where $0 \leqslant a_j < a_{j+1} \leqslant T$ and $A_j = \{\tau = a_j\} \in \mathcal{F}_{a_j}$. In this way we get that

$1_{]\tau,T]}\Phi$ is an elementary process since

$$\begin{aligned}
1_{]\tau,T]}(s)\Phi(s) &= \sum_{m=0}^{k-1} \Phi_m 1_{]t_m,t_{m+1}]\cap]\tau,T]}(s) \\
&= \sum_{m=0}^{k-1}\sum_{j=0}^{n} 1_{A_j}\Phi_m 1_{]t_m,t_{m+1}]\cap]a_j,T]}(s) \\
&= \sum_{m=0}^{k-1}\sum_{j=0}^{n} \underbrace{1_{A_j}\Phi_m}_{\mathcal{F}_{t_m\vee a_j}\text{-measurable}} 1_{]t_m\vee a_j,t_{m+1}\vee a_j]}(s)
\end{aligned}$$

and concerning the integral we are interested in, we obtain that

$$\begin{aligned}
&\int_0^t 1_{]0,\tau]}(s)\Phi(s)\,\mathrm{d}W(s) = \int_0^t \Phi(s)\,\mathrm{d}W(s) - \int_0^t 1_{]\tau,T]}(s)\Phi(s)\,\mathrm{d}W(s) \\
&= \sum_{m=0}^{k-1} \Phi_m\big(W(t_{m+1}\wedge t) - W(t_m\wedge t)\big) \\
&\qquad - \sum_{m=0}^{k-1}\sum_{j=0}^{n} 1_{A_j}\Phi_m\Big(W\big((t_{m+1}\vee a_j)\wedge t\big) - W\big((t_m\vee a_j)\wedge t\big)\Big) \\
&= \sum_{m=0}^{k-1} \Phi_m\big(W(t_{m+1}\wedge t) - W(t_m\wedge t)\big) \\
&\qquad - \sum_{m=0}^{k-1}\sum_{j=0}^{n} 1_{A_j}\Phi_m\Big(W\big((t_{m+1}\vee \tau)\wedge t\big) - W\big((t_m\vee \tau)\wedge t\big)\Big) \\
&= \sum_{m=0}^{k-1} \Phi_m\big(W(t_{m+1}\wedge t) - W(t_m\wedge t)\big) \\
&\qquad - \sum_{m=0}^{k-1} \Phi_m\Big(W\big((t_{m+1}\vee \tau)\wedge t\big) - W\big((t_m\vee \tau)\wedge t\big)\Big) \\
&= \sum_{m=0}^{k-1} \Phi_m\Big(W(t_{m+1}\wedge t) - W(t_m\wedge t) \\
&\qquad - W\big((t_{m+1}\vee \tau)\wedge t\big) - W\big((t_m\vee \tau)\wedge t\big)\Big) \\
&= \sum_{m=0}^{k-1} \Phi_m\Big(W(t_{m+1}\wedge\tau\wedge t) - W(t_m\wedge\tau\wedge t)\Big) = \int_0^{t\wedge\tau} \Phi(s)\,\mathrm{d}W(s).
\end{aligned}$$

Step 2: Now we consider the case that Φ is still an elementary process while τ is an arbitrary stopping time with $P(\tau \leqslant T) = 1$.

Then there exists a sequence

$$\tau_n = \sum_{k=0}^{2^n-1} T(k+1)2^{-n} 1_{]Tk2^{-n},T(k+1)2^{-n}]} \circ \tau, \quad n \in \mathbb{N},$$

of simple stopping times such that $\tau_n \downarrow \tau$ as $n \to \infty$ and because of the continuity of the stochastic integral we get that

$$\int_0^{\tau_n \wedge t} \Phi(s)\, \mathrm{d}W(s) \xrightarrow{n\to\infty} \int_0^{\tau\wedge t} \Phi(s)\, \mathrm{d}W(s) \quad P\text{-a.s.}$$

Besides, we obtain (even for non-elementary processes Φ) that

$$\left\| 1_{]0,\tau_n]}\Phi - 1_{]0,\tau]}\Phi \right\|_T^2 = E\left(\int_0^T 1_{]\tau,\tau_n]}(s) \|\Phi(s)\|^2_{L_2^0}\, \mathrm{d}s \right) \xrightarrow{n\to\infty} 0,$$

which by the definition of the integral implies that

$$E\left(\left\| \int_0^t 1_{]0,\tau_n]}(s)\Phi(s)\, \mathrm{d}W(s) - \int_0^t 1_{]0,\tau]}(s)\Phi(s)\, \mathrm{d}W(s) \right\|^2 \right) \xrightarrow{n\to\infty} 0$$

for all $t \in [0,T]$. As by Step 1

$$\int_0^t 1_{]0,\tau_n]}(s)\Phi(s)\, \mathrm{d}W(s) = \int_0^{\tau_n\wedge t} \Phi(s)\, \mathrm{d}W(s), \quad n \in \mathbb{N},\ t \in [0,T],$$

the assertion follows.

Step 3: Finally we generalize the statement to arbitrary $\Phi \in \mathcal{N}_W^2(0,T)$:

If $\Phi \in \mathcal{N}_W^2(0,T)$ then there exists a sequence of elementary processes Φ_n, $n \in \mathbb{N}$, such that

$$\|\Phi_n - \Phi\|_T \xrightarrow{n\to\infty} 0\,.$$

By the definition of the stochastic integral this means that

$$\int_0^{\cdot} \Phi_n(s)\, \mathrm{d}W(s) \xrightarrow{n\to\infty} \int_0^{\cdot} \Phi(s)\, \mathrm{d}W(s) \quad \text{in } \mathcal{M}_T^2\,.$$

Hence it follows that there is a subsequence n_k, $k \in \mathbb{N}$, and a P-null set $N \in \mathcal{F}$ independent of $t \in [0,T]$ such that

$$\int_0^t \Phi_{n_k}(s)\, \mathrm{d}W(s) \xrightarrow{k\to\infty} \int_0^t \Phi(s)\, \mathrm{d}W(s) \quad \text{on } N^c$$

for all $t \in [0,T]$ and therefore we get for all $t \in [0,T]$ that

$$\int_0^{\tau\wedge t} \Phi_{n_k}(s)\, \mathrm{d}W(s) \xrightarrow{k\to\infty} \int_0^{\tau\wedge t} \Phi(s)\, \mathrm{d}W(s) \quad P\text{-a.s.}$$

In addition, it is clear that

$$\|1_{]0,\tau]}\Phi_n - 1_{]0,\tau]}\Phi\|_T \underset{n\to\infty}{\longrightarrow} 0$$

which implies that for all $t \in [0,T]$

$$E\left(\left\|\int_0^t 1_{]0,\tau]}(s)\Phi_n(s)\,\mathrm{d}W(s) - \int_0^t 1_{]0,\tau]}(s)\Phi(s)\,\mathrm{d}W(s)\right\|^2\right) \xrightarrow{n\to\infty} 0.$$

As by Step 2

$$\int_0^t 1_{]0,\tau]}(s)\Phi_{n_k}(s)\,\mathrm{d}W(s) = \int_0^{\tau\wedge t} \Phi_{n_k}(s)\,\mathrm{d}W(s) \quad P\text{-a.s.}$$

for all $k \in \mathbb{N}$ the assertion follows. □

Therefore, for $m < n$ on $\{\tau_m \geqslant t\} \subset \{\tau_n \geqslant t\}$

$$\begin{aligned}\int_0^t 1_{]0,\tau_n]}(s)\Phi(s)\,\mathrm{d}W(s) &= \int_0^{\tau_m\wedge t} 1_{]0,\tau_n]}(s)\Phi(s)\,\mathrm{d}W(s)\\ &= \int_0^t 1_{]0,\tau_m]}(s)1_{]0,\tau_n]}(s)\Phi(s)\,\mathrm{d}W(s) = \int_0^t 1_{]0,\tau_m]}(s)\Phi(s)\,\mathrm{d}W(s) \quad P\text{-a.s.},\end{aligned}$$

where we used Lemma 2.3.9 for the second equality. Hence the definition is consistent.

Remark 2.3.10. *In fact it is easy to see that the definition of the stochastic integral does not depend on the choice of τ_n, $n \in \mathbb{N}$. If σ_n, $n \in \mathbb{N}$, is another sequence of stopping times such that $\sigma_n \uparrow T$ as $n \to \infty$ and $1_{]0,\sigma_n]}\Phi \in \mathcal{N}_W^2$ for all $n \in \mathbb{N}$ we also get that*

$$\int_0^t \Phi(s)\,\mathrm{d}W(s) = \lim_{n\to\infty} \int_0^t 1_{]0,\sigma_n]}(s)\Phi(s)\,\mathrm{d}W(s) \quad P\text{-a.s. for all } t \in [0,T].$$

Proof. Let $t \in [0,T]$. Then we get that on the set $\{\tau_m \geqslant t\}$

$$\begin{aligned}\int_0^t \Phi(s)\,\mathrm{d}W(s) &= \int_0^t 1_{]0,\tau_m]}(s)\Phi(s)\,\mathrm{d}W(s)\\ &= \lim_{n\to\infty}\int_0^{t\wedge\sigma_n} 1_{]0,\tau_m]}(s)\Phi(s)\,\mathrm{d}W(s)\\ &= \lim_{n\to\infty}\int_0^{t\wedge\tau_m} 1_{]0,\sigma_n]}(s)\Phi(s)\,\mathrm{d}W(s)\\ &= \lim_{n\to\infty}\int_0^{t} 1_{]0,\sigma_n]}(s)\Phi(s)\,\mathrm{d}W(s) \quad P\text{-a.s..}\end{aligned}$$

□

2.4. Properties of the stochastic integral

Let T be a positive real number and $W(t)$, $t \in [0,T]$, a Q-Wiener process as described at the beginning of the previous section.

Lemma 2.4.1. *Let Φ be a L_2^0-valued stochastically integrable process, $(\tilde{H}, \| \, \|_{\tilde{H}})$ a further separable Hilbert space and $L \in L(H, \tilde{H})$.*

Then the process $L\big(\Phi(t)\big)$, $t \in [0,T]$, is an element of $\mathcal{N}_W(0,T;\tilde{H})$ and

$$L\Big(\int_0^T \Phi(t)\, \mathrm{d}W(t)\Big) = \int_0^T L\big(\Phi(t)\big)\, \mathrm{d}W(t) \quad P\text{-a.s.}$$

Proof. Since Φ is a stochastically integrable process and

$$\Big\|L\big(\Phi(t)\big)\Big\|_{L_2(U_0,\tilde{H})} \leqslant \|L\|_{L(H,\tilde{H})} \|\Phi(t)\|_{L_2^0},$$

it is obvious that $L\big(\Phi(t)\big)$, $t \in [0,T]$, is $L_2(U_0, \tilde{H})$-predictable and

$$P\Big(\int_0^T \Big\|L\big(\Phi(t)\big)\Big\|^2_{L_2(U_0,\tilde{H})} \mathrm{d}t < \infty\Big) = 1.$$

Step 1: As the first step we consider the case that Φ is an elementary process, i.e.

$$\Phi(t) = \sum_{m=0}^{k-1} \Phi_m 1_{]t_m,t_{m+1}]}(t), \quad t \in [0,T],$$

where $0 = t_0 < t_1 < \cdots < t_k = T$, $\Phi_m : \Omega \to L(U,H)$ $\mathcal{F}_{t_m}$-measurable with $\big|\Phi_m(\Omega)\big| < \infty$ for $0 \leqslant m \leqslant k$. Then

$$\begin{aligned} L\Big(\int_0^T \Phi(t)\, \mathrm{d}W(t)\Big) &= L\Big(\sum_{m=0}^{k-1} \Phi_m\big(W(t_{m+1}) - W(t_m)\big)\Big) \\ &= \sum_{m=0}^{k-1} L\Big(\Phi_m\big(W(t_{m+1}) - W(t_m)\big)\Big) = \int_0^T L\big(\Phi(t)\big)\, \mathrm{d}W(t). \end{aligned}$$

Step 2: Now let $\Phi \in \mathcal{N}_W^2(0,T)$. Then there exists a sequence Φ_n, $n \in \mathbb{N}$, of elementary processes with values in $L(U,H)_0$ such that

$$\|\Phi_n - \Phi\|_T = \bigg(E\Big(\int_0^T \|\Phi_n(t) - \Phi(t)\|^2_{L_2^0}\, \mathrm{d}t\Big)\bigg)^{\frac{1}{2}} \xrightarrow{n\to\infty} 0.$$

Then $L(\Phi_n)$, $n \in \mathbb{N}$, is a sequence of elementary processes with values in $L(U,\tilde{H})_0$ and

$$\big\|L(\Phi_n) - L(\Phi)\big\|_T \leqslant \|L\|_{L(H,\tilde{H})} \|\Phi_n - \Phi\|_T \xrightarrow{n\to\infty} 0.$$

By the definition of the stochastic integral, Step 1 and the continuity of L we get that there is a subsequence n_k, $k \in \mathbb{N}$, such that

$$\begin{aligned}
\int_0^T L\big(\Phi(t)\big)\,\mathrm{d}W(t) &= \lim_{k\to\infty} \int_0^T L\big(\Phi_{n_k}(t)\big)\,\mathrm{d}W(t) \\
&= \lim_{k\to\infty} L\bigg(\int_0^T \Phi_{n_k}(t)\,\mathrm{d}W(t)\bigg) = L\bigg(\lim_{k\to\infty}\int_0^T \Phi_{n_k}(t)\,\mathrm{d}W(t)\bigg) \\
&= L\bigg(\int_0^T \Phi(t)\,\mathrm{d}W(t)\bigg) \quad P\text{-a.s.}
\end{aligned}$$

Step 3: Finally let $\Phi \in \mathcal{N}_W(0,T)$.

Let τ_n, $n \in \mathbb{N}$, be a sequence of stopping times such that $\tau_n \uparrow T$ as $n \to \infty$ and $1_{]0,\tau_n]}\Phi \in \mathcal{N}_W^2(0,T,H)$. Then $1_{]0,\tau_n]}L(\Phi) \in \mathcal{N}_W^2(0,T,\tilde{H})$ for all $n \in \mathbb{N}$ and we obtain by Remark 2.3.10 and Step 2 (selecting a subsequence if necessary)

$$\begin{aligned}
\int_0^T L\big(\Phi(t)\big)\,\mathrm{d}W(t) &= \lim_{n\to\infty} \int_0^T 1_{]0,\tau_n]}(t) L\big(\Phi(t)\big)\,\mathrm{d}W(t) \\
&= \lim_{n\to\infty} L\bigg(\int_0^T 1_{]0,\tau_n]}(t)\Phi(t)\,\mathrm{d}W(t)\bigg) = L\bigg(\lim_{n\to\infty}\int_0^T 1_{]0,\tau_n]}(t)\Phi(t)\,\mathrm{d}W(t)\bigg) \\
&= L\bigg(\int_0^T \Phi(t)\,\mathrm{d}W(t)\bigg) \quad P\text{-a.s.}
\end{aligned}$$

$\square$

Lemma 2.4.2. *Let $\Phi \in \mathcal{N}_W(0,T)$ and f an $(\mathcal{F}_t)$-adapted continuous H-valued process. Set*

$$\int_0^T \big\langle f(t), \Phi(t)\,\mathrm{d}W(t)\big\rangle := \int_0^T \tilde{\Phi}_f(t)\,\mathrm{d}W(t) \tag{2.4.1}$$

with

$$\tilde{\Phi}_f(t)(u) := \big\langle f(t), \Phi(t)u\big\rangle,\ u \in U_0.$$

Then the stochastic integral in (2.4.1) is well-defined as a continuous $\mathbb{R}$-valued stochastic process. More precisely, $\tilde{\Phi}_f$ is a $\mathcal{P}_T/\mathcal{B}(L_2(U_0,\mathbb{R}))$-measurable map from $[0,T]\times\Omega$ to $L_2(U_0,\mathbb{R})$,

$$\|\tilde{\Phi}_f(t,\omega)\|_{L_2(U_0,\mathbb{R})} = \|\Phi^*(t,\omega)f(t,\omega)\|_{U_0}$$

for all $(t,\omega) \in [0,T]\times\Omega$ and

$$\int_0^T \|\tilde{\Phi}_f(t)\|^2_{L_2(U_0,\mathbb{R})}\,\mathrm{d}t \leqslant \sup_{t\in[0,T]}\|f(t)\| \int_0^T \|\Phi(t)\|^2_{L_2^0}\,\mathrm{d}t < \infty \quad P\text{-a.e.}.$$

Proof. Since f is continuous, $\tilde{\Phi}_f$ is clearly predictable. Let e_k, $k \in \mathbb{N}$, be an orthonormal basis of U_0. Then for all $(t, \omega) \in [0, T] \times \Omega$

$$\begin{aligned}
\|\tilde{\Phi}_f(t,\omega)\|^2_{L_2(U_0,\mathbb{R})} &= \sum_{k=1}^{\infty} \langle f(t,\omega), \Phi(t,\omega)e_k\rangle^2 \\
&= \sum_{k=1}^{\infty} \langle \Phi^*(t,\omega)f(t,\omega), e_k\rangle^2_{U_0} \\
&= \|\Phi^*(t,\omega)f(t,\omega)\|^2_{U_0} \\
&\leqslant \|\Phi^*(t,\omega)\|^2_{L(H,U_0)} \|f(t,\omega)\|^2_H \\
&\leqslant \|\Phi^*(t,\omega)\|^2_{L_2(H,U_0)} \|f(t,\omega)\|^2_H \\
&= \|\Phi(t,\omega)\|^2_{L_2^0} \|f(t,\omega)\|^2_H,
\end{aligned}$$

where we used Remark B.0.6(i) in the last step. Now all assertions follow. □

Lemma 2.4.3. *Let $\Phi \in \mathcal{N}_W(0,T)$ and $M(t) := \int_0^t \Phi(s)\, \mathrm{d}W(s)$, $t \in [0,T]$. Define*

$$\langle M\rangle_t := \int_0^t \|\Phi(s)\|^2_{L_2^0}\, \mathrm{d}s,\ t \in [0,T].$$

Then $\langle M\rangle$ is the unique continuous increasing $(\mathcal{F}_t)$-adapted process starting at zero such that $\|M(t)\|^2 - \langle M\rangle_t$, $t \in [0,T]$, is a local martingale. If $\Phi \in \mathcal{N}^2_W(0,T)$, then for any sequence

$$I_l := \{0 = t_0^l < t_1^l < \ldots < t_{k_l}^l = T\},\ l \in \mathbb{N},$$

of partitions with

$$\max_i (t_i^l - t_{i-1}^l) \to 0 \text{ as } l \to \infty$$

$$\lim_{l\to\infty} E\left(\left|\sum_{t_{j+1}^l \leqslant t} \|M(t_{j+1}^l) - M(t_j^l)\|^2 - \langle M\rangle_t\right|\right) = 0.$$

Proof. For $n \in \mathbb{N}$ let τ_n be as in (2.3.1) and τ an $\mathcal{F}_t$-stopping time with $P[\tau \leqslant T] = 1$. Then by Lemma 2.3.9 for $\sigma := \tau \wedge \tau_n$, $t \in [0,T]$

$$\begin{aligned}
E\left(\left\|\int_0^{t\wedge\sigma} \Phi(s)\, \mathrm{d}W(s)\right\|^2\right) &= E\left(\left\|\int_0^t 1_{]0,\sigma]}\Phi(s)\, \mathrm{d}W(s)\right\|^2\right) \\
&= E\left(\int_0^t \|1_{]0,\sigma]}\Phi(s)\|^2_{L_2^0}\, \mathrm{d}s\right) \\
&= E\left(\int_0^{t\wedge\sigma} \|\Phi(s)\|^2_{L_2^0}\, \mathrm{d}s\right),
\end{aligned}$$

and the first assertion follows, because the uniqueness is obvious, since any real-valued local martingale of bounded variation is constant.
To prove the second assertion we fix an orthonormal basis $\{e_i | i \in \mathbb{N}\}$ of H and note that by the theory of real-valued martingales we have for each $i \in \mathbb{N}$

$$\lim_{l \to \infty} E\left(\Big| \sum_{t^l_{j+1} \leqslant t} \langle e_i, M(t^l_{j+1}) - M(t^l_j) \rangle_H^2 - \int_0^t \|\Phi(s)^* e_i\|_{U_0}^2 \, \mathrm{d}s \Big| \right) = 0, \quad (2.4.2)$$

since by the first part of the assertion and Lemmas 2.4.1 and 2.4.2

$$\left\langle \int_0^t \langle e_i, \Phi(s) \, \mathrm{d}W(s) \rangle_H \right\rangle_t = \int_0^t \|\Phi(s)^* e_i\|_{U_0}^2 \, \mathrm{d}s, \ t \in [0,T].$$

Furthermore, for all $i \in \mathbb{N}$

$$\begin{aligned}
&E\left(\Big| \sum_{t^l_{j+1} \leqslant t} \langle e_i, M(t^l_{j+1}) - M(t^l_j) \rangle_H^2 - \int_0^t \|\Phi(s)^* e_i\|_{U_0}^2 \, \mathrm{d}s \Big| \right) \\
&\leqslant \sum_{t^l_{j+1} \leqslant t} E\left[\left(\int_{t^l_j}^{t^l_{j+1}} \langle e_i, \Phi(s) \, \mathrm{d}W(s) \rangle_H \right)^2 \right] + E\left(\int_0^t \|\Phi(s)^* e_i\|_{U_0}^2 \, \mathrm{d}s \right) \\
&= \sum_{t^l_{j+1} \leqslant t} E\left(\int_{t^l_j}^{t^l_{j+1}} \|\Phi(s)^* e_i\|_{U_0}^2 \, \mathrm{d}s \right) + E\left(\int_0^t \|\Phi(s)^* e_i\|_{U_0}^2 \, \mathrm{d}s \right) \\
&\leqslant 2E\left(\int_0^t \|\Phi(s)^* e_i\|_{U_0}^2 \, \mathrm{d}s \right)
\end{aligned} \qquad (2.4.3)$$

which is summable over $i \in \mathbb{N}$. Here we used the isometry property of Int in the second to last step. But

$$\begin{aligned}
&E\left(\Big| \sum_{t^l_{j+1} \leqslant t} \|M(t^l_{j+1}) - M(t^l_j)\|^2 - \int_0^t \|\Phi(s)\|_{L_2^0}^2 \, \mathrm{d}s \Big| \right) \\
&= E\left(\Big| \sum_{i=1}^{\infty} \Big(\sum_{t^l_{j+1} \leqslant t} \langle e_i, M(t^l_{j+1}) - M(t^l_j) \rangle_H^2 - \int_0^t \|\Phi(s)^* e_i\|_{U_0}^2 \, \mathrm{d}s \Big) \Big| \right) \\
&\leqslant \sum_{i=1}^{\infty} E\left(\Big| \sum_{t^l_{j+1} \leqslant t} \langle e_i, M(t^l_{j+1}) - M(t^l_j) \rangle_H^2 - \int_0^t \|\Phi(s)^* e_i\|_{U_0}^2 \, \mathrm{d}s \Big| \right)
\end{aligned}$$

where we used Remark B.0.6(i) in the second step. Hence the second assertion follows by Lebesgue dominated convergence theorem from (2.4.2) and (2.4.3).
□

2.5. The stochastic integral for cylindrical Wiener processes

Until now we have considered the case that $W(t)$, $t \in [0,T]$, was a standard Q-Wiener process where $Q \in L(U)$ was nonnegative, symmetric and with finite trace. We could integrate processes in

$$\mathcal{N}_W := \Big\{ \Phi : \Omega_T \to L_2(Q^{\frac{1}{2}}(U), H) \mid \Phi \text{ is predictable and } P\left(\int_0^T \|\Phi(s)\|^2_{L_2^0}\, ds < \infty \right) = 1 \Big\}.$$

In fact it is possible to extend the definition of the stochastic integral to the case that Q is not necessarily of finite trace. To this end we first have to introduce the concept of cylindrical Wiener processes.

2.5.1. Cylindrical Wiener processes

Let $Q \in L(U)$ be nonnegative definite and symmetric. Remember that in the case that Q is of finite trace the Q-Wiener process has the following representation:

$$W(t) = \sum_{k \in \mathbb{N}} \beta_k(t) e_k, \quad t \in [0,T],$$

where e_k, $k \in \mathbb{N}$, is an orthonormal basis of $Q^{\frac{1}{2}}(U) = U_0$ and β_k, $k \in \mathbb{N}$, is a family of independent real-valued Brownian motions. The series converges in $L^2(\Omega, \mathcal{F}, P; U)$, because the inclusion $U_0 \subset U$ defines a Hilbert–Schmidt embedding from $(U_0, \langle\ ,\ \rangle_0)$ to $(U, \langle\ ,\ \rangle)$. In the case that Q is no longer of finite trace one looses this convergence. Nevertheless, it is possible to define the Wiener process.

To this end we need a further Hilbert space $(U_1, \langle\ ,\ \rangle_1)$ and a Hilbert–Schmidt embedding

$$J : (U_0, \langle\ ,\ \rangle_0) \to (U_1, \langle\ ,\ \rangle_1).$$

Remark 2.5.1. *$(U_1, \langle\ ,\ \rangle_1))$ and J as above always exist; e.g. choose $U_1 := U$ and $\alpha_k \in]0, \infty[$, $k \in \mathbb{N}$, such that $\sum_{k=1}^\infty \alpha_k^2 < \infty$. Define $J : U_0 \to U$ by*

$$J(u) := \sum_{k=1}^\infty \alpha_k \langle u, e_k \rangle_0\, e_k, \quad u \in U_0.$$

Then J is one-to-one and Hilbert–Schmidt.

Then the process given by the following proposition is called a *cylindrical Q-Wiener process* in U.

Proposition 2.5.2. *Let e_k, $k \in \mathbb{N}$, be an orthonormal basis of $U_0 = Q^{\frac{1}{2}}(U)$ and β_k, $k \in \mathbb{N}$, a family of independent real-valued Brownian motions. Define $Q_1 := JJ^*$. Then $Q_1 \in L(U_1)$, Q_1 is nonnegative definite and symmetric with finite trace and the series*

$$W(t) = \sum_{k=1}^{\infty} \beta_k(t) J e_k, \quad t \in [0,T], \tag{2.5.1}$$

converges in $\mathcal{M}_T^2(U_1)$ and defines a Q_1-Wiener process on U_1. Moreover, we have that $Q_1^{\frac{1}{2}}(U_1) = J(U_0)$ and for all $u_0 \in U_0$

$$\|u_0\|_0 = \|Q_1^{-\frac{1}{2}} J u_0\|_1 = \|J u_0\|_{Q_1^{\frac{1}{2}} U_1},$$

i.e. $J : U_0 \to Q_1^{\frac{1}{2}} U_1$ is an isometry.

Proof. **Step 1:** We prove that $W(t)$, $t \in [0,T]$, defined in (2.5.1) is a Q_1-Wiener process in U_1.
If we set $\xi_j(t) := \beta_j(t) J(e_j)$, $j \in \mathbb{N}$, we obtain that $\xi_j(t)$, $t \in [0,T]$, is a continuous U_1-valued martingale with respect to

$$\mathcal{G}_t := \sigma\Big(\bigcup_{j \in \mathbb{N}} \sigma(\beta_j(s) | s \leqslant t)\Big),$$

$t \in [0,T]$, since

$$E(\beta_j(t) \mid \mathcal{G}_s) = E(\beta_j(t) \mid \sigma(\beta_j(u) | u \leqslant s)) = \beta_j(s) \quad \text{for all } 0 \leqslant s < t \leqslant T$$

as $\sigma\big(\sigma(\beta_j(u)|u \leqslant s) \cup \sigma(\beta_j(t))\big)$ is independent of

$$\sigma\Big(\bigcup_{\substack{k \in \mathbb{N} \\ k \neq j}} \sigma(\beta_k(u) | u \leqslant s)\Big).$$

Then it is clear that

$$W_n(t) := \sum_{j=1}^{n} \beta_j(t) J(e_j), \quad t \in [0,T],$$

is also a continuous U_1-valued martingale with respect to $\mathcal{G}_t$, $t \in [0,T]$. In addition, we obtain that

$$\begin{aligned} E\left(\sup_{t \in [0,T]} \|\sum_{j=n}^{m} \beta_j(t) J(e_j)\|_1^2\right) &\leqslant 4 \sup_{t \in [0,T]} E\left(\|\sum_{j=n}^{m} \beta_j(t) J(e_j)\|_1^2\right) \\ &= 4T \sum_{j=n}^{m} \|J(e_j)\|_1^2, \quad m \geqslant n \geqslant 1. \end{aligned}$$

Note that $\|J\|^2_{L_2(U_0,U_1)} = \sum_{j\in\mathbb{N}} \|J(e_j)\|_1^2 < \infty$. Therefore, we get the convergence of $W_n(t)$, $t \in [0,T]$, in $\mathcal{M}_T^2(U_1)$, hence the limit $W(t)$, $t \in [0,T]$, is P-a.s. continuous.

Now we want to show that $P \circ (W(t) - W(s))^{-1} = N(0, (t-s)JJ^*)$. Analogously to the second part of the proof of Proposition 2.1.6 we get that $\langle W(t) - W(s), u_1\rangle_1$ is normally distributed for all $0 \leqslant s < t \leqslant T$ and $u_1 \in U_1$. It is easy to see that the mean is equal to zero and concerning the covariance of $\langle W(t) - W(s), u_1\rangle_1$ and $\langle W(t) - W(s), v_1\rangle_1$, $u_1, v_1 \in U_1$, we obtain that

$$
\begin{aligned}
&E(\langle W(t) - W(s), u_1\rangle_1 \langle W(t) - W(s), v_1\rangle_1)\\
&= \sum_{k\in\mathbb{N}} (t-s)\langle Je_k, u_1\rangle_1 \langle Je_k, v_1\rangle_1\\
&= (t-s)\sum_{k\in\mathbb{N}} \langle e_k, J^*u_1\rangle_0 \langle e_k, J^*v_1\rangle_0\\
&= (t-s)\langle J^*u_1, J^*v_1\rangle_0 = (t-s)\langle JJ^*u_1, v_1\rangle_1.
\end{aligned}
$$

Thus, it only remains to show that the increments of $W(t)$, $t \in [0,T]$, are independent but this can be done in the same way as in the proof of Proposition 2.1.10.

Step 2: We prove that $\operatorname{Im} Q_1^{\frac{1}{2}} = J(U_0)$ and that $\|u_0\|_0 = \|Q_1^{-\frac{1}{2}} Ju_0\|_1$ for all $u_0 \in U_0$.

Since $Q_1 = JJ^*$, by Corollary C.0.6 we obtain that $Q_1^{\frac{1}{2}}(U_1) = J(U_0)$ and that $\|Q_1^{-\frac{1}{2}} u_1\|_1 = \|J^{-1}u_1\|_0$ for all $u_1 \in J(U_0)$. We now replace u_1 by $J(u_0)$, $u_0 \in U_0$, to get the last assertion, because $J : U_0 \to U_1$ is one-to-one. $\square$

2.5.2. The definition of the stochastic integral for cylindrical Wiener processes

We fix $Q \in L(U)$ nonnegative, symmetric but not necessarily of finite trace. After the preparations of the previous section we are now able to define the stochastic integral with respect to a cylindrical Q-Wiener process $W(t)$, $t \in [0,T]$.

Basically we integrate with respect to the standard U_1-valued Q_1-Wiener process given by Proposition 2.5.2. In this sense we first get that a process $\Phi(t)$, $t \in [0,T]$, is integrable with respect to $W(t)$, $t \in [0,T]$, if it takes values in $L_2(Q_1^{\frac{1}{2}}(U_1), H)$, is predictable and if

$$
P\left(\int_0^T \|\Phi(s)\|^2_{L_2(Q_1^{\frac{1}{2}}(U_1),H)}\, ds < \infty\right) = 1.
$$

But in addition, we have by Proposition 2.5.2 that $Q_1^{\frac{1}{2}}(U_1) = J(U_0)$ and that

$$\langle Ju_0, Jv_0 \rangle_{Q_1^{\frac{1}{2}}(U_1)} = \langle Q_1^{-\frac{1}{2}} Ju_0, Q_1^{-\frac{1}{2}} Jv_0 \rangle_1 = \langle u_0, v_0 \rangle_0$$

for all $u_0, v_0 \in U_0$ (by polarization). In particular, it follows that Je_k, $k \in \mathbb{N}$, is an orthonormal basis of $Q_1^{\frac{1}{2}}(U_1)$. Hence we get that

$$\Phi \in L_2^0 = L_2(Q^{\frac{1}{2}}(U), H) \Longleftrightarrow \Phi \circ J^{-1} \in L_2(Q_1^{\frac{1}{2}}(U_1), H)$$

since

$$\begin{aligned} \|\Phi\|_{L_2^0}^2 &= \sum_{k \in \mathbb{N}} \langle \Phi e_k, \Phi e_k \rangle \\ &= \sum_{k \in \mathbb{N}} \langle \Phi \circ J^{-1}(Je_k), \Phi \circ J^{-1}(Je_k) \rangle = \|\Phi \circ J^{-1}\|^2_{L_2(Q_1^{\frac{1}{2}}(U_1), H)} \end{aligned}$$

Now we define

$$\int_0^t \Phi(s) \ \mathrm{d}W(s) := \int_0^t \Phi(s) \circ J^{-1} \ \mathrm{d}W(s), \quad t \in [0, T]. \tag{2.5.2}$$

Then the class of all integrable processes is given by

$$\mathcal{N}_W = \Big\{ \Phi : \Omega_T \to L_2^0 \mid \Phi \text{ predictable and } P\Big(\int_0^T \|\Phi(s)\|_{L_2^0}^2 \, ds < \infty \Big) = 1 \Big\}$$

as in the case where $W(t)$, $t \in [0, T]$, is a standard Q-Wiener process in U.

Remark 2.5.3.

1. *We note that the stochastic integral defined in* (2.5.2) *is independent of the choice of* $(U_1, \langle\ ,\ \rangle_1)$ *and* J. *This follows by construction, since by* (2.5.1) *for elementary processes* (2.5.2) *does not depend on* J.

2. *If* $Q \in L(U)$ *is nonnegative, symmetric and with finite trace the standard* Q*-Wiener process can also be considered as a cylindrical* Q*-Wiener process by setting* $J = I : U_0 \to U$ *where* I *is the identity map. In this case both definitions of the stochastic integral coincide.*

Finally, we note that since the stochastic integrals in this chapter all have a standard Wiener process as integrator, we can drop the predictability assumption on $\Phi \in \mathcal{N}_W$ and just assume progressive measurability, i.e. $\Phi|_{[0,t]\times\Omega}$ is $\mathcal{B}([0,t]) \otimes \mathcal{F}_t / \mathcal{B}(L_2^0)$-measurable for all $t \in [0, T]$, at least if $(\Omega, \mathcal{F}, P)$ is complete (otherwise we consider its completion) (cf. [WW90, Theorem 6.3.1]).

We used the above framework so that it easily extends to more general Hilbert-space-valued martingales as integrators replacing the standard Wiener process. Details are left to the reader.